Economic Theory **15**, 279–296 (2000)

Economic Theory

Bargaining, coalitions and competition*

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Received: April 20, 1999; revised version: May 6, 1999

Summary. We study a Gale-like matching model in a large exchange economy, in which trade takes place through non-cooperative bargaining in coalitions of finite size. Under essentially the same conditions of core equivalence, we show that the strategic equilibrium outcomes of our model coincide with the Walrasian allocations of the economy. Our method of proof makes use of the theory of the core. With respect to previous work, our positive implementation result applies to a substantially larger class of economies: the model relaxes differentiability and convexity of preferences, and also admits an arbitrary number of divisible and indivisible goods.

Keywords and Phrases: Finite coalitions and Edgeworthian theory of exchange, Marginal rates of substitution and Jevonsian theory of exchange, Matching and bargaining, Core, Walrasian equilibrium.

JEL Classification Numbers: D51, D41, C78.

1 Introduction

The Walrasian or competitive equilibrium is the central solution concept in economics. However, from its definition it is not clear what trading procedures lead

* We thank seminar audiences at Barcelona-Jocs, Brown, Harvard/MIT, NYU, Penn State and Yale, and participants at the European Meetings on General Equilibrium (Louvain) and NSF-NBER conference on decentralization (Penn State) for their comments. We thank Douglas Gale for help and encouragement. We also received detailed comments from Daron Acemoglu, Andreu Mas-Colell, Andrew McLennan, Ariel Rubinstein, Myrna Wooders and anonymous referees. Serrano and Volij acknowledge a Salomon research award from Brown University. Serrano also thanks Universitat Pompeu Fabra, Barcelona (Spain) for its hospitality and acknowledges research support from Spain's Ministry of Education and Science under grant SAB95-0186 and from the Alfred P. Sloan Foundation.

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to Walrasian outcomes. In contrast to what many economists may think, the original account of the theory (Walras, 1874) does not rely on the existence of the so-called Walrasian auctioneer.[1] Nonetheless, the usual formal presentation of the model includes the auctioneer implicitly due to a lack of explanation for the formation of equilibrium prices.

Negishi (1989) distinguishes two major schools in the analysis of markets. One of them considers prices as part of the economic mechanism in and out of equilibrium. This school is attributed to Cournot (1838) and Walras (1874). A second school, associated with Jevons (1879) and Edgeworth (1881), attempts to consider decentralized trading mechanisms and answer the question of whether equilibrium prices will emerge as the consequence of agents' trading actions.

We can distinguish at least two major approaches in the Jevons-Edgeworth school of decentralized trading. One of them, which today is referred to as the core equivalence literature, has its origins in Edgeworth (1881). This approach finds conditions under which core and Walrasian allocations are equivalent. If one conceives the core as a decentralized mechanism, these results give insights about the Walrasian allocations that were not provided by Walras. The shortcoming of this approach is that, although the core captures a natural idea of coalitional stability, it does not specify the trading procedure either. Here is relevant the work on the non-cooperative implementation of the core in finite games and economies (see, for example, Perry and Reny, 1994; Serrano, 1995; Serrano and Vohra, 1997).

A second approach models the trading procedure more explicitly and studies its strategic equilibria. As part of a recent literature that starts with Rubinstein (1982) and Rubinstein and Wolinsky (1985) [henceforth, RW], many researchers have turned to models where a decentralized trading procedure is made explicit in a bargaining extensive form.[2] RW (1985) analyze a market for an indivisible good and claim that in a frictionless economy the strategic equilibria need not be Walrasian. This claim was challenged later by the classic paper of Gale (1986a), who constructed an alternative bargaining procedure in a continuum economy with divisible commodities, in which strategic and Walrasian equilibria coincide. Gale's work was generalized in some respects by McLennan and Sonnenschein (1991) [McLS in the sequel]. Finally, Gale (1987) studies a market for a single indivisible good and also obtains a Walrasian result.

Combining the above strands of the literature, in this paper we study a matching model in a large exchange economy, in which trade takes place through non-cooperative bargaining in coalitions of finite size. Under essentially the con-

[1] *'The markets which are best organized from a competitive standpoint are those in which purchases and sales are made by auctions ... Besides these markets, there are others, such as the fruit, vegetables and poultry markets, where competition, though not so well organized, functions fairly effectively and satisfactorily. City streets with their stores and shops of all kinds —baker's, butcher's, grocer's, taylor's, shoemaker's, etc.— are markets where competition, though poorly organized, nevertheless operates quite adequately... '.* (Walras, 1874, pp. 83–84).

[2] See Diamond and Maskin (1979), and also Osborne and Rubinstein (1990, Chap. 6) and the references therein for earlier models of decentralized trade in pairwise meetings where each pair uses the Nash bargaining solution to split the gains from trade.

ditions of core equivalence, we show that the strategic equilibrium outcomes of our model coincide with the Walrasian allocations of the economy. With respect to the papers of the previous paragraph, our model relaxes differentiability and convexity of preferences, and also admits an arbitrary number of divisible and indivisible goods. By considering multilateral meetings in which trade takes place, we are able to use the full power of the theory of the core. In doing so, it becomes apparent that the driving force behind core equivalence and Gale–like results is similar: core equivalence takes now a non-cooperative dimension.

From an implementation theoretic point of view, we construct a mechanism that fully implements the Walrasian allocations of an economy with a continuum of agents. In contrast to all previous work, though, implementation is achieved over a substantially larger class of economies. Moreover, our mechanism can be seen as an ϵ-perturbation of Gale's, since meetings involving more than two agents can be assumed to have negligible probability. Thus, our results demonstrate that fixing a correspondence of interest (the Walrasian in this case), by perturbing slightly a successful mechanism one may create very positive effects on the domain over which implementation takes place. It is interesting to compare this insight to the virtual implementation results, where for a fixed domain, the slight perturbation of the solution concept (approximate rather than exact implementation) has also very positive effects on the set of correspondences that a planner can (approximately) implement (see Matsushima, 1988; Abreu and Sen, 1991).

Like going from Jevons to Edgeworth, our theory of exchange departs from the one based on pairwise meetings. That is, free of the differentiability assumption, our model differs from the preceeding ones in allowing for trade to take place in coalitions. This trade-off in the modelling choice leads to two distinct methods of proof and captures different insights. The proofs of the existing results (Gale, 1986a,c; McLS, 1991) rely crucially on the existence of the marginal rate of substitution at every bundle in order to get the unique (to that bundle) supporting price. Since in the strategic equilibria of these models there cannot be any pair of agents with positive gains from trade, by differentiability, at the equilibrium bundles every two agents have the same marginal rate of substitution. Consequently, this must hold for all agents. Feasibility of the equilibrium outcome (zero aggregate excess demand) and some extra technical assumptions then take care of the rest of the argument. In contrast to this method of proof, Theorem 1 uses the theory of the core to show that every strategic equilibrium yields a Walrasian allocation. The existence of marginal rates of substitution is not necessary. Since we allow for finite coalitions to meet, in our strategic equilibria there cannot be any such coalition with positive gains from trade. Then, a powerful lemma due to Hammond, Kaneko and Wooders (1989) and core equivalence take us the rest of the way without the intermediate aid of differentiability.[3]

Unlike our first result just referred to, its converse, Theorem 2, does not use in its proof the theory of the core. That is, although we allow for finite coalitions, it

[3] Gale (1987) does not rely on differentiability, but the arguments there are confined to a market for a single good.

turns out that all equilibrium outcomes of our game can be supported by strategies according to which all trade takes place in pairs. Moreover, if all gains from trade in the economy can be exhausted in coalitions bounded by a given finite size k [e.g. assignment markets, like in RW (1985) or Gale (1987), in which $k = 2$], one can restrict the matching process to meetings of size $k + 1$. Thus, in some cases only very small coalitions are needed for the result.

The paper is organized as follows: Sect. 2 describes the underlying economic model. The non-cooperative bargaining game is described in Sect. 3. The main result is presented in Sect. 4. In Sect. 5 we show that every Walrasian allocation can be supported by a strategic equilibrium of the game, and Sect. 6 concludes and compares our results with the previous literature.

2 Description of the economy

Let $(A, \mathscr{A}, \mu)$ be a measure space, where A is the set of agents, $\mathscr{A}$ is the set of measurable subsets of A, and μ is an atomless measure. We denote by C the set of agents' characteristics. An element $c \in C$ is a pair $c = (u, e)$, where $u : X \to \mathbb{R}$ is a utility function, X is the consumption set and $e \in X$ is an endowment. The consumption set X is assumed to be identical for all agents and is of the form $\mathbb{R}_+^D \times \mathbb{N}^I$, where $\mathbb{N}$ is the set of non-negative integers. The consumption set includes $|D|$ divisible goods and $|I|$ indivisible goods; we assume that $|D| \geq 1$.

An economy $\mathscr{E}$ is a measurable map $\mathscr{E} : A \to C$. Let $S \subseteq A$ be a coalition. An S-allocation f is a measurable map $f : S \to X$ that satisfies $\int_S f(a) d\mu \leq \int_S e(a) d\mu$. A-allocations are called simply allocations. We denote by F the set of allocations of the economy. From now on, and whenever there is no danger of confusion, the domain of integration A of the set of all agents will be omitted.

A coalition S can improve upon an allocation f if S has a positive measure and there exists an S-allocation $g : S \to X$ such that almost everywhere in S $u_a(g(a)) > u_a(f(a))$. The core of an economy is the set of all allocations that no coalition can improve upon (see Aumann, 1964).

We shall make the following assumptions on the utility functions of the agents. The assumptions can be grouped into two classes. First, we need the assumptions of Hammond, Kaneko and Wooders (1989) that guarantee the validity of their core equivalence theorem, different from Aumann's, as is based on a core with respect to finite coalitions.

A0 All the commodities are present in the economy: $\int e(a) d\mu \gg 0$.

For all $c = (u, e) \in C$:

A1 The utility function u is continuous, strictly increasing in the divisible commodities, and non-decreasing in the indivisible commodities;
A2 for all $(x_D, x_I) \in \mathbb{R}_+^D \times \mathbb{N}^I$, there exists $y_D \in \mathbb{R}_+^D$ such that $u(y_D, 0_I) > u(x_D, x_I)$; and
A3 for all $x_I \in \mathbb{N}^I, u(e) > u(0_D, x_I)$.

A0 is merely technical. A1 states that preferences are continuous and monotone. A2 expresses the basic desirability of divisible goods (money, for example). Finally, A3 implies that an agent's initial endowment contains some divisible goods.

In addition, we need assumptions that are necessary to deal with the possibly random outcome of the strategic bargaining process. A4 is mainly a technical assumption, while A5 has an interesting economic meaning: it guarantees that the aggregate economy's feasible utility set is convex, even though individual agents may not be risk averse.

A4 The utility functions are bounded: For all $c = (u, e) \in C$: there exists a number k_u such that $u(x) \leq k_u$ for all $x \in X$.

A5 The economy satisfies strong aggregate risk aversion in the individually rational domain (defined in the following paragraphs.)

Aggregate risk aversion

Gale (1986a, b, c) assumes that all individuals have strictly concave utility functions. This assumption is used in order to prove that the outcome of a strategic equilibrium of the market is a degenerate lottery. The "individual risk aversion" assumption excludes the possibility that the consumption sets include indivisible goods. This makes the comparison of his model with the earlier ones (e.g. RW, 1985) difficult. The work of McLS (1991) dispenses with the convexity assumption, but requires differentiability. Therefore, also in their framework indivisible goods cannot be studied. On the other hand, one should expect that the convexification effects of large numbers may help relax the assumption of individual risk aversion of preferences. These arguments motivate our assumption of aggregate risk aversion.

A lottery on allocations is a probability measure on the Borel σ–algebra of allocations. We shall say that the economy satisfies *weak aggregate risk aversion* if for every lottery L on allocations there exists an allocation g such that for almost all agents $u_a(g(a)) \geq \int_F u_a(f(a))dL$.

We say that the economy satisfies weak aggregate risk aversion in the individually rational domain if it satisfies weak aggregate risk aversion when considering only lotteries that assign an expected utility at least as high as the utility of the endowment.

A lottery L is *degenerate* if for almost every agent a there exists a constant k_a such that $u_a(f(a)) = k_a$ for L-almost all $f \in F$. The economy satisfies *strong aggregate risk aversion* if it satisfies weak aggregate risk aversion and for every lottery which is not degenerate there exists an allocation g that satisfies for almost all $a \in A$ $u_a(g(a)) \geq \int_F u_a(f(a))dL$, and there exists a set of agents of positive measure for whom the inequality is strict. Strong aggregate risk aversion in the individually rational domain is defined just like its weak counterpart.

Aggregate risk aversion means that society cannot gain from lotteries over allocations. The strong version of the property means that if the lottery is non-degenerate, society actually will lose from having the lottery.

Although this is a property on the aggregate, we find conditions on individual preferences that are weaker than concavity of the utility functions that imply the corresponding aggregate risk aversion properties.

First we consider an individual with a utility function $u : X \to \mathbb{R}$. We define *the quasiconcave cover* of u as $\hat{u} : \hat{X} \to \mathbb{R}$, where $\hat{X}$ is the convex hull of X:

$$\hat{u}(x) = \max\{u(y) : x \in \hat{R}(y)\},$$

where $\hat{R}(y)$ is the convex hull of the set of all bundles that are weakly preferred to y. The assumption that the consumption set X is bounded below and assumption A1 on the utility function u ensure the existence of $\hat{u}$, as shown in Einy and Shitovitz (1997, Lemma 3.3). See also Starr (1969, Appendix 3) for an early treatment of non-convex preferences.

The first condition we consider is simply that $\hat{u}$ is concave for almost all agents. The proof of the next result is in the appendix.

Proposition 1 *If the quasiconcave covers of the utility functions of almost all agents are concave, then the economy satisfies weak aggregate risk aversion.*

A slightly stronger assumption on individual preferences is that the quasiconcave cover of the utility function is almost strictly concave. A function $v : X \to \mathbb{R}$ is *almost strictly concave* if for all $x_1, x_2 \in X$ such that $v(x_1) \neq v(x_2)$ and for all $\lambda \in (0, 1)$ we have that $v(\lambda x_1 + (1 - \lambda)x_2) > \lambda v(x_1) + (1 - \lambda)v(x_2)$. The difference between almost strict concavity and strict concavity is that the former requires that $v(x_1) \neq v(x_2)$ and not simply that $x_1 \neq x_2$. This difference is crucial for our purposes, because if a cover is strictly concave, the original function is strictly concave as well, whereas functions that are not even quasiconcave may have an almost strictly concave cover.

Proposition 2 *If the quasiconcave covers of the utility functions of almost all agents are almost strictly concave, then the economy satisfies strong aggregate risk aversion.*

This proposition is also proved in the appendix. It follows that if the quasiconcave covers (restricted to the individually rational domain) of the utility functions of almost all agents are almost strictly concave, then the economy satisfies strong aggregate risk aversion in the individually rational domain, which is our assumption A5.

An example will be useful to clarify our last assumption A5.

Example 1 Consider an individual who may consume two goods: one of them is perfectly divisible and the other is indivisible, (as in RW, 1985). The agent wants to consume at most one unit of the indivisible good and his reservation price in terms of the divisible commodity for the first unit of the indivisible good is 1. More formally, the commodity space is $\mathbb{R}_+ \times \mathbb{N}$. His utility function is $u(x_1, x_2) = v(x_1 + I(x_2))$ where $I(x_2) = 1$ if $x_2 \geq 1$ and $I(x_2) = 0$ otherwise, and v is a strictly increasing, strictly concave and bounded function from $\mathbb{R}_+$ to $\mathbb{R}$.

Suppose now there are many agents with these preferences. They differ only in their endowments, i.e., each buyer holds only one unit of x_1 and each seller holds one unit of x_2. Since the consumption set contains indivisible goods, there is no clear notion of risk aversion of preferences in this setting. However, the quasiconcave cover of u, restricted to the individually rational domain, is almost strictly concave: $\hat{u}(x_1, x_2) = v(x_1 + \min\{x_2, 1\})$. This implies that if there is a continuum of such agents, the economy cannot gain from introducing lotteries over bundles.

3 Description of the game

The set A of all agents is present from the outset. Time runs discretely from 1 to infinity. In each round the agents are matched at random into coalitions of finite size. At every round t there is a proportion $\alpha \in (0, 1)$ that is left unmatched. For each round t and each size $n \geq 1$, there is a positive probability $p(n)$ of being matched in an n-person coalition. Thus, $p(1) = \alpha$.[4]

Matches are made randomly and for a fixed $n \geq 2$ the probability of being matched to any $n-1$ agents chosen from $n-1$ sets is proportional to the product of the measures of these sets. Existence of such a random matching process is guaranteed by the same conditions as in models with only pairwise meetings. For possible treatments, see McLS (1991, footnote 4) and the references therein.

When a coalition S meets, there is a ‘cheap talk’ phase in which every agent announces a bundle.[5] In addition, an order is chosen at random with equal probability. Denote by x_j the bundle held by agent $j \in S$ at the beginning of this meeting. The first agent in the order becomes the proposer, who makes a public offer consisting of a trade $(z_j)_{j\in S}$ in which $\sum_{j\in S} z_j = 0$ and for all $j \in S, x_j + z_j \in X$. Responses are also public and occur sequentially following the order. They can be one of two possible actions: ‘yes,’ and ‘no.’ The trade proposed to the coalition takes place if and only if every responder agrees to it. Every agent who is matched in round t can, if he so wishes, leave the market and consume his bundle after the bargaining session ends. Agents who are not matched in round t cannot leave the market in that round. In the next round, all agents who chose to leave abandon the market place and consume their current bundle. All other agents continue as active traders ready to be matched again.

In each round each agent recognizes the economic characteristics of the agents with whom he is matched. These consist of the current bundle each of them holds and their utility functions. However, they do not have information about their

[4] All our results go through if we assume that for every $n \geq 3$, $p(n)$ is arbitrarily small. This makes our model an ϵ-perturbation of Gale’s procedure.

[5] The arguments in the characterization theorem (Theorem 1) are entirely independent from this phase. However, its introduction simplifies greatly the proof of Theorem 2, which could be complicated due to the fact that our model allows for demand correspondences.

histories: each agent remembers only his own, but not the others'. They do not know anything about meetings that do not include them.

The restrictive information available to traders requires us to endow agents with beliefs about what happens elsewhere in the market. This must be done in order to have well defined expected utility computations. We will take care of these details in the next section, where we present our equilibrium notion.

The payoff to a typical trader in this market is the utility of the bundle with which he leaves the market. Thus, there is no discounting. On the other hand, if an agent never leaves the market, his utility is the utility corresponding to the zero bundle. All agents are expected utility maximizers when evaluating lotteries over bundles.

4 The equilibrium notion and the main result

A *strategic equilibrium* is a particular type of perfect Bayesian equilibrium (see Fudenberg and Tirole, 1991, for the general notion of PBE). Namely, it is a strategy/belief profile such that, given the beliefs explained below, every agent plays a best response to the others at every information set. On the equilibrium path, we shall assume that beliefs are derived from the equilibrium strategies using Bayes' rule. On the other hand, off the equilibrium path we shall assume that each agent believes that a full measure of the agents continues to play according to the equilibrium. This equilibrium concept is motivated and formally defined in the sequel.

Since each agent is an entity of measure 0 in the continuum and since each of them has met only a finite number of agents in all the rounds up to round t, we can define the variable of the "state of the market." That is, a fixed profile of strategies played by the continuum of agents determines the state of the market in round t as a distribution over characteristics. This happens with independence of the actions of a set of measure 0 (the history of an agent at a given point). Notice that the distribution over characteristics that we refer to as the 'state of the market' need not be supported by an allocation of the economy. Such an example can be constructed following the one found in Kannai (1970). However, a distribution is all an expected utility maximizing agent needs in order to make his calculations.

Alternatively (and more rigorously from a technical point of view), we could take the approach based on distributions like in Hart, Hildenbrand and Kohlberg (1974) instead of the name-based approach. We should then assume (like Gale, 1986a,b,c; Osborne and Rubinstein, 1990; McLS, 1991) that any two agents with identical characteristics and histories play the same strategy. This would enable us to employ the machinery developed by McLS (1991, Sect. 3.3) in order to establish that for any given strategies the "state of the market" in round t in the sense of distribution of agents' characteristics is deterministic. See also Osborne and Rubinstein (1990, pp. 160-161), who show this for the finite type pure strategy case. More recently, Al-Najjar (1995) claims to have proved an equivalence theorem that by-passes the difficulty. However, this is far from being the case, as pointed out by Khan and Sun (1999). Ultimately, we agree with Khan and Sun (1999) on the need to look for alternative ways other than the continuum to model mass interactions. After all, the continuum should be just an idealization, and we should avoid the technical complications that do exist and divert attention from the economic content of the model.

Now we can state more formally the equilibrium concept based on Osborne and Rubinstein (1990) as follows:

Definition 1 A strategic equilibrium is a pair of functions (σ, β), that assign to each agent a strategy and a belief, such that:

(i) The beliefs β are the "state of the market" beliefs induced by σ both on and off the equilibrium path,
(ii) For each agent a and for each of his information sets, $\sigma(a)$ prescribes a best response to $\sigma(A\backslash a)$ given the beliefs $\beta(a)$.

Our characterization result follows.

Theorem 1 *Suppose that the economy satisfies assumptions A0–A5. In every strategic equilibrium there exists a core allocation f such that almost every agent a eventually leaves the market with a bundle $g(a)$ such that $u(g(a)) = u(f(a))$.*

Note that in equilibrium an agent may receive a lottery over different bundles (due to the random matching process and possibly to mixed strategies, the same equilibrium comprises different paths). However, all of these bundles belong to the same indifference surface. If there were several core allocations that assigned the same utility to almost every agent, then the outcome could be a lottery over these core allocations.

Proof. Consider a strategic equilibrium. All of our statements are relative to this equilibrium and to histories in which at most a set of agents of measure 0 has deviated. Since each agent's history is private information and as the matching process treats all agents alike, two agents with the same characteristics and beliefs must get the same payoff regardless of their histories. If not, the agent with the lower payoff would simply imitate the behavior of the other and get the same probability distribution over outcomes. Recall that all agents have the same beliefs about the "state of the market" independent of their histories.

All agents at the beginning of round t before their match has been determined believe that the "state of the market" corresponds to the equilibrium. Thus in the equilibrium all such agents that in addition share the same characteristics have the same expected utility. We denote this utility by $V(c,t)$. For each $c = (u,e)$, we define $w(c) = u(e)$.

Step 1: $V(c,t) \geq w(c)$ for all values c and t.

To see this, notice that every agent with characteristics c in period t can adopt the following strategy: whenever matched, propose the zero trade, reject any trade, and leave the market. Since with probability 1 he will be matched in finite time, this strategy guarantees him a payoff of $w(c)$.

Step 2: $V(c,t) \geq V(c,t+1)$ for all values of c and t.

This assertion follows from the fact that by proposing the null trade and rejecting every offer and staying in the market, any agent in the market in round t is sure to be in the market in round $t+1$ with the same bundle as in round t.

We shall say that an agent is 'about to leave the market' if he has already reached an information set at which his strategy tells him to leave.

Step 3: For an agent of characteristic c who is about to leave the market in round t, we have that $V(c,t+1) = w(c)$.

The proof of this step is also simple. By step 1, we have that $V(c,t+1) \geq w(c)$. If $V(c,t+1) > w(c)$ and given that this agent is about to leave the market, he would be better off by deviating and staying in the market until round $t+1$.

Step 4: At some round t there is a positive measure of agents who are about to leave the market.

To prove this step, we argue by contradiction. Suppose that no positive measure of agents ever leaves the market. In this case the utility of almost all agents is that of the zero bundle. On the other hand, at any point in time there is a positive measure of agents who hold a bundle different from the zero bundle. This contradicts step 1.

At this point our method of proof departs crucially from Gale's. Instead of proceeding through a sequence of lemmas based on the existence of the marginal rates of substitution, we employ the insights of the theory of the core. This different technique yields a simpler proof.

Step 5: There does not exist a coalition $S \in \mathscr{A}$ with $\mu(S) > 0$, that has an S-allocation g for which $u_a(g(a)) > V(c(a),1)$ for almost all $a \in S$, where $c(a)$ is the initial characteristics of agent a.

To prove this step, we also argue by contradiction. Assume there exist such a coalition and such an allocation. Then, by Hammond, Kaneko and Wooders (1989, Claim 1) there exists a partition of this coalition into $h+1$ coalitions $S_0, S_1, \ldots, S_h$ such that $\mu(S_1) = \mu(S_2) = \cdots = \mu(S_h) > 0$ and a list of trades $z_1, \ldots, z_h$ such that for all $a \in S_m, m = 1, \ldots, h \quad u_a(e(a) + z_m) > u_a(g(a))$ and $\sum_m z_m \leq 0$.

Informally, this means that there are 'many' h-person (finite) coalitions that can improve upon the S-allocation g.

By step 4, at some round t a positive measure of agents is about to leave the market. We will show now that, under the contradiction hypothesis we are making, i.e., the existence of the improving coalition S, step 3 would be violated.

Recall that in every round t a positive measure of agents did not trade yet and thus keep their initial endowment. In particular, there is a positive measure of agents that can be chosen from each of the above coalitions $S_1, \ldots, S_h$. It follows from step 2 that in every round t the probability of an agent to be matched in an $h+1$-person coalition such that his h partners constitute an improving coalition is positive: if $\{i_1, \ldots, i_h\}$, $i_1 \in S_1$, ..., $i_h \in S_h$ was an improving coalition in round 1, so it continues to be in round t.

But then each person who is about to leave the market can adopt the following strategy: to stay in the market and whenever being a proposer in such an improving coalition, to offer them an improving trade z which gives the proposer higher amounts of some divisible goods without giving away any amount of the others; in all other situations, he holds on to his bundle and leaves the market at some finite date.

The proposal z will be unanimously accepted by the members of the improving coalition since for each of them with characteristic $c = (u, e)$ we have that:

$$V((u, e + z_c), t + 1) \geq u(e + z_c) > V(c, 1) \geq V(c, t + 1),$$

where the first inequality follows from step 1, the second from the existence of the improving trade z for the group of responders and the third from step 2.

Clearly, this deviation gives the deviating agent a higher expected utility than the utility of his current bundle, which contradicts step 3.

Step 6: In a strategic equilibrium, there exists a core allocation f such that almost every agent a eventually leaves the market with a bundle $g(a)$ such that $u(g(a)) = u(f(a))$.

We prove this step using a couple of arguments. [a]: By the previous step applied to the set of all agents A, it is not true that A can improve upon the strategic equilibrium outcome. This implies that almost every agent is receiving a degenerate lottery over bundles: otherwise, since the economy satisfies strong aggregate risk aversion, step 5 would be violated. And [b]: In addition, by applying step 5 to any other coalition of positive measure, the equilibrium outcome satisfies all the core conditions.

In order to choose the core allocation that is utility equivalent to the equilibrium outcome, note that almost every path in the extensive form associated with the equilibrium strategies constitutes a core allocation. Thus, it suffices to choose any such path. $\|$

Corollary 1 *Suppose that the economy satisfies assumptions A0–A5. In every strategic equilibrium there exists a Walrasian price p such that almost all agents eventually leave the market with a bundle that maximizes their utility on the budget set corresponding to the price p and to their initial endowment e.*

Proof. By Hammond, Kaneko and Wooders (1989, Theorem 2), A0–A3 imply that core allocations are Walrasian. $\|$

Remark: If agents are allowed to observe the state of the market every period, Theorem 1 and its proof go through without change when the solution concept used is the unrestricted set of perfect Bayesian equilibria.

5 Supporting Walrasian allocations by strategic equilibria

In Corollary 1 we have shown that all strategic equilibrium outcomes are Walrasian. Next we show the converse. That is, for every Walrasian allocation we find strategies that support it as a strategic equilibrium. In the proof below, no use will be made of the fact that the Walrasian allocation f is also a core allocation. This departs from the papers on implementation of the core in finite games and economies (e.g., Perry and Reny, 1994; Serrano, 1995; Serrano and Vohra, 1997). There, the strategies that support each core allocation are of the following form: the allocation in question is proposed to the grand coalition, and it is unanimously accepted. Obviously, such a construction is not possible in a continuum of agents where only sets of measure 0 meet to trade.

Theorem 2 *Let f be a Walrasian allocation corresponding to an equilibrium price p and suppose that all agents have a maximizer over every budget set corresponding to the price vector p. Then, there exists a strategy profile that supports f as a strategic equilibrium outcome of the game.*

The assumption in Theorem 2 is needed in our model as we can have Walrasian equilibria with some prices equal to 0. In this case, it could be that some agents (that constitute a set of measure 0) do not have an optimal bundle in their budget sets even in the equilibrium allocation. Of course, if equilibrium prices are all positive, or the model is a single market, (e.g. like in RW, 1985), the assumption in Theorem 2 is satisfied. In the models of Gale (1986a,b,c) and McLS (1991), such an assumption is also needed since they work with open consumption sets.

The next proof builds on Gale (1986b). In particular, in order to support each Walrasian allocation by a strategic equilibrium, it suffices to have trade only in pairs. However, we cannot simply invoke Gale's results, as our model allows agents to have set-valued demand correspondences.

Proof. For each agent a let $h(a,e)$ be a function that assigns to each agent a with holdings e a bundle selected from the agents' demand correspondence with respect to the price p and the income pe. The selection h is restricted so that $h(a,e) = f(a)$ if $pe = pe(a)$. Now consider the following strategy profile:

(1) In all meetings every trader announces during the 'cheap talk' phase the bundle assigned to him by the selection h.
(2) In multilateral meetings (those with at least three agents), the proposer offers the 0 trade.
(3) In bilateral meetings, the proposer offers a trade according to the trading rule g defined below (which is based on Gale, 1986b).
(4) In all meetings, every responder a who currently holds e that did not achieve the bundle $h(a,e)$ accepts a trade if and only if his income (the value of his holdings evaluated at the prices p) does not decrease. If he already achieved the bundle $h(a,e)$, then he accepts if and only if his income increases.

(5) Every agent leaves the market as soon as he achieves the bundle $h(a,e)$.

On the equilibrium path and as in Gale (1986b), we shall distinguish between the behavior of one of the divisible commodities (say, commodity 1) and that of the other $|D|+|I|-1$ goods. While according to the trading rule g an agent's excess demand in commodities other than 1 is non-increasing, commodity 1 serves to balance the budget whenever there is no pure coincidence of wants.

In order to define the trading rule g, we shall denote the proposer by a_0 and the responder by a_1. Subscripts denote agents and superscripts denote commodities. Let z_0 be $h(a_0, e(a_0,t)) - e(a_0,t)$, where $e(a_0,t)$ are the holdings of a_0 in round t, and similarly let z_1 be $h(a_1, e(a_1,t)) - e(a_1,t)$. Define the set $B(z_0,z_1)$ as the set of vectors $x \in \mathbb{R}^D \times \mathbf{Z}^I$ satisfying the following conditions (where $\mathbf{Z}$ denotes the set of integers):

(i) $|x^k| \leq |z_i^k|, \quad i = 0,1, \quad k \geq 2$
(ii) $0 \leq (-1)^i x^k z_i^k, \quad i = 0,1, \quad k \geq 2$
(iii) $e(a_i) + (-1)^i x \in \mathbb{R}_+^D \times \mathbb{N}^I$ and $px = 0 \quad i = 0,1.$

If the net trade x is proposed and accepted, the proposer's new endowment $e(a_0,t+1) = e(a_0,t) + x$ and the responder's $e(a_1,t+1) = e(a_1,t) - x$.

Now we are ready to present the trading rule proposed in all bilateral meetings. The trade proposed is denoted by:

$$g(z_0,z_1) = \arg\max\{-\sum_{k\geq 2} \exp(-|x^k|) : x \in B(z_0,z_1)\}.$$

Denote by $u^*(c,p)$ the maximum utility that an agent with characteristics c can achieve over the budget set determined by his endowment and the prices p. We will show that if every agent behaves according to the specified strategies, almost every agent of characteristic c achieves a bundle (corresponding to the allocation f) that yields $u^*(c,p)$ in finite time.

Notice first, as in Gale (1986b), that if agents follow the specified strategies, it is not possible for an agent to increase his income as evaluated by the prices p. Next we will show that an agent a ends up at the bundle $f(a)$ in finite time with probability 1. This will show that the proposed strategies are a strategic equilibrium. That is, given that there is no way to increase one's income, the proposed strategies induce a random path that takes each agent to his chosen maximizer $f(a)$ over the budget set determined by e and p.

For each agent a and for prices p define the excess demand as follows (for convenience and given that the Walrasian prices p are fixed throughout, we shall drop p from the expressions below): $\phi(a,t) = f(a) - e(a,t)$, where $e(a,t)$ are the holdings of agent a at round t. Notice that, given the strategies specified above, every agent travels along the frontier of his budget set which means that $f(a)$ continues to be a utility maximizer for agent a.

As we said above, we shall distinguish between the behavior of commodity 1 and that of the others. The trading rule g is constructed so that the absolute value of the excess demand of every agent in all goods but 1 does not increase. On the other hand, good 1 serves to balance the budget whenever there is no pure coincidence of wants. Thus we define for each agent a the following number: $\beta(a,t) = \sum_{k\geq 2} |\phi^k(a,t)|$. That is, for each agent a the statistic $\beta(a,t)$ indicates the sum of absolute values of excess demands in all goods but 1 in round t. We will next show that over time the distribution of $\beta(a,t)$ converges weakly in measure to a degenerate distribution concentrated on 0.

Recall that μ denotes the measure of characteristics in the economy, i.e., the measure of characteristics of all agents who are active in the market plus that of the agents who already left the market. The random matching process and the specified strategies lead to new distributions of characteristics at every round, and hence to new distributions of the statistic $\beta(a,t)$. We shall concentrate on an arbitrary path determined by a particular realization of the different random variables at play (the coalitions that meet and the roles of each agent in each meeting). We show then that, along this path, the distribution of $\beta(a,t)$ converges in measure to the degenerate distribution concentrated on 0.

The space of characteristics C at each round t is the Cartesian product of a fixed space of utility functions with $\mathbb{R}_+^D \times \mathbb{N}^I$. The evolution of the economy is thus described by a sequence of measurable maps from the set of agents A to the set C. Given that the set of utility functions is fixed throughout the model, any Cauchy sequence of such measurable maps must converge to a measurable map from A to C. To see this, notice that, after having fixed the utility functions, the marginal of characteristics c on agents' endowments e allows us to consider a Cauchy sequence of integrable maps from A to $\mathbb{R}_+^D \times \mathbb{N}^I$. Endowing this space of integrable maps with the topology induced by the supremum norm, it is easy to see that such a sequence converges to an integrable map into $\mathbb{R}_+^D \times \mathbb{N}^I$ as this is a complete space. Notice that the marginal of the measure μ_t on utility functions is constant. We can then abuse notation slightly and denote by μ_t round t's measure on the agents' endowments and not on characteristics. Thus, using Hildenbrand (1974, p.50) the sequence of measures $\{\mu_t\}$ is tight and has a convergent subsequence to μ^*. Without loss of generality, suppose the sequence itself converges to μ^*.

By the properties of the trading rule g, we must have that for a given constant $\tau > 0$, $\int_{\beta(a,t)\geq\tau} \mu(a,t)$ converges to 0 as time goes to infinity. To see this, we argue by contradiction. Suppose that the limiting measure μ^* is not the one concentrated at 0. Since μ^* is the limiting measure, it must be the case that the measure of agents trading positive amounts of goods when the distribution of $\beta(a,t)$ is approximately μ^* must be arbitrarily close to 0: For all $\epsilon > 0$ there exists a T such that for all $t > T$ we have that $\int[\mu_t - \mu^*] < \epsilon$. If μ^* is not the one concentrated at 0, there must exist a positive measure of agents whose characteristics satisfy that $\phi^k(c) > 0$ for some good k. By Walras' law which holds at each step of this time path, there must also exist a positive fraction of agents whose characteristics satisfy that $\phi^k(c) < 0$. Since the matching process is random, there exists a positive probability that agents in these two situations will meet. Finally, given the trading rule g, these agents will trade at least in good k, which is a contradiction, i.e., there exists $\epsilon > 0$ such that for all T there exists $t > T$ with $\int[\mu_t - \mu^*] > \epsilon$ as $g(z_0, z_1)$ stays bounded away from 0 for a positive fraction of meetings.

As for convergence in finite time, the arguments are identical to those in Gale (1986b, Sect. 7). ||

6 Discussion

By using the insights of the theory of the core, this paper presents a model of trade through non-cooperative bargaining in coalitions of finite size. This allows us to obtain equivalence results among core, Walrasian and strategic equilibrium allocations for a wide class of large exchange economies, including non-differentiable non-convex preferences and an arbitrary number of divisible and indivisible goods.

Our results are robust to several extensions of the model. First, we could allow for any entry process (not necessarily one-time entry) as long as the measure of the total entering population is finite. Second, different bargaining procedures in the coalitional meetings could be adopted: for example, veto power can be given only to those responders who are offered a non-zero trade vector. Third, we only need to assume that an agent's probability of meeting a coalition of size n be positive. In particular, we could assume that the probability of meeting a coalition of more than two agents be arbitrarily small and all our results would go through (because the probability of meeting a coalition of a given size always ends up being 1). Thus, Gale's model can be viewed as the "limit" of ours as the probabilities of multilateral meetings vanish. This poses the important open question of lower hemicontinuity of the equilibrium payoff correspondence, i.e., which of the extra assumptions made by Gale are really needed to obtain the result using only trade in pairs. Fourth, in a model with discounting, the conclusions of our main result (Theorem 1 and Corollary 1) extend as discounting is removed.

A separate dimension along which our results are more robust than the previous ones found in the literature is the class of economies to which they apply. We discuss this in length in the following paragraphs, especially comparing our results to the works of Gale (1986a, b, c) and McLS (1991).

Existence of equilibrium

Thanks to considering finite coalitions in the procedure, our paper yields the equivalence between strategic and Walrasian equilibria under essentially the same assumptions as those needed for the core equivalence theorem. Moreover, our assumptions also guarantee the existence of a Walrasian equilibrium, as opposed to Gale (1986a, b, c) and McLS (1991), which deal with open consumption sets.

Limited applicability of the previous models

As discussed in the introduction, we regard the relaxation of differentiability as a crucial conceptual departure from Gale's and McLS's work. From an applied view-point, the differentiability assumption by itself that Gale (1986a, c) and McLS (1991) make is not very restrictive: many models in economics incorporate it in order to allow for a closed solution and for the performance of comparative statics exercises. However, the proofs of the above mentioned authors rely on additional strong assumptions, that exclude most applied models. Gale (1986a) assumes that for each utility function the support of the endowments compatible with it is the entire consumption set. This assumption excludes the possibility of a finite type economy. Gale (1986c), who assumes a finite number of types, uses a bounded curvature assumption, thereby excluding, for example, Cobb–Douglas utility functions on the non-negative orthant or its interior. McLS (1991) make either a bounded curvature assumption similar to Gale's (1986c) or a restriction on the equilibrium which seems to require an assumption similar to Gale's (1986a) on the primitives of the economy. In contrast, our model, which applies to very general economies, also applies to these standard cases.

Feasibility in and out of equilibrium

We assume, like Gale (1986a, b, c), that the flow of agents entering the market constitutes an economy, i.e., they sum up to a finite measure. In addition, we also assume that short sales are not allowed. These two assumptions together ensure that the flow of agents out of the market is consistent with the feasibility constraint of the economy. Suppose, like McLS (1991), that the total measure of agents is finite, but short sales are allowed. In this case, nothing assures that feasibility is met. Consider an arbitrary assignment of bundles to agents, and the following strategies (that do not constitute an equilibrium in McLS's game with short sales). Each proposer asks for the bundle assigned arbitrarily to him and each responder accepts any proposal; agents leave the market as soon as they reach their assigned bundle. Clearly, these strategies guarantee that each agent will get with probability 1 the assigned bundle. The problem stays even if we restrict attention to the equilibria of their game. Indeed, the strategic equilibrium that McLS propose (pp. 1395–1396) to support a Walrasian equilibrium is a strategic equilibrium for any prices. That is, for an arbitrary price vector, their strategic equilibrium gives the outcome that every agent maximizes over the corresponding budget set, but the market clearing conditions may be violated. The problem is a consequence of assuming unlimited short sales. See Dagan, Serrano and Volij (1998) for other related comments on the McLS model.

Strict concavity and indivisible goods

McLS (1991) note the restrictiveness of Gale's (1986a, b, c) assumption of strictly concave utility functions in a continuum setting. One should expect that the convexifying effects of large numbers could be helpful to relax this assumption. McLS do not make any assumption regarding the concavity of utility functions; instead, they allow for short sales, which enables them to prove that outcomes of the strategic equilibria are not random. McLS also maintain the differentiability assumption, which precludes non-convexities arising from the existence of indivisible commodities.

One difference between the underlying economy and the strategic model is that in the latter the outcome (at least for an individual agent) may be random and thus preferences on random outcomes must play a role. Gale (1986a) uses the strict concavity assumption only to ensure that the introduction of lotteries does not enlarge the set of possible utilities of the agents. Thus, what is needed is a property of risk aversion in the aggregate. We impose a condition on the quasiconcave covers of the utility functions that ensures the sufficient degree of aggregate risk aversion. This assumption is compatible with having indivisible commodities as well as other kinds of non-convexities. Thus, our assumptions allow for a unified treatment of assignment markets (markets for an indivisible good) à la RW (1985) and classical exchange economies with divisible commodities à la Gale (1986a, b, c). We should stress that our assumption of aggregate risk aversion is sufficient to obtain Gale's results as well (of course, within his restricted subdomain).

Appendix

This appendix contains the proofs of Propositions 1 and 2.

Proof of Proposition 1. Let L be a lottery on allocations. We define $EU_a(L) = \int_F u_a(f(a))dL$. We also define $E_a(L) = \int_F f(a)dL$ and $h : A \to \mathbb{R}_+^{D \cup I}$ as the function that assigns to each agent a the bundle $E_a(L)$. It is easy to see that h is integrable. Now let $\phi(a) = \{x \in \mathbb{R}_+^{D \cup I} : \hat{u}_a(x) \geq \hat{u}_a(h(a))\}$. Clearly, $\int h(a) \in \int \phi(a)$ since it is true for every a. Let $\psi(a) = \{x \in X : u_a(x) \geq \hat{u}_a(h(a))\}$. It follows from the definition of $\hat{u}$ that $\hat{\psi}(a) = \phi(a)$, where $\hat{\psi}(a)$ is the convex hull of $\psi(a)$. Now consider $\int \psi(a)$. It follows from Liapunov's theorem that this integral is convex, and thus $\int \psi(a) = \int \phi(a)$. Therefore, $\int h(a) \in \int \psi(a)$. Thus, there exists an allocation g such that $u_a(g(a)) \geq \hat{u}_a(h(a))$ for almost all $a \in A$. It follows from the fact that all the covers $\hat{u}$ are concave that almost all agents weakly prefer the allocation g to the lottery L. ||

Proof of Proposition 2. Let L be a non-degenerate lottery over allocations. First it follows from Proposition 1 that there exists an allocation g that satisfies that $u_a(g(a)) \geq \hat{u}_a(h(a))$ for almost all $a \in A$. Since the lottery is non-degenerate, there is a positive measure of agents such that each of them is not indifferent among almost all bundles in the support of L. Now for each agent a in this set, we can have two cases:

1. There does not exist k such that for L-almost all $f \in F$, $\hat{u}_a(f(a)) = k$. In this case, it follows from almost strict concavity of $\hat{u}$ that $\hat{u}_a(h(a)) > EU_a(L)$.

2. There exists k such that for L-almost all $f \in F$, we have that $\hat{u}_a(f(a)) = k$. Since the agent is not indifferent among almost all bundles, there exists a set of allocations that are assigned positive probability by L such that $\hat{u}_a(f(a)) > u_a(f(a))$. Thus, $\hat{u}_a(h(a)) = k > EU_a(L)$. ‖

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