On Risk Aversion and Bargaining Outcomes*

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Proposed Running Head: Risk Aversion and Bargaining

Abstract

We revisit the well-known result that asserts that an increase in the degree of one's risk aversion improves the position of one's opponents. To this end, we apply Yaari's dual theory of choice under risk both to Nash's bargaining problem and to Rubinstein's game of alternating offers. Under this theory, unlike under expected utility, risk aversion influences the bargaining outcome only when this outcome is random, namely, when the players are risk-lovers. In this case, an increase in one's degree of risk aversion increases one's share of the pie. *Journal of Economic Literature* Classification Numbers: C70, C78.

1 Introduction

One of the results most frequently quoted in the bargaining literature asserts that increasing risk aversion reduces a player's share in the bargaining outcome and increases that of his opponent. This result[1] has appeared in different variations including both the cooperative and the non-cooperative frameworks. (See Kannai (1977), Khilstrom, Roth, and Schmeidler (1981), Sobel (1981), Thomson (1988), Osborne (1984), Roth (1985), and Roth (1989).)

This result has gained popularity partly because it seems very intuitive. The assumptions under which the result holds, however, are somewhat restrictive. For instance, Roth and Rothblum (1982) (see also Safra, Zhou, and Zilcha (1990)) show that if the set of feasible agreements includes lotteries, then an increase in one's opponent's risk aversion might be disadvantageous. Safra and Zilcha (1993) show that if the agents' risk-preferences belong to a wide family of non-expected utility preferences, then the effect of changes in the degree of risk aversion on the bargaining outcome is not conclusive. In particular, they show an example where an agent is hurt by an increase of his opponent's risk aversion.

Even under the assumptions where the result holds, there are some problems with its interpretation. Risk aversion affects the bargaining outcome in cases (like the framework of the Nash bargaining problem) where the underlying outcomes involve no lotteries and thus no risk at all. Indeed, Roth (1985) uses the term "strategic risk" when referring to these cases. He interprets this risk as arising from each player's ignorance of his opponent's actions, thus resulting in a subjective probability that no agreement will be reached during the negotiations that are underway. But these probabilities are outside the model and their effect on the bargaining outcome is therefore dubious.

When the underlying outcomes include lotteries, there is still a problem with the interpretation of the result. Consider a situation where two risk-averse expected utility maximizers bargain over one perfectly divisible dollar. Even when the agents can agree on a random division of the money, the Nash bargaining solution selects a non-random division. Since the outcome in this case is not random, one would expect that changes in the degree of the agents' risk aversion did not affect the outcome. As is well known, however, the Nash bargaining solution awards increasing shares of the dollar to the agent whose opponent becomes more risk-averse. The result is even more striking when we consider the case of two risk-loving agents who bargain over a dollar. In this case, the Nash bargaining solution selects the random outcome that assigns the whole dollar to each of the agents with probability 1/2, *independently of their degree of risk loving.* In this case,

[1]Henceforth we will refer to the result mentioned above as "the result."

even though the outcome is a non-degenerate lottery, changes in the degree of risk loving have no effect whatsoever on the outcome.

Instead of considering, as Safra and Zilcha (1993) do, a large class of non-expected utility preferences, this paper restricts attention to one particular model of choice under risk. Specifically, we analyze the comparative statics of changes in risk aversion when agents' preferences follow the dual theory of choice under risk proposed by Yaari (1987). Under this theory, the objects of choice are lotteries over monetary outcomes. The term *dual* refers to the feature that when this theory is compared to the expected utility theory, the roles of monetary outcomes and probabilities are reversed. In particular, while the expected utility functional is linear in probabilities and not necessarily linear in monetary outcomes, the dual theory utility functional is linear in monetary outcomes and not necessarily linear in probabilities.

By restricting attention to the dual theory we are able to obtain some positive results, which can be compared to the predictions of the model with expected utility maximizers. Further, these results do not suffer from the interpretational difficulties that we mentioned above. Another reason for concentrating on the dual theory is that it proved to be useful in economic applications (see for example Demers and Demers (1990), Hadar and Seo (1995) and Volij (1999)). We believe that the prominence of expected utility theory is derived not only from the fact that it is supported by an elegant axiomatic characterization (a property that is shared by the dual theory) but also from its usefulness in economic applications.

Risk aversion is usually defined as aversion to mean-preserving spreads. Under expected utility maximization, this is equivalent to the concavity of the von Neumann–Morgenstern utility function. Under the dual theory, risk aversion is equivalent to the convexity of the "dual function" with which probabilities are evaluated. As a result, the concept of risk aversion is not entangled with the agent's attitude towards wealth. In particular, preferences over sure outcomes give no information about the agent's degree of risk aversion. One immediate consequence is that if the underlying set of allocations involves no lotteries, the bargaining solution will be invariant to changes in risk attitude. As we pointed out earlier, this is not the case under the expected utility framework.

If the dual theory resulted in essentially the same predictions concerning the relation between risk aversion and bargaining outcomes, there would be little point in carrying out this exercise. The same would be true if the predictions were ambiguous. The interesting message, however, is that whenever changes in risk aversion affect the bargaining outcome according to the dual theory, they affect it in precisely the opposite direction compared with expected utility theory. Namely, more risk-loving reduces a player's payoff and increases that of his opponent. Our analysis includes both Nash's (cooperative) framework and Rubinstein's non-cooperative framework. In both cases, the set of underlying bargaining outcomes is defined in the standard way, i.e., as consisting of all allocations of a divisible physical unit (money) plus lotteries on such allocations.

The paper is organized as follows. Section 2 gives a brief review of Yaari's dual theory. After presenting a simple bargaining situation in Section 3, Section 4 formulates the corresponding Nash bargaining problem in terms of this theory. Our first result argues that if both players are risk-averse, then changes in risk aversion do not affect the Nash bargaining allocation. Then we take up the case of risk-loving players. We show that if both players are risk-loving, players who become more risk-loving are worse off while their opponents are better off. Section 5 considers the strategic framework where two impatient players play a game of alternating offers. The outcome of this game is consistent with the results of the cooperative approach. Finally, Section 6 shows that another commonly used strategic model, one where the players are not impatient but face a probability of a negotiations breakdown, yields very different results. Moreover, its predictions are not consistent with the outcomes of the cooperative approach. One should note that, in contrast, the two strategic models are strategically equivalent if the players satisfy the expected utility axioms.

2 A Short Review of the Dual Theory of Choice under Risk

Let r be a random variable that takes values in the unit interval. Denote by G_r its *decumulative distribution function* (DDF), which is defined by

$$G_r(x) = \Pr\{r > x\}, \qquad 0 \leq x \leq 1.$$

It is known that G_r is non-increasing, right-continuous and satisfies $G_r(1) = 0$. The random variable r represents a lottery over monetary outcomes. Two random variables, r and s, are comonotonic if for every pair of states, ω and ω', $(r(\omega) - r(\omega'))\,(s(\omega) - s(\omega')) \geq 0$. In words, r and s are comonotonic if, when going from state ω to ω' in Ω, both random variables move (weakly) in the same direction.

The primitive of the dual theory is the set Δ of all non-increasing, right-continuous functions $G : [0,1] \to [0,1]$ that satisfy $G(1) = 0$. This set is interpreted as the set of all DDF's associated with some random variable defined on some sufficiently rich probability space Ω and taking values in $[0,1]$.

Let $\succeq$ be a complete preference relation on Δ. Yaari (1987) imposes the following axioms on

$\succeq$:

1. continuity (with respect to L_1-convergence),
2. monotonicity: if $G_r \geq G_s$ then $G_r \succeq G_s$, and
3. dual independence: if r, s and t are pairwise comonotonic and $G_r \succeq G_s$, then $G_{\alpha r+(1-\alpha)t} \succeq G_{\alpha s+(1-\alpha)t}$.

Continuity is a technical requirement. Monotonicity requires that if G_r stochastically dominates G_s then $G_r \succeq G_s$. The dual independence axiom is where the dual theory departs from the traditional expected utility theory. It deals with portfolios of comonotonic random variables. Dual independence requires that whenever r, s, and t are pairwise comonotonic and $G_r \succeq G_s$, then any portfolio containing a proportion α of r and $1-\alpha$ of t should be weakly preferred to a portfolio containing α of s and $1-\alpha$ of t.

Yaari (1987) uses the above axioms in the following representation theorem.

Theorem 1 A complete preference relation $\succeq$ satisfies continuity, monotonicity, and dual independence if and only if there exists a continuous and non-decreasing real function g, defined on the unit interval, such that for all G_r and G_s belonging to Δ,

$$G_r \succeq G_s \Leftrightarrow \int_0^1 g(G_r(t))\,dt \geq \int_0^1 g(G_s(t))\,dt.$$

Moreover, the function g is unique up to a positive affine transformation.

The function g is analogous to the von Neumann–Morgenstern utility function and, in a sense, we can say that g "represents" the agent's preferences. Graphically, an agent whose preferences are described by the dual theory evaluates random variables according to the area under a suitable transformation (the function g) of their DDF.

Given a function g that represents an agent's preference, for any random variable r, let

$$U(r) = \int_0^1 g(G_r(x))\,dx.$$

Theorem 1 says that agents will choose among random variables so as to maximize U. Further, Yaari (1987) also shows that U is "linear" in payments. Formally, let $a > 0$ and b be two real numbers and let v and $av+b$ be two random variables taking values in the unit interval. Then $U(av+b) = aU(v) + b$.

One of the appealing features of the dual theory is that, unlike expected utility theory, the agent's attitude towards risk is not entangled with his attitude towards wealth. More specifically, under the dual theory, marginal utility of wealth is constant and this feature is consistent with any attitude towards risk. In particular, Yaari (1987) shows that a preference relation $\succeq$ that satisfies the dual theory's axioms exhibits risk aversion if and only if the function g that represents $\succeq$ is convex. In this paper, we will be interested in the effect of changes in the degree of an agent's risk aversion on bargaining outcomes. For this purpose, it is necessary to understand what it means for one agent to be *more risk-averse than* another. Following Yaari (1986), since risk aversion is characterized by the convexity of the function g, it would be natural to define an agent to be more risk-averse than another if and only if the former's g is more convex than the latter's.

Definition 1 Let $\succeq_1$ and $\succeq_2$ be two preference relations that satisfy the dual theory's axioms and that are represented by the functions g_1 and g_2, respectively. We say that $\succeq_1$ is more risk averse than $\succeq_2$ if and only if there exists a convex function h, defined on the unit interval, such that $g_1 = h \circ g_2$.

For $p \in [0,1]$ let $\langle 1, p \rangle$ denote the lottery that assigns 1 with probability p and 0 with probability $1-p$. The above definition is equivalent to requiring that for every random variable r, if $\langle 1, p_1 \rangle \sim_1 r$ and $\langle 1, p_2 \rangle \sim_2 r$, then $p_1 \geq p_2$. In words, for every lottery r, 1's "probability equivalent" is at least as high as 2's "probability equivalent." Under the dual theory, this is stronger than requiring that for all lotteries, 1's certainty equivalent be lower than 2's certainty equivalent. Under expected utility, however, this is a weaker requirement.[2]

The characterization of the risk attitude by means of the convexity of a univariate non-decreasing function makes it relatively easy to analyze the effect of risk aversion on the outcome of bargaining situations. This is done in the following sections.

3 The Bargaining Situation

Two individuals bargain over one unit of a single commodity (money). Assume that both bargainers' preferences over risky prospects satisfy the axioms of the dual theory of choice under risk and are represented by the functions f and g, respectively. Without loss of generality f and g are chosen so that $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$.

[2]The interested reader should consult Yaari (1986) for various and equivalent interpretations of Definition 1.

Any non-negative division of the single commodity is feasible if both agents agree on it. Otherwise they get 0. As a result, the set of physical outcomes is

$$X = \{(x_1, x_2) : x_1 + x_2 \le 1,\ x_1, x_2 \ge 0\},$$

and the disagreement physical outcome is $D = (0, 0)$.

We shall assume that the bargainers are not constrained to agree on certain (as opposed to random) outcomes, but that they can agree upon any lottery over elements of X. Let $(\Omega, \mathcal{B}, \lambda)$ be the probability space that consists of the unit square, its Borel sets and the Lebesgue measure. A *lottery* is an random variable $\ell : \Omega \to X$ that assigns one physical outcome to each element $\omega \in \Omega$. Note that we can think of a lottery ℓ as two random variables (ℓ_1, ℓ_2) such that $\ell_i : \Omega \to [0, 1]$ and $\ell_1(\omega) + \ell_2(\omega) \le 1$ for all $\omega \in \Omega$. These are the random variables of each of the agent's payoffs. We shall denote by F_ℓ the decumulative distribution function of the first agent's payoffs and by G_ℓ the DDF of the second agent's payoffs. Formally, $F_\ell(t) = \lambda(\{\omega \in \Omega : \ell_1(\omega) > t\})$ and $G_\ell(t) = \lambda(\{\omega \in \Omega : \ell_2(\omega) > t\})$. The set of lotteries will be denoted by $\mathcal{L}$. Consequently, the utility levels that the agents get from lottery ℓ are given, respectively, by

$$U_1(\ell) = \int_0^1 f[F_\ell(t)]\, dt \qquad \text{and} \qquad U_2(\ell) = \int_0^1 g[G_\ell(t)]\, dt. \tag{1}$$

A lottery $\ell \in \mathcal{L}$ is *efficient* if there is no other lottery $\ell' \in \mathcal{L}$ that is weakly preferred to ℓ by both agents and strictly preferred to ℓ by at least one agent. For the record, note that agent 1's and 2's expected monetary payoffs associated with a given lottery, ℓ, are given by

$$\int_0^1 F_\ell(t)\, dt \quad \text{and} \quad \int_0^1 G_\ell(t)\, dt,$$

respectively. Moreover, if ℓ is an efficient lottery, then $\int_0^1 F_\ell(t)\, dt + \int_0^1 G_\ell(t)\, dt = 1$.

4 The Cooperative Approach

In this section, we analyze the effect of changes in the agents' degree of risk aversion on the outcome of a simple bargaining problem, when the agents' preferences satisfy the axioms of the dual theory of choice under risk. Recall that Nash (1950) defined a *bargaining problem* to be a pair $\langle S, d \rangle$ where S is a compact convex subset of $\mathbb{R}^2$ such that $d \in S$ and there is a point $s \in S$ such that $s \gg d$. The set S is the set of feasible utility pairs and d is the utility pair that coresponds to the disagreement outcome. In our case the bargaining problem is given by the pair

$\langle S, d\rangle$ where

$$S = \{(U_1(\ell), U_2(\ell)) : \ell \in \mathcal{L}\}, \text{ and } d = (U_1(0,0), U_2(0,0)) = (0,0).$$

It follows from the linearity and monotonicity of preferences in payments that the set S is comprehensive, that is, if $s \in S$ and $0 \leq s' \leq s$ then $s' \in S$. Also, since $U_i(\ell) \in [0,1]$ for $i = 1,2$, S is bounded. Using Helly's selection theorem and the fact that f and g are continuous functions, it can be shown that S is closed as well. Further, given two lotteries $\ell^1 = (\ell_1^1, \ell_2^1)$ and $\ell^2 = (\ell_1^2, \ell_2^2)$, we can find two lotteries $\hat{\ell}^1 = (\hat{\ell}_1^1, \hat{\ell}_2^1)$ and $\hat{\ell}^2 = (\hat{\ell}_1^2, \hat{\ell}_2^2)$ such that $\hat{\ell}_i^1$ and $\hat{\ell}_i^2$ are comonotonic, for $i = 1,2$ and such that $F_{\ell^j} = F_{\hat{\ell}^j}$ and $G_{\ell^j} = G_{\hat{\ell}^j}$, for $j = 1,2$. Therefore, $U_i(\hat{\ell}^j) = U_i(\ell^j)$, for $i,j = 1,2$. Consider the lottery $\ell^* = \alpha\hat{\ell}^1 + (1-\alpha)\hat{\ell}^2$. Since $\hat{\ell}^1$ and $\hat{\ell}^2$ are comonotonic, we have $U_i(\ell^*) = \alpha U_i(\hat{\ell}^1) + (1-\alpha)U_i(\hat{\ell}^2)$. This shows that S is convex. Therefore, $\langle S, d\rangle$ is indeed a bargaining problem.

A *bargaining solution* is a function that takes a bargaining problem $\langle S, d\rangle$ as an input and returns a point in S. The *Nash bargaining solution* is the function $\mathcal{N}$ that returns the point $s^* = (s_1^*, s_2^*) \in S$ that satisfies $s^* \geqq d$ and $(s_1^* - d_1)(s_2^* - d_2) \geq (s_1 - d_1)(s_2 - d_2)$ for all $s = (s_1, s_2) \in S$ with $s \geqq d$. Nash (1950) shows that $\mathcal{N}$ is the only bargaining solution that satisfies the properties of Pareto optimality, symmetry, independence of irrelevant alternatives, and invariance with respect to positive affine transformations. Pareto optimality requires that no feasible agreement is preferred by both agents to the selected agreement. Symmetry dictates that the selected outcome yield the same utility level for both agents whenever the problem is symmetric, namely whenever one cannot distinguish one agent from the other by just looking at the problem $\langle S, d\rangle$. Independence of irrelevant alternatives requires that the selected outcome not change if the feasible alternatives are reduced to a smaller set that still contains the selected outcome. Lastly, invariance requires that if $\langle S', d'\rangle$ is obtained from $\langle S, d\rangle$ by means of the transformations $s_i \to \alpha_i s_i + \beta_i$, for $i = 1,2$, where $\alpha_i > 0$ and $\beta_i \in \mathbb{R}$, then $s_i^{*\prime} = \alpha_i s_i^* + \beta_i$, for $i = 1,2$, where $s^{*\prime}$ and s^* are the points selected for $\langle S', d'\rangle$ and $\langle S, d\rangle$, respectively. This axiom is motivated by the idea that the selected outcome should depend only on the preferences of the players and not on their utility representations. Recall that two von Neumann-Morgenstern utility functions represent the same expected utility preferences if and only if one is a positive affine transformation of the other. Therefore $\langle S, d\rangle$ and $\langle S', d'\rangle$ can be interpreted as the images of the feasible agreements and disagreement outcomes, under two pairs (one for each agent) of equivalent utility representations. Note, however, that the invariance axiom dictates that $s_i^{*\prime} = \alpha_i s_i^* + \beta_i$, for $i = 1,2$, even if the transition from $\langle S, d\rangle$ to $\langle S', d'\rangle$ is not due to a change in the utility representations but to a more significant change in the feasible agreements. We shall elaborate on this soon after we give the motivation of the invariance axiom under the dual preferences.

The axioms of Pareto optimality, symmetry and independence of irrelevant alternatives are uncontroversial, in the sense that if they are reasonable when agents' risk preferences satisfy the von Neumann-Morgenstern axioms, they are also reasonable when agents satisfy Yaari's axioms. For instance, if it is reasonable that a bargaining solution should not select inefficient outcomes in general, the same requirement remains reasonable when the agents' risk preferences satisfy the dual theory axioms. The invariance axiom is the only one that requires some justification under the dual theory. Recall that according to Yaari's representation theorem, the agents' dual functions f and g are unique up to positive affine transformations. Further, if $f' = a_1 f + b_1$ and $g' = a_2 g + b_2$ where a_1 and a_2 are positive reals and, b_1 and b_2 are arbitrary numbers, then

$$U_1'(\ell) = \int_0^1 f'(F_\ell(t))\,dt = a_1 \int_0^1 f(F_\ell(t))\,dt + b_1 = a_1 U_1(\ell) + b_1$$

and

$$U_2'(\ell) = \int_0^1 g'(G_\ell(t))\,dt = a_2 \int_0^1 g(G_\ell(t))\,dt + b_2 = a_2 U_2(\ell) + b_2.$$

As a result, the bargaining problem $\langle S', d' \rangle$ where

$$S' = \{(U_1'(\ell), U_2'(\ell)) : \ell \in \mathcal{L}\}, \text{ and } d' = (U_1'(0,0), U_2'(0,0))$$

is obtained from $\langle S, d \rangle$ where

$$S = \{(U_1(\ell), U_2(\ell)) : \ell \in \mathcal{L}\}, \text{ and } d = (U_1(0,0), U_2(0,0))$$

by means of the transformations $s \to a_i s + b_i$ for $i = 1, 2$. In other words, $\langle S, d \rangle$ and $\langle S', d' \rangle$ can be interpreted as the images of the feasible agreements and disagreement outcomes, under two pairs (one for each agent) of equivalent cardinal representations. Therefore, if we would like a bargaining solution to depend on the agents' risk preferences but not on their respective cardinal representations, the bargaining solution should satisfy the axiom of invariance to positive affine transformations.

We said above that the invariance axiom dictates that $s_i^{*\prime} = \alpha_i s_i^* + \beta_i$, for $i = 1, 2$, even if the transition from $\langle S, d \rangle$ to $\langle S', d' \rangle$ is not due to a change in the utility representations but to a more significant change in the feasible agreements.[3] To be specific, consider a situation in which two agents bargain over 50 (perfectly divisible) poker "chips". Assume that each chip awards its owner \$0.01, and that both agents' preferences satisfy the dual theory axioms. In case of disagreement, both agents get no chips. This situation induces a bargaining problem $\langle S, d\rangle$. Consider now a similar situation where both agents bargain over the same 50 chips but now, agent 2 gets \$0.02 for each chip while agent 1 still gets \$0.01 per chip. Given that the dual preferences are linear in money, the new situation induces a bargaining problem $\langle S', d'\rangle$ where $S' = \{(s_1', s_2') : (s_1', s_2') = (s_1, 2s_2), \text{ for some } (s_1, s_2) \in S\}$ and $(d_1', d_2') = (d_1, 2d_2)$. Since the bargaining problem $\langle S', d'\rangle$ is obtained from $\langle S, d\rangle$ by means of a positive affine transformation, any bargaining solution that satisfies the invariance axiom selects the same division of chips in both situations. One could argue that this result is reasonable because preferences are linear in money prices. The motivation of the invariance axiom, however, has nothing to do with the above change in the bargaining situation because in both situations the same cardinal representations of the agents' preferences were used. In any case, the invariance axiom implies that in both situations the division of the chips be the same.

This feature of the invariance axiom, of having stronger implications than just the ones suggested by its motivation, is not related to the fact that in the above example agents have dual preferences. One can easily build a dual example that shows that the invariance axiom has analogous consequences under expected utility. To see this, assume again that 2 agents bargain over the same 50 chips. Now, as opposed to the previous situation, both agents are expected utility maximizers. Further, each chip awards its owner not \$0.01 but a 1% chance of winning \$1. That is, if an agent gets x chips, he is awarded a lottery that awards \$1 with probability $x/100$ and \$0 with the complementary probability. Normalizing utility so that $u_i(0) = 0$ and $u_i(1) = 1$, for $i = 1, 2$, we get that the set of feasible utilities is $S = \{(s_1, s_2) : s_1 + s_2 \leq 0.50\}$ and that the disagreement point is $d = (0, 0)$. Now, consider a similar situation where agent 2 gets a 2% chance of winning the dollar for each chip he gets while agent 1 gets 1% as before. The corresponding utilities possibilities set is $S' = \{(s_1', s_2') : (s_1', s_2') = (s_1, 2s_2), \text{ for some } (s_1, s_2) \in S\}$ and again, any bargaining solution that satisfies invariance should select the same division of the chips in both situations. One could argue that this result is reasonable because preferences are linear in probabilities. The above change in the bargaining situation, however, has nothing to do with the motivation of the axiom because there has been no change in the cardinal representations of the agents' preferences.

[3]We thank an anonymous referee for suggesting this discussion.

Summarizing, the axiomatic foundation of the Nash bargaining solution is as justified in the context of the dual theory as it is in the context of expected utility. Therefore, we can now turn to the analysis of changes in risk aversion on the bargaining outcomes.

The case of risk-averse agents

When the agents are risk-averse, both f and g are convex functions. As a result, we have the following:

Proposition 1 Assume that both agents are risk-averse. Then $S = X$. Further, if in addition one agent is strictly risk-averse, then a lottery is efficient only if it assigns probability one to a certain outcome.

Proof : By definition $X \subseteq S$. We shall show that $S \subseteq X$ as well. Let $(s_1, s_2) \in S$. Then there is a lottery $\ell \in \mathcal{L}$ such that

$$s_1 = \int_0^1 f(F_\ell(t))\,dt \quad \text{and} \quad s_2 = \int_0^1 g(G_\ell(t))\,dt.$$

Since $f \geq 0$ and $g \geq 0$, we have that $(s_1, s_2) \geq (0, 0)$. Let

$$s_1' = \int_0^1 F_\ell(t)\,dt \quad \text{and} \quad s_2' = \int_0^1 G_\ell(t)\,dt.$$

Namely, s_1' and s_2' are the expected monetary payoffs that lottery ℓ awards to agent 1 and agent 2, respectively. Therefore, $(s_1', s_2') \in X$. But since both agents are risk-averse, f and g are convex functions, which implies that $s_1 \leq s_1'$ and $s_2 \leq s_2'$. Since X is comprehensive, $(s_1, s_2) \in X$. This proves the first part of the claim.

Assume now that ℓ is an efficient lottery. Then, since $S = X$ we have

$$1 = s_1' + s_2' \geq s_1 + s_2 = 1.$$

Consequently, as $s_1 \leq s_1'$ and $s_2 \leq s_2'$, we have

$$s_1 = \int_0^1 f(F_\ell(t))\,dt = \int_0^1 F_\ell(t)\,dt = s_1' \tag{2}$$

and

$$s_2 = \int_0^1 g(G_\ell(t))\,dt = \int_0^1 G_\ell(t)\,dt = s_2'.$$

Assume now that one agent is strictly risk-averse. Without loss of generality, assume that it is agent 1. Then it follows from equation (2) that $f(F_\ell(t)) = F_\ell(t)$ for all $t \in [0, 1]$, which in turn implies that $F_\ell(t)$ is either 0 or 1, that is, ℓ is a degenerate lottery. $\square$

As an immediate corollary of Proposition 1, we obtain the following:

Theorem 2 Let $\langle S, d\rangle$ be the bargaining problem induced by two risk-averse agents. The Nash bargaining solution awards a utility level of 1/2 to each agent. In particular, changes in the degree of risk aversion do not affect the outcome.

Note that when both players are risk-averse, the only agreement that yields the utility pair $(1/2, 1/2)$ is equal division with certainty. Since the outcome recommended by the Nash bargaining solution does not involve randomness, it is perfectly reasonable that the degree of agents' risk aversion has no influence. An increase in the agents' risk aversion means that lotteries are less attractive. But since, by Proposition 1, when agents are risk-averse every utility pair that can be obtained by means of a lottery can also be obtained by means of a certain outcome, changes in risk aversion cannot have any effect on the final agreement.

Theorem 2 should be compared with Theorem 1 in Khilstrom, Roth, and Schmeidler (1981) where it is shown that when the two agents are risk-averse expected utility maximizers, an increase in the degree of risk aversion of one agent is beneficial to the opponent. This is true even though the utility pair singled out by the Nash bargaining solution always corresponds to a certain outcome.

The case of risk-loving agents

As soon as one of the agents is not risk-averse, efficient outcomes may involve nondegenerate lotteries. In this case, an increase in the degree of risk loving (a decrease in the degree of risk aversion) of one agent may affect the outcome selected by the Nash bargaining solution, as the following theorem shows.

Theorem 3 Let $\langle S, d\rangle$ be a bargaining problem and let ℓ^* be a lottery that attains the utility levels determined by the Nash bargaining solution. Assume further that ℓ^* assigns probability p to the outcome $(1, 0)$ and probability $(1 - p)$ to the outcome $(0, 1)$. The utility which the Nash bargaining solution assigns to a player does not decrease as his opponent becomes more risk-loving. That is $\mathcal{N}_2(\widetilde{S}, d) \geq \mathcal{N}_2(S, d)$ where $\langle \widetilde{S}, d \rangle$ is obtained from $\langle S, d \rangle$ by replacing agent 1 with a more risk-loving agent.

Proof : Since agent 1 has become more risk-loving, there is a concave and non-decreasing function h, defined on the unit interval, such that agent 1's new preferences are represented by the function $h \circ f$ and, therefore, his new utility function is given by $V_1(\ell) = \int_0^1 h(f[F_\ell(t)])\, dt$. Further, since ℓ^* is a lottery that assigns probability p to the outcome $(1,0)$ and probability $(1-p)$ to the outcome $(0,1)$, we have that $U_1(\ell^*) = f(p)$ and

$$V_1(\ell^*) = h[f(p)] = h[U_1(\ell^*)]. \tag{3}$$

Now let $\tilde{\ell}$ be a lottery such that $U_2(\tilde{\ell}) < U_2(\ell^*)$. We need to show that $\tilde{\ell}$ cannot attain the utility levels selected by the Nash bargaining solution for the bargaining problem $\langle \widetilde{S}, d \rangle$. By the way ℓ^* was selected, we have $U_1(\ell^*)U_2(\ell^*) > U_1(\tilde{\ell})U_2(\tilde{\ell})$. Therefore,

$$U_1(\ell^*) > U_1(\tilde{\ell})\frac{U_2(\tilde{\ell})}{U_2(\ell^*)}.$$

Since h is non-decreasing,

$$h[U_1(\ell^*)] \geq h[U_1(\tilde{\ell})\frac{U_2(\tilde{\ell})}{U_2(\ell^*)}].$$

By the concavity of h,

$$h[U_1(\ell^*)] \geq h[U_1(\tilde{\ell})]\frac{U_2(\tilde{\ell})}{U_2(\ell^*)}.$$

By Jensen's inequality,

$$h[U_1(\ell^*)] \geq V_1(\tilde{\ell})\frac{U_2(\tilde{\ell})}{U_2(\ell^*)},$$

which, together with (3) implies that $V_1(\ell^*)U_2(\ell^*) \geq V_1(\tilde{\ell})U_2(\tilde{\ell})$. Since $(V_1(\ell^*), U_2(\ell^*)) \neq (V_1(\tilde{\ell}), U_2(\tilde{\ell}))$ we conclude that $(V_1(\tilde{\ell}), U_2(\tilde{\ell})) \neq \mathcal{N}(\widetilde{S}, d)$. $\square$

Note that the above theorem does not assume anything about the agents' risk-loving except for the fact that it should be enough to induce the Nash bargaining solution to select a non-degenerate lottery whose prices are only the two extreme outcomes. The next result shows that a sufficient condition for this kind of lottery to be selected is that both agents be risk-loving. Specifically, it is shown that the utility possibilities frontier corresponds to lotteries that assign some probability to the outcome $(1,0) \in X$ and the remaining probability to the outcome $(0,1) \in X$.

Proposition 2 Assume both agents are risk-lovers and let ℓ be an efficient lottery. Then there exists $p \in [0,1]$ such that a lottery, denoted by ℓ^*, that assigns probability p to the outcome $(1,0)$ and probability $(1-p)$ to the outcome $(0,1)$ is utility-equivalent to ℓ. If in addition one of the agents is strictly risk-loving, then $\ell = \ell^*$.

Proof : Let ℓ be an efficient lottery. Therefore, we have

$$\int_0^1 F_\ell(t)\,dt + \int_0^1 G_\ell(t)\,dt = 1.$$

Let $p = \int_0^1 F_\ell(t)\,dt$ and consequently $(1-p) = \int_0^1 G_\ell(t)\,dt$. Consider a lottery ℓ^* that awards the commodity to agent 1 with probability p and to agent 2 with probability $1-p$. By construction:

$$\int_0^1 F_{\ell^*}(t)\,dt = \int_0^1 F_\ell(t)\,dt \tag{4}$$

$$\int_0^1 G_{\ell^*}(t)\,dt = \int_0^1 G_\ell(t)\,dt. \tag{5}$$

Also by construction:

$$\int_0^T F_{\ell^*}(t)\,dt \geq \int_0^T F_\ell(t)\,dt, \quad \forall T \in [0,1] \tag{6}$$

$$\int_0^T G_{\ell^*}(t)\,dt \geq \int_0^T G_\ell(t)\,dt, \quad \forall T \in [0,1]. \tag{7}$$

Since both agents are risk-loving, equations (4)-(5) and inequalities (6)-(7) imply that

$$U_1(\ell^*) \geq U_1(\ell) \text{ and } U_2(\ell^*) \geq U_2(\ell).$$

But since ℓ is efficient we conclude that ℓ^* and ℓ are utility-equivalent, which proves the first part of the proposition. If one of the agents, say agent 1, is strictly risk-loving, then $\ell^* \neq \ell$, equation (4) and inequality (7) would imply that $U_1(\ell^*) > U_1(\ell)$, which is impossible. Therefore, $\ell^* = \ell$. □

The following result is an immediate corollary of the above proposition and Theorem 3.

Corollary 1 Let $\langle S, d\rangle$ be a bargaining problem where both agents are risk-lovers. The utility which the Nash bargaining solution assigns to a player increases as his opponent becomes more risk-loving. That is $\mathcal{N}_1(\widetilde{S}, d) \geq \mathcal{N}_1(S, d)$ where $\langle \widetilde{S}, d\rangle$ is obtained from $\langle S, d\rangle$ by replacing agent 2 with a more risk-loving agent.

When the agents are risk-lovers, the Nash bargaining solution awards a utility pair that can be achieved only by a non-degenerate lottery. Consequently, it is reasonable that changes in the agents' risk loving affect the outcome. The reason why an increase in the degree of risk loving of one's opponent is beneficial is analogous to the reason why, under expected utility maximization, an increase in the degree of risk aversion of one's opponent is beneficial. Under expected utility maximization, an increase in one agent's degree of risk aversion increases his utility from any *certain outcome* in such a way that the equality-mindedness of the Nash bargaining solution has no other choice but to "tax" him and transfer some of the surplus to the other agent. This transfer of utility is implemented by means of a bigger share of the pie. Similarly, under the dual theory an increase in one agent's degree of risk loving increases his utility from any given *lottery* in such a way that the equality-mindedness of the Nash bargaining solution has no other choice but to "tax" the agent who benefits from the increased utility and to make some stochastic transfer to the other agent.

Again, it is instructive to compare Corollary 1 to the case of two risk-loving expected utility maximizers. In this case, by choosing von Neumann–Morgenstern utility functions such that $u_i(0) = 0$ and $u_i(1) = 1$ for $i = 1, 2$, the utility possibilities set is given by X. As a result, the Nash bargaining solution assigns an expected utility of $1/2$ to each of the bargainers, *independently of their degree of risk loving.* This utility pair can be achieved, however, only by the lottery that assigns the whole pie to each agent with probability $1/2$.

Summarizing, Theorem 2 and Theorem 3 show that according to the dual theory the Nash bargaining solution is sensitive to changes in risk attitude only when the Pareto-efficient agreements (and thus also the Nash solution) involve lotteries. This is in contrast to the expected utility theory, where the solution is sensitive to changes in the players' risk attitudes precisely in cases where Pareto efficiency implies no lotteries.

5 The Strategic Approach

In this section, we analyze the effect of risk aversion on the outcome of the well-known game of alternating offers. Consider the following game, denoted by Γ. Time is divided into periods. In odd-numbered periods, player 1 proposes a lottery $\ell \in \mathcal{L}$ to which player 2 responds either by accepting it or rejecting it. In even-numbered periods, player 2 proposes a lottery and player 1 responds. Payoffs are as follows. If proposal ℓ is accepted in period t, then player 1 gets $U_1(\ell)\delta_1^{t-1}$ and player 2 gets $U_2(\ell)\delta_2^{t-1}$, where for $i = 1, 2$, $\delta_i \in (0, 1)$ is player i's discount factor, and U_i is defined in equation 1. If no player ever accepts, they both get 0.

The utility functions used here deserve a comment. Within periods, the function U_i represents agent i's risk-preferences. Across periods, the function U_i together with the discount factor δ_i represent agent i's time-preferences. In Rubinstein (1982), the set of feasible agreements is an abstract set of outcomes, which are not necessarily lotteries. Hence, the issue of changes in risk aversion does not arise. There, changes in U_i affect player i's time preferences. Here, since the set of feasible agreements consists of lotteries, changes in the utility function also affect the within periods risk preferences. We are not deriving these joint preferences from basic axioms on individual behavior but, as can be easily checked, the within periods risk-preferences satisfy all the dual theory axioms and the across periods time-preferences satisfy all the properties required by Rubinstein (1982). In particular, preferences are stationary and exhibit impatience.

We can state now the following result.

Proposition 3 Assume either that both players are risk-averse, with at least one of them strictly so, or that both are risk-lovers, with at least one of them strictly so. Then, the game Γ has a subgame perfect equilibrium. Further, there is a unique pair of efficient lotteries $\hat{\ell}^1$ and $\hat{\ell}^2$ such that in every subgame perfect equilibrium:

1. player 1 proposes $\hat{\ell}^1$, accepts $\hat{\ell}^2$ and all lotteries ℓ such that $U_1(\ell) > U_1(\hat{\ell}^2)$ and rejects all lotteries ℓ such that $U_1(\ell) < U_1(\hat{\ell}^2)$;

2. player 2 proposes $\hat{\ell}^2$, accepts $\hat{\ell}^1$ and all lotteries ℓ such that $U_2(\ell) > U_2(\hat{\ell}^1)$ and rejects all lotteries ℓ such that $U_2(\ell) < U_2(\hat{\ell}^1)$.

The pair $(\hat{\ell}^1, \hat{\ell}^2)$ is the only pair of efficient lotteries that solves the system

$$U_1(\ell_2) = \delta_1 U_1(\ell_1) \qquad \text{and} \qquad U_2(\ell_1) = \delta_2 U_2(\ell_2). \tag{8}$$

Proof : Consider the system of equations (8). Proposition 1 tells us that when both agents are risk-averse, system (8) becomes

$$\begin{cases} y = \delta_1(1 - x) \\ x = \delta_2(1 - y) \end{cases} \tag{9}$$

where x is the share of the pie that player 1 offers to player 2 and y is the share of the pie that player 2 offers to player 1. It is immediately apparent that system (9) has a unique solution. Similarly, when both agents are risk-lovers, it follows from Proposition 2 that system (8) becomes

$$\left\{\begin{array}{lcl} f(q) & = & f(1-p)\delta_1 \\ g(p) & = & g(1-q)\delta_2, \end{array}\right. \tag{10}$$

where p is the probability with which player 2 gets the whole object and $1-p$ is the probability with which player 1 gets the whole object, according to player 1's equilibrium proposal, and similarly q and $1-q$ are the probabilities with which player 1 and 2 get the whole object, respectively, according to player 2's proposal. Since both f and g are concave functions, the system of equations (10) has a unique solution as well. Therefore, under the proposition's premises, our game satisfies the assumptions of Osborne and Rubinstein (1994), Proposition 122.1, which guarantees that the statement of our proposition is true.[4] $\square$

Theorem 4 If both players are risk-averse, with one strictly so, then an increase in the risk aversion of one player does not have any influence on the outcome.

Proof : If both players are risk-averse, the unique equilibrium outcome is the solution to (9), which is independent of the players' degree of risk aversion. $\square$

Recall that if the players were risk-averse expected utility maximizers, a player's equilibrium share of the object increases as his opponent becomes more risk-averse.

Theorem 5 Let Γ be a game of alternating offers where both players are risk-lovers (one of them strictly so) and let $\widehat{\Gamma}$ be the game of alternating offers that is obtained from Γ by replacing player 2 with a more risk-loving player. Denote by ℓ and $\hat{\ell}$ the corresponding equilibrium lotteries. Then $F_{\hat{\ell}} \geq F_{\ell}$. Namely, player 1's payoff distribution when his opponent is more risk-loving stochastically dominates player 1's payoffs when his opponent is less risk-loving.

[4]Our game does not satisfy Osborne and Rubinstein's Assumption A1 but it does satisfy a weaker version of it, which is all that is needed for their proof to work.

Proof : If both players are risk-lovers, the unique equilibrium outcome of Γ is the solution to

$$\begin{cases} f(q) &= f(1-p)\delta_1 \\ g(p) &= g(1-q)\delta_2, \end{cases}$$

where both f and g are concave functions. Similarly, the unique equilibrium outcome of $\widehat{\Gamma}$ is the solution to

$$\begin{cases} f(q) &= f(1-p)\delta_1 \\ \hat{g}(p) &= \hat{g}(1-q)\delta_2, \end{cases}$$

where $\hat{g} = h \circ g$ for some concave function $h : [0,1] \rightarrow [0,1]$. Letting (p,q) and $(\hat{p},\hat{q})$ be the solutions to the above two systems of equations, respectively, it is well known that $p > \hat{p}$. See for example Roth (1985) or Roth (1989). Recalling that the subgame perfect equilibrium outcome of Γ ($\widehat{\Gamma}$) is a lottery that assigns the whole object to player 1 and player 2 with probability $1-p$ ($1-\hat{p}$) and p ($\hat{p}$), respectively (see Proposition 3), the random variable of player 1's payoffs stochastically increases when player 2 is replaced by a more risk-loving player. $\square$

Note that if the agents were risk-loving expected utility maximizers, then an increase in the players' risk loving would have no effect whatsoever on the equilibrium of the game of alternating offers.

6 On the Relation between the Strategic and the Cooperative Approaches

6.1 A limit result

When the players are expected utility maximizers, the predictions of the cooperative and the strategic approaches are related in the following sense: In the limit, as the players' impatience disappears, the subgame perfect equilibrium outcome of the game of alternating offers converges to the Nash solution of the associated cooperative bargaining problem. See for example Binmore (1987) and Binmore, Rubinstein, and Wolinsky (1986). Not surprisingly, the same result holds when the players' risk preferences satisfy the dual theory axioms. This is stated formally by the following proposition.

Proposition 4 Consider two agents whose preferences over lotteries are represented by the dual theory and assume that either both agents are risk-averse or both are risk-lovers. Let $\langle S, d\rangle$ be the bargaining problem induced by these two agents and let $(\hat{\ell}_1(\delta), \hat{\ell}_2(\delta))$ be the pair of lotteries identified in Proposition 3 for the variant of the game where both agents share the same discount factor δ. Then, $\lim_{\delta\to 1}(U_1(\hat{\ell}^i(\delta)), U_2(\hat{\ell}^i(\delta))) = \mathcal{N}(S,d)$, for $i = 1, 2$.

The proof is standard (see Osborne and Rubinstein (1994), Proposition 310.3 for instance) and thus it is omitted.

6.2 A model with exogenous risk of breakdown

In Section 5, we analyzed a game of alternating offers where the players have a degree of time preference represented by a discount factor. Binmore, Rubinstein, and Wolinsky (1986) discuss an alternative to this game, where agents do not present time preference but there is a fixed probability of breakdown after a player rejects a proposal. Specifically, consider a modification of the game presented in Section 5, such that there is no discount factor but after every rejection there is a probability, $1-\delta$, that the game ends, in which case each bargainer receives 0. Thus, a typical path of play consists of rejections until period t (which can be ∞), at which point there will be agreement on the division (x_1, x_2), unless the game ended before. This path of play generates the lottery that assigns probability δ^{t-1} to the division (x_1, x_2) and probability $1-\delta^{t-1}$ to the outcome $(0, 0)$. The corresponding utility levels for players 1 and 2 are, respectively, $x_1 f(\delta^{t-1})$ and $x_2 g(\delta^{t-1})$, where again f and g are the "probability-evaluation" functions which represent player 1's and 2's preferences, respectively. It is well known that if the players were risk averse expected utility maximizers, this game would be strategically identical to the one in which players have a common discount factor δ. As a consequence, both games would have the same comparative statics properties with respect to changes in the degree of risk aversion. Further, the subgame-perfect equilibrium agreement converges to the Nash agreement as δ goes to 1. In our case, however, when players behave according to the dual theory of choice under risk, the two games are no longer strategically equivalent. First of all, players' preferences will not be stationary in general, which means that one of Rubinstein's (1982) basic assumptions fails to hold. If one wants the preferences to be stationary, one needs to assume that the functions f and g are of the form p^α. Second, even if both players' preferences are represented by the same potential function, namely $f(p) = g(p) = p^\alpha$, still the subgame perfect equilibrium of the game exhibits comparative statics properties that are opposite to those of the strategic game with discount factor as well as to those of the Nash bargaining solution. Moreover, as δ tends to 1, the subgame perfect equilibrium of this game does not converge to the Nash outcome of the corresponding Nash bargaining problem. To see this, assume that both players have the same preferences over lotteries, that are represented by the function $f(p) = g(p) = p^{\alpha}$. For concreteness, assume that agents are risk-averse, namely $\alpha > 1$. As pointed out above, every pair of strategies in the game of alternating offers yields a lottery that assigns probability δ^{t-1} to a division (x_1, x_2) and probability $1 - \delta^{t-1}$ to the outcome $(0,0)$. Denote this lottery by $[t,(x_1,x_2)]$. Note now that player 1 prefers $[t,(x_1,x_2)]$ to $[t',(y_1,y_2)]$ if and only if $x_1(\delta^{t-1})^{\alpha} > y_1(\delta^{t'-1})^{\alpha}$, if and only if $x_1^{1/\alpha}\delta^{t-1} > y_1^{1/\alpha}\delta^{t'-1}$. But this means that player 1's behavior is identical to the behavior of a risk-averse expected utility maximizer whose von Neumann–Morgenstern utility function is $u(x) = x^{1/\alpha}$. A similar reasoning shows the same conclusion for player 2. Since we have shown that the predictions of the Nash bargaining solution for agents whose preferences satisfy Yaari's axioms are different from the predictions for expected utility maximizers, we conclude that the game of alternating offers with a fixed probability of breakdown is essentially different from the game with impatient players. In particular, while an increase in one agent's degree of risk aversion does not influence the outcome predicted by the Nash bargaining solution when both agents are risk-averse, it does reduce that player's share in the subgame perfect equilibrium of the game with a fixed probability of breakdown.

7 The Ordinal Nash Solution

The primitives of our model consist of a set of physical outcomes X, preferences over the set of lotteries on X and the disagreement outcome D. Rubinstein, Safra, and Thomson (1992) propose a solution concept that is intended to be applied to bargaining problems, like ours, that are described in physical terms. They call their solution concept the *ordinal* Nash bargaining solution for the good reason that when agents' risk preferences satisfy the expected utility axioms, it induces the (cardinal) Nash bargaining solution on the induced cardinal bargaining problem. It is only natural then to inquire on the relation between the ordinal Nash bargaining solution and the cardinal Nash bargaining solution as we applied it to our problem.

Let $\langle p, \ell \rangle$ be the compound lottery that assigns lottery ℓ with probability p and the disagreement outcome, D, with probability $1-p$. An *ordinal Nash outcome* is defined to be a lottery $\ell^* \in \mathcal{L}$ such that for all $p \in [0,1]$, for all $\ell \in \mathcal{L}$, and for $i = 1,2$ and $j = 3-i$, it satisfies that if $\langle p, \ell \rangle \succ_i \ell^*$ then $\langle p, \ell^* \rangle \succeq_j \ell$. The interpretation is as follows. If a player is willing to run a risk of negotiations breakdown in order to get lottery ℓ instead of ℓ^* with certainty, then his opponent is willing to run the same risk of breakdown in negotiations to get ℓ^* rather than getting ℓ with certainty. Rubinstein, Safra, and Thomson (1992) call their solution the ordinal Nash bargaining solution for the good reason that when the agents' risk preferences satisfy the expected utility axioms, and therefore can be represented by von Neumann-Morgenstern utility functions u_i for $i = 1, 2$, then ℓ^* is such that

$$E_{\ell^*}(u_1)E_{\ell^*}(u_2) \geq E_\ell(u_1)E_\ell(u_2) \qquad \forall \ell \in \mathcal{L},$$

where $E_\ell(u_i)$ is agent i's expected utility of lottery ℓ.

When agents' preferences satisfy the dual theory axioms, the ordinal Nash solution does not induce the Nash bargaining solution on our induced cardinal problem $\langle S, d \rangle$. Namely, it is not true that

$$U_1(\ell^*)U_2(\ell^*) \geq U_1(\ell)U_2(\ell), \qquad \forall \ell \in \mathcal{L}.$$

One would like to know if there is an ordinal dual solution that induces the Nash bargaining solution in our framework. The solution we are looking for is defined as follows:

Definition 2 A lottery $\ell^* \in \mathcal{L}$ is said to be a *dual ordinal Nash outcome* if for all $\alpha \in [0, 1]$, for all $\ell \in \mathcal{L}$, and for $i = 1, 2$ and $j = 3 - i$, we have

$$\text{if } \alpha\ell \succ_i \ell^*, \text{ then } \alpha\ell^* \succeq_j \ell.$$

The interpretation is as follows. If a player prefers a proportion α of lottery ℓ to lottery ℓ^*, then his opponent prefers the same proportion of lottery ℓ^* to the whole lottery ℓ. An argument similar to the one used by Rubinstein, Safra, and Thomson (1992) shows that if both agents' preferences satisfy the dual theory axioms, then a dual ordinal Nash outcome ℓ^* is such that

$$U_1(\ell^*)U_2(\ell^*) \geq U_1(\ell)U_2(\ell), \qquad \forall \ell \in \mathcal{L}.$$

Therefore, the dual ordinal Nash solution induces the Nash bargaining solution when the agents' preferences satisfy the dual theory axioms.

The difference between the two ordinal solution concepts lies on the basic mixture operations used in their respective definitions. The ordinal Nash solution is defined by means of *probability* mixtures — the ones that appear in the independence axiom. The dual ordinal Nash solution on the other hand, is defined by means of *wealth* mixtures or portfolios —the ones that appear in the dual independence axiom.

Following the insights of Binmore, Rubinstein, and Wolinsky (1986) and the idea behind the dual ordinal Nash solution, Dagan, Volij, and Winter (2001) define the *time-preference* Nash solution, which turns out to be closely related to the cardinal Nash bargaining solution. Further, the time-preference Nash solution places no constraint on the risk preferences of the agents and the comparative statics properties of the temporal Nash solution are consistent with the results of this paper.

As a corollary of Section 5, we see that there is a discrepancy between the ordinal Nash solution and the subgame perfect equilibrium outcome of the game of alternating offers. In particular, as the discount factor approaches 1, the equilibrium outcomes do not converge to the Nash ordinal solution. In fact, it can be shown that they converge to the dual ordinal Nash solution. It is interesting to note that Rubinstein Safra and Thomson's ordinal solution is the limit of the subgame perfect equilibrium outcome of the strategic game with a risk of breakdown analyzed in Section 6.2.

8 Conclusions

We have shown that under the dual theory of choice under risk, standard comparative static results on risk attitude in bargaining are reversed. Further, within the strategic approach, we showed that the very details of the game may drastically affect the comparative statics properties of the subgame perfect equilibrium.

This paper does not argue that one theory is more relevant than the other in the context of bargaining. This is an empirical question that can be answered only by means of empirical or experimental research. Unfortunately, this evidence is very small and weak. Murnighan, Roth, and Schoumaker (1988) for example, report results on several experiments that try to check the effects of risk aversion on bargaining outcomes. The results give mild support to the idea that risk aversion is disadvantageous. While the strategic game they used allows agents to bargain freely over a simple class of lotteries, it has a strict deadline after which a disagreement outcome is implemented. This deadline makes the game very similar to the one analyzed in Section 6.2. At the very last phases of the game, a player who must respond to an offer knows that if he gives a counteroffer there is a chance that the deadline is reached. As shown in Section 6.2, when there is a risk of breakdown, the comparative statics of the model with expected utility maximizers and with dual utility maximizers are qualitatively the same.

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Footnotes

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1. Henceforth we will refer to the result mentioned above as "the result."

2. The interested reader should consult Yaari (1986) for various and equivalent interpretations of Definition 1.

3. We thank an anonymous referee for suggesting this discussion.

4. Our game does not satisfy Osborne and Rubinstein's Assumption A1 but it does satisfy a weaker version of it, which is all that is needed for their proof to work.