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PAPER

Effect of bremsstrahlung radiation emission on fast electrons in plasmas

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Abstract

Bremsstrahlung radiation emission is an important energy loss mechanism for energetic electrons in plasmas. In this paper we investigate the effect of spontaneous bremsstrahlung emission on the momentum–space structure of the electron distribution, fully accounting for the emission of finite–energy photons by modeling the bremsstrahlung interactions with a Boltzmann collision operator. We find that electrons accelerated by electric fields can reach significantly higher energies than predicted by the commonly used radiative stopping–power model. Furthermore, we show that the emission of soft photons can contribute significantly to the dynamics of electrons with an anisotropic distribution by causing pitch–angle scattering at a rate that increases with energy.

Energetic electrons are ubiquitous in plasmas, and bremsstrahlung radiation is one of their most important energy loss mechanisms [1, 2]. At sufficiently high electron energy, around a few hundred megaelectronvolts in hydrogen plasmas, the energy loss associated with the emission of bremsstrahlung radiation dominates the energy loss by collisions. Bremsstrahlung emission can also strongly affect electrons at lower energies, particularly in plasmas containing highly charged ion species.

An important electron acceleration process, producing energetic electrons in both space and laboratory plasmas, is the runaway mechanism [3]. In the presence of an electric field which exceeds the minimum to overcome collisional friction [4], a fraction of the charged particles can detach from the bulk population and be accelerated to high energies, where radiative losses become important. Previous studies of laboratory plasmas [5, 6] and lightning discharges [7] have shown that the energy carried away by bremsstrahlung radiation is important in limiting the energy of runaway electrons. The effect of bremsstrahlung radiation loss on energetic-electron transport has also been considered in astrophysical plasmas, for example in the context of solar flares [8]. However, only the average bremsstrahlung friction force on test particles has been considered in these studies. In this paper, we present the first quantitative kinetic study of how bremsstrahlung emission affects the runaway-electron distribution function.

Starting from the Boltzmann electron transport equation, we derive a collision operator representing bremsstrahlung radiation reaction, fully accounting for the finite energies and emission angles of the emitted photons. We implement the operator in a continuum kinetic-equation solver [9], and use it to study the effect of bremsstrahlung on the distribution of electrons in 2D momentum space. We find significant differences in the distribution function when bremsstrahlung losses are modeled with a Boltzmann equation (referred to as the 'Boltzmann' or 'full' bremsstrahlung model), compared to the model where only the average friction force is accounted for (the 'mean-force' model). In the former model, the maximum energy reached by the energetic electrons is significantly higher than is predicted by the latter. In previous treatments which considered average energy loss [5–7] or isotropic plasmas [2], the emission of soft (low-energy) photons did not influence the electron motion. We show that in the general case, emission of soft photons contributes significantly to angular deflection of the electron trajectories.

Kinetic description of bremsstrahlung losses

We will treat bremsstrahlung as a binary interaction ('collision') between two charged particles, resulting in the emission of a photon [1]. We shall describe the effect of such collisions on the rate of change of the distribution function $f_a(t,\,\mathbf{x},\,\mathbf{p})$ of some particle species a at time t, position $\mathbf{x}$ and momentum $\mathbf{p}$, defined such that $n_a(t,\,\mathbf{x}) = \int \mathrm{d}\mathbf{p}\, f_a(t,\,\mathbf{x},\,\mathbf{p})$ is the number density of species a at $\mathbf{x}$. In what follows we suppress the time- and space dependence of all functions, as the collisions will be assumed local in space-time, and we shall consider only spatially homogeneous plasmas.

The collision operator $C^{\mathrm{B}}_{ab}\{f_a,\,f_b\}$ describing the rate of change of the distribution function due to bremsstrahlung interactions between species a and b is given by $C^{\mathrm{B}}_{ab} = (\partial f_a/\partial t)_{c,ab} = \int (\mathrm{d}n_a)_{c,ab}/\mathrm{d}t\mathrm{d}\mathbf{p}$. The integration is to be carried out over target-particle momenta and scattering angles, and the differential change $(\mathrm{d}n_a)_{c,ab}$ in the phase-space density due to collisions in a time interval $\mathrm{d}t$ is given by [10, 11]

$$(\mathrm{d}n_a)_{c,ab} = f_a(\mathbf{p}_1)f_b(\mathbf{p}_2)\bar{g}_{\o}\,\mathrm{d}\bar{\sigma}_{ab}\mathrm{d}\mathbf{p}_1\mathrm{d}\mathbf{p}_2\mathrm{d}t - f_a(\mathbf{p})f_b(\mathbf{p}')g_{\o}\,\mathrm{d}\sigma_{ab}\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{p}'\mathrm{d}t. \tag{1}$$

Here, $\mathrm{d}\sigma_{ab} = \mathrm{d}\sigma_{ab}(\mathbf{p}_1,\,\mathbf{p}_2,\,\mathbf{k};\,\mathbf{p},\,\mathbf{p}')$ is the differential cross-section for a particle a of momentum $\mathbf{p}$ and a particle b of momentum $\mathbf{p}'$ to be taken to momentum $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively, while emitting a photon of momentum $\mathbf{k}/c$. We have also introduced the Møller relative speed [10] $g_{\o} = \sqrt{(\mathbf{v}-\mathbf{v}')^2 - (\mathbf{v}\times\mathbf{v}')^2/c^2}$. The barred quantities $\mathrm{d}\bar{\sigma}$ and $\bar{g}_{\o}$ are defined likewise, but with $(\mathbf{p},\,\mathbf{p}')$ and $(\mathbf{p}_1,\,\mathbf{p}_2)$ exchanged. Equation (1) accounts only for the effect on the distribution of the spontaneous emission of photons; interactions with existing photons by absorption and stimulated bremsstrahlung emission will be neglected here. The correction to the collision operator by these processes is described in [12]; the effect is negligible when $\phi(t,\,\mathbf{x},\,\mathbf{p}) \ll 2/h^3$, where h is Planck's constant and ϕ is the distribution function of photons. An estimate of the photon distribution function shows that the corrections are important for sufficiently dense, or large, plasmas; however, for the special case of electron runaway during tokamak disruptions, which is of particular concern, the corrections may be safely neglected. In other scenarios it is primarily bremsstrahlung processes involving low-energy photons that may be affected.

The collision operator then takes the form

$$C^{\mathrm{B}}_{ab}(\mathbf{p}) = \int \mathrm{d}\mathbf{p}_1 f_a(\mathbf{p}_1)\int \mathrm{d}\mathbf{p}_2\,\bar{g}_{\o}f_b(\mathbf{p}_2)\frac{\partial\bar{\sigma}_{ab}}{\partial\mathbf{p}} - f_a(\mathbf{p})\int \mathrm{d}\mathbf{p}'\,g_{\o}f_b(\mathbf{p}')\sigma_{ab}, \tag{2}$$

where $\sigma_{ab} = \int \mathrm{d}\mathbf{p}_1\,(\partial\sigma_{ab}/\partial\mathbf{p}_1)$ is the total bremsstrahlung cross-section. A significant simplification to (2) occurs if (i) target particles can be assumed stationary, $f_b(\mathbf{p}) = n_b\delta(\mathbf{p})$; and (ii) the plasma is cylindrically symmetric (and spin unpolarized), $f_a(\mathbf{p}) = f_a(p,\,\cos\theta)$, where $\cos\theta = p_{\|}/p$ and $p_{\|}$ is the Cartesian component of $\mathbf{p}$ along the symmetry axis. Then the differential cross-section $\partial\bar{\sigma}_{ab}/\partial\mathbf{p}$, for an electron to scatter from momentum $\mathbf{p}$ into $\mathbf{p}_1$ with the emission of a photon, depends only on p, p_1 and $\cos\theta_s = \mathbf{p}_1\cdot\mathbf{p}/p_1p$. The resulting operator can be conveniently expressed in terms of an expansion in Legendre polynomials P_L. We write $f_a(\mathbf{p}) = \sum_L f_L(p)P_L(\cos\theta)$ and $C^{\mathrm{B}}_{ab}(\mathbf{p}) = \sum_L C^{\mathrm{B}}_L(p)P_L(\cos\theta)$, and obtain

$$C^{\mathrm{B}}_L(p) = n_b\int \mathrm{d}p_1\left[p_1^2 v_1 f_L(p_1)2\pi\int_{-1}^{1}\mathrm{d}\cos\theta_s P_L(\cos\theta_s)\frac{\partial\bar{\sigma}_{ab}}{\partial\mathbf{p}}\right] - n_b v f_L(p)\sigma_{ab}(p). \tag{3}$$

The integration limits in p_1 are determined by the conservation of energy, giving $m_ec\sqrt{(\gamma + k/m_ec^2)^2 - 1} < p_1 < \infty$. In this work we use the differential cross-section $\partial\bar{\sigma}/\partial\mathbf{p}$ for scattering in a static Coulomb field in the Born approximation, integrated over photon emission angles. This expression was first derived by Racah [13], with a misprint later corrected in [14]. For the Boltzmann model this full cross-section is employed, while for the mean-force model we use the high-energy limit as in [5–7].

A useful approximation to the collision operator (3) is obtained by noting that the radiation emitted by runaway electrons will be strongly focused in the forward direction by relativistic beaming ('the headlight effect'), and the dominant contribution to the integral then originates from scattering angles $\theta_s \lesssim 1/\gamma$. For small angles, the Legendre polynomials take the asymptotic form $P_L(\cos\theta_s) \sim 1 - L(L+1)\theta_s^2/4$. Consequently, when $L \ll 2\gamma_1$ the angular integral in (3) can be replaced with

$$\int \mathrm{d}\cos\theta_s\; P_L(\cos\theta_s)\partial\bar{\sigma}_{ab}/\partial\mathbf{p} \approx \int \mathrm{d}\cos\theta_s\;\partial\bar{\sigma}_{ab}/\partial\mathbf{p} \equiv 1/(2\pi p^2)\partial\bar{\sigma}/\partial p.$$

This approximation leads to a bremsstrahlung collision operator of the form

$$C^{\mathrm{B}}_{ab}(\mathbf{p}) \approx n_b\int \mathrm{d}p_1\; v_1 f_a(p_1,\,\cos\theta)\frac{\partial\bar{\sigma}}{\partial p}(p;\,p_1) - n_b v f_a(p,\,\cos\theta)\sigma(p). \tag{4}$$

This is a one-dimensional integral operator acting only on the energy variable, and involves the integrated cross-section which is well known (it is related to the cross-section in photon energy by $\partial\bar{\sigma}/\partial p = (p/\gamma)\partial\bar{\sigma}/\partial k$) and is given analytically for example in equation (14) of [1].

Low-energy photon contribution—The bremsstrahlung cross-section has an infrared divergence; for low photon energies k, it diverges logarithmically as $\mathrm{d}\sigma \propto 1/k$. The total energy loss rate is however finite, indicating that a large number of photons carrying negligible net energy are emitted. A consequence of this behavior is that the two terms in the Boltzmann operator (2) are individually infinitely large, necessitating the introduction of a photon cut-off energy k_0, below which the bremsstrahlung interactions are ignored in (3) and (4). We can however proceed analytically to evaluate the effect of the low-energy photons. While they carry little energy, they may contribute to angular deflection, analogously to the small-angle collisions associated with elastic scattering. Taylor expanding (3) in small photon energy $k = \gamma_1 - \gamma$ yields to leading order

$$C_L^{\mathrm{small-k}} = -n_b v f_L(p) \int_{k_c}^{k_0} \mathrm{d}k \int_{-1}^{1} \mathrm{d}\cos\theta_s \left[1 - P_L(\cos\theta_s)\right] \frac{\partial \bar{\sigma}}{\partial k \partial \cos\theta_s}. \tag{5}$$

Since $P_0(\cos\theta_s) \equiv 1$, the angle-averaged electron distribution (represented by the $L = 0$ term) is not directly affected by the low-energy photons, reflecting the fact that the photons carry negligible energy, consistent with the description by Blumenthal and Gould [2] for the isotropic case. Due to the logarithmic divergence of the cross-section, however, a significant contribution to angular deflection (represented by the $L \neq 0$ terms) is possible. Inspection of the integrand in (5) further reveals that significant contributions originate from large-angle scatterings, indicating that a Fokker–Planck approximation is inappropriate. Indeed the bremsstrahlung cross-section $\partial\sigma^{\mathrm{low-}k}/\partial\cos\theta_s = \int_{k_c}^{k_0} \mathrm{d}k \partial\bar{\sigma}/\partial k \partial\cos\theta_s$, integrated over small photon energies, behaves for small angles ($\theta_s^2 \ll k_0^2/\gamma^4$) as $1/\theta_s^2$, which can be compared to the elastic Coulomb cross-section proportional to $1/\theta_s^4$. This weaker singularity of the bremsstrahlung cross-section means that the contribution from small-angle collisions will be negligible compared to those from the large-angle deflections, and therefore a Boltzmann model must be used to account for these events. While it may seem counter-intuitive that low-energy photon emissions contribute to large-angle collisions, note that due to the large mass ratio between electron and ion, large momentum transfers to the nucleus are allowed even without any energy transfer. For very energetic electrons, however, when the kinetic energy exceeds the ion rest energy, ion recoil effects would need to be accounted for in deriving (5).

We can quantify the importance of the low-energy photons by calculating the $L = 1$ term of (5)–giving the loss rate of parallel momentum—and comparing it to the corresponding term of the elastic-scattering collision operator given in [9]. Carrying out the integration, one obtains the ratio

$$\frac{C_1^{\mathrm{small-k}}}{C_1^{\mathrm{elastic}}} = \alpha \frac{2}{\pi} \frac{\ln\Lambda_{\mathrm{B}}}{\ln\Lambda} \left[\left(\ln\frac{2p}{m_e c} - 1 \right)^2 + 1 \right], \tag{6}$$

with a relative error of magnitude $O(m_e^2 c^2/p^2) + O(k_0/pc)$, and where $\alpha = e^2/4\pi\varepsilon_0\hbar c \simeq 1/137$ is the fine-structure constant. Here, we have introduced a bremsstrahlung logarithm $\ln\Lambda_{\mathrm{B}} = \ln(k_0/k_c)$, which arises in a way similar to the Coulomb logarithm $\ln\Lambda$ for elastic collisions, and is due to cutting off the logarithmically diverging integral at some lowest photon energy k_c.

Various mechanisms may suppress the bremsstrahlung interactions at low photon energy, such as multiple scattering, photon interactions with the medium, pair production and more [15]. Most important in dilute ionized gases, in the energy range we are interested in, is the photon interaction with the medium; the effect may be viewed as coherent forward Compton scattering on the target, causing destructive interference in the emitted radiation due to the induced phase shift in the emission. The analysis, originally due to Ter-Mikaelian [16], shows that the suppression can be accounted for by multiplying the cross-section with a suppression factor S, given by the ratio of in-medium to vacuum formation lengths $l_{\mathrm{B}} = \hbar/(p_{\parallel} - p_{1\parallel} - \sqrt{\epsilon/\epsilon_0}k_{\parallel}/c)$, with ϵ the dielectric constant of the medium. The formation length is approximately the distance over which the interaction amplitudes add coherently, and $\parallel$ here denotes the direction of the incident electron. Evaluating the ratio yields the suppression factor $S = k^2/(k^2 + k_p^2)$ where $k_p \sim \hbar\omega_p$ is the photon energy corresponding to radiation at the plasma frequency, suggesting an effective lower cut-off $k_c = k_p$ of our collision operator.

This gives a bremsstrahlung logarithm $\ln\Lambda_{\mathrm{B}} \approx 21 + \ln(k_0/(m_e c^2\sqrt{n_{20}}))$, where n_{20} is the electron density in units of 10^{20} m^{-3}. Assuming a plasma with $\ln\Lambda = 15$, $n_{20} = 1$ and choosing $k_0 = 0.01p$, the ratio (6) is of order 10% at 30 MeV, 50% at 2 GeV and 100% at 30 GeV, demonstrating that angular deflection caused by the emission of low-energy photons can contribute significantly to the motion of highly energetic electrons.

The bremsstrahlung collision operator has been implemented in the initial-value continuum kinetic-equation solver CODE (COllisional Distribution of Electrons) [9]. For this study we use CODE to solve the equation

$$\frac{\partial f_e}{\partial t} - eE_{\parallel}\frac{\partial f_e}{\partial p_{\parallel}} = C^{\mathrm{FP}}\{f_e\} + C^{\mathrm{B}}\{f_e\}, \tag{7}$$

which in a magnetized plasma represents the gyro-averaged kinetic equation, with the parallel direction given by the magnetic field **B**. The equation is also valid for an unmagnetized plasma which is cylindrically symmetric

around the electric field $\mathbf{E}$. Elastic collisions are accounted for by the linearized relativistic Fokker–Planck operator for Coulomb collisions C^{FP}, and C^{B} is the bremsstrahlung operator C^{B}_{eb} summed over all particle species b in the plasma. Both thermal and fast electrons are resolved simultaneously, allowing runaway generation as well as the slowing-down of the fast population to be accurately modeled.

We will compare the effect of bremsstrahlung radiation losses on the momentum-space distribution of fast electrons using several models. The contribution from the emission of large-energy photons (with $k > k_0$) are accounted for by either the Boltzmann operator in (3) or its approximation without angular deflection (4), while the low-energy photon contribution ($k < k_0$) is described by (5). For the numerical solutions we choose an energy-dependent cut-off $k_0 = m_e c^2(\gamma - 1)/1000$. We have found that this is sufficiently small that the results are not sensitive to the choice of this cut-off parameter. The cut-off, which determines when the emitted photons will be counted as 'low energy', and when the interaction is treated as elastic using the operator in (5), generally produces a relative error in the solution of order $k_0/[m_e c^2(\gamma - 1)]$.

The Boltzmann models will be compared to the mean-force model where the bremsstrahlung losses are accounted for by an isotropic force term in the kinetic equation, defined as $\mathbf{F}_{\mathrm{B}} = -\hat{\mathbf{p}}\sum_b n_b \int_0^{m_e c^2(\gamma-1)} \mathrm{d}k\, k\partial\sigma_{eb}/\partial k$, which is chosen to produce the correct average energy-loss rate [1].

Numerical results

To characterize the effect of bremsstrahlung on the electron distribution, we investigate quasi-steady-state numerical solutions of the kinetic equation (7). These are obtained by evolving the distribution function in time until an equilibrium is reached, typically after a few seconds at density $n_{20} = 1$ if an initial seed of fast electrons is provided (this equilibration time is directly proportional to n_e). This means that if, in reality, the duration of near-constant acceleration is shorter than this equilibration time scale, the amplitude of the runaway tail will be smaller than reported here. The qualitative features of the runaway distribution are however set up on a shorter time scale of a few hundred milliseconds at $n_{20} = 1$, and can be representative of a wider range of realistic scenarios.

We investigate a range of electric-field values near the minimum electric field $E_c = 4\pi \ln\Lambda n_e r_0^2 m_e c^2/e$ to overcome collisional friction [4], using plasma parameters characteristic of tokamak-disruption experiments with massive gas injection. We assume accumulated impurity densities to be of order $n_Z \sim 10^{20}\,\mathrm{m}^{-3}$ and that, for the ultrarelativistic electrons in the far tail of the distribution, the binding energy of the bound electrons is negligible. The electron density n_e then denotes the full electron density $n_e \sim n_{\mathrm{free}} + n_{\mathrm{bound}}$.

Figure 1 shows the electron distribution function in momentum space, calculated using CODE, with full Boltzmann bremsstrahlung effects included (black, solid); neglecting angular deflections in the large-k contribution (yellow, dash-dotted); also neglecting the small-k contribution (blue, dashed); and finally using the mean-force model (red, solid). Non-monotonic features form in the mean-force as well as the Boltzmann models, but their characteristics are significantly different. With the Boltzmann models, an extended tail forms in the electron distribution. In contrast, the mean-force model produces a sharp feature, located where the energy gain due to the electric-field acceleration balances friction and bremsstrahlung losses. The addition of low-k scatterings (5), which lead to large-angle deflections, causes a subpopulation of fast electrons with significant perpendicular momentum to form. Furthermore, (3) and (4) appear to generally produce the same qualitative features, indicating that scatterings involving large-energy photons are well approximated by neglecting the angular deflection of the electron.

Inclusion of synchrotron radiation losses associated with the gyromotion of electrons in a straight magnetic field has been shown to be an important energy-loss mechanism [17–21]. Figure 1(b) shows that, in conjunction with bremsstrahlung losses, synchrotron losses (modeled as in [17]) shift the distribution towards lower energies but does not change its qualitative features. The difference between the Boltzmann and mean-force models is reduced in such cases, as the extent of the distribution when full bremsstrahlung effects are included is reduced by the synchrotron effect. When bremsstrahlung losses are ignored, and synchrotron emission alone is responsible for the energy loss by radiation, a non-monotonic runaway tail can also form (solutions to this problem have been characterized in [17, 20]). However, for the present values of density, magnetic and electric fields this occurs at the significantly higher momentum [6, 20]

$$p \sim \frac{3n_e \ln\Lambda m_e^2 c}{2\epsilon_0 B^2 (Z+1)} \frac{E}{E_c}\left(\frac{E}{E_c} - 1\right) \approx 300 m_e c,$$

corresponding approximately to an energy of 150 MeV.

Angle-averages of the electron distribution functions in figure 1 are shown in figure 2 as a function of electron kinetic energy $W = m_e c^2(\gamma - 1)$. The bulk population ($W < 1\,\mathrm{MeV}$) has been excluded from the figure in order to highlight the differences in the shape of the tail of the runaway distribution, which is where the

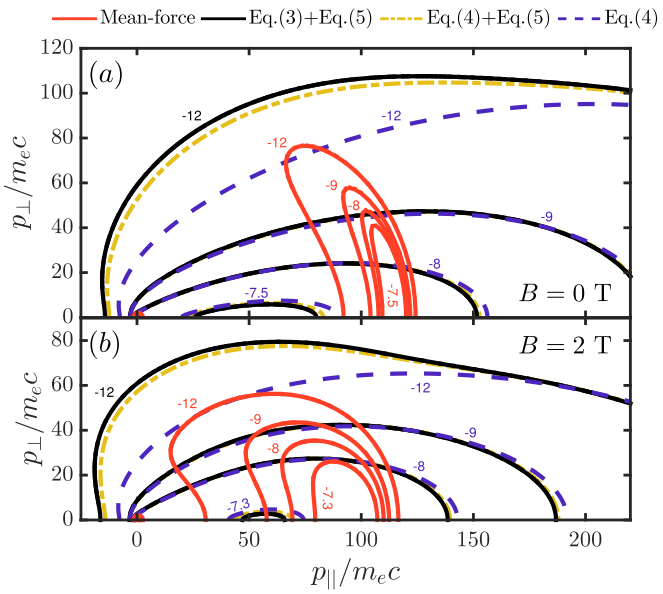

Figure 1. Steady-state electron 2D distribution functions; (*a*) with no magnetic field, (*b*) with $B = 2$ T. Electric field $E = 2E_c$, plasma parameters $Z_{\rm eff} = 10$, $n_{20} = 30$ and $T_e = 10$ keV. Contours show $\log_{10} F$, where $F = (2\pi m_e T_e)^{3/2} f_e / n_e$.

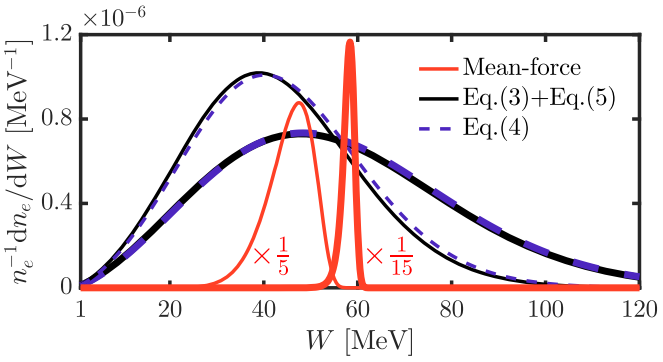

Figure 2. Angle-averaged tail of the electron distributions in figure 1, with $B = 0$ (thick lines) and $B = 2$ T (thin lines).

greatest variation between the different radiation-loss models can be seen. When there are no synchrotron losses present, the difference between the two Boltzmann models for bremsstrahlung losses is seen to be insignificant when considering the angle-averaged distribution. In the presence of effects which are sensitive to the angular distribution of electrons, such as synchrotron radiation losses (which are proportional to $p_\perp^2$), the difference is somewhat enhanced as angular deflection amplifies the dissipation.

To quantify the width in energy of the fast-electron tail, figure 3 shows the fraction of total plasma kinetic energy carried by electrons with energy greater than W, for a range of plasma compositions and electric fields, neglecting synchrotron losses. Again, the steady-state solutions are considered, and the energy ratio is calculated as $\int_W^\infty {\rm d}W\ W\,({\rm d}n_e/{\rm d}W)/W_{\rm tot}$. When normalized to the energy W_0 which solves the energy-balance equation $eE_\parallel - eE_c + F_{\rm B} = 0$ (accounting for collisional and bremsstrahlung energy loss), the behavior is seen to be insensitive to electric field and effective charge. The Boltzmann model consistently predicts that a fraction of the electron population reaches significantly higher energies than in the mean-force model, where all electrons have energy near W_0. For instance, in the Boltzmann model 5% of the plasma energy is carried by electrons with energy more than $2W_0$.

Summary

We have developed a kinetic description of the effect of spontaneous bremsstrahlung emission on energetic electrons in plasmas. By treating bremsstrahlung emission as a discrete process, we have shown that electrons

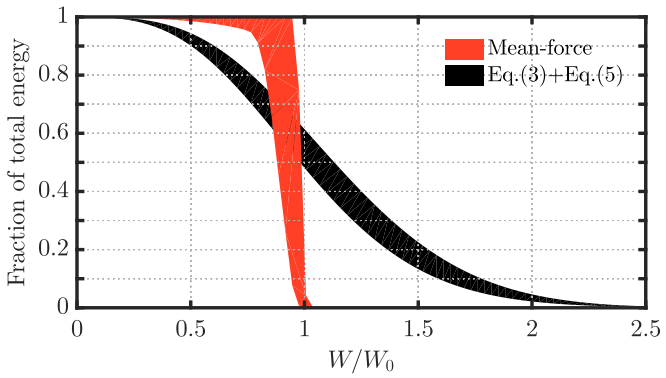

Figure 3. Fraction of plasma kinetic energy carried by electrons of energy greater than W. The values are calculated from numerical solutions of the kinetic equation, and are shown as a function of normalized electron kinetic energy. Filled regions indicate the values spanned when Z_{eff} and the normalized electric field $(E/E_c - 1)/(Z_{\mathrm{eff}} + 1)$ are varied between 1 and 35, and 0.05 and 0.25, respectively.

may be accelerated to significantly higher energies than would be predicted by energy balance alone, with a significant fraction of particles reaching at least twice the expected energy. This effect has important implications for the interpretation of experimental observation of fast electron beams in plasmas where bremsstrahlung losses are important, such as those in magnetic-confinement fusion. Since we have furthermore demonstrated that the features of the bremsstrahlung-loss dominated distribution function are insensitive to plasma composition and electric field, our findings may also be important in the study of other scenarios where runaway occurs, such as in lightning discharges and solar flares. The explanation for the increased maximum energy can be intuitively understood in the single-particle picture, where the new model allows some electrons to suddenly lose a large fraction of their energy in one emission, whereas other electrons may be accelerated for a long time before a bremsstrahlung reaction occurs, thereby allowing higher maximum energies to be reached.

Furthermore, new effects are revealed in our treatment, as the emission of soft photons is found to contribute to angular deflection of the electron trajectory at a rate that increases with electron energy. This effect shifts part of the momentum-space distribution function towards higher perpendicular momenta, which in turn has implications for e.g. the destabilization of kinetic instabilities or the level of synchrotron radiation loss in magnetized plasmas.

In order to resolve the logarithmically divergent contribution from low-energy photons, the bremsstrahlung collision operator is split into two contributions by introducing a cut-off photon energy $k_0 \ll m_e c^2(\gamma - 1)$. In the contribution from photons with energy $k < k_0$, the energy carried by the photons may be neglected, and the corresponding term in the kinetic equation is given by the elastic collision operator given in (5). However, both contributions must be treated with a Boltzmann collision operator in order to accurately capture the dynamics of the fast electrons. For the $k < k_0$ contribution, it is required as those interactions are dominated by large-angle deflections of the electron orbit, while the $k > k_0$ part requires it as the emitted photon causes a large change of the electron energy in each emission.

A computationally efficient representation of the bremsstrahlung collision operator has been obtained using an expansion in Legendre polynomials, with which the operator is reduced to a set of one-dimensional energy integrals. This allows for rapid evaluation of the self-consistent electron distribution function in the presence of bremsstrahlung losses derived from the full Boltzmann operator.

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