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The Efficiency and Evolution of R&D Networks

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Abstract

This work introduces a new model to investigate the efficiency and evolution of networks of firms exchanging knowledge in R&D partnerships. We first examine the efficiency of a given network structure in terms of the maximization of total profits in the industry. We show that the efficient network structure depends on the marginal cost of collaboration. When the marginal cost is low, the complete graph is efficient. However, a high marginal cost implies that the efficient network is sparser and has a core-periphery structure. Next, we examine the evolution of the network structure when the decision on collaborating partners is decentralized. We show the existence of multiple equilibrium structures which are in general inefficient. This is due to (i) the path dependent character of the partner selection process, (ii) the presence of knowledge externalities and (iii) the presence of severance costs involved in link deletion. Finally, we study the properties of the emerging equilibrium networks and we show that they are coherent with the stylized facts of R&D networks.

Key words: R&D networks, technology spillovers, network efficiency, network formation
JEL classification: D85, L24, O33

1. Introduction

R&D partnerships have become a widespread phenomenon characterizing technological dynamics, especially in industries (Hagedoorn, 2002) with rapid technological development such as, for instance, the pharmaceutical, chemical and computer industries (see Ahuja, 2000; Pammolli and Riccaboni, 2002; Powell et al., 2005; Roijakkers and Hagedoorn, 2006). In those industries, firms have become more specialized on specific domains of a technology and they tend to combine their knowledge with the one of other firms that are specialized in different domains (Ahuja, 2000; Powell et al., 1996).

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In this paper, we build a model in which firms innovate by recombining their knowledge with that of other firms in the industry, via a network of costly R&D collaborations. Within this framework, we first study the efficiency of a given network structure in terms of maximization of total profits in the industry. We characterize the topology of the efficient structure for any level of the marginal cost of collaborations in the relevant range. Next, we study the emergence of pairwise stable structures by employing the notion of "improving path" (cf. Jackson and Watts, 2002), and assuming that link deletion is subject to severance costs. We show the existence of multiple stable structures. In addition, we study the relation between network stability and efficiency. Finally, we investigate equilibrium selection under a two-sided myopic link dynamics and we show that the model is able to generate stable structures that match the properties of empirically observed R&D networks.

Our research is motivated by two different, albeit related, streams of literature on R&D collaborations. On the one hand, the increasing importance of R&D partnerships has spurred research, both theoretical and empirical, on the consequences of a given structure of the network of R&D collaborations for technology innovation and diffusion (see among many others Ahuja, 2000; Cowan and Jonard, 2004, 2007; Letterie et al., 2008). To this regard, an important and still unsettled debate concerns the relation between the position of a firm in the network and its performance, and, in particular, whether a densely interconnected network is more conducive to knowledge diffusion and innovation than a network with structural holes (i.e. displaying the presence of hubs indirectly connecting many firms which have no direct link across them). Indeed, clusters of densely and directly connected firms might be seen as fostering collaboration efforts among participants by generating trust and punishment of opportunistic behaviors, and a common language and problem solving heuristics (see e.g. Ahuja, 2000; Coleman, 1988; Cowan and Jonard, 2007; Walker et al., 1997). Conversely, by creating a structural hole in the network firms may have access to different sources of knowledge spillovers at the same time economizing on the costs of direct collaborations (cf. Burt, 1992; Gargiulo and Benassi, 2000; Rowley et al., 2000).

On the other hand, another body of contributions has investigated the salient features of empirically observed R&D networks (see e.g. Ahuja, 2000; Fleming et al., 2007; Hanaki et al., 2007; Powell et al., 2005; Roijakkers and Hagedoorn, 2006). These empirical studies have identified three main structural properties of innovation networks that are invariant across the different industries examined: (i) Networks are sparse, that is, from all possible connections between firms, only a small subset is realized. (ii) Networks are highly clustered, that is, they are locally dense. In clusters firms are closely interconnected but between different clusters there exist only few connections. (iii) The distribution of links over the firms tends to be highly heterogeneous with only few firms being connected to many others. Following this wave of empirical research, theoretical models have explored the emergence of R&D networks in a framework with firms being allowed to form any arbitrary pattern of bilateral R&D agreements (see Goyal and Joshi, 2003; Goyal and Moraga-Gonzalez, 2001, for an equilibrium approach, and Cowan et al., 2006, for an agent-based approach) . However, these models lead to network structures that are too simple to account for the stylized facts listed above.

Our paper contributes to the foregoing literature along several dimensions. *First*, we show that the network structure maximizing industry welfare (measured as the sum of firms' profits) is a function of the marginal cost of collaborations. In particular, we show that the efficient graph always belongs to a specific class of graphs (the class of nested-split graphs, see Definition (2) and Proposition (3)). Furthermore, when the marginal cost is low the efficient graph graph coincides with the complete graph, i.e. the one maximizing the number of direct ties . As marginal costs increase, it is efficient for the industry to organize into networks having a core-periphery structure. More precisely, at high level of collaboration costs, efficient networks display the presence of hubs, indirectly linking cliques of firms to otherwise disconnected nodes in the network. *Second*, and relatedly, we show that if marginal collaboration costs and the size of the industry are large enough, the efficient structure for the industry is characterized by significant inequality in profits across firms. In particular, firms having less (more) direct connections are also the ones displaying higher (lower) profits. In addition, profits inequality increases both in the number of firms and in the marginal cost of collaboration. *Third*, we study the relation between efficiency and equilibrium networks in our model. We show that multiple equilibrium structures for the same level of collaboration costs do arise in our model. In particular, we demonstrate that, for the same level of collaboration cost, both the spanning star (i.e. the star encompassing all nodes in the network), as well as the graph composed by disconnected cliques of the same size are possible equilibrium networks. The existence of multiple equilibria implies that efficiency is not necessarily met by equilibrium structures in our model. In addition, we identify the conditions on industry size and collaboration costs under which the efficient network never belongs to the set of possible equilibria. *Finally*, we study the properties of equilibrium structures in our model and we compare them with those of empirical R&D networks. More precisely we investigate the emergence equilibrium structures under a two-sided link formation/deletion mechanism (see Vega-Redondo, 2007, p. 212) in which firms are stochastically selected to revise their collaboration strategies. In this dynamics, firms decide to form a link if the link did not exist before and the link is beneficial to both of them, and decide to delete a link if the link existed before and deletion is beneficial to at least one of the agents selected. We show that under this dynamics the model is able to generate equilibrium structures matching the stylized facts of empirical R&D networks.

As we mentioned above, the possibility of recombining different knowledge stocks to introduce innovations in the industry is the rationale for R&D collaborations in our model (see Ahuja, 2000; Kogut and Zander, 1992; Powell et al., 1996; Weitzman, 1998). We formalize this idea by assuming that the arrival rate of innovations is proportional to the growth rate in the knowledge stock of the firm, and that firm's knowledge growth is a linear combination of the idiosyncratic knowledge stocks of the firm and the knowledge of its R&D partners. In the model, firm's expected profits are a linear function of the expected number of innovations per period and of the costs of R&D collaborations. Each R&D collaboration requires a fixed investment over each period. Total costs of collaboration are thus proportional to the number of collaborations (the degree) of the firm. Moreover, if the period over which collaborations are evaluated is long enough, the expected number of innovations per period turns out to be proportional to the largest eigenvalue of the adjacency matrix associated with the connected component to which the firm belongs (see Proposition (1) and Corollary (1)). This has several implications. First, as the largest eigenvalue is the same for all firms in the same component, the formation/deletion of a collaboration by a firm has a strong non-rival external effect on all its direct and indirect neighbors. Second, the magnitude of the change in eigenvalue, resulting from creating/severing a collaboration, varies with the topology of the network and the position of the two firms involved in the collaboration, thus implying a strong path-dependent character of partner's choice decisions. Finally, it can be shown that the largest eigenvalue is related to the number of *all* walks connecting firms in a given component. .

Our model can be related to the models in the network formation literature in which agents face a trade-off between the benefit they get from accessing the network and the cost of forming links with other agents (see e.g. Bala and Goyal, 2000; Carayol et al., 2008; Haller and Sarangi, 2005; Jackson and Wolinsky, 1996; Vega-Redondo and Goyal, 2007). To this regard, our model shares many similarities and differences with the "connections" model introduced in Jackson and Wolinsky (1996) and with the linear "two-way flow" model without decay introduced in Bala and Goyal (2000). For instance, similar to both models, the benefit an agent receives from the network derives also from indirect connections. In addition, such a benefit is non-rival [1] (see in particular Equation (10) and discussion thereafter). However, differently from both models, link deletion involves severance costs. Furthermore, differently from Jackson and Wolinsky's model the benefit the agent receives from the network does not only depend on the shortest path existing between the agent and its direct and indirect neighbors but it accounts for all possible walks existing among them. Next, differently from Bala and Goyal's linear model, link-formation is two-sided. In addition, the payoff of the agent does not only depend on the number of direct and indirect neighbors that can be reached by the agent with its existing connections, but also on *how* each neighbor can be reached. Incorporating all walks and severance costs in the network formation process has several implications for the results obtained. First, as efficiency is concerned, similar to Jackson and Wolinsky's model (and differently from Bala and Goyal's model) we obtain that the complete graph and the empty graph are efficient for, respectively, very low and very high values of the marginal cost of link formation. By contrast, differently from both models, first, the efficient graph for intermediate levels of the marginal cost of collaboration (i.e. the *nested-split graph*), is in general not minimally connected (i.e. more than one path exists between any two agents in the efficient graph). Second, stable graphs are not necessarily connected as they can consists of several disconnected components. Similar to both models, we obtain the spanning star as possible equilibrium network for intermediate levels of marginal cost of collaboration. However, differently from both models, this equilibrium coexists with an equilibrium consisting of the class of graphs composed of disconnected cliques of the same size.

The paper is organized as follows. Section 2 contains the description of the model, starting with the definition of the network of R&D collaborations across firms, and then moving to explain how firms profit from R&D collaborations, and the relations between our model and the others proposed in the literature. Section 3 is devoted to the analysis of the efficiency of R&D network structures and to the relation between efficiency and inequality in firms' profits. Network dynamics, the emergence of equilibrium networks and their properties are analyzed in Section 4. Finally, Section 5 concludes. All proofs can be found in the appendix.

[1] See Vega-Redondo and Goyal (2007) for a model where benefits from indirect connections are rival.

2. The Model

We consider an industry in which firms engage in pairwise R&D collaborations with other firms. Collaborations allow the growth of knowledge within the firm and an increase in the probability to introduce innovations that yield profits to the firm. We first define the network of R&D collaborations. Next, we characterize how the R&D network influences knowledge growth, innovation and profits of the firms. Finally, we briefly discuss the relations between our model and the relevant literature.

2.1. *The Network*

Consider an industry populated by n firms. The *network* [2] G is the pair (N, E) consisting of the set of nodes $N(G) = \{1, ..., n\}$, representing the population of firms and a set of edges $E(G)$, representing R&D collaborations among the firms [3] (for simplicity we may just write N and E where it is obvious to which network G the sets refer). An edge $ij \in E$, represents the existence of an R&D collaboration between firm i and j, which are said to be *adjacent*. A *subgraph* of G is a pair $G' = (N', E')$ such that $N' \subseteq N$, $E' \subseteq E$. The number of nodes is $|N| = n$ and the number of edges $|E| = m$. A *complete graph* K_n is a graph in which all n nodes are pairwise adjacent. The graph in which no pair of nodes is adjacent is the empty graph $\bar{K}_n$. A *clique* $K_{n'}$, $n' \leq n$, is a complete subgraph of the network G. In contrast to the clique, an *independent set* $\bar{K}_{n'}$ is a subgraph in which all n' nodes are not pairwise adjacent.

The *neighborhood* of i is the set $N_i = \{j \in N : ij \in E\}$. The *degree* of a node i in G, written by d_i, is the number of edges incident to i. Clearly, $d_i = |N_i|$. The maximum degree is $\Delta(G)$ and the minimum degree is $\delta(G)$. The *clustering coefficient* $\mathcal{C}_i$ of firm i is the proportion of links between the firms within its neighborhood N_i divided by the number of links that could possibly exist between them, i.e.

$$\mathcal{C}_i = \frac{2|\{jk : j, k \in N_i \wedge jk \in E\}|}{d_i(d_i - 1)}. \quad (1)$$

The total clustering coefficient is the sum of the clustering coefficients for each firm, $\mathcal{C} = \sum_{i=1}^{n} \mathcal{C}_i$.

A *walk* W_k of length k connecting firm i_1 and i_k is a sequence of firms $(i_1, i_2, ..., i_k)$ such that $i_1 i_2, i_2 i_3, ..., i_{k-1} i_k \in E$. A walk is *closed* if the first and last firm in the sequence are the same, and *open* if they are different. A *path* is a walk in which no firm is visited twice. A closed path encompassing n nodes is a *cycle*, denoted by C_n.

A *connected component* in G is a maximal set of firms such that there exists a path between any two of them. We will say that two components are *disconnected* if there is no path between them. A *connected graph* is a graph consisting of only one connected component.

Let $\mathbf{A}(G)$ be the symmetric $n \times n$ *adjacency matrix* of the R&D network G. The element $a_{ij} \in \{0, 1\}$ indicates if there exists a link between agent i and j such that $a_{ij} = 1$ if $ij \in E$ and $a_{ij} = 0$ if $ij \notin E$. The *eigenvalues* of the adjacency matrix $\mathbf{A}$ are

[2] In this paper we will use the terms graph and network interchangeably. The same holds for links and edges.

[3] We consider undirected graphs only.

the numbers λ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ has a nonzero solution vector, which is an *eigenvector* associated with λ. The term λ_{PF} denotes the *largest real eigenvalue* of $\mathbf{A}$ (the *Perron-Frobenius eigenvalue*, cf. Horn and Johnson, 1990; Seneta, 2006), i.e. all eigenvalues λ of $\mathbf{A}(G)$ satisfy $|\lambda| \leq \lambda_{\mathrm{PF}}$ and there exists an associated nonnegative eigenvector $\mathbf{v} \geq 0$ such that $\mathbf{A}\mathbf{v} = \lambda_{\mathrm{PF}}\mathbf{v}$. For a connected graph G the adjacency matrix $\mathbf{A}(G)$ has a unique largest real eigenvalue λ_{PF} and a positive associated eigenvector $\mathbf{v} > 0$.

Finally, for a graph with n nodes there are $\binom{n}{2}$ possible links and accordingly there are $2^{\binom{n}{2}}$ possible graphs on n nodes. We denote with $G(n, p)$ the *random graph* with n nodes, in which each of the possible links occurs independently with probability p. Similarly, $G(n, m)$ is the random graph with n nodes and m edges.

2.2. *Innovation and Profits from R&D Collaborations*

Firms exploit R&D collaborations to introduce innovations in the industry. Decisions over R&D partners are taken at discrete times $t = T, 2T, 3T, ...$ where the length of a period is given by $T > 0$. Innovations are introduced during each period $(t, t + T]$. The rewards from each innovation are assumed to be appropriable so that an innovation returns a value equal to the constant $V > 0$. Following the theoretical literature on innovation and endogenous technical change (see e.g. Aghion and Howitt, 1998; Reinganum, 1983, 1985; Winter, 1984), we assume that the introduction of innovations by a firm $i \in N$ is governed by a non-homogeneous Poisson process with arrival rate equal to $h_i(\tau)$, where $\tau \geq 0$ indicates the time variable within a period. Thus, the probability that an innovation is introduced by firm i in the interval $d\tau$, is equal to $h_i(\tau)d\tau$. Moreover we assume that, for any firm i, the arrival rate of innovations is proportional to the growth rate $\rho_i(\tau)$ of knowledge [4]

$$h_i(\tau) = b\rho_i(\tau), \quad b > 0. \tag{2}$$

In other words, the higher the growth rate of knowledge, the more likely it is that the firm will be able to innovate. Expected revenues of firm i in a period $(t, t+T]$ are given by the value V of each innovation times the expected number of innovations in the period. Note that in Equation (2) the innovation process starts anew at the beginning of every period $(t, t + T]$, taking as initial condition the stock of knowledge at the end of the previous period $(t-1, t-1+T]$. In addition, let us set $\tau \in (0, T]$. From Equation (2) the expected number of innovations in a period $(t, t + T]$ can be written as

$$\int_0^T h_i(\tau)d\tau = b\int_0^T \rho_i(\tau)d\tau. \tag{3}$$

In turn, the growth rate of knowledge is affected by the network of collaborations as follows. In each period $(t, t+T]$, new knowledge is generated by recombining the existing knowledge stocks of firms in the economy via the existing network of R&D collaborations (see Kogut and Zander, 1992; Weitzman, 1998). More precisely, let us denote by $x_i(\tau)$ the stock of knowledge of firm i at time $\tau \in (t, t + T]$. Then new knowledge within firm i is generated according to:

[4] Note that both, the innovation arrival rate $h_i(\tau)$ and the growth rate of knowledge $\rho_i(\tau)$ are flow variables and are measured per unit of time.

$$\dot{x}_i(\tau) = \sum_{j=1}^{n} a_{ij}(t) x_j(\tau), \tag{4}$$

where $a_{ij}(t)$ are the elements of the adjacency matrix $\mathbf{A}(G(t))$ (defined in Section 2.1) corresponding to the network of R&D collaborations[5]. In vector-matrix notation Equation (4) reads $\dot{\mathbf{x}}(\tau) = \mathbf{A}(G(t))\mathbf{x}(\tau)$. Note also that in Equation (4) for non-negative initial values of $\mathbf{x}(0) \geq 0$, we have that $\dot{\mathbf{x}}(\tau) \geq 0$ as well as $\mathbf{x}(\tau) \geq 0$.

The growth rate of knowledge of firm i, $\rho_i(\tau) = \dot{x}_i(\tau)/x_i(\tau)$, in Equation (4) is directly affected by the growth rate of knowledge of its neighbors, whose growth rate is affected by the growth rate of their neighbors, and so on. It turns out that the topology of the whole network of R&D collaborations (including all direct and indirect paths along which knowledge can flow between the firms), influences the innovation process within the firm.

Collaborations also imply a cost for firms. Within a period $(t, t+T]$ each collaboration involves a cost per unit of time equal to $\tilde{c}$. Moreover, we assume that firms are risk-neutral. Finally, if we denote by $G_i(t)$ the connected component to which firm i belongs in the period, then expected profits for the firm at the beginning of the period can be written as

$$\tilde{\pi}_i(G_i(t), c, t) = bV \int_0^T \rho_i(\tau) d\tau - \tilde{c} T d_i(t), \tag{5}$$

where $d_i(t)$ is the degree of the firm at time t and during the period.

The timing of events in each period $(t, t+T]$ runs as follows: at the beginning t of the period the network of R&D collaborations is determined (only one link is added or removed at time t), and remains fixed throughout the period $(t, t+T]$. During the period $(t, t+T]$, firms recombine their knowledge stocks through the network while they also bear the costs of their collaborations. As a result, innovations are introduced and the rents accrue to the firm.

The expression for expected profits in Equation (5) can be directly related to the structure of the network of collaborations. For this purpose, the next proposition establishes a relation between, on one hand, the asymptotic growth rate of ideas, the asymptotic relative stock of knowledge and the rate of convergence, and, on the other hand, the eigenvalues and eigenvectors of the adjacency matrix $\mathbf{A}(G_i(t))$ of the connected component of firm i.

Proposition 1 *Consider the eigenvalues $\lambda_{PF} = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ associated with the adjacency matrix $\mathbf{A}(G_i(t))$ of the connected component $G_i(t)$ of firm $i \in N(G_i(t))$. Then the following results hold:*

(i) The asymptotic knowledge growth rate of a firm i is constant and equal to the largest real eigenvalue (Perron-Frobenius eigenvalue) of the adjacency matrix $\mathbf{A}(G_i(t))$

$$\lim_{\tau \to \infty} \rho_i(\tau) = \lambda_{PF}(G_i(t)). \tag{6}$$

The rate of convergence is $\mathcal{O}\left(e^{-[\lambda_{PF}(G_i(t)) - \lambda_2(G_i(t))]\tau}\right)$ as $\tau \to \infty$.

(ii) The asymptotic value of a firm i's relative knowledge stock equals the element v_i of the eigenvector associated with the eigenvalue $\lambda_{PF}(G_i(t))$

[5] In Equation (4) we are assuming the process of creation of ideas at the firm level is cumulative, in that larger knowledge stocks (of the firm and of its collaborators) lead to higher knowledge growth. This property of knowledge dynamics has often been emphasized in innovation studies (see e.g. Dosi, 1988).

$$\lim_{\tau \to \infty} \frac{x_i(\tau)}{\sum_{j=1}^{n} x_j(\tau)} = v_i. \tag{7}$$

Item (i) of the above proposition states that the knowledge dynamics defined in Equation (4) converges, for a given R&D network, to a steady state characterized by a constant growth rate of ideas. In addition, such a constant growth rate depends on the topology of the connected component which the firm belongs to (through the largest eigenvalue $\lambda_{PF}(G_i(t))$). This implies that, in the steady state, the arrival rate of an innovation is constant and equal to $b\lambda_{PF}$. Moreover, item (ii) implies that the topology of the connected component $G_i(t)$ determines the distribution of relative values of the knowledge stocks of firms in the same component. Finally, the rate of convergence to the steady state is determined by the eigenvalues of[6] $A(G_i)$.

An important assumption of our model is that the growth of knowledge is much faster than the formation of R&D collaborations. This is equivalent to saying that τ is measured in time units much smaller than those used to measure t. In other words, $t = k\tau$, with k large. Under this assumption, the expected number of innovations per unit of time can be approximated (taking the limit $k \to \infty$) with the largest real eigenvalue of a firm's connected component.

Corollary 1 *The expected number of innovations of firm i per unit of time in a period $(t, t+T]$ tends to a limit proportional to the largest real eigenvalue of firm i's connected component G_i.*

$$\lim_{k \to \infty} \frac{b}{kT} \int_0^{kT} \rho_i(\tau) d\tau = b\lambda_{PF}(G_i(t)). \tag{8}$$

Expected profits of the firm at beginning of the period $(t, t+T]$ can now be written as

$$\tilde{\pi}_i(G_i(t), c, t) = b\lambda_{\mathrm{PF}}(G_i(t))VT - \tilde{c}d_i(t)T. \tag{9}$$

Applying an affine transformation to the above equation, we finally obtain expected profits per unit of time in the period between t and $t+T$,

$$\pi_i(G_i(t), c, t) = \lambda_{\mathrm{PF}}(G_i(t)) - cd_i(t), \tag{10}$$

where $c = \frac{\tilde{c}}{bV}$ is the marginal cost of link formation (rescaled by the factor $1/bV$)[7]. Since in Equation (10) the largest eigenvalue $\lambda_{\mathrm{PF}}(G_i(t))$ is the same for all firms in the same connected component, the expected revenues from R&D collaborations will be the same for all the members of G_i. Nonetheless, profits from R&D collaborations vary, in general, across firms, since each firm may have a different number of collaborations. The following lemma[8] characterizes the relation between the largest eigenvalue of a connected component and the creation or removal of R&D collaborations.

[6] In general, the convergence in a connected component to its largest real eigenvalue is always guaranteed. In addition, more dense networks are characterized by a faster convergence (see the proof of Proposition (1) in the appendix). However, the convergence can be slow for sparse networks and particular network topologies. For a recent application and discussion of the convergence properties of the social network matrices see Jackson and Golub (2007).

[7] The introduction of linear and homogeneous in-house R&D activities in Equation (4) for the dynamics of knowledge stocks would not alter the functional form of profits (up to a constant).

[8] A proof of the foregoing lemma can be found in Cvetkovic et al. (1995).

Lemma 1 *Denote $G' = (N', E')$ the graph obtained from the graph $G = (N, E)$ by the addition or removal of an edge. Then*

(i) $\lambda_{PF}(G') \geq \lambda_{PF}(G)$ *if* $ij \notin E$ *and* $\lambda_{PF}(G') \leq \lambda_{PF}(G)$ *if* $ij \in E$.

(ii) $\lambda_{PF}(G') \leq \lambda_{PF}(K_n) = n - 1$.

(iii) $|\lambda_{PF}(G') - \lambda_{PF}(G)| \leq 1$

Thus, the largest real eigenvalue in a component is a non decreasing function of the number of links. In addition, it is a bounded function, since its value can never be higher than the one associated with the complete graph K_n. Finally, the change in the eigenvalue is itself a bounded function, since its value must be less than one. The preceding observations deliver two central properties of the model. First, since the probability of innovation is the same for all the firms in a given connected component and it is affected by each link, the creation (deletion) of a collaboration by one firm has a positive (negative) non-rival external effect on all its direct and indirect neighbors in the component. As we will discuss in Section 4, this property is at the origin of the fact that the network can evolve into equilibria that are socially inefficient. Second, the marginal revenue from R&D collaborations is always a positive (albeit bounded) function of the number of links. This means that the creation (deletion) of a new R&D collaboration increases (decreases) the probability of an innovation and thus the expected revenue. Moreover, the revenue itself is a bounded function of the number of links. The last property does not imply that the revenue is also a concave function of the number of links [9]. However, as we will show in Section 4, it implies that, as the network grows in the number of links, the highest marginal revenue that can *actually* be obtained from the creation of a new link or from the removal of an existing link can become very small. It turns out that, when the highest marginal revenue from a collaboration that can be obtained is smaller than the marginal cost of collaboration, the network reaches an equilibrium, and this may happen well before the network has grown to a fully connected graph.

2.3. *Relation to the Literature on Network Formation*

The profit Equation (10) can be compared to other similar utility functions in the literature that feature a dependence on the position of a firm in the network. For instance, the utility function proposed in the "connections" model in Jackson and Wolinsky (1996) is given by

$$u_i = \sum_{j=1}^{n} \delta^{d(i,j)} - cd_i, \tag{11}$$

where $0 < \delta < 1$ and $d(i, j)$ is the length of the shortest path from node i to node j.

The difference between the profit function in (10) and the utility function in (11) becomes apparent in the benefit term. While Equation (11) considers the shortest path between firm i and j only, our model instead takes into account all possible walks from firm i to the other firms in the connected component [10]. Recall that, in our model, a

[9] Incidentally, note that λ_{PF} is not even determined as a function of m, because, for a given m, there are many different ways to arrange the links among the nodes, resulting in different values of λ_{PF}.

[10] It has been argued that in several settings paths that are not the shortest may have a big impact on the information that is transmitted from one agent to another (see e.g. Stephenson and Zelen, 1989; Wasserman and Faust, 1994). Moreover, empirical studies on R&D networks (cf. Powell et al., 2005) bring

walk represents a sequence of recombination of the knowledge of the firms along that walk. Not any recombination of knowledge might translate into a successful innovation. However, the more walks there are in the component, the higher is the number of possible knowledge recombinations available. It turns out that the likelihood for a successful innovation is increased. Indeed, the largest eigenvalue $\lambda_{\mathrm{PF}}(G_i)$ of the adjacency matrix of a connected component G_i, is related to the number of possible walks in that component (more precisely, the growth rate in the number of walks of length k tends to $\lambda_{\mathrm{PF}}(G_i)$; this property has been further elaborated in König et al. (2008)). Thus, the larger is $\lambda_{\mathrm{PF}}(G_i)$, the larger is the number of possible knowledge recombinations via direct and indirect R&D collaborations. From Equation (10) we can conclude that profits of firm i grow with the number of walks in the connected component to which firm i belongs. On the other hand, profits decrease with the degree d_i of the firm. Therefore, it is best for a firm to be able to reach the other firms through many walks but to have not too many links to pay for. This observation becomes apparent if one considers the following simple example. The revenues of the hub in a star $K_{1,n-1}$ and a node in a complete graph K_n in Equation (11) are identical, because the shortest paths to all the other nodes are one link long in both cases. This is not the case in our model where these two graphs generate very different revenues. A node in the complete graph can reach the other nodes through many different paths and this generates a much higher revenue than the one of the hub in a star.

In their linear "two-way flow" model (without decay) Bala and Goyal (2000) introduce a utility function of the form

$$u_i = |G_i| - cd_i, \tag{12}$$

where $|G_i|$ is the size of the connected component of firm $i \in N(G_i)$. This means that the utility of firm i grows with the number of all firms in the network who can be reached by firm i across at least one path. The number of links and the number of paths between i and the other firms do not matter because the benefit flow across the network is assumed to be independent of its topology. In contrast, in our model, the topological properties of the component the firm belongs to are critical for the profits of the firm. Consider the following simple examples. According to Equation (12), revenues for a firm in the complete graph K_n, in the clique $K_{1,n-1}$ and in the cycle C_n, are identical. However, in our model the revenues a firm earns from being part of a clique K_n are higher than in a star $K_{1,n-1}$, which in turn are higher than in a cycle C_n for the same number of collaborations (see also Table (1), and the discussion in Section 4.2). This ranking can be understood if one considers the possible walks in these graphs. The number of walks is highest in the complete graph K_n while it is smallest in the cycle C_n (that contains only one walk). While all these graphs encompass the same number of firms, they differ significantly in the way the links are arranged among the firms.

3. Efficiency

In the model presented in the previous section, firms face a trade-off between increasing the probability to innovate by forming R&D collaborations and the cost of sharing knowl-

support to the claim that firms are forming R&D collaborations in a way that increases the number of walks in the network.

edge with other firms in the industry. In this section we investigate how this trade-off can be managed in order to yield the best outcome from the industry point of view. First, we show that there exists an interval of the marginal cost of link formation, $c \in [0, 1]$, in which the network that maximizes social welfare, that is the efficient graph, is a connected graph. We will show in Section 4 that this interval is the one of main interest since for values above this interval, $c > 1$, firms do not have the incentive to form any additional collaboration.

We then investigate the topology of the efficient graph, and we show that it belongs to a well defined class of connected graphs, the "nested split graphs". In particular, for c small enough, the efficient graph is the complete graph. On the other hand, for higher values of c and a larger number of firms, the efficient graph is sparser and shows a strong degree heterogeneity. In addition, we show that it is characterized by significant inequality in profits.

3.1. *Efficient Networks*

Following Jackson and Wolinsky (1996), we define industry welfare as the sum of firms' individual profits

$$\begin{aligned}\Pi(G,c) &= \textstyle\sum_{i=1}^{n} \pi_i(G_i) \\ &= \textstyle\sum_{i=1}^{n} (\lambda_{\mathrm{PF}}(G_i) - cd_i) \\ &= \textstyle\sum_{i=1}^{n} \lambda_{\mathrm{PF}}(G_i) - 2mc.\end{aligned} \tag{13}$$

We are interested in the solution of the following social planner's problem. Let $\mathcal{G}(n)$ denote the set of all possible graphs with n nodes. For a given value of cost c, the social planner's solution is given by

$$G^* = \underset{G \in \mathcal{G}(n)}{\operatorname{argmax}} \quad \Pi(G,c). \tag{14}$$

A graph G^* solving the maximization problem (14), will be denoted as "efficient".In order to solve this problem we begin by identifying an interval for the marginal costs c in which industry welfare is increased by connecting two disconnected components of the network. The following lemma can be stated.

Lemma 2 *Consider a graph G consisting of two disconnected components G_1 and G_2, with n_1, n_2 nodes, m_1, m_2 edges, eigenvalues $\lambda_{PF}(G_1)$, $\lambda_{PF}(G_2)$ and total profits $\Pi(G_1) = n_1\lambda_{PF}(G_1) - 2m_1c$, $\Pi(G_2) = n_2\lambda_{PF}(G_2) - 2m_2c$. We further assume that $c \in [0, 1]$. Then there exists a connected graph G' with $n = n_1 + n_2$ nodes that has higher total profits than G, that is $\Pi(G') \geq \Pi(G) = \Pi(G_1) + \Pi(G_2)$.*

Thus, for $c \in [0, 1]$, connecting two previously disconnected components of the graph yields total profits larger than the respective total profits of the disconnected components. From this it follows immediately that the efficient network is connected.

Proposition 2 *Let $\mathcal{H}(n, m)$ denote the set of connected graphs having n nodes and m links. If $c \in [0, 1]$ then $G^* \in \mathcal{H}(n, m)$.*

This means that, in order to guarantee efficiency, each firm must have (direct or indirect) access to the knowledge of all other firms in the industry.

Since the efficient graph is connected, Equation (13) for total profits simplifies to

$$\Pi(G, c) = n\lambda_{\mathrm{PF}}(G) - 2mc. \tag{15}$$

This implies that, for any given values of n and m, the efficient graph is also the one with maximal λ_{PF}. In other words, for $c \in [0, 1]$, the efficient graph G^* belongs to the set of connected graphs that maximize $\lambda_{\mathrm{PF}}(G)$, denoted by $\mathcal{H}^*(n, m)$. As a result, the efficient graph belongs to a special class of graphs characterized by well defined topological properties[11]. In order to fully describe these properties we first need to introduce some basic definitions.

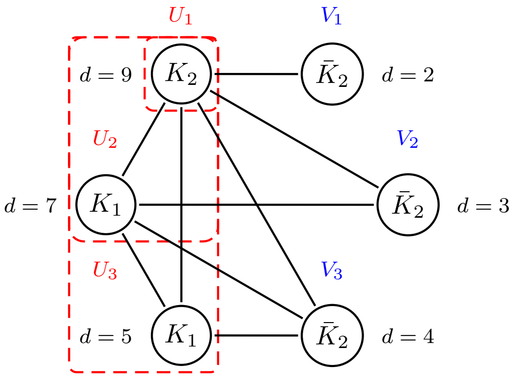
Brualdi and Solheid (1986) show that the graphs in the set $\mathcal{H}^*(n, m)$ have a stepwise adjacency matrix $\mathbf{A}$, defined as follows:

Definition 1 *In a stepwise matrix* $\mathbf{A}$*, the elements* a_{ij} *satisfy the following condition. If* $i < j$ *and* $a_{ij} = 1$*, then* $a_{hk} = 1$ *whenever* $h < k \leq j$ *and* $h \leq i$*.*

The above definition says that if the adjacency matrix has an element equal to one, $a_{ij} = 1$, then also the element above in the matrix is one, $a_{i,j-1} = 1$, and the element to the left in the matrix is one, $a_{i-1,j} = 1$. Consequently, all the preceding elements to the left and above are one. In this way, the one-elements are separated from the zero-elements in the adjacency matrix along a line which has the form of a step function. This fact has brought about the name stepwise matrix. An example of a stepwise matrix is shown in Figure (1, right). The graphs associated with a stepwise adjacency matrix are called *nested split graphs* (Aouchiche et al., 2008). Nested split graphs have a nested neighborhood structure: the set of neighbors of each node is contained in the set of neighbors of the next higher degree nodes. However, before providing the formal definition of these graphs, we provide an intuition of its structure with the help of the representation in Figure (1, left). In particular, we consider a nested split graph that is also connected, since this will be our focus later on. First, the nodes in a nested split graph can be partitioned in subsets of nodes with different properties. In Figure (1, left) each circle represents a subset of nodes (and not an actual node of the network). Furthermore, we denote the partition of the graph as $\mathcal{P} = U \cup V$, where U and V consist of subsets, $U = \{U_1, U_2, ..., U_k\}$ and $V = \{V_1, V_2, ..., V_k\}$ respectively. Recall the notation from Section 2.1 in which K_n denotes the complete graph with n nodes and $\bar{K}_n$ the empty graph with n isolated nodes. Then, for example, in Figure (1, left) the sets are $U_1 = K_2, U_2 = K_2 \cup K_1$ and $U_3 = K_2 \cup K_1 \cup K_1$ and $V_1 = \bar{K}_2, V_2 = \bar{K}_2$ and $V_3 = \bar{K}_2$ respectively. Of course, the subgraph K_2 is simply a complete graph since it contains only two nodes, and even more so K_1 is a subgraph consisting of a single node only.

The subsets U_i and V_i differ in the fact that in U_i all nodes are connected to each other while in V_i there exist no links between the nodes. Moreover, there exist also links between nodes belonging to different subsets. Indeed, the neighborhood of the nodes in each set V_i is precisely the set U_i. In Figure (1, left) a line between two subsets indicates that there exists a link between each node in one subgraph to each node in the other subgraph. For example the nodes in V_1 at the top right of the figure are all connected to the nodes in U_1 at the top left. Similarly, the nodes in V_2 are connected to the nodes in U_2 and the nodes in V_3 are connected to the nodes in U_3. Additionally, the set U as well

[11] The efficient networks are similar to those obtained in the model of Ballester et al. (2006). More precisely, within that model Corbo et al. (2006) show that the networks that maximize welfare are given by the graphs with maximal eigenvalue.



$$\mathbf{A} = \begin{pmatrix} 0&1&1&1&1&1&1&1&1&1\\ 1&0&1&1&1&1&1&1&1&1\\ 1&1&0&1&1&1&1&1&0&0\\ 1&1&1&0&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0 \end{pmatrix}$$

Fig. 1. Representation of a connected nested split graph (left) and the associated adjacency matrix (right) with $n = 10$ nodes. A nested split graph can be partitioned into subsets of nodes with the same degree (each subset is represented as circle, the degree d of the nodes in the subset is indicated). A line connecting two subsets indicates that there exists an edge between each node in one set and all the nodes in the other set. The boxes around the sets indicate the cliques $U_1 = K_2, U_2 = K_2 \cup K_1$ and $U_3 = K_2 \cup K_1 \cup K_1$. In the matrix to the right the zero-entries are separated from the one-entries by a stepfunction.

as any union of the subsets in U form a complete subgraph or clique. Similarly, any union of the sets in V form an independent set. Notice also, that all the nodes in one set have the same degree. Next to the sets in Figure (1, left) the degree of the nodes in a subset is indicated. The degree of a node in a set can be easily derived from the adjacency matrix shown in Figure (1, right) by counting the number of ones in a row corresponding to a particular node in a set. For example the set K_2, top left in the figure, corresponds to two nodes whose links are indicated in the first two rows of the adjacency matrix.

With the preceding discussion in mind, we can now give a more formal definition of a nested split graph(see Cvetkovic et al., 2007).

Definition 2 *In a nested split graph, the set of nodes have a partition $\mathcal{P} = U \cup V$ with the following properties.*

(i) U induces a clique, and V induces an independent set. This also holds for any union of subsets in U and V.

(ii) U has subsets $U_1, ..., U_k$ such that $U_1 \supset ... \supset U_k$ and the neighborhood of each node in V_i is U_i, for any $i = 1, ..., k$.

If a nested split graph is connected we call it a *connected nested split graph*. As we mentioned above, the representation and the adjacency matrix shown in Figure (1) actually show a connected nested split graph. From the stepwise property of the adjacency matrix it follows that a connected nested split graph contains at least one spanning star, that is, there is at least one node that is connected to all other nodes. This property can also be seen in Figure (1), where the first row of the adjacency matrix that is entirely filled with ones indicates the presence of a spanning star[12].

We have shown that G^* is connected and we know that G^* has a stepwise adjacency matrix. From the above discussion we can further conclude that G^* is a connected nested split graph and it contains at least one spanning star as a subgraph.

[12] A related type of graph, so called *inter-linked stars*, has been introduced by Goyal and Joshi (2003). However, this notion does not specify the nested neighbourhood structure that characterizes nested split graphs.

The determination of the exact topology of G^*, for given n and c, is simple for $c \in [0, 1/2]$ (see Proposition 3). In contrast, for $c > 1/2$ this problem requires the determination of the graph with the largest eigenvalue among all graphs in $\mathcal{H}^*(n, m)$ (for a fixed n and arbitrary m, with $n-1 \leq m \leq n(n-1)/2$). This is still an unresolved research problem in Spectral Graph Theory (Aouchiche et al., 2008). However, it turns out that the value of total profits associated with the efficient graph G^* can be approximated by the total profits associated with a special type of a connected nested split graph. Following Bell (1991) we denote this graph by $F_{n,d}$.

Definition 3 *$F_{n,d}$ is the graph obtained from the complete graph K_d with d nodes and a subset of $n-d$ disconnected nodes, by adding $n-d$ links connecting one node in K_d to each of the $n-d$ disconnected nodes.*

Notice that the complete graph and the spanning star are particular cases of connected nested split graphs (and of the graph $F_{n,d}$): the star is $K_{1,n} = F_{n,1}$ and the complete graph is $K_n = F_{n,n}$. Figure (3) shows several examples of this type of graph for $n = 10$. Moreover, the number of edges in $F_{n,d}$ is given by $m = \binom{d}{2} + (n-d)$.

As discussed in more detail in the proof of the next proposition, the maximum relative discrepancy of total profits between $F_{n,d}$ and the efficient graph G^* is considerably small and vanishes for large n. For example with $n = 100$ we get get an error below 2%, while for $n = 200$ the error is below 1% (see also Figure 4 top left). The higher is the number n of firms, the more total profits of $F_{n,d}$ get close to total profits of G^*. Thus, in order to determine the efficient network G^*, if n is small one can search through all connected nested split graphs and identify the one with highest total profits, while for large n, one can use $F_{n,d}$ as a good approximation.

Bringing the above results together, we can state the following proposition which characterizes the topology of the efficient graph G^* with n firms in the industry as a function of the marginal cost of collaboration $c \in [0, 1]$.

Proposition 3 *Let G^* be the efficient graph for a given number n of firms and $F_{n,d}$ be the graph introduced in Definition (3).*

(i) If $c \in [0, 1]$ then G^* is a connected nested split graph.

(ii) Denote the relative error in total profits between the the efficient graph and the graph $F_{n,d}$ as $\epsilon = (\Pi(G^*) - \Pi(F_{n,d}))/\Pi(F_{n,d})$. If $c \in [0, 1]$, then the relative error is bounded from above as follows

$$\epsilon \leq \frac{2c(2c-1)n - 5c^2}{n^2 + 2c(1-2c)n + 9c^2}, \tag{16}$$

and vanishes for large n, i.e. $\lim_{n\to\infty} \epsilon = 0$.

(iii) If $c \in [0, 0.5]$ then G^* is the complete graph K_n.

(iv) If $c > n$ then G^* is the empty graph $\bar{K}_n$.

Figure (2) gives a graphical representation of the results on network efficiency in Proposition (3). In Figure (3) the efficient graphs for values of cost $c \in [0, 1]$ and system size $n = 10$ are shown [13]. We observe that, with increasing marginal cost, the efficient network becomes more sparse and the degree heterogeneity is increasing. For any value of

[13] For the particular case of $n = 10$ the connected graph with maximal eigenvalue is known (Aouchiche et al., 2008) and so is the efficient network G^*. In this case, the efficient graph is $F_{n,d}$ itself (without any approximation).

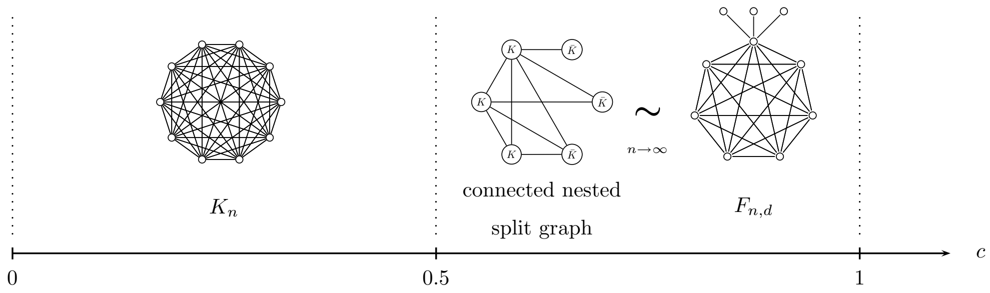


Fig. 2. Illustration of the range of efficient graphs as a function of the cost of collaboration. For costs $0 \leq c \leq 0.5$ the efficient graph is the complete graph K_n. In the region $0.5 < c \leq 1$ the efficient graph is a connected nested split graph. As indicated in the figure, for n large, $F_{n,d}$ can be seen as an approximation of G^* (with a vanishing relative error in total profits).

cost larger than 0.5 the efficient network consists of a densely connected cluster (clique) and one node that acts as a hub (star) and connects the remaining nodes to the cluster.

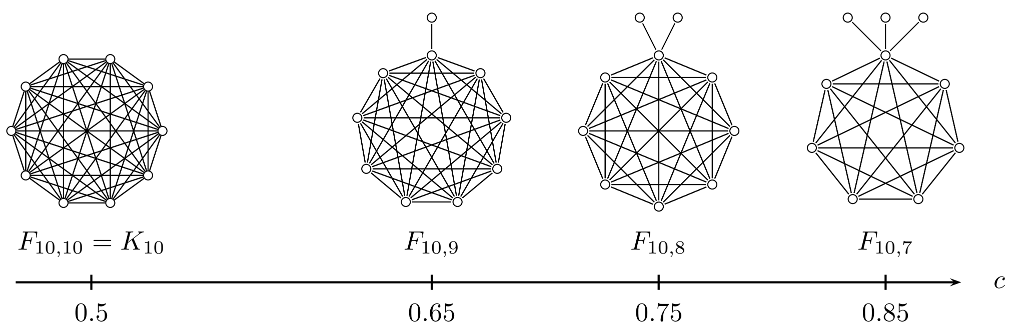


Fig. 3. Efficient graphs for values of cost $c = 0.5, 0.65, 0.75, 0.85$ and $n = 10$. The density of the efficient graph is decreasing and the degree heterogeneity is increasing with increasing cost.

If we consider $F_{n,d}$ as the efficient network, we can make the following observation. From the topological structure of $F_{n,d}$ it follows that, when marginal cost of link formation is high, it is efficient to concentrate knowledge creation in a small and dense cluster with one firm acting as a hub that connects all the peripheral firms to the cluster. As the marginal cost of link formation decreases, knowledge recombination becomes cheaper [14] and it is efficient that a larger fraction of firms takes part in the densely connected cluster. Finally, in the region of small marginal cost, $0 \leq c \leq 0.5$, it is efficient that all firms take part in a densely connected cluster, therewith establishing as many collaborations as possible. In this case, the fully connected graph encompassing all firms is the one in which total profits in the industry attain their highest possible value.

[14] Note that for $c = 0$ the problem in (14) can be reduced to the problem of maximizing total knowledge growth in the steady state for a given number of firms, in which case the complete graph is the solution.

An important final remark concerns the relation to the efficient graphs found in models akin to ours. Similar to both Jackson and Wolinsky (1996) and Bala and Goyal (2000), we find that the efficient graph is always connected and that it includes, depending on the cost c, the star and the complete graph. However, differently from the model of Bala and Goyal, the efficient graph is in general not minimally connected (removing one link does not necessarily make the graph disconnected). Moreover, differently from the model of Jackson and Wolinsky, in our model the set of efficient graphs is not limited to the star and the complete graph, but it includes a whole class of graphs that can be seen as intermediate graphs between these two extreme cases.

3.2. *Efficiency and Profits Distribution*

Former works on R&D networks (see Cowan and Jonard, 2004) have emphasized the emergence of a trade-off between efficiency (in terms of knowledge diffusion) and inequality (in terms of knowledge levels). A similar trade-off between efficiency and profits inequality emerges also in this model if the marginal cost of link formation and the number of firms operating in the industry are high enough. We measure inequality in profits in terms of profit variance.

In Proposition (2) we have shown that for $c \in [0, 1]$ the efficient graph is connected, $G^* \in \mathcal{H}(n, k)$. Thus, the returns from collaborations in an efficient graph are identical for all firms (since they have the same largest real eigenvalue) but the cost is different and is proportional the degree of the firm. More formally, let us define by σ_π^2 the variance of profits associated with the graph G. It follows for a graph $G \in \mathcal{H}(n, m)$ that

$$\sigma_\pi^2(G) = c^2 \sigma_d^2(G), \tag{17}$$

where σ_d^2 is the degree variance. Since degree is by definition homogeneous in a complete graph, from Proposition (3) it follows that for $c \leq 0.5$ profits inequality is zero, and no tension between efficiency and equality arises.

For higher values of costs, we can take $F_{n,d}$ as a sufficient approximation to the efficient network G^*, and from the properties of $F_{n,d}$ we can conclude that the efficient network is characterized by considerable degree heterogeneity and profits inequality. More precisely, the following proposition can be stated.

Proposition 4 *Let $F_{n,d}$ be the graph defined in (3) and $\bar{d} = 2m/n$ the average degree. Then*

(i) The degree variance is growing quadratically with the number of firms, i.e.

$$\sigma_d^2(F_{n,d}) = \mathcal{O}(n^2). \tag{18}$$

(ii) Let $c > 0.5$. For large n the coefficient of variation of degree, $c_v(F_{n,d}) = \sigma_d(F_{n,d})/\bar{d}$, tends to a constant depending on the cost,

$$\lim_{n \to \infty} c_v(F_{n,d}) = \sqrt{2c - 1}. \tag{19}$$

(iii) Consider the random graph $G(n, m)$ with n nodes and m links. For large n, the degree variance of the graph $F_{n,d}$ is larger by a factor n than the variance of a random graph with equal number of nodes and links

$$\sigma_d^2(F_{n,d})/\sigma_d^2(G(n, m)) = \mathcal{O}(n). \tag{20}$$

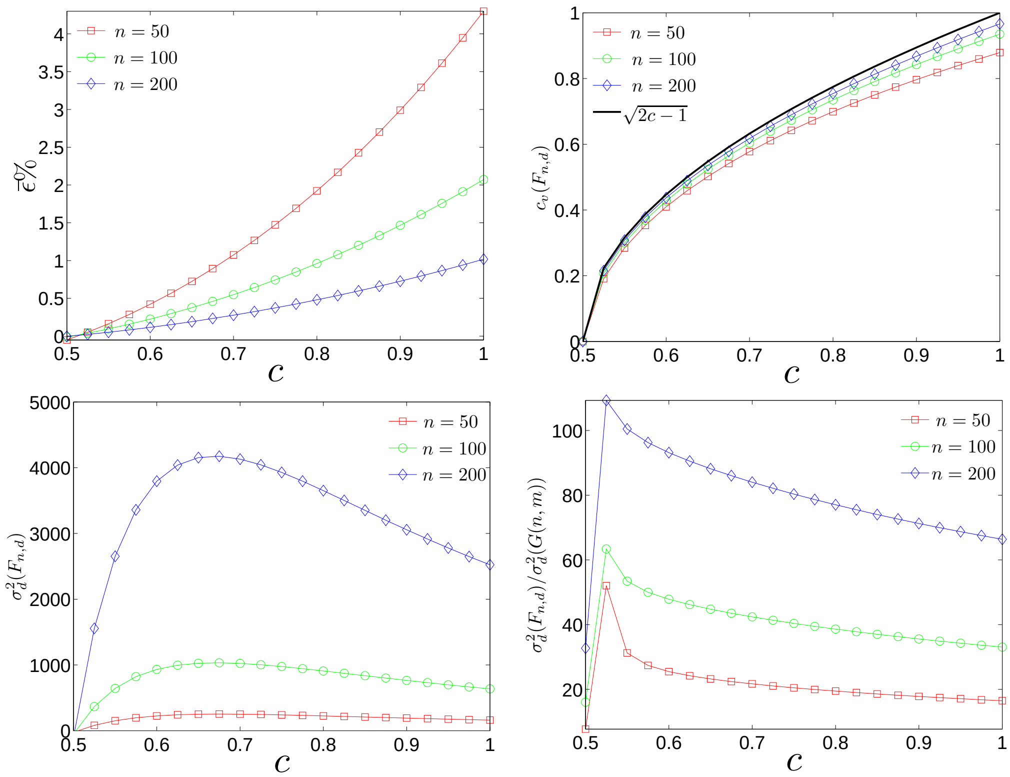


Fig. 4. Properties of the graph $F_{n,d}$ as a function of the cost of collaboration. Upper bound $\bar{\epsilon}$ on the relative error ϵ in the approximation of the efficient graph G^* (top, left); degree variance $\sigma_d^2(F_{n,d})$ (bottom, left); degree coefficient of variation $c_v(F_{n,d})$ (top, right); ratio of degree variance of $F_{n,d}$ and degree variance of a random graph $G(n,m)$ of the same size and density, $\sigma_d^2(F_{n,d})/\sigma_d^2(G(n,m))$ (bottom, right) for $n = 50$, $n = 100$, $n = 200$ and cost $c \in [0.5, 1]$.

The results of this proposition are illustrated in Figure (4). The coefficient of variation of degree, c_v, increases with increasing cost (Figure (4), top-left). It also increases with the number of firms up to the finite limit of $\sqrt{2c-1}$ for large n. Moreover, the degree variance of $F_{n,d}$ is many times larger than the degree variance of a random graph $G(n,m)$ with the same density (Figure (4), bottom-left). Since, Equation (17) links the variance in profits to the variance in degree it follows that, for higher values of marginal cost $0.5 < c \leq 1$, the industry displays an inequality in profits significantly larger than the one that could be observed if collaborations would be formed at random.

4. Network Evolution

The analysis contained in the previous section assumes that the structure of the network is fixed. In this way, it is possible to study which network topologies maximize industry welfare. In this section we depart from this static network perspective, and we investigate how the structure of the network evolves whenever firms are allowed to endogenously choose the partners with whom they want to collaborate.

Following Jackson and Wolinsky (1996) we consider a network formation process in which the creation of a new link requires the bilateral agreement of the two parties involved. However, the deletion of a link requires the unilateral decision of one of the two firms only. Consistently, as network equilibrium criterion, we adopt the definition of pairwise stability, as in Jackson and Wolinsky (1996). Based on this definition of stability, we derive the conditions on the value of cost for which structures like the empty graph, the complete graph or the star are stable. Among the possible stable graphs, we find also a disconnected graph consisting of multiple cliques of the same size. A first important finding here is the co-existence of multiple equilibrium networks for the *same* value of cost.

However, these relatively simple structures are not the only stable networks emerging in our model. Since it is increasingly difficult to derive general proofs of stability for more complex structures, we follow the argument in Vega-Redondo (cf. 2007, p. 208) and we perform a dynamic study of network stability. We model explicitly the evolution process in which, at the beginning of each period, a pair of firms decides whether to form or delete a link, based on the expected profits this action brings about. This investigation, performed through computer simulation, shows that there exist a multitude of complex structures which are pairwise stable. Remarkably, these networks display topological properties that are consistent with the stylized facts of R&D networks in a region of the parameters of the model.

4.1. *Improving Paths and Equilibrium Networks*

We consider a process of network evolution in which firms form or delete one link at a time based on the marginal profits they expect from that action. In other words, new links are created whenever the increase in the probability of innovation, i.e. the marginal revenue of a new collaboration, is greater than the marginal cost of a collaboration, with the gain being strict for at least one of the firms in the selected pair. Likewise, link deletion occurs whenever the saving in marginal cost from removing a collaboration are enough to compensate for the decrease in marginal revenue. However, given its unilateral nature, we assume that removing a collaboration involves severance costs [15] so that the savings in marginal costs from removing a collaboration is reduced by a factor α.

Following Jackson and Watts (2002), we call *improving path*, a sequence of networks $\{G(t)\}_{t\in\mathbb{N}_+}$ such that (i) any two consecutive networks, $G(t)$ and $G(t+1)$, differ by one link only, (ii) if the link is added, both firms benefit from the new link, at least one of them strictly, and (iii) if a link is deleted, at least one of the two firms strictly benefit from the deletion .

Improving paths emanating from any initial network must either lead to an equilibrium network structure, in which no pair of firms has an incentive to form a link, and no single firm has an incentive to remove a link, or to a cycle, in which a finite number of networks is repeatedly visited (see Lemma (1) in Jackson and Watts, 2002). In this section we investigate the existence of both equilibrium networks and cycles.

[15] These severance costs can be associated with the legal procedures needed to unilaterally bring a contract to an end, or it can have a different nature, e.g. they can be associated with the loss of reputation for managers breaking long-lasting collaborations.

Let G denote the current graph $G(t)$ at time t. Further, denote by $G + ij$ the graph obtained from G by adding the edge ij. Similarly, let $G-ij$ denote the graph obtained by removing the edge ij. Denote by $\lambda_i(G)$ the largest eigenvalue $\lambda_{\mathrm{PF}}(G_i)$ of the connected component G_i to which the firm i belongs. Note that, although link deletion implies that the degree of i is reduced by one (and so is the cost for firm i), the firm saves only a fraction of the cost due to the presence of the severance costs $v(c) = (1 - \alpha)c$. Thus, the change in profits of firm i induced by the removal of a link are given by:

$$
\begin{aligned}
\pi_i(G - ij) - \pi_i(G) &= \lambda_i(G - ij) - (d_i - 1)c - v(c) - (\lambda_i(G) - d_i c) \\
&= c - v(c) - (\lambda_i(G) - \lambda_i(G - ij)) \\
&= \alpha c - (\lambda_i(G) - \lambda_i(G - ij))
\end{aligned}
\tag{21}
$$

where $\alpha \in [0, 1]$. Obviously, the firm will only remove a link if this action increases her profits. With the above notation we can now give the definition of a pairwise stable network.

Definition 4 *The graph G is pairwise stable if*

(i) $\forall ij \in E(G)$, $\pi_i(G) \geq \pi_i(G - ij)$ *and* $\pi_j(G) \geq \pi_j(G - ij)$ *or, equivalently,* $\forall ij \in E(G)$, $\lambda_i(G) - \lambda_i(G - ij) \geq \alpha c$ *and* $\lambda_j(G) - \lambda_j(G - ij) \geq \alpha c$

(ii) $\forall ij \notin E(G)$, *if* $\pi_i(G + ij) > \pi_i(G)$ *then* $\pi_j(G + ij) < \pi_j(G)$, *and, if* $\pi_j(G + ij) > \pi_j(G)$ *then* $\pi_i(G+ij) < \pi_i(G)$ *or, equivalently,* $\forall ij \notin E(G)$, *if* $\lambda_i(G+ij)-\lambda_i(G) > c$ *then* $\lambda_j(G+ij)-\lambda_j(G) < c$, *and, if* $\lambda_j(G+ij)-\lambda_j(G) > c$ *then* $\lambda_i(G+ij)-\lambda_i(G) < c$

Before moving to the analysis of the existence of stable graphs, we give an explanation about why in our model the network might stop evolving along an improving path and finally reach an equilibrium. Let us consider an improving path along which the number m of links is increasing from $m_1 = 0$, corresponding to the empty graph, to at most $m_2 = n(n - 1)/2$, corresponding to the complete graph K_n. Figure (5) shows some instances of improving paths and their corresponding densities (in terms of the number of links) for different values of the cost c, but with severance cost $\alpha = 0$. Note that a vanishing value of α implies that no links are removed since then the severance cost exceeds any potential gains that could be realized by saving the cost for that link .

For comparison, the figure shows also the straight line with slope $\frac{2}{n}$, equal to the average increase of λ_{PF} going from the empty graph to the complete graph. In contrast, along any improving path the trajectory of $\lambda_{\mathrm{PF}}(m)$ starts off above such straight line. This is stated in the following Lemma and has an important implication.

Lemma 3 *Along any improving path in which the number m of links is increasing, $\lambda_{PF}(m)$ increases with m faster than $\frac{2}{n}$, in a set of integers $I = \{0, 1, 2, ..., m_0\}$ with $m_0 < n(n - 1)/2$.*

Since, in addition, the sum of the increments of $\lambda_{\mathrm{PF}}(m)$ has to be constant (cf. Lemma (1), item (ii)), this means that any improving path has to cross the straight line at some point before reaching the complete graph. In other words, for some value of m the marginal revenue becomes smaller than the marginal cost (for any value of cost and n large enough), implying that the evolution stops. This is stated more precisely in the following proposition.

Proposition 5 *Along an improving path in which the number m of links increases,*

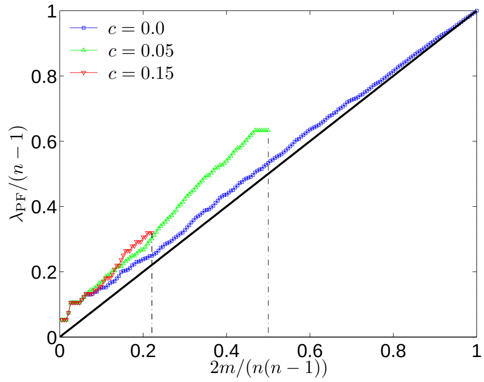


Fig. 5. Largest real eigenvalue λ_{PF} of a network of $n = 20$ firms as a function of the number m of links, along three specific improving paths for costs $c = 0.0$, $c = 0.05$, $c = 0.15$ and severance cost parameter $\alpha = 0$. The number of links m and the largest real eigenvalue λ_{PF} are normalized to their maximum values (attained by the complete graph). The improving path for cost $c = 0.15$ reaches 22% of the density of a complete graph before it arrives at a stable network while the improving path for cost $c = 0.05$ reaches already 50% of the maximum density. Obviously the improving path for $c = 0$ reaches the complete graph.

marginal profits become negative for some value of $m^* \leq n(n-1)/2$, for any value of cost c and n large enough.

In light of the foregoing results [16] we now proceed to investigate the stability of specific network structures. From a straightforward application of the properties of the marginal revenues from collaboration (cf. item (iii) in Lemma (1)) it follows that, on one hand, when marginal costs are zero, $c = 0$, links will always be created and no existing link will be deleted. Then the unique equilibrium is the complete graph K_n.

Proposition 6 *If costs are zero, $c = 0$, then the complete graph K_n is the unique stable network.*

On the other hand, when the difference between marginal costs c and severance costs $v(c)$ is larger than one, it is profitable to remove any link and the only equilibrium is the empty graph $\bar{K}_n$.

Proposition 7 *For cost $c' = \alpha c > 1$ the empty graph $\bar{K}_n$ is the unique stable network.*

Besides the foregoing extreme situations, the determination of stable networks becomes quite involved. This is because, in general, the marginal revenue from a collaboration depends on the *topology* of the graph. In addition, for a given topology, it varies with the *position* of the firm which is chosen to create or delete a link. Starting from an initial graph G_0 this property implies that different network trajectories can be explored, according to the particular pair of firms that is allowed to revise its collaboration strategy at the beginning of each period. Thus, improving paths have a strong path dependent character in this model and multiple equilibrium networks might be possible for the same level of marginal costs. In what follows we show that, on one hand, multiple pairwise stable

[16] In Section 4.2 we will show with Proposition (12) that for large enough networks and costs smaller than 1/2, no improving path will ever reach the complete graph.

networks exist for the same value of marginal cost $c \in (0,1)$ and severance costs $v(c)$. On the other hand, we identify a region of costs in the same interval in which stable networks do not exist and a sequence of networks is repeatedly visited [17] . In the following proposition, we show that a set of disconnected cliques of the same size can be a stable network, if their size falls within a certain interval that depends on the marginal cost of collaboration c and on the severance cost parameter α.

Proposition 8 *Consider costs c, $c' = \alpha c$ and $\alpha \in [0,1]$. If the network G consists of a set of k equally sized, disconnected cliques $K_n^1, K_n^2, ..., K_n^k$ (G having kn nodes in total) then G is stable if*[18]

$$\lceil \frac{1+c(1-c)}{c} \rceil \leq n \leq \lfloor \frac{2-c'(1-c')}{c'} \rfloor. \tag{22}$$

From Proposition (8) it follows immediately that for a given value of cost c there exist multiple integer values n (the size of the clique) that fit into the interval spanned by the upper and lower bounds in Equation (8). This is discussed in more details in the proof of Proposition (8) (see appendix) and implies that multiple equilibrium networks exist for a given value of marginal cost c and severance cost $v(c)$.

Moreover, note that the homogeneous size of the cliques is only a sufficient condition for stability but it is not necessary. Indeed, the equilibrium networks obtained with computer simulations show clearly that there exist also equilibria with disconnected cliques of different sizes (see e.g. Figure (9), bottom-right). The requirement of having cliques of the same size appears in Proposition (8) only to allow for an analytical treatment.

Equally sized disconnected cliques are not the only possible stable networks structures in the interval $c \in (0,1)$ and $\alpha \in [0,1]$. The next proposition shows that the spanning star, i.e. the star encompassing all nodes, can be pairwise stable as well, if the size of the star (and therewith the number of firms in the industry) falls within a certain region that depends on the cost c and on the severance cost parameter α.

Proposition 9 *Consider costs c, $c' = \alpha c$, $\alpha \in [0,1]$. The network G consisting of a spanning star $K_{1,n-1}$ with $\lceil \frac{2}{c} \rceil \leq n \leq \lfloor \frac{1+c'^2(6+c'^2)}{4c'^2} \rfloor$ is stable.*

The foregoing results have two important implications in relation to the literature. First, stable graphs are not necessarily connected. Second, in general they are not minimally connected. Indeed, the multiple clique equilibrium is a disconnected graph in which each component is complete and thus not minimally connected. This is an important feature that for instance distinguishes our model from the “connections” model in Jackson and Wolinsky (1996) and from the linear “two-way flow” model Bala and Goyal (2000). In both such models, the equilibrium networks are always connected, while in the latter they are also minimally connected. Furthermore, both models find that the spanning star is stable for intermediate values of the cost of collaboration. However, differently from both models, in our model the spanning star is *never* the unique stable network. Indeed, the next proposition combines together the results of the previous two propositions, the conditions under which the link formation dynamics defined in (5) may lead to two different pairwise stable network topologies for the same level of marginal cost c and

[17] This is a cycle in the space of network trajectories, to not confuse with the specific graph called cycle.
[18] In the following, $\lceil x \rceil$, where x is a real valued number $x \in \mathbb{R}$, denotes the smallest integer larger or equal than x (the ceiling of x). Similarly, $\lfloor x \rfloor$ the largest integer smaller or equal than x (the floor of x).

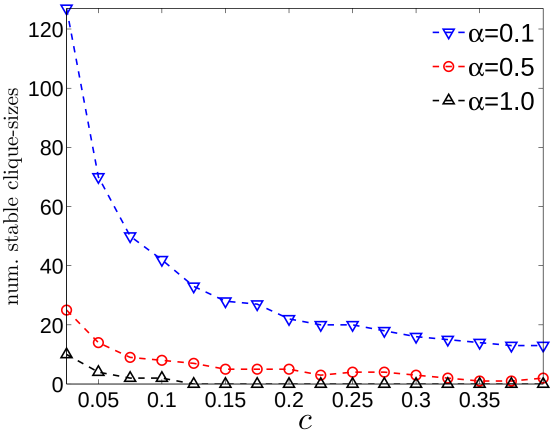


Fig. 6. Number of stable clique sizes when the spanning star $K_{1,n-1}$ is an equilibrium as well (for $\alpha = 0.1$, $\alpha = 0.5$ and $\alpha = 1.0$). If this number is positive then we have a spanning star $K_{1,n-1}$ and (at least one) set of disconnected cliques $K_k^1, ..., K_k^d$ as equilibrium networks for the same level of cost c.

severance cost parameter α, namely (i) the set of disconnected equally sized cliques or (ii) the spanning star.

Proposition 10 *Consider costs c, $c' = \alpha c$, $\alpha \in [0, 1]$ and the network G with n nodes such that $\lceil \frac{2}{c} \rceil \leq n \leq \lfloor \frac{1+c'^2(6+c'^2)}{4c'^2} \rfloor$. If there exists an integer $k \leq n, \mod (n, k) = 0$ such that $\lceil \frac{1+c(1-c)}{c} \rceil \leq k \leq \lfloor \frac{2-c'(1-c')}{c'} \rfloor$ then G can be stable for at least two cases.*
(i) G consists of disconnected cliques $K_k^1, ..., K_k^d$, $n = kd$ or
(ii) G consists of a spanning star $K_{1,n-1}$.
There are at least two stable networks for the same level of marginal cost c (degenerate cost region).

The multiplicity of equilibria stated in the above proposition is illustrated in Figure (6). The plot shows the number of different values of size of cliques when the configuration of multiple cliques and the spanning star are both stable. One can see that for smaller values of α the number of stable networks increases. Furthermore, Figure (7) shows two examples of possible equilibrium networks obtained with $n = 20$, $c = 0.3$ and $\alpha = 0.1$.

Not all values of marginal cost c and severance cost parameter α lead to pairwise stable networks. Consistently with the concept of improving path (cf. Jackson and Watts, 2002, Lemma (1)) the next proposition shows that in the interval $(0.586, 0.618)$, there exists a *cycle* of repeatedly visited networks.

Proposition 11 *For values of cost $2 - \sqrt{2} = 0.586 < c < \frac{1}{2}\left(\sqrt{5} - 1\right) = 0.618$ and $\alpha \in [0.707, 1]$, the improving path leads to a cycle of networks. In such cycle, a sequence of paths $(P_2, \{P_2, P_2\}, P_4, P_3, P_2)$ is repeatedly visited.*

The graphs which are repeatedly visited are illustrated in Figure (8). The fact that for some values of the parameters of the model there exists no stable network has also been found by Jackson and Watts (2002) and by Haller et al. (2007); Haller and Sarangi (2005).

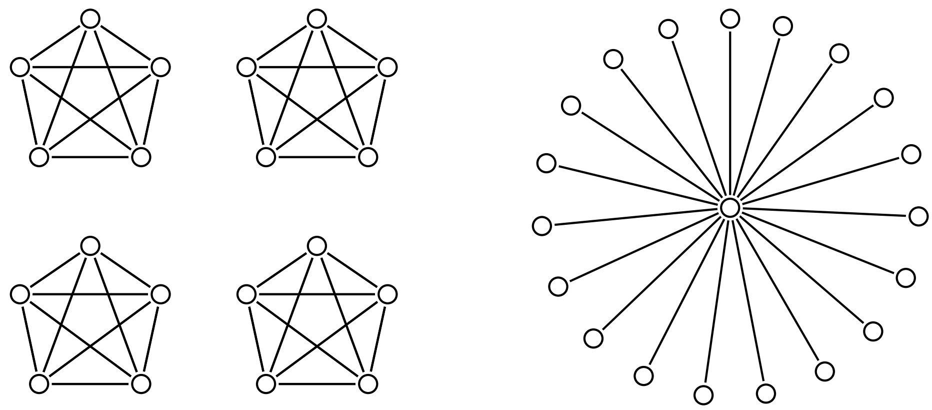


Fig. 7. An example of two possible (different) equilibrium networks for cost $c = 0.3$ and $\alpha = 0.1$ with $n = 20$ firms. A set of disconnected cliques of the same size (left) and a spanning star (right)

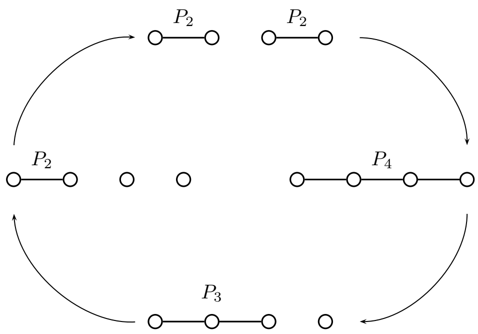


Fig. 8. Cycle $C = (P_2, \{P_2, P_2\}, P_4, P_3)$ of repeatedly visited graphs in which one graph is improved by the next in the sequence.

4.2. *Stability vs. Efficiency*

We have shown that for the same level of marginal cost there exist multiple equilibrium structures associated with different values of total profits. This indicates that stable networks can, in general, be inefficient. In particular, we have shown that in the marginal cost interval $c \in [0, 1]$, graphs that are not connected can be stable (cf. Propositions (8) and (10)), while in that cost region the efficient graph is always connected (cf. Proposition (2)). The possible inefficiency of the network evolution process stems from the externalities inherent to the process of knowledge recombination, described in Section 2.2. Indeed, when a firm decides to create or delete a link it takes into account its private marginal revenue from collaboration (given by the change in the largest eigenvalue of its connected component), but neglects social marginal revenues inherent to that decision. The latter is equal to the sum of changes in the largest eigenvalue of all firms belonging to the same connected component. Thus, it may well be that creating a link is not profitable for the individual firm although it would be profitable from the industry point of view.

Furthermore, the efficient network may not even belong to the set of equilibria, as it is shown in the next proposition.

Proposition 12 *Consider a network of size $n \geq \frac{2}{c}$. For cost $c < \frac{1}{2}$ the equilibrium network is not efficient.*

This result can be explained in the following way. Proposition (3) states that, when the marginal cost of link formation is less or equal to 1/2, the complete graph is the efficient graph. However, if the number n of firms in the industry is large enough, the individual marginal revenue of a collaboration is bounded from above by a value decreasing with n (see the proof of the Proposition (12) in the appendix). In particular, for $n \geq \frac{2}{c}$ the upper bound is always smaller than the marginal cost c. Therefore the complete graph is not stable [19].

An exhaustive discussion of efficiency and stability would require the determination of individual and total profits of firms under all possible network configurations. Both require the computation of the largest eigenvalue. Unfortunately, there is no general closed form solution available for any graph. However, one can provide general results for some special classes of graphs. Based on these findings Table (1) summarizes the results on efficiency and stability discussed so far and compares them with results for other well known classes of graphs in the literature.

Three graphs in the table deserve a special attention. The first is the empty graph, which is never stable nor efficient in the interval $[0, 1)$. The second one is the complete graph, which is efficient in $[0, 0.5]$, but is never stable for $c > 0$ (see Proposition (5) and Proposition (12)). The third graph is the star, which can be stable but is never efficient [20] in $[0, 1]$. In other words, both the star and the complete graph are never stable and efficient at the same time. This is a first important difference with respect to the literature, e.g. the models in Jackson and Wolinsky (1996) and Bala and Goyal (2000), where, at least in an interval of the parameters considered, the star (or, respectively, the complete graph) can be efficient and stable. In our model the tension between efficiency and stability is more pronounced. We were not able to find any efficient graph which is also stable, except from the trivial case of $c = 0$ in which, due the absence of collaboration cost, the complete graph is both stable and efficient.

Moreover, it is interesting to review the properties of the other graphs listed in the table and their mutual relations. A k-regular graph, i.e. a graph in which all nodes have the same degree, yields a revenue proportional to the degree of the nodes, regardless of the size of the graph. This means that when the degree is small the performance in terms of aggregate profits of this graph is rather poor. However, the complete graph is a particular case of regular graph in which all nodes have degree $n - 1$. In this case, the regular graph can be efficient.

The set of cliques of the same size, is stable for particular values of their size d, depending on the level of costs. It can also be efficient, in the particular case of one

[19] Another (degenerate) region of the parameter space in which the network dynamics leads to inefficient equilibrium outcomes is the one in which marginal cost is in the open interval $(2 - \sqrt{2}, \frac{1}{2}\left(\sqrt{5} - 1\right) = (0.586, 0.618)$. In that case (cf. Proposition (11)), for any number of firms in the industry the dynamics gets stuck into a cycle of networks, none of which is efficient.

[20] One can show that for $c < \frac{n}{n-1+\sqrt{n-1}} \sim 1$ for $n \to \infty$, K_n has a higher performance than $K_{1,n-1}$. E.g. for $n = 100$ we get $c < 0.918$.

set containing all firms, i.e. the complete graph. In this case however, it is never stable, as noted above. The set of identical cliques is also a particular case of k-regular graph, because the nodes in each clique have the same degree. In a path, the degree of the nodes is 2, except from the two nodes at the beginning and at the end of the path. In this sense, the graph is similar to a 2-regular graph. Indeed its eigenvalue is a little smaller than the one of 2-regular graph. When the network evolves starting from an empty graph, the first connected graph that is formed is indeed a path of length 2, possibly followed by a path of length 3 (see the proof of Lemma (3) in the appendix). In such transition, the largest eigenvalue of the component jumps from 0 to 1 and then to $2\cos\left(\frac{\pi}{4}\right) > 1$. Instead, when the graph is almost complete, the addition of a new link yields a negligible increase in the eigenvalue. Notice that the path of length 3 is also a star with one hub and two peripheral nodes.

A cycle is a closed path and it is in particular a 2-regular graph. In a cycle there is only one walk, which yields a revenue independent of the number of participating firms. In particular, because of this in our model the path is never an efficient graph. As we already noticed in Section 2.3, this is a consequence of the payoff function which differs in this respect from the one used in other models in the literature (e.g. Bala and Goyal, 2000).

We also list in the table the bipartite graph because of its relation to the notion of structural holes (cf. Burt, 1992). In a bipartite graph, nodes can be grouped in two separate classes so that links connect only nodes of one class to nodes of the other class. Consider for example a network consisting of few hubs, disconnected among them, and of many peripheral nodes, connected only to one or more hubs. The hubs fill the structural holes among the the peripheral nodes. This network is also a bipartite graph, since the hubs and the peripheral nodes form two separate classes of nodes. Notice that the star is a particular case of a bipartite graph. In our model, the bipartite graph is not efficient nor stable.

Finally, concerning the largest eigenvalue of $F_{n,d}$ an exact solution (see Bell, 1991) is given by the largest root of the cubic polynomial $x^3-(d-2)x^2-(n-1)x+(d-2)(n-d)$. From this exact solution, one can show that for a fixed value of d, $\lim_{n\to\infty}\Delta\lambda_{\mathrm{PF}} = 0$ and thus it is always profitable to remove a link if n is large (however large the severance cost or small the marginal cost may be).

4.3. *Topological Properties of Stable Networks*

The empirical research on R&D partnerships has investigated in depth the topological patterns of networks of knowledge exchange. From this literature (see e.g. Ahuja, 2000; Fleming et al., 2007; Hanaki et al., 2007; Powell et al., 2005), three features emerge as robust stylized facts: (i) R&D networks are sparse, that is the number of actual links is much less than the number of possible links. (ii) Networks are highly clustered, where clusters consist of highly interconnected firms, but different clusters are only sparsely connected. (iii) The distribution of links over the firms is characterized by high dispersion, with few firms being connected to many others.

The analytical study of equilibrium networks in Section 4 has pointed to the existence of equilibrium networks that match some of the stylized facts mentioned above. Indeed,

Graph Class	Eigenvalue	Total Profits	Efficiency	Stability
empty graph $G = \bar{K}_n$	$\lambda_{\mathrm{PF}} = 0$	$\Pi = 0$	$c > n$	$c > 1$
complete graph $G = K_n$	$\lambda_{\mathrm{PF}} = n - 1$	$\Pi = (1-c)n(n-1)$	$c \leq \frac{1}{2}$	$c = 0$
k-regular graph	$\lambda_{\mathrm{PF}} = k - 1$	$\Pi = n(k-1)(1-c)$	if $k = n$ see K_n	see cliques
path $G = P_n$	$\lambda_{\mathrm{PF}} = 2\cos\left(\frac{\pi}{n+1}\right)$	$\Pi = 2\cos\left(\frac{\pi}{n+1}\right) - (n-1)c$	no	no
star $G = K_{1,n-1}$	$\lambda_{\mathrm{PF}} = \sqrt{n-1}$	$\Pi = n\sqrt{n-1} - 2(n-1)c$	not in $0 < c < 1$	$\lceil \frac{2}{c} \rceil \leq n \leq \lfloor \frac{1+c'^2(6+c'^2)}{4c'^2} \rfloor$
cycle $G = C_n$	$\lambda_{\mathrm{PF}} = 2$	$\Pi = 2n(1-c)$	no	no
bipartite graph $G = K_{n_1,n_2}$	$\lambda_{\mathrm{PF}} = \sqrt{n_1 n_2}$	$\Pi = (n_1 + n_2)\sqrt{n_1 n_2} - n_1 n_2 c$	no	no
$G = F_{n,d}$	$\lambda_{\mathrm{PF}} \geq d - 1$	$\Pi = \lambda_{\mathrm{PF}}(F_{n,d}) - 2c\left(\binom{n}{2} + (n-d)\right)$	with good approx.[a]	no
cliques[b] $G = \{K_d^1, ..., K_d^l\}$	$\lambda_{\mathrm{PF}} = d - 1$	$\Pi = n(d-1)(1-c)$	if $l = 1, d = n$, see K_n	$\lceil \frac{1+c(1-c)}{c} \rceil \leq d \leq \lfloor \frac{2-c'(1-c')}{c'} \rfloor$

Table 1. Summary of the largest real eigenvalue, total profits, efficiency and stability for different types of networks.

[a] $\forall c$, and for large n, total profits of this graph tends to the one of efficient graph, $\lim_{n\to\infty} \epsilon = 0$

[b] We have l cliques of identical size d.

equally sized cliques are characterized by a high clustering, while the spanning star shows high degree heterogeneity. All these networks belong to the set of possible equilibria structures in our model.

In this section we define an explicit process of network evolution that is a particular case of improving path and we analyze by means of computer simulations the structural properties of stable networks in our model. In this way we explore the existence of more complex stable network structures, beyond those described in the previous section. Furthermore, we investigate whether our model is also able to generate pairwise stable structures that feature, at the same time, all the stylized facts of R&D networks.

There are several possible processes which would be consistent with the definition of an improving path. In this work, we investigate a stochastic process in which all pairs of firms have the same probability to be selected to revise their R&D collaboration strategy (cf. Vega-Redondo, 2007, p. 212).

Definition 5 (Myopic Pairwise Dynamics) *Let G denote the current graph $G(t)$ at time t. We define the network formation process $\Gamma(G)$ as follows. At the beginning of each period (at times $t = 0, T, 2T, ...$) a single pair of firms, i and j, is uniformly selected at random from the set N of firms.*

(i) If the link ij does not currently exist, $ij \notin E(G)$, then it is created whenever neither firm is harmed by the creation and at least one of them strictly gains, i.e.

$$\begin{aligned} &\pi_i(G+ij,c) \geq \pi_i(G,c) \wedge \pi_j(G+ij,c) \geq \pi_j(G,c) \wedge \\ &\pi_i(G+ij,c) > \pi_i(G,c) \vee \pi_j(G+ij,c) > \pi_j(G,c). \end{aligned} \tag{23}$$

or equivalently

$$\begin{aligned} &\lambda_i(G+ij) - \lambda_i(G) \geq c \wedge \lambda_j(G+ij) - \lambda_j(G) \geq c \wedge \\ &\lambda_i(G+ij) - \lambda_i(G) > c \vee \lambda_j(G+ij) - \lambda_j(G) > c. \end{aligned} \tag{24}$$

(ii) If the link ij is currently in place, $ij \in E(G)$, then it is removed whenever at least one of the firms strictly gains from the change, with link deletion involving the severance cost $v(c) = (1-\alpha)c$, and $\alpha \in [0,1]$. More formally

$$\pi_i(G-ij,c,v) > \pi_i(G,c,v) \vee \pi_j(G-ij,c,v) > \pi_j(G,c,v), \tag{25}$$

or equivalently:

$$\lambda_i(G) - \lambda_i(G-ij) < \alpha c \vee \lambda_j(G) - \lambda_j(G-ij) < \alpha c \tag{26}$$

Note that, in the evolution of the network defined above, the only element of stochasticity is the sequence of the pairs of firms chosen to create or delete links.

We study stable network structures arising from this process in computational experiments [21] conducted in a large region of the model's parameter space. More precisely, we carried out multiple (50 repetitions for each parameter choice) computer simulations of

[21] When simulating the network evolution discussed in Section 4 the largest real eigenvalue of the network has to be computed many times. Since the largest real eigenvalue of a graph can be computed in polynomial time (Hong, 1993) our model is well suited for numerical investigations.

the network dynamics defined in (5) with a fixed number n of firms in the industry [22] ($n = 50$), starting each from an empty network $\bar{K}_n$. For each simulation we selected a value for the marginal cost c in the interval $[0, 1]$ and a value for the severance cost parameter α in the interval [23] $[0, 0.5]$. As the number of chosen values were respectively 12 for the marginal cost and 5 for the severance cost parameter, the total number of computer simulations summed up to 3000. The results of the aforementioned Monte-Carlo experiments are shown in the Figures from (9) to (11).

The plots in Figure (9) show typical equilibrium networks obtained in simulations for marginal cost of link formation equal to 0.15 and different values of the severance cost parameter α. Recall that severance cost are equal to $v = (1 - \alpha)c$, and thus are inversely related to the parameter α. As the plots reveal, in this region of the parameter space the dynamics in our model is able to generate equilibrium structures displaying the complex features that characterize empirically observed R&D networks (see e.g. Fleming et al., 2007). In particular, for very high severance costs the equilibrium network contains a giant component with a high degree heterogeneity. On the other hand, as severance costs associated with link deletion fall down (increasing values of α), we observe a significant increase in the cliquishness of the network, and a reduction of degree heterogeneity.

The insights coming from the foregoing qualitative study are confirmed by a more quantitative analysis of the topological properties of equilibrium graphs. The plots in Figure (10) display respectively the mean and the variance of the network degree distribution as functions of the marginal cost c and severance cost parameter α. The mean degree is inversely related to the sparseness of the graph, while degree variance captures the degree heterogeneity. As the plots in the figure make clear, higher cost of R&D collaboration lead to graphs that are more sparse. On the other hand, degree heterogeneity reaches a peak for values of marginal cost close to 0.1, and then falls down as collaboration costs increase. In addition, degree heterogeneity increases with severance costs (decreasing α).

The presence of clusters of highly interconnected firms is a key feature of empirically observed R&D networks (cf. stylized fact number (iii)). As the plots in Figure (11) show, this feature is also a characteristic for the equilibrium networks generated by the model. In particular, the average clustering coefficient (Figure (11), top-left) is close to one in a wide region of the explored parameter space ($c \in (0, 0.5)$, $\alpha > 0$). Moreover, it is a decreasing function of severance costs $v = (1 - \alpha)c$. Finally, note that clustering becomes zero for values of costs greater or equal to 0.7. Further information on the topological features of R&D clusters can be gathered by looking at the average number of connected components, at their average size and the concentration of their size (Figure (11), top-right, bottom-left and bottom-right respectively). As the plots in the figure indicate, the number of connected components is an increasing function of collaboration costs while the average size and its concentration variables are negatively related to collaboration costs. In addition, as the costs of link severance increase the number of components increases, while component size and its concentration decrease.

[22] Choosing a different, possibly higher, number of firms would have not altered the results, as only the size (the number of firms) of the connected components, and not the total size of the system matters for the dynamics.

[23] Preliminary simulation studies with values of α greater than 0.5 for the severance cost did not reveal the presence of any striking difference in the results.

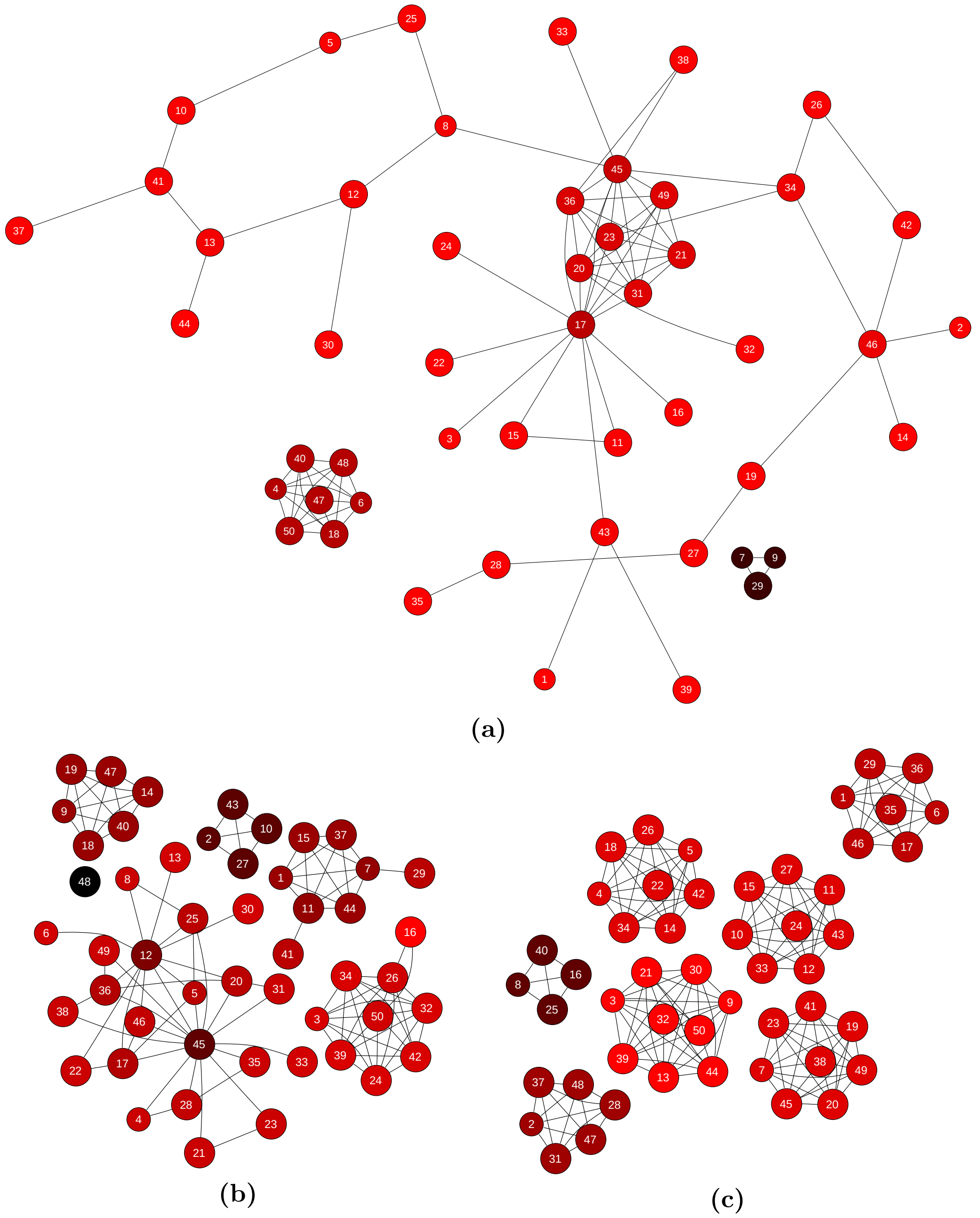


Fig. 9. Equilibrium networks for $n = 50$, $c = 0.15$, (a) $\alpha = 0.0$, (b) $\alpha = 0.1$, and (c) $\alpha = 1.0$ starting from an empty network. Relative profits (compared to the firm with highest profits in the network) are indicated with different shades, meaning that nodes representing firms with higher relative profits are shown in a lighter shade. The network plots use the Fruchterman-Reingold algorithm (Fruchterman and Reingold, 1991).

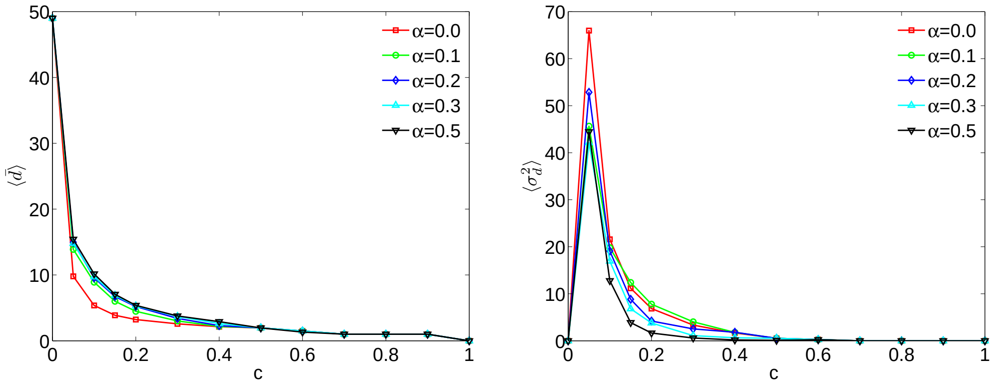


Fig. 10. Average degree $\langle \bar{d} \rangle$ and degree variance $\langle \sigma_d^2 \rangle$ in the equilibrium network for $n = 50$, $c \in [0, 1]$, starting from an empty network (averaged over 50 simulations).

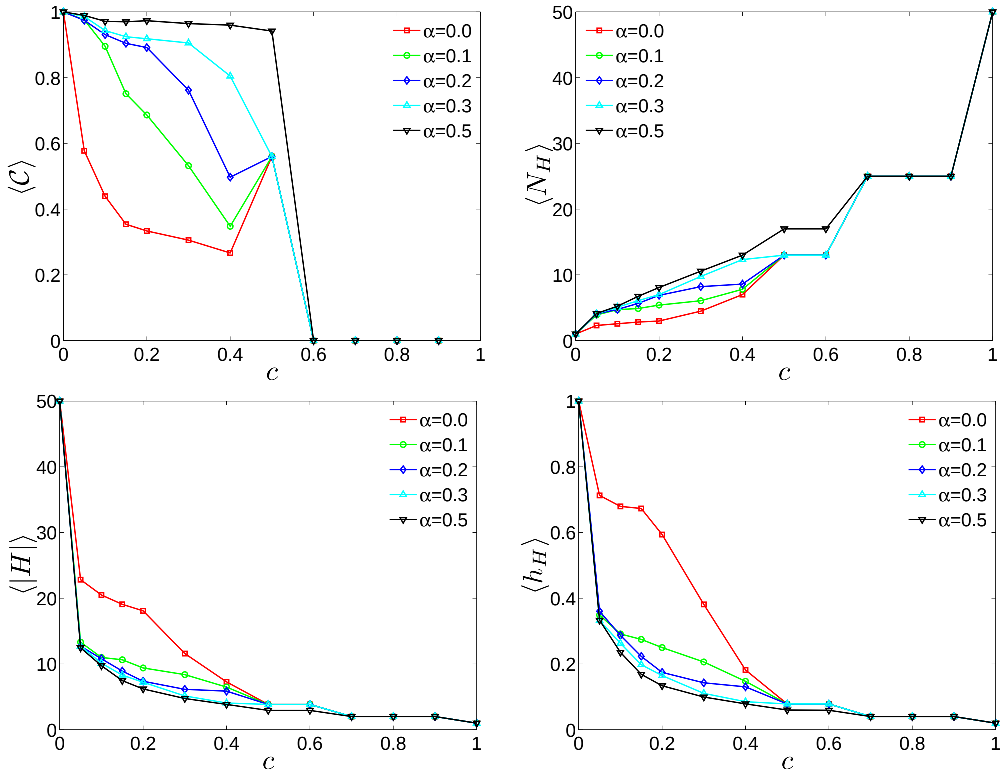


Fig. 11. Average clustering coefficient $\langle \mathcal{C} \rangle$ (top, left), average number of components $\langle N_H \rangle$ (top, right), average size of components $\langle |H| \rangle$ (bottom, right) and average Herfindal index of components size concentration $\langle h_H \rangle$ (bottom, right) in the equilibrium network for $n = 50$, $c \in [0, 1]$, starting from an empty network (averaged over 50 simulations).

Joining together the foregoing results we can conclude that sparse equilibrium networks organized in clusters of highly interconnected firms are a distinctive feature of the network dynamics in our model. Moreover, low values of R&D collaboration costs and high values of the costs of link severance lead to equilibrium structures characterized by a small number of large components, with a highly dispersed degree distribution. As collaboration costs increase and as link severance costs decrease, we observe that equilibrium networks tend to be more and more organized in size homogeneous cliques having only few connections among them.

5. Conclusions

In this paper, we have investigated the efficiency and the evolution of networks of knowledge exchange across firms. We developed a model in which firms recombine their knowledge stock with the stocks of knowledge of other firms in the industry, in order to introduce innovations in the market. Since each collaboration is costly for firms they face a trade-off between the benefits of new collaborations (in terms of an increase in the expected number of innovations per period) and the costs associated with them. Furthermore, we showed that under mild conditions on the horizon over which the performance of R&D collaborations is evaluated, the benefit the firm receives from the network depends on the growth rate of *all* walks existing across firms in their connected component. To this end, our model can be seen as extending other popular models in the network formation literature (cf. the "connections" model in Jackson and Wolinsky, 1996, and the linear "two-way flow" model without decay in Bala and Goyal, 2000).

Within the foregoing framework, we characterized the topology of the efficient graph for any level of the marginal cost of collaboration. We showed that, when the marginal cost of maintaining collaborations is low, the efficient network is the complete graph. Thus, when collaboration costs are low, a network of densely connected firms maximizes total profits in the industry. On the other hand, as the marginal cost of collaboration increases it is better for the industry network to display the presence of structural holes. In particular, for intermediate costs of collaboration the efficient graph belongs to the class of nested split graphs, characterized by the presence of a hub linking a clique to a set of disconnected firms. Furthermore, we showed that nested split graphs are characterized by significant cross-firm profit inequality, increasing both in collaboration costs and size of the industry. Finally, we showed that for very large costs of collaboration the empty graph is efficient.

We then studied the existence of equilibrium graphs in the model, and the relation between equilibrium and efficiency. For this purpose, we employed the notion of "improving path" (cf. Jackson and Watts, 2002), and we assumed that the deletion of existing connections involves a severance cost. In line with the concept of improving path, we identified regions of collaboration and severance costs in which there exist either pairwise stable graphs or a closed cycles of networks. As far as pairwise stable networks are concerned, we showed that different network structures are stable for the same level of costs. In particular, we identified regions of the collaboration and severance costs in which (i) the spanning star (i.e. the star encompassing all firms in the network), and (ii) the class of size-homogeneous disconnected cliques are stable. In turn, the source of multiplicity of equilibria lies in (i) the strong path dependency involved in partner selection decisions,

(ii) in the presence of external effects affecting marginal revenue of collaborations for firms belonging to the same connected component and (iii) the inertia arising from the presence of a severance cost associated with link deletion. The presence of multiple stable structures for the same level of collaboration costs implies that, in general, efficient structures are not attained in our model. Furthermore, we identified a region of the size of the industry and of costs in which the efficient graph is *never* attained.

Finally, we investigated the topological characteristics of pairwise stable graphs in our model, to see whether they are able to replicate the stylized facts on empirically observed R&D networks. To this end, we studied via computer simulations the properties of equilibria generated under a two-sided myopic pairwise dynamics (cf. Vega-Redondo, 2007, p. 212). The results of our simulations show the existence of a region of low marginal costs of collaboration and high costs of link deletion in which the aforementioned dynamics is able to select pairwise stable structures matching the stylized facts on R&D networks.

The present work could be extended at least in three ways. First, the model could be extended to account for industry demand, for example like in Goyal and Moraga-Gonzalez (2001). In this way, one could then study how the efficiency and dynamics of network structure may change when firms operate in markets that are interdependent. Second, one could investigate whether the foregoing results about the properties of stable networks are robust to different link updating algorithms. For example, one could study the effect on the network dynamics of introducing firms pursuing different strategies, for instance of the kind explored in Bala and Goyal (2000). Similarly, one could depart from the strong assumptions we made on the knowledge firms have about the network. In this respect, one could study instead the efficiency and emergence of network structures when firms follow more simple rules of behavior, for example of the kind suggested in the empirical work by Powell et al. (2005). Third, in the present model we assumed that the knowledge bases of firms in the industry were sufficiently homogeneous to be transferred across firms. However, the process of knowledge transfer across firms is likely to be shaped by its degree of tacitness, as well as by the existing technological complementarities across sectors and firms' knowledge bases. A further analysis of R&D network dynamics and efficiency should therefore embed all the foregoing ingredients related to industry technology, and try to investigate how they may affect the revenues and costs of the process of knowledge recombination.

Acknowledgements

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Appendix

In the appendix we give the proofs of the propositions and lemmas stated in the preceding sections.

Proof of Proposition (1) The adjacency matrix $\mathbf{A}(G)$ is diagonalizable (Haemers, 2006) and thus, the general solution of (4) can be written as (Zwillinger, 1998)

$$\mathbf{x}(t) = \sum_{j=1}^{n} c_j \mathbf{v}_j e^{\lambda_j t}, \tag{27}$$

where c_i are unknown constants, that are determined by the initial values $\mathbf{x}(0) = \sum_{j=1}^{n} c_j \mathbf{v}_j$, $\lambda_{\mathrm{PF}} = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ are the real eigenvalues of $\mathbf{A}$ and $\mathbf{v}_1, ..., \mathbf{v}_n$ the corresponding eigenvectors. In Equation (27) only those eigenvalues and corresponding eigenvectors of the adjacency matrix of the connected component G_i of firm i appear. All other eigenvalues have vanishing eigenvector components and do not contribute to the trajectory. This is intuitively clear since firms in disconnected components have decoupled equations of the form (4) and their trajectories can be computed independently. We get

$$\begin{aligned} \lambda_{\mathrm{PF}} - \tfrac{\dot{x}_i(t)}{x_i(t)} &= \tfrac{\lambda_{\mathrm{PF}} x_i(t) - \dot{x}_i(t)}{x_i(t)} \\ &= \tfrac{\sum_{j=1}^{n} c_j v_{ji} e^{\lambda_j t} (\lambda_{\mathrm{PF}} - \lambda_j)}{x_i(t)} \\ &= \tfrac{\sum_{j=2}^{n} c_j v_{ji} e^{\lambda_j t} (\lambda_{\mathrm{PF}} - \lambda_j)}{\sum_{j=1}^{n} c_j v_{ji} e^{\lambda_j t}}. \end{aligned} \tag{28}$$

In the numerator of Equation (28) we obtain a sum of exponentials with one exponential term less than in the denominator, namely the one with the largest real eigenvalue in the exponent. We have that the sum of exponentials converges to the exponential with the largest real eigenvalue. Consider for example $ae^{\lambda_1 t} + be^{\lambda_2 t} = ae^{\lambda_1 t}\left(1 + \frac{b}{a} e^{(\lambda_2 - \lambda_1)t}\right) \sim ae^{\lambda_1 t}$ for large t. Thus we get

$$\lambda_{\mathrm{PF}} - \lim_{t \to \infty} \frac{\dot{x}_i(t)}{x_i(t)} = \lim_{t \to \infty} \frac{c_2 v_{2i} e^{\lambda_2 t} (\lambda_{\mathrm{PF}} - \lambda_2)}{c_1 v_{1i} e^{\lambda_{\mathrm{PF}} t}} \propto \lim_{t \to \infty} e^{-(\lambda_{\mathrm{PF}} - \lambda_2)t} = 0. \tag{29}$$

In what follows we compute a lower bound for the difference $\lambda_{\mathrm{PF}} - \lambda_2$ and thus the order of convergence. Consider the real eigenvalues $\lambda_{\mathrm{PF}} = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ of the adjacency matrix $\mathbf{A}$. We have that $\sum_{j=1}^{n} \lambda_i^2 = \mathrm{tr}(\mathbf{A}^2) = 2m$ (Bollobas, 1998). Thus, we get

$$\begin{aligned} \lambda_2^2 &= 2m - \lambda_{\mathrm{PF}}^2 - \textstyle\sum_{j=3}^{n} \lambda_i^2 \\ &\leq 2m - \lambda_{\mathrm{PF}}^2 \\ &\leq 2m - \left(\tfrac{2m}{n}\right)^2 \\ &= \tfrac{2m(n^2 - 2m)}{n^2}. \end{aligned} \tag{30}$$

Here we use the fact that $\lambda_{\mathrm{PF}} \geq \frac{2m}{n}$ (Bollobas, 1998). Therefore we get

$$\lambda_{\mathrm{PF}} - \lambda_2 \geq \frac{2m - \sqrt{2m(n^2 - 2m)}}{n}, \tag{31}$$

which is positive and a monotonic increasing function for $n^2/4 < m \leq n(n-1)/2$.

Proof of Corollary (1) The proof follows directly from an application of the following lemma.

Lemma 4 *Consider a continuous function $f : [0, \infty) \to \mathbb{R}$ that converges to a finite value λ, i.e.* $\lim_{t\to\infty} f(t) = \lambda < \infty$. *Then*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)dt = \lambda. \tag{32}$$

Proof of Lemma (4) Denote $F(T) = \frac{1}{T}\int_0^T f(t)dt$. We can write

$$F(T) = \underbrace{\frac{1}{T}\int_0^{\tau'} f(t)dt}_{\leq c\tau'} + \frac{1}{T}\int_{\tau'}^T f(t)dt. \tag{33}$$

The first integral in the above expression is finite since any continuous function on the compact set $[0, \tau']$ has a maximum denoted by c. Since $f(t)$ converges to λ, for any ϵ' we can find a $\tau'(\epsilon')$ such that for all $t \geq \tau'$ we have $|f(t) - \lambda| < \epsilon'$. Thus we get

$$\begin{aligned} |F(T) - \lambda| &= \left|\tfrac{1}{T}\left(\int_0^{\tau'} f(t)dt + \int_{\tau'}^T f(t)dt - \lambda T\right)\right| \\ &\leq \tfrac{1}{T}\left(|c|\tau' + \left|\int_{\tau'}^T f(t)dt - \lambda T\right|\right) \\ &\leq \tfrac{1}{T}\left(|c|\tau' + \int_{\tau'}^T |f(t) - \lambda|dt + (T - \tau' - T)\lambda\right) \\ &\leq \tfrac{1}{T}\left(|c|\tau' + \epsilon'(T - \tau') - \tau'\lambda\right) \\ &= \tfrac{(|c|-\lambda)\tau'}{T} + \tfrac{T-\tau'}{T}\epsilon' \\ &\leq \tfrac{|c|\tau'}{T} + \epsilon'. \end{aligned} \tag{34}$$

We define

$$\begin{aligned} \epsilon &= \tfrac{|c|\tau'}{T} + \epsilon' \\ \tau &= \tfrac{|c|\tau'}{\epsilon - \epsilon'}. \end{aligned} \tag{35}$$

Since $\frac{\partial \epsilon}{\partial T} = -\frac{|c|\tau'}{T^2} < 0$ we have that $|F(T) - \lambda| < \epsilon$ for $T > \tau$. For any $\epsilon > 0$ we can find an $\epsilon' < \epsilon$ (e.g. $\epsilon' = \epsilon/2$) and the corresponding $\tau'(\epsilon')$ from which we compute $\tau(\epsilon)$ such that $|F(T) - \lambda| < \epsilon$ for all $T > \tau(\epsilon)$. This means that $\lim_{T\to\infty} F(T) = \lambda$.

Proof of Lemma (2) Since G_1 and G_2 are connected, we have that $m_1 \geq n_1 - 1$ and $m_2 \geq n_2 - 1$ (West, 2001). We now consider different cases for the number of edges in the components.

(i) $m_1 \geq n_1$ and $m_2 \geq n_2$: Assume that the largest eigenvalue of G_1 is $\lambda_{\text{PF}}(G_1) \geq \lambda_{\text{PF}}(G_2)$. Let G' be the graph obtained as follows: for each node in G_2 we rewire one incident edge to a node in G_1. In this way, all nodes in G_2 are connected to G_1. The number of rewired edges is n_2 (and there are at least that many edges since $m_2 \geq n_2$ by assumption). There exists a relationship between the largest real eigenvalue of a graph and those of its subgraphs (Cvetkovic et al., 1995): if H is a subgraph of G,

$H \subseteq G$, then $\lambda_{\mathrm{PF}}(H) \leq \lambda_{\mathrm{PF}}(G)$. Therefore, $\lambda_{\mathrm{PF}}(G') \geq \lambda_{\mathrm{PF}}(G_1) \geq \lambda_{\mathrm{PF}}(G_2)$. Total profits of G' are

$$\begin{aligned}\Pi(G') &= (n_1+n_2)\lambda_{\mathrm{PF}}(G') - 2(m_1+m_2)c \\ &\geq n_1\lambda_{\mathrm{PF}}(G_1) + n_2\lambda_{\mathrm{PF}}(G_2) - 2(m_1+m_2)c \\ &= \Pi(G_1) + \Pi(G_2).\end{aligned} \tag{36}$$

(ii) $m_1 \geq n_1$ and $m_2 = n_2 - 1$: If $m_2 = n_2 - 1$ then the largest real eigenvalue of G_2 is at most the one of the star K_{1,n_2-1} with $\lambda_{\mathrm{PF}}(G_2) \leq \sqrt{n_2-1}$ (Hong, 1993).

We construct the graph G by connecting all nodes of K_{1,n_2-1} to a single node in G_1 and including the remaining isolated node by adding one more edge. The graph G has an eigenvalue $\lambda_{\mathrm{PF}}(G) \geq \lambda_{\mathrm{PF}}(K_{1,n_1+n_2-1}) = \sqrt{n_1+n_2-1}$. Otherwise, the edges in G are redistributed to form a star K_{1,n_1+n_2-1} and the remaining edges are attached at random. Since λ_{PF} is an increasing function of the number of edges in the graph the inequality follows. We obtain

$$\begin{aligned}&\Pi(G) = (n_1+n_2)\lambda_{\mathrm{PF}}(G) - 2(m+1)c \\ &\Pi(G_1)+\Pi(G_2) = n_1\lambda_{\mathrm{PF}}(G_1) + n_2\sqrt{n_2-1} - 2\underbrace{(m_1+(n_2-1))}_{m}c\end{aligned} \tag{37}$$

Thus, we get

$$\begin{aligned}\Pi(G) - (\Pi(G_1)+\Pi(G_2)) &= \Pi(G) - (\Pi(G_1)+\Pi(K_{1,n_2-1})) \\ &= n_1\underbrace{(\lambda_{\mathrm{PF}}(G) - \lambda_{\mathrm{PF}}(G_1))}_{\geq 0} \\ &\quad + n_2(\lambda_{\mathrm{PF}}(G) - \sqrt{n_2-1}) - 2c \\ &\geq n_2(\lambda_{\mathrm{PF}}(G) - \sqrt{n_2-1}) - 2c.\end{aligned} \tag{38}$$

With $\lambda_{\mathrm{PF}}(G) \geq \sqrt{n_1+n_2-1} \geq \sqrt{n_2+1}$ if $n_1 \geq 2$ (by assumption). If the last inequality above is large than 0, we have that

$$\sqrt{n_2+1} - \sqrt{n_2-1} \geq \frac{2c}{n_2}. \tag{39}$$

If $0 \leq c \leq 1$, this inequality is true if $n_2 \geq 2$ (by assumption).

(iii) $m_1 = n_1 - 1$ and $m_2 = n_2 - 1$: If $m_1 = n_1 - 1$ and $m_2 = n_2 - 1$, then both components are stars, K_{1,n_1-1} and K_{1,n_2-1} with eigenvalues $\sqrt{n_1-1}$ and $\sqrt{n_2-1}$. Construct the graph G by connecting n_2-1 nodes from K_{1,n_2-1} to the central node in K_{1,n_1-1}. Then attach an edge to the remaining isolated node to obtain a star $G = K_{1,n_1+n_2-1}$.

$$\Pi(G) = \Pi(K_{1,n_1+n_2-1}) = (n_1+n_2)\sqrt{n_1+n_2-1} - 2c(n_1+n_2-1) \tag{40}$$

For the difference we get

$$
\begin{aligned}
\Pi(G) - (\Pi(K_{1,n_2-1}) + \Pi(K_{1,n_2-1})) &= n_1(\sqrt{n_1+n_2-1} - \sqrt{n_1-1}) \\
&\quad + n_2(\sqrt{n_1+n_2-1} - \sqrt{n_2-1}) - 2c \\
&\geq (n_1+n_2)(\sqrt{n_1+n_2-1} - \sqrt{n_1-1}).
\end{aligned} \tag{41}
$$

W.l.o.g. we have assumed that $n_1 \geq n_2$. The expression above is larger or equal than 0 if

$$
\underbrace{(n_1+n_2)}_{\geq n_1+2}(\underbrace{\sqrt{n_1+n_2-1} - \sqrt{n_1-1}}_{\geq \sqrt{n_1+1}-\sqrt{n_1-1}}) \geq 2 \geq 2c, \tag{42}
$$

with $n_2 \geq 2$. We get

$$
\sqrt{n_1+1} - \sqrt{n_1-1} \geq \frac{2}{n_1+2}, \tag{43}
$$

and the last inequality holds for $n_1 \geq 2$.

(iv) $n_1 \geq 2$ and $n_2 = 1$: We have one isolated node and a connected graph G_1. Total profits are $\Pi(G) = n_1 \lambda_{\mathrm{PF}}(G_1) - 2m_1 c$. Denote the graph G' obtained by connecting the isolated node to G_1. Then

$$
\begin{aligned}
\Pi(G') &= (n_1+1)\lambda_{\mathrm{PF}}(G') - 2(m_1+1)c \\
&\geq \Pi(G) + (\lambda_{\mathrm{PF}}(G') - 2c),
\end{aligned} \tag{44}
$$

We now consider three more cases:

(1) If $n_1 \geq 4$, then $m_1 \geq n_1 - 1$ (since G_1 is connected by assumption). We can construct a star K_{1,n_1-1} plus additional edges from G_1 and connect the isolated node to it. Denote the resulting graph G'. Then, $\lambda_{\mathrm{PF}}(G') \geq \lambda_{\mathrm{PF}}(K_{1,n_1}) = \sqrt{n_1} \geq 2$. Thus, $\Pi(G') - \Pi(G) \geq 0$ if $\lambda_{\mathrm{PF}}(G') \geq 2 \geq 2c$ for $c \in [0,1]$.

(2) If $n_1 = 3$, then G_1 is either a path P_3 of length 3 or a cycle C_3 containing 3 nodes. We connect the isolated node to G_1. In the case of $G_1 = P_3$ we get

$$
\begin{aligned}
\Pi(G') - \Pi(G) &= \underbrace{4\sqrt{3} - 6c}_{\Pi(G')} - \underbrace{(3\sqrt{2} - 4c)}_{\Pi(G)} \\
&= 2.69 - 2c > 0,
\end{aligned} \tag{45}
$$

where the last inequality follows from $c \in [0,1]$. In the case of $G_1 = C_3$ we obtain

$$
\begin{aligned}
\Pi(G') - \Pi(G) &= \underbrace{42.17 - 8c}_{\Pi(G')} - \underbrace{(32 - 6c)}_{\Pi(G)}, \\
&= 2.68 - 2c > 0
\end{aligned} \tag{46}
$$

again, using $c \in [0,1]$.

(3) For $n_1 = 2$ we connect the isolated node to $G_1 = P_2$ and again denote the resulting connected graph G'. We have that

$$
\begin{aligned}
\Pi(G') - \Pi(G) &= \underbrace{3\sqrt{2} - 4c}_{\Pi(G')} - \underbrace{(21 - 2c)}_{\Pi(G)}, \\
&= 2.24 - 2c > 0
\end{aligned} \tag{47}
$$

with $c \in [0,1]$.

(v) $n_1 = 1$ and $n_2 = 1$: We have two isolated nodes with total profits $\Pi(G) = 0$. If we connect the nodes via an edge we have $\Pi(G') = 2(1-c)$. Since $0 \leq c \leq 1$ total profits in the connected graph G' are higher.

The above cases consider all possible cases of disconnected graphs and show that total profits Π can be increased by connecting them.

Proof of Proposition (2) For a contradiction assume that the efficient graph G is disconnected (and all connected graphs have smaller total profits than G). Since G is disconnected then it has at least two components. With Proposition (2) each pair of components can be connected, resulting in a graph with higher total profits. Ultimately all components of G can be connected, yielding a connected graph G' with at least the total profits of G. This is a contradiction to the assumption that the efficient graph is disconnected.

Proof of Proposition (3) We prove each claim of the proposition as follows.

(i) From Lemma (2) we know that the efficient graph is connected. Moreover, (Brualdi and Solheid, 1986) have shown that among the connected graphs, the graphs with maximal eigenvalue have a stepwise adjacency matrix. We have mentioned already that these graphs are referred to connected nested split graphs (Aouchiche et al., 2008).

(ii) We have introduced the graph $F_{n,d}$ in Section 3.1. In order to prove the claim, we derive a lower bound for the total profits of $F_{n,d}$, as well as an upper bound for the total profit of the efficient graph G^*. We then show that, if one chooses d appropriately, the relative difference between the two bounds vanishes for large n. Let us start with the lower bound. Recall that $F_{n,d}$ is the graph obtained from a complete graph K_d of d nodes and $n-d$ isolated nodes by connecting each isolated node to one and the same node of K_d via one link. The number of links m in this graph is determined by the size d of the clique, $m(d) = \binom{n}{2} + (n-d)$. Since $F_{n,d}$ contains K_d as a subgraph, the largest real eigenvalue of $F_{n,d}$ is larger or equal to the one of K_d, which is $\lambda_{\mathrm{PF}}(K_d) = d-1$. Therefore, total profits of the graph $F_{n,d}$ are bounded from below as follows:

$$\Pi(F_{n,d}) = n\lambda_{PF}(F_{n,d}) - 2m(d)c \geq n(d-1) - 2m(d)c. \tag{48}$$

Since the inequality above is valid for any integer d, such that $1 \leq d \leq n$, we are interested in the value of d that maximizes the right hand side of Equation (48), that is

$$d = \underset{1 \leq k \leq n}{\operatorname{argmax}} \{n(k-1) - 2m(k)c\}, \tag{49}$$

where $m(k) = \binom{n}{2} + (n-k)$ and $k \in \mathbb{N}_+$. By computing the first and second derivative of the objective function $n(k-1) - 2m(k)c$ with respect to k, one finds that its maximum occurs for $k = \frac{n+3c}{2c}$. For simplicity, one can take d as the closest integer to this value [24]. Notice that, as a consequence, d converges to $\frac{n}{2c}$ for large n.

[24] The results on the relative error that we obtain later in this proof are still valid under this assumption.

Replacing $d = \frac{n+3c}{2c}$ in Equation (48), we obtain a lower bound, which is independent of d, and given by

$$\Pi(F_{n,d}, c) \geq \frac{n^2 + n(2c - 8c^2) + 9c^2}{4c}. \tag{50}$$

We now derive an upper bound for total profits of the efficient network G^*. The largest real eigenvalue of a connected graph is at most $\sqrt{2m - n + 1}$ (Hong, 1993) and from this it follows immediately that total profits of G^* are bounded by

$$\Pi(G^*, c) \leq n\sqrt{2m - n + 1} - 2mc. \tag{51}$$

We have shown already that for cost $c \leq 1/2$ the efficient graph is complete. Therefore, we are interested in values of cost $c > 1/2$. Assuming that $c > 0.5$, the number m of edges that maximize the right hand side of Equation (51) is $m = \frac{n^2+4nc^2-4c^2}{8c^2}$.

Replacing such value of m, we obtain an upper bound that is independent on the number m of edges,

$$\Pi(G^*, c) \leq \frac{n^2 - 4nc^2 + 4c^2}{4c}. \tag{52}$$

At this point, combining Equation (50) and (52), we obtain that the relative difference ϵ in the total profits of the graph $F_{n,d}$ and the graph G^* is bounded from above by

$$\epsilon = \frac{\Pi(G^*, c) - \Pi(F_{n,d}, c)}{\Pi(F_{n,d}, c)} \leq \frac{2c(2c-1)n - 5c^2}{n^2 + 2c(1-2c)n + 9c^2}. \tag{53}$$

The expression on the right hand side of the above inequality converges to zero for n large, and therefore the relative difference in total profits vanishes for n large.

(iii) Since for the complete graph K_n, $\lambda_{\mathrm{PF}} = n - 1$ and $m = \frac{n(n-1)}{2}$, its total profits are given by

$$\Pi(K_n) = n(n-1) - 2\frac{n(n-1)}{2}c = n(n-1)(1-c). \tag{54}$$

On the other hand, the largest real eigenvalue λ_{PF} of a graph G with m edges is bounded from above so that $\lambda_{\mathrm{PF}} \leq \frac{1}{2}(\sqrt{8m+1} - 1)$ (Stanley, 1987) [25]. For total profits we then have

$$\begin{aligned} \Pi &= \textstyle\sum_{i=1}^{n} \lambda_{\mathrm{PF}}(G_i) - 2mc \\ &\leq n \max\nolimits_{1\leq i\leq n} \lambda_{\mathrm{PF}}(G_i) - 2mc \\ &\leq \tfrac{n}{2}(\sqrt{8m+1} - 1) - 2cm \\ &= b(n, m, c), \end{aligned} \tag{55}$$

with $n \leq m \leq \binom{n}{2}$. For fixed cost c and number of nodes n, the number of edges maximizing Equation (55) is given by $m^* = \frac{n^2-c^2}{8c^2}$ if $\frac{n^2-c^2}{8c^2} < \binom{n}{2}$ and $m^* = \frac{n(n-1)}{2}$

[25] Notice that a similar result can be obtained using an alternative bound for connected graphs, $\lambda_{\mathrm{PF}} \leq \sqrt{2m - n + 1}$ due to (Hong, 1993)

if $\frac{n^2-c^2}{8c^2} > \binom{n}{2}$. The graph with the latter number of edges is the complete graph K_n. Inserting m^* into Equation (55) yields

$$b(n, m^*, c) = \begin{cases} \frac{n}{2}(\sqrt{\frac{n^2-c^2}{c^2}+1}-1) - \frac{n^2-c^2}{4c}, & \text{if } c > \frac{n}{2n-1}, \\ n(n-1)(1-c) = \Pi(K_n), & \text{if } c < \frac{n}{2n-1}. \end{cases} \tag{56}$$

The bound for $c \leq \frac{n}{2n-1} \sim \frac{1}{2}$ in the limit of large n coincides with total profits of

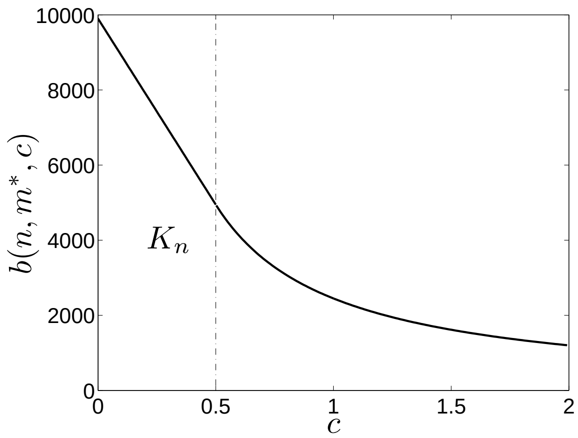

Fig. 12. Upper bound $b(n, m^*, c)$ of Equation (56) for $n = 100$ and varying costs c. For $c \leq \frac{n}{2n-1}$ the upper bound corresponds with the complete graph K_n.

the complete graph K_n. Therefore K_n is the efficient graph in that region of cost.

(iv) If $c = n$ then the number of edges maximizing Equation (55) is given by $m^* = 0$ and the efficient graph is the empty graph $\bar{K}_n$.

Finally, in Figure (13) we show total profits and the number of links of $F_{n,d}$ for $n = 10$. Note that in this case $F_{n,d}$ is not an approximation but the exact efficient network.

Proof of Proposition (4) The three claims of the proposition are addressed in sequence.

(i) With $\sum_i d_i = 2m$ we can write the degree variance as follows

$$\sigma_d^2 = \frac{1}{n}\sum_{i=1}^n \left(d_i - \frac{2m}{n}\right)^2 = \frac{1}{n}\sum_{i=1}^n d_i^2 - \left(\frac{2m}{n}\right)^2. \tag{57}$$

Using then the fact that the graph $F_{n,d}$ contains one node with degree $n-1$ (the hub), $d-1$ nodes with degree $d-1$ (those in the clique) and $n-d$ nodes with degree 1, we get

$$\sigma_d^2(F_{n,d}) = \frac{1}{n}\left((n-1)^2 + (d-1)^3 + (n-d)\right) - \left(\frac{2m}{n}\right)^2. \tag{58}$$

We now replace in the equation above the value of d that maximizes total profits for the graph $F_{n,d}$, $d = \frac{n+3c}{2c}$, as found from Equation (49), as well as the corresponding value of m, given by $m(d) = \frac{n^2+8c^2n-9c^2}{8c^2}$. As a result, one obtains the degree variance and this expression is of quadratic order in n, $\sigma_d^2 = \mathcal{O}(n^2)$.

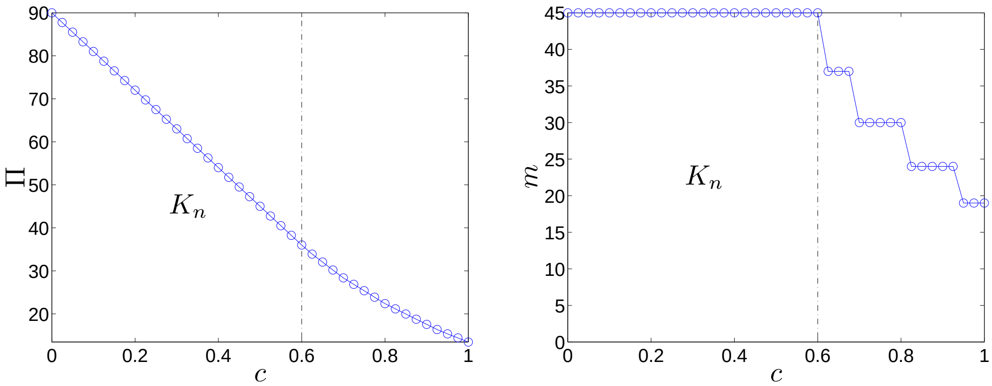

Fig. 13. Total profits Π (left) and number of links m (right) of the efficient networks $F_{n,d}$, with $n = 10$. The plot on the right shows that, with this value of n, the efficient network coincides with the complete graph ($m = n(n-1)/2 = 45$) for $c \in [0, 0.6]$, while it differs significantly from the complete graph for higher value of c. Note that in Proposition (3,(iii)), $c \leq 0.5$ is a sufficient condition for the efficient graph to be complete, but not a necessary one.

(ii) The coefficient of variation of the degree is defined as $c_v = \sigma_d/\bar{d}$. Recalling that the average degree is $\bar{d} = 2m/n$ and replacing, as above, the value of d that maximizes total profits, $d = \frac{n+3c}{2c}$, and the corresponding value of m, one obtains an expression in n and c. The limit of large n for this expression is well defined and equal to

$$\lim_{n\to\infty} c_v = \sqrt{2c-1}. \tag{59}$$

(iii) Gutman and Paule (2002) have shown that the degree variance of a random graph $G(n, m)$ with n nodes and m links is given by

$$\sigma_d^2(G(n,m)) = \frac{2m(n^2 - n - 2m)}{n^3 + n^2}. \tag{60}$$

Replacing m as in (ii), the expression above turns out to be of order $\mathcal{O}(n)$, and consistently, the ratio of (58) and (60) is of order $\mathcal{O}(n)$.

Proof of Lemma (3) The assumptions on the improving path require that we add one link at a time. Starting from an empty network, the first link added yields a pair (i.e. a path[26] P_2 of length 2), with an eigenvalue of $\lambda_{\mathrm{PF}} = 1 > \frac{2}{n}$ for $n > 2$. If a second link is added to one of the nodes of the pair by attaching another node to them, a path of three nodes is formed, with associated largest real eigenvalue $\lambda_{\mathrm{PF}} = 2\cos(\pi/4) = 1.41$ (see Table (1)). It is $1.41 > \frac{4}{n}$ for $n \geq 3$. Therefore, we can always find an integer $m_3 \geq 1$, such that $\lambda_{\mathrm{PF}}(m) > \frac{4}{n}$ for $m \in [0, m_3] \cap \mathbb{N}_+$, with $m_3 \leq n(n-1)/2$.

Proof of Proposition (5) Along an improving path the number m of links can vary only by one or zero in absolute value. Here, we restrict ourselves to the improving paths in which the number of links increases from $m_1 = 0$ to at most $m_2 = n(n-1)/2$. From

[26] The term *path* refers to a particular type of graph, as we have discussed in Section 2.1. On the other hand, in Section 4.1 we have introduced the term improving path which refers to a sequence in the space of graphs.

Proposition (1) we know that the largest real eigenvalue is bounded, $\lambda_{\mathrm{PF}}(m) \leq n-1$. Taking the average between the extreme values $\lambda_{\mathrm{PF}}(m_1) = 0$ and $\lambda_{\mathrm{PF}}(m_2) = n-1$ we get an average increase of $\lambda_{\mathrm{PF}}(m)$ per additional link of $\frac{2}{n}$. This fact is depicted by the straight line in Figure (5) which has a slope of $\frac{2}{n}$ and intersects the origin. Moreover, Figure (5) shows an example of an improving path that reaches 50% of the density of a complete graph before it arrives at a stable network. Let us now define $y_m = \lambda_{\mathrm{PF}}(m) - \frac{2}{n}m$. y_m is just the difference between the largest real eigenvalue of the improving path and the straight line in Figure (5). Since we have that $y_{m_1} = 0$ and $y_{m_2} = 0$, it obviously holds that the sum of the increments $\Delta y_m = y_m - y_{m-1}$ in $I = [m_1, m_2] \cap \mathbb{N}_+$, have to be zero, that is

$$\sum_{m=m_1+1}^{m_2} \Delta y_m = 0. \tag{61}$$

However, Lemma 3 ensures that along the improving path, y_m starts off positive. Therefore, we can always find an integer $m_3 \geq 1$, such that (1) $y_{m_3} = b > 0$ and (2) $I = I_1 \cup I_2$, with $I_1 = [m_1, m_3] \cap \mathbb{N}_+$ and $I_2 = [m_3, m_2] \cap \mathbb{N}_+$. The condition (61) on the increments of y implies that

$$\sum_{m \in I_2} \Delta y_m = -\sum_{m \in I_1} \Delta y_m = -b. \tag{62}$$

Denoting by $\langle \Delta y \rangle_{I_2}$ the average increment in the set I_2, we have

$$\sum_{m \in I_2} \Delta y_m = \langle \Delta y \rangle_{I_2} (m_2 - m_3). \tag{63}$$

There must be some increments that are smaller or equal to the average increment. Hence, there must exists a value m^* such that

$$\Delta y_{m^*} \leq -\frac{b}{m_2 - m_3} < 0, \tag{64}$$

or, equivalently, there exists an m^* such that

$$\Delta \lambda_{\mathrm{PF}}(m^*) < \frac{2}{n}. \tag{65}$$

For any given cost c, we can find an n large enough and an m^* such that $\Delta \lambda_{\mathrm{PF}}(m^*) < c$. This means that the marginal revenue is smaller than the cost, for some value of m^*. This concludes the proof.

Proof of Proposition (6) If costs are zero, $c = 0$, then the change in eigenvalue equals the change in profits. Since (in a connected graph G) each link created strictly increases λ_{PF} (Horn and Johnson, 1990) and accordingly profits, the complete graph K_n is reached eventually.

Proof of Proposition (7) There exists the following bound on the change in eigenvalue by the removal and creation of a link (Cvetkovic et al., 1995): If the graphs G, G' differ in one edge only then $|\lambda_{\mathrm{PF}}(G') - \lambda_{\mathrm{PF}}(G)| \leq 1$. A link is created if $\Delta \lambda_{\mathrm{PF}} > c$. Thus, no link is created if $c = 1$. On the other hand, a link is removed, if $\Delta \lambda_{\mathrm{PF}} < c'$. And thus, all links are removed if $c' > 1$ and we obtain an empty graph $\bar{K}_n$.

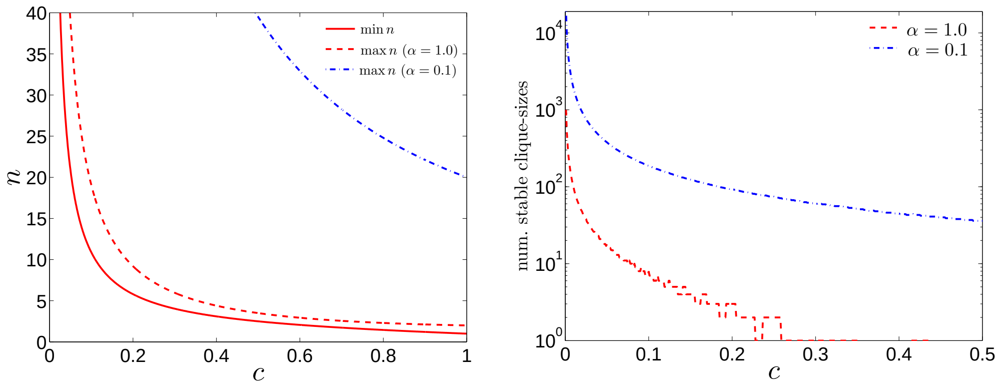

Fig. 14. Upper and lower bounds for the size of the stable cliques (left) and the number of different sizes of stable cliques (right) for $\alpha = 1$ and $\alpha = 0.1$.

Proof of Proposition (8) The structure of the proof is as follows. We want to show that the graph G consisting of k cliques of the same size is stable, that is, no link is removed or created. For the removal, we can focus on links between nodes in the same clique, since these are the only links in G. Thus, in Proposition (13) we show that, for any pair of nodes in the same clique, the link is not removed as long as the size n of the clique is smaller than a given bound b_r. In particualar, we will show that, $b_r = \lfloor \frac{2-c'(1-c')}{c'} \rfloor$.

For the creation of links, we can focus on links between nodes in different cliques, since these are the only new links that can be added to the graph. Thus, in Proposition (14) we show that for any pair of firms belonging to different cliques, a link between them is not created as long as the size n of the clique is larger than another bound b_c, and we will show that $b_c = \lceil \frac{1+c(1-c)}{c} \rceil$.

It turns out that the bound for the removal, b_r, is larger than the bound for the creation, b_c, for any value of $c \in [0, 1]$, as it is shown in Figure (14). However, since the size n of the clique has to be an integer, the interval $[b_c, b_r]$ needs to contain at least one integer. This can be done constructively. We explored the interval $c \in [0, 1]$ with cost increments of 10^{-3} and we counted the number of integer values that fall within $[b_c, b_r]$. As it is shown in Figure (6), for $c < 0.35$, there is always at least one integer in between the two bounds, while for $c < 0.2$, there are always several integers falling in between the two bounds.

This is a remarkable finding as it implies that for the values of cost given above, there exists a multiplicity of equilibria. Indeed, for a given value of cost, the stable graphs are all the configurations with cliques of the same size n, where n varies among the integers included in the interval $[b_c, b_r]$.

Propositions (8) and (13), used for this proof, are given below.

Proposition 13 *Consider a clique K_n and denote by $K_n - ij$ the graph obtained from K_n by removing an edge ij. Then $\lambda_{PF}(K_n) - \lambda_{PF}(K_n - ij) > c'$ if $n \leq \lfloor \frac{2-c'(1-c')}{c'} \rfloor$.*

Proof of Proposition (13) Denote the matrix obtained from the adjacency matrix $\mathbf{A}$ of $K_n - ij$, and subtracting the variable λ on the diagonal of $\mathbf{A}$ by $\mathbf{M} = \mathbf{A} - \lambda\mathbf{I}$. $\mathbf{M}$ is a block matrix of the form

$$\mathbf{M} = \begin{pmatrix} \mathbf{K} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{D} \end{pmatrix}, \tag{66}$$

with submatrices[27]

$$\mathbf{K} = \begin{pmatrix} -\lambda & 1 & \cdots\cdots & 1 \\ 1 & -\lambda & & \vdots \\ \vdots & & \ddots & \\ & & & 1 \\ 1 & \cdots & 1 & -\lambda \end{pmatrix}_{(n-2)\times(n-2)}, \tag{67}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdots \end{pmatrix}_{2\times n}, \tag{68}$$

$$\mathbf{D} = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}_{2\times 2}. \tag{69}$$

Since $\mathbf{M}$ is a block-matrix (Horn and Johnson, 1990) we can write

$$\det(\mathbf{M}) = \det(\mathbf{K})\det(\mathbf{P}). \tag{70}$$

We have the following lemma.
Lemma 5

$$\det \begin{pmatrix} a & 1 & \cdots\cdots & 1 \\ 1 & a & & \vdots \\ \vdots & & \ddots & \vdots \\ & & & 1 \\ 1 & \cdots\cdots & 1 & a \end{pmatrix}_{n\times n} = ((n-1)+a)(a-1)^{n-1}. \tag{71}$$

Proof of Lemma (5) The above determinant can be written as $\det(\mathbf{U} - (1-a)\mathbf{I})$, where $\mathbf{U}$ is a matrix consisting of all ones, $u_{ij} = 1$ $i, j = 1, ..., n$ and $\mathbf{I}$ is the identity matrix. Hence, the eigenvalues of the above matrix are minus $1-a$ the eigenvalues of $\mathbf{U}$. $\mathbf{U}$ has eigenvalues n and 0 with multiplicities 1 and $n-1$ respectively (Horn and Johnson, 1990). Therefore, we can write for the determinant $(n-(1-a))(0-(1-a))^{n-1} = ((n-1)+a)(a-1)^{n-1}$.

Thus, we get for the determinant of $\mathbf{K}$

$$\det\mathbf{K} = -((n-1)-\lambda)(1+\lambda)^{n-1}. \tag{72}$$

[27] The numbers at the bottom right of the matrix indicate the dimension of the matrix.

The Schur complement is $\mathbf{P} = \mathbf{D} - \mathbf{B}\mathbf{K}^{-1}\mathbf{B}^T$. Multiplying the inverse of $\mathbf{K}$ with $\mathbf{B}$ from the left and $\mathbf{B}^T$ from the right we obtain

$$\mathbf{B}\mathbf{K}^{-1}\mathbf{B}^T = ||\mathbf{K}^{-1}||_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{73}$$

where $||\mathbf{K}^{-1}||_1$ is the sum of all elements in the matrix $\mathbf{K}^{-1}$ (the l_1 norm of the matrix $\mathbf{K}^{-1}$ (Horn and Johnson, 1990)). By computing $\mathbf{K}^{-1}\mathbf{K} = \mathbf{I}$ one can verify that

$$\mathbf{K}^{-1} = \begin{pmatrix} \frac{n-4-\lambda}{(\lambda-(n-3))(1+\lambda)} & -\frac{1}{(\lambda-(n-3))(1+\lambda)} & \cdots \\ -\frac{1}{(\lambda-(n-3))(1+\lambda)} & \ddots & \\ \vdots & & \end{pmatrix}. \tag{74}$$

And, by summation over the elements in $\mathbf{K}^{-1}$, we obtain $||\mathbf{K}^{-1}||_1 = \frac{n-2}{(n-3)-\lambda}$. Consequently, the determinant of the Schur complement $\mathbf{P}$ is given by

$$\det(\mathbf{P}) = (1+\lambda)^{n-3}\lambda\left(\lambda^2 - (n-3)\lambda - 2(n-2)\right). \tag{75}$$

The largest real eigenvalue of $K_n - ij$ is given by the root of

$$\lambda^2 - (n-3)\lambda - 2(n-2) = 0. \tag{76}$$

Thus we get

$$\lambda_{\mathrm{PF}} = \frac{1}{2}\left(n - 3 + \sqrt{n^2 + 2n - 7}\right). \tag{77}$$

For the change in eigenvalue $\Delta\lambda_{\mathrm{PF}} = \lambda_{\mathrm{PF}}(K_n) - \lambda_{\mathrm{PF}}(K_n - ij)$ we obtain

$$\Delta\lambda_{\mathrm{PF}} = \frac{1}{2}\left(n + 1 - \sqrt{n^2 + 2n - 7}\right), \tag{78}$$

since $\lambda_{\mathrm{PF}}(K_n) = n - 1$. This is a decreasing function in n. Then for $n \in \mathbb{N}$, $\Delta\lambda_{\mathrm{PF}} > c'$ if

$$n \leq \lfloor \frac{2 - c'(1-c')}{c'} \rfloor. \tag{79}$$

For $c' = 2 - \sqrt{2} = 0.586$ we have $n \leq 3$ and for $c' = 1$ we obtain $n \leq 2$.

Proposition 14 *Denote the graph consisting of two disconnected cliques by G and the graph obtained from G by connecting the two cliques in G via an edge by G'. Then for $n \geq \lceil \frac{1+c(1-c)}{c^2} \rceil$ we have $\lambda_{PF}(G') - \lambda_{PF}(G) < c$.*

Proof of Proposition (14) Denote the adjacency matrix of the graph obtained by connecting two complete subgraphs K_n and K_n via and edge, see Figure (15), by $\mathbf{A}$. And denote the matrix obtained by subtracting the variable λ on the diagonal of $\mathbf{A}$ by $\mathbf{M} = \mathbf{A} - \lambda\mathbf{I}$. The eigenvalues of $\mathbf{A}$ are given by the roots of the determinant of $\mathbf{M}$. $\mathbf{M}$ has the form of a block matrix with the submatrices $\mathbf{K}$ and $\mathbf{B}$. We have

$$\mathbf{M} = \begin{pmatrix} \mathbf{K} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{K} \end{pmatrix}, \tag{80}$$

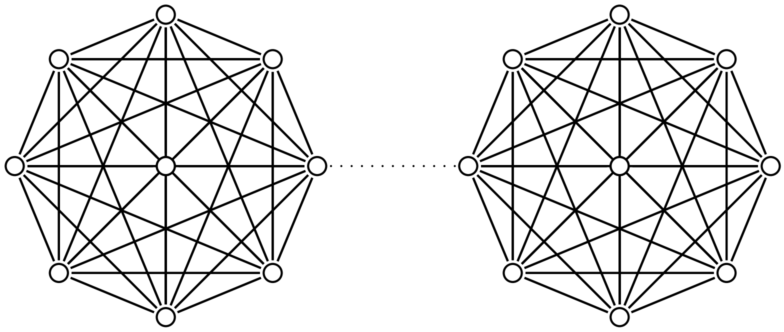

Fig. 15. Two complete graphs, K_9, connected through an edge.

$$\mathbf{K} = \begin{pmatrix} -\lambda & 1 & \cdots\cdots & 1 \\ 1 & -\lambda & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & -\lambda \end{pmatrix}_{n\times n} . \tag{81}$$

Due to the symmetry of the graph we can consider a matrix of the following form, where we have put the one on the diagonal indicating the link between the cliques,

$$\mathbf{B} = \begin{pmatrix} 0 \cdots & & \cdots 0 \\ \vdots \ddots & & \vdots \\ & 0 & \\ & 0\ 1\ 0 & \\ & 0 & \\ \vdots & & \ddots \vdots \\ 0 \cdots & & \cdots 0 \end{pmatrix}_{n\times n} , \tag{82}$$

with the Schur complement

$$\mathbf{P} = \mathbf{K} - \mathbf{B}^T\mathbf{K}^{-1}\mathbf{B}. \tag{83}$$

For the determinant of $\mathbf{M}$ we have $\det \mathbf{M} = \det(\mathbf{K})\det(\mathbf{P})$. The determinant of $\mathbf{K}$ is given by

$$\det(\mathbf{K}) = (1+\lambda)^{n-3}(\lambda - n + 3). \tag{84}$$

The inverse of $\mathbf{K}$ is already given in (74) [28]. W.l.o.g. the Schur complement $\mathbf{P}$ is given by

[28] Note that here the matrix $\mathbf{K}$ has dimension $n \times n$

$$
\mathbf{P} = \begin{pmatrix} -\lambda & 1 & \cdots & \cdots & & 1 \\ 1 & -\lambda & & & & \vdots \\ \vdots & & \ddots & & & \\ & & & -\lambda & & 1 \\ 1 & \cdots & & 1 & & -\lambda + \frac{\lambda-(n-2)}{(\lambda-(n-1))(1+\lambda)} \end{pmatrix}. \tag{85}
$$

In the next step we make use of the following lemma.

Lemma 6

$$
\det \begin{pmatrix} b & 1 & \cdots & 1 \\ 1 & a & & \vdots \\ \vdots & & \ddots & \\ & & a & 1 \\ 1 & \cdots & 1 & a \end{pmatrix}_{n\times n} = (1-n+(n-2)b+ab)\,(a-1)^{n-2}. \tag{86}
$$

Proof of Lemma (6) We give a proof by induction. For $n=2$ we get

$$
\det \begin{pmatrix} b & 1 \\ 1 & a \end{pmatrix}_{2\times 2} = ab-1 = (b(2-2)+ab-(2-1))(a-1)^0. \tag{87}
$$

For $n=3$ we get

$$
\det \begin{pmatrix} b & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix}_{3\times 3} = a^2b-2a+2-b = (b+ab-2)(a-1). \tag{88}
$$

For the induction step we apply a Laplace expansion of the determinant in (86) into Minors.

$$
b\det \begin{pmatrix} a & 1 & \cdots & \cdots & 1 \\ 1 & a & & & \vdots \\ \vdots & & \ddots & & \\ & & & a & 1 \\ 1 & \cdots & & 1 & a \end{pmatrix}_{(n-1)\times(n-1)} - (n-1)\det \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & a & & & \vdots \\ \vdots & & \ddots & & \\ & & & a & 1 \\ 1 & \cdots & & 1 & a \end{pmatrix}_{(n-1)\times(n-1)} \tag{89}
$$

For the first determinant we can use Lemma (5) and for the second the induction hypothesis in order to obtain

$$
b\,((n-2)+a)\,(a-1)^{n-2} - (n-1)(a-1)^{n-2}. \tag{90}
$$

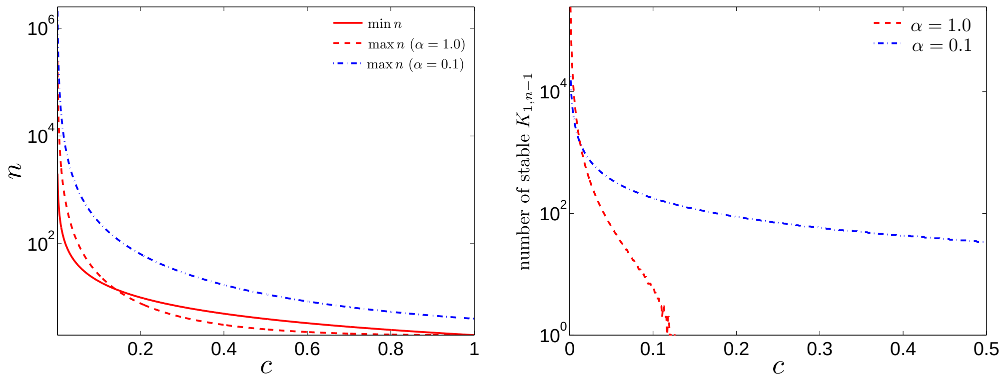

Fig. 16. Upper and lower bounds for the sizes of the stars (left) and the number of stars of different sizes (right) for $\alpha = 1$ and $\alpha = 0.1$. One can see that for $\alpha = 1$ and cost larger than 0.4 a stable spanning star does not exist.

Now we can compute the determinant of the Schur complement $\mathbf{P}$

$$\begin{aligned} \det \mathbf{P} &= -\left(1 - n + (n-2)q - \lambda q\right)(1+\lambda)^{n-2} \\ q &:= \tfrac{-(n-2)+\lambda}{(-(n-1)+\lambda)(1+\lambda)} - \lambda. \end{aligned} \tag{91}$$

λ_{PF} is given by the largest root of $\det \mathbf{P} = 0$. We obtain $\lambda_{\mathrm{PF}} = \frac{1}{2}(n - 1 + \sqrt{n^2 - 2n + 5})$. The change in the largest real eigenvalue is

$$\begin{aligned} \Delta\lambda_{\mathrm{PF}} &= \tfrac{1}{2}(n - 1 + \sqrt{n^2 - 2n + 5}) - (n-1) \\ &= \tfrac{1}{2}(1 - n + \sqrt{(n-2)n + 5}), \end{aligned} \tag{92}$$

since $\lambda_{\mathrm{PF}}(K_n) = n - 1$. Thus, $\Delta\lambda_{\mathrm{PF}} < c$ if

$$n \geq \lceil \frac{1 + c(1-c)}{c} \rceil. \tag{93}$$

For costs $c = 0.5$ we get $n \geq 2$ and for $c = 1$ we get $n \geq 1$.

Proof of Proposition (9) In order to proof the stability of the spanning star $K_{1,n-1}$, that connects all nodes in the network, we have to consider two cases: (i) the creation of a link and (ii) the removal of a link (see Figure (17)).

(i) We consider the creation of a link between the nodes in the star. The normalized eigenvector associated with the largest real eigenvalue λ_{PF} is given by

$$\frac{1}{\sqrt{2(n-1)}}(1, ..., 1, \sqrt{n-1}, 1, ..., 1)^T. \tag{94}$$

Maas (1987) found an upper bound for the largest real eigenvalue λ_{PF} and the corresponding eigenvector $\mathbf{x}$ of an undirected graph G if an edge ij is added

$$\lambda_{\mathrm{PF}}(G + ij) - \lambda_{\mathrm{PF}}(G) < 1 + \delta - \frac{\delta(1+\delta)(2+\delta)}{(x_i + x_j)^2 + \delta(2 + \delta + 2x_i x_j)}, \tag{95}$$

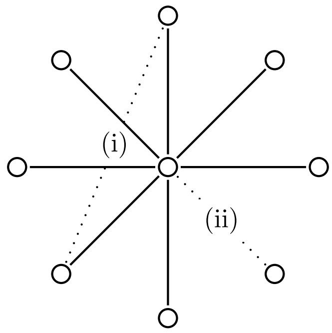

Fig. 17. A star $K_{1,8}$ and (i) the creation of a link or (ii) the removal of a link.

where δ denotes the minimum degree in the graph G [29]. Applying Equation (95) to the star $K_{1,n-1}$ gives $\Delta\lambda_{\text{PF}} = \lambda_{\text{PF}}(K_{1,n-1} + ij) - \lambda_{\text{PF}}(K_{1,n-1}) < \frac{2}{n}$. The link ij is not created if $\Delta\lambda_{\text{PF}} < c$ or equivalently

$$n > \frac{2}{c}. \tag{96}$$

This is a decreasing function in c.

(ii) The change in eigenvalue by removing a link from $K_{1,n-1}$ is given by $\Delta\lambda_{\text{PF}} = \lambda_{\text{PF}}(K_{1,n-1}) - \lambda_{\text{PF}}(K_{1,n-2}) = \sqrt{n-1} - \sqrt{n-2}$. A link is not removed from the star if $\Delta\lambda_{\text{PF}} > c'$ or equivalently

$$2 < n < \frac{1 + c'^2(6 + c'^2)}{4c'^2}. \tag{97}$$

Putting the bounds obtained in (i) and (ii) together we get the desired proposition. The number of different sizes of stars that are stable are shown in Figure (5).

Proof of Proposition (11) A link between two disconnected firms is created if the largest real eigenvalue of the connected component of the firms after the link is created increases more then the cost, i.e. $\Delta\lambda_{\text{PF}} > c$. Similarly an existing link is removed if the largest real eigenvalue of the connected component of the firms after the link is removed does not decrease more then the cost, i.e. $|\Delta\lambda_{\text{PF}}| < c' = \alpha c$, with $c, c', \alpha \in [0, 1]$. We therefore have to consider the change in eigenvalue by the creation or removal of a link and compare it to the cost.

The proof of Proposition (11) is composed of two steps. (i) We show that in every period t in the network formation process $\Gamma(G) = G(0), G(1), \ldots$ the network $G(t)$, $t \geq 1$, consists only of graphs from the set $S = \{\emptyset, P_2, P_3, P_4\}$, where $\emptyset$ denotes the set of isolated nodes. (ii) We show that there exists a cycle, i.e. a sequence of repeatedly visited graphs, $C = (P_2, \{P_2, P_2\}, P_4, P_3)$, in which each graph is an improvement over the previous graph in the sequence C (Jackson and Watts, 2002). Since all the graphs in the set S can be found in the cycle C, starting from any of the graphs in S, the network formation process will proceed to the next graph in the cycle C. Therefore, for the given values of α and cost c, c' respectively, we can infer that there does not exist a pairwise stable equilibrium network.

[29] Equation (95) is an upper bound and the number of stable stars derived from it may actually be higher.

(i) We give a proof by induction on the periods $t \geq 1$ of the network formation process $\Gamma(G)$. The induction basis is period $t = 1$. The network $G(1)$ is obtained from the empty network $G(0) = \bar{K}_n$ (initial network) by the formation of a link and thus contains only a P_2 and isolated nodes, both graphs are contained in the set S. Now we assume that the network at time $t > 1$ consist only of graphs in the set S (induction hypothesis). The induction step consists in showing from $G(t)$ to $G(t+1)$, no other graphs than the ones in the set S will be created. This will conclude this part of the proof. In order to prove the induction step, we observe that in the network formation process $\Gamma(G)$, at time t, a pair of nodes, say i and j, is selected at random. Either i and j are already connected in $G(t)$ or they are not. In any case, they both belong by assumption to one of the graphs in S. All the possible cases can be grouped as follows.

(a) Both nodes are isolated. We show that the empty graph evolves into a P_2. The creation of a link between two isolated nodes results in $\Delta\lambda_{\mathrm{PF}} = 1$. Since by assumption $c < 1$, the link is indeed created.

(b) At least one of the nodes, say i, is part of a P_2. In this case, we show that the only possible evolution step is from two P_2 to one P_4.

(i) Link creation: Figure (18) shows all possible distinct graphs that can be obtained depending on which graph belongs the second node, j, and in which position. Each of these possible graphs is named with a number in the following way. For instance, when j is in another P_2, the possible positions in that P_2 result both in one same graph labelled as 1. When j is in a P_3, there are two possible distinct resulting graphs, labelled as 2.1 and 2.2. Similarly, we label the graphs resulting in the remaining case that j is in a P_4. Table (2) report the increase of the largest eigenvalue of the graph when the link is created in all the possible cases. For instance, consider the graph 2.1 resulting from a P_2 and a P_3 with the creation of a link. Before the creation of the link, node i is in a P_2 which has $\lambda_{PF} = 1$ and node j is in a P_3 graph which has $\lambda_{PF} = \sqrt{2}$. Since the link formation rule requires that both nodes will benefit after the creation of the link, we have to consider the worst case for the initial graph, which means the highest of the two values, i.e. $\lambda_{PF} = \sqrt{2} = 1.414$. For the resulting graph 2.1 we have $\lambda'_{PF} = \sqrt{3} = 1.732$, and therefore an increase of eigenvalue $\Delta\lambda_{PF} = 0.318$ which is smaller than the cost $c = 0.586$. It follows that this link will not be created. After analyzing all the other cases, we can see that only the case 1 results in an increase in the largest real eigenvalue $\Delta\lambda_{\mathrm{PF}} = 0.618$ that is higher than the lower bound of the cost $c > 2 - \sqrt{2} = 0.586$. This implies that the only possible evolution step at this point is the formation of one P_4 starting from two P_2. Notice that P_4 is in the set S.

(ii) Link deletion: A link is deleted if this beneficial to at least one of the two firms concurrent to the link, or, equivalently, if $|\Delta\lambda_{\mathrm{PF}}| < c' = c\alpha$. In the case we are considering, by assumption at least one of the nodes is in a P_2 and we examine the deletion of a link. This implies that the two nodes form a P_2, which has $\lambda_{\mathrm{PF}} = 1$. The deletion of the link implies to evolve into an empty graph which has $\lambda_{\mathrm{PF}} = 0$, yielding $|\Delta\lambda_{\mathrm{PF}}| = 1$. Since, by assumption we have that $c > 2-\sqrt{2}$, the case of $c\alpha > 1$ implies

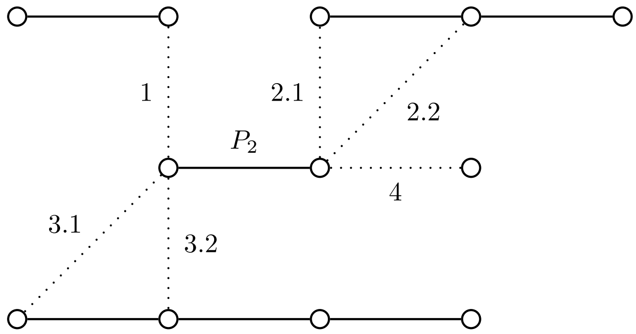

Fig. 18. All possible graphs for link creation when at least one of the selected nodes is part of a P_2. We have labeled all the possible cases or links respectively with numbers shown next to the dashed links.

	1	2.1	2.2	3.1	3.2	4
λ_{PF}	1	1.414		1.618		1
λ_{PF}'	1.618	1.732	1.848	1.802	1.932	1.414
$\Delta\lambda_{\mathrm{PF}}$	0.618	0.318	0.434	0.184	0.314	0.414

Table 2
Change in eigenvalue for link creation when at least one of the selected nodes is part of a P_2. The numbers in the first row in the table refer to the possible links indicated by the same numbers in Figure (18). The maximum increase in the largest real eigenvalue is given by the creation of a link between the two pairs, indicated by 1 in Figure (18).

that the link is removed only if $\alpha > \frac{1}{2-\sqrt{2}} = 1.707$. But we have assumed that $\alpha \in [0, 1]$ and therefore the link is not removed.

(c) At least one of the nodes is part of a P_3. In this case, we show that if $\alpha \in [0.707, 1]$ then the only possible evolution step is from one P_3 to one P_2 and one isolated node.
   (i) Link creation: Figure (19) shows all possible graphs that can be obtained by adding a link when at least one of the selected nodes is part of a P_3. Table (3) shows the increase in eigenvalue for all these possible graphs. From the Table we can see that in none of the cases the increase in eigenvalue is higher than the lower bound of the cost. Thus, no link is created.
   (ii) Link deletion: The removal of a link from P_3 results in a change in eigenvalue of $\Delta\lambda_{\mathrm{PF}} = \sqrt{2} - 1 = 0.414$. The lower bound for the cost is $c > 2 - \sqrt{2}$. We have that $|\Delta\lambda_{\mathrm{PF}}| \le c' = \alpha c$ if $\alpha \ge \frac{\sqrt{2}-1}{2-\sqrt{2}} = 0.707$. Therefore, if we restrict the values of α to the interval $[0.707, 1]$ then the link is removed and we obtain a single connected pair P_2 and one isolated node. Both are contained in the set of graphs $S = \{\emptyset, P_2, P_3, P_4\}$.

(d) At least one of the nodes is part of a P_4. In this case, we show that if $\alpha \in [0.707, 1]$ then the only possible evolution step is from one P_4 to one P_3 and an isolated node.

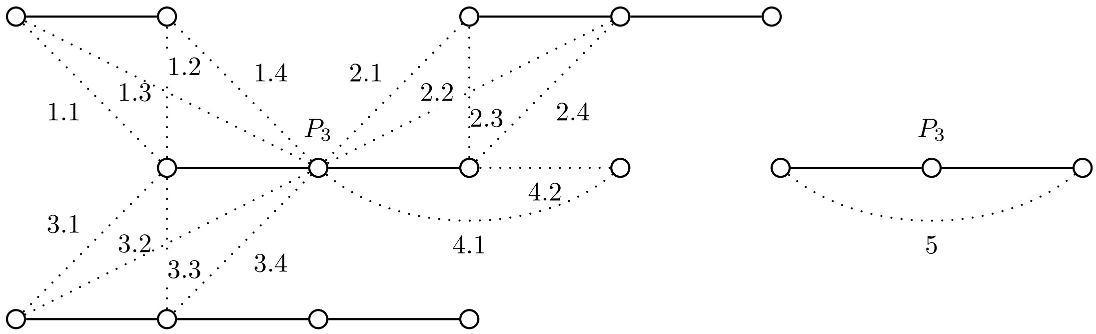

Fig. 19. All possible cases for link creation when at least one of the selected nodes is part of a P_3. We have labeled all the possible cases or links respectively with numbers shown next to the dashed links.

	1.1	1.2	1.3	1.4
λ_{PF}	1.414			
λ_{PF}'	1.732	1.848	1.732	1.848
$\Delta\lambda_{\mathrm{PF}}$	0.318	0.434	0.318	0.434

	2.1	2.2	2.3	2.4
λ_{PF}	1.414			
λ_{PF}'	1.902	1.802	2	1.902
$\Delta\lambda_{\mathrm{PF}}$	0.488	0.388	0.586	0.488

	3.1	3.2	3.3	3.4
λ_{PF}	1.618			
λ_{PF}'	1.848	2	1.932	2.053
$\Delta\lambda_{\mathrm{PF}}$	0.23	0.382	0.314	0.435

	4.1	4.2	5
λ_{PF}	1.414		
λ_{PF}'	1.732	1.618	2
$\Delta\lambda_{\mathrm{PF}}$	0.318	0.204	0.586

Table 3

Change in eigenvalue for link creation when at least one of the selected nodes is part of a P_3. The numbers in the first row in the table refer to the possible links indicated by the same numbers in Figure (19). The maximum increase in the largest real eigenvalue is given by the creation of a triangle, indicated by 5 in Figure (19). However, does not exceed the minimum value of cost $c \geq 0.586$ and so the corresponding firms do not form this link.

(i) Link creation: Figure (20) shows the possible graphs that can be obtained by adding a link when at least one of the selected nodes is part of a P_4. Table (3) shows the corresponding increase in eigenvalue. From the Table we can see that in none of the cases the increase in eigenvalue is higher than the lower bound of the cost. Thus, no link is created.

(ii) Link deletion: The change in eigenvalue is either (A) $\Delta\lambda_{\mathrm{PF}} = 0.204$ if the first or last link in P_4 is removed or (B) it is $\Delta\lambda_{\mathrm{PF}} = 0.618$ if the second link in the middle of P_4 is removed.

In case (A) the change in eigenvalue is given by $\Delta\lambda_{\mathrm{PF}} = \sqrt{2} - \frac{1}{2}(1 + \sqrt{5})$. If $c' = \alpha c \geq \sqrt{2} - \frac{1}{2}(1 + \sqrt{5})$, then $|\Delta\lambda_{\mathrm{PF}}| \leq c' = \alpha c$ and the link is removed. This means that, for $c > 2 - \sqrt{2}$ we must have that $\alpha \geq \frac{|\sqrt{2} - \frac{1}{2}(1+\sqrt{5})|}{2-\sqrt{2}} = 0.348$ which is certainly true since we have assumed that $\alpha \geq \frac{\sqrt{2}-1}{2-\sqrt{2}} = 0.707$. Thus, the link is removed under the above made assumptions on cost and α and we obtain a path of length three,

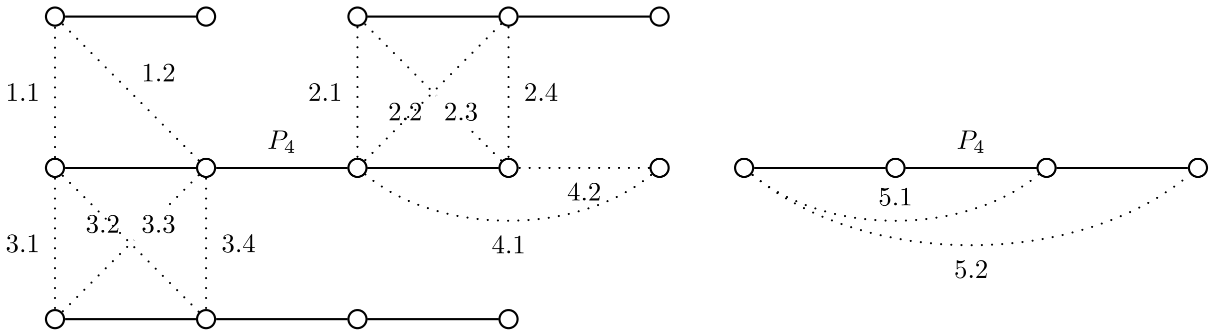

Fig. 20. All possible cases for link creation when at least one of the selected nodes is part of a P_4.

	1.1	1.2
λ_{PF}	1.618	
λ_{PF}'	1.802	1.932
$\Delta\lambda_{\mathrm{PF}}$	0.184	0.314

	2.1	2.2	2.3	2.4
λ_{PF}	1.618			
λ_{PF}'	1.970	1.848	2.053	1.932
$\Delta\lambda_{\mathrm{PF}}$	0.352	0.23	0.435	0.314

	3.1	3.2	3.3	3.4
λ_{PF}	1.618			
λ_{PF}'	1.879	1.989	1.989	2.095
$\Delta\lambda_{\mathrm{PF}}$	0.261	0.371	0.371	0.477

	4.1	4.2
λ_{PF}	1.618	
λ_{PF}'	1.848	1.732
$\Delta\lambda_{\mathrm{PF}}$		

	5.1	5.2
λ_{PF}	1.618	
λ_{PF}'	2.170	2
$\Delta\lambda_{\mathrm{PF}}$	0.552	0.382

Table 4
Change in eigenvalue for link creation when at least one of the selected nodes is part of a P_4.

P_3, which is in the set of graphs $\{\emptyset, P_2, P_3, P_4\}$.

In case (B) we have that $\Delta\lambda_{\mathrm{PF}} = 1 - \frac{1}{2}(1+\sqrt{5})$. This link is removed for $|\Delta\lambda_{\mathrm{PF}}| \leq c' = c\alpha$ implying that $\alpha \geq \frac{|1-\frac{1}{2}(1+\sqrt{5})|}{2-\sqrt{2}} = 1.055$. Since we have assumed that $\alpha \in [0,1]$ this cannot be true. Therefore, the link is not removed.

Notice that in all the cases the graphs created belong to the set S, as we wanted to prove.

(ii) From the preceding analysis we can infer two facts: First, the individual profits of the firms involved in the creation or removal of a link always increase along the closed sequence of graphs $C = (P_2, \{P_2, P_2\}, P_4, P_3.P_2)$, as it is illustrated in Figure (8). Therefore, this is an improving path (Jackson and Watts, 2002) which is cyclical and never reaches an equilibrium. Notice that, the firms responsible for the creation or deletion of the links along the sequence are different and individual profits of a given firm are not increasing at every step. Along the improving path, the individual profits of the firms involved in the link creation or removal increase, while the profits of the others may decrease. This highlights the effects of the externalities inherent in our model on the individual profits of the firms.

Second, since at every step of the sequence there is only one possible network evolution step and since all the non-empty graphs of the set S are also in the

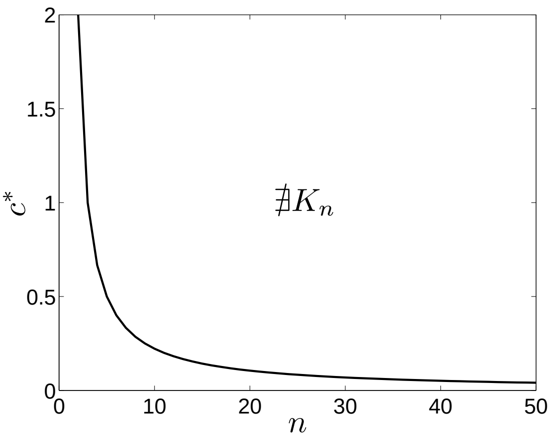

Fig. 21. Maximal value of cost c for which the complete graph K_n can be obtained as an equilibrium network.

cycle C, we can conclude that C is the only improving path in the given range of parameters.

Proof of Proposition (12) The change in the largest real eigenvalue, $\Delta\lambda_{\mathrm{PF}}$ of a graph G with m edges and n nodes, by adding one edge to the graph is bounded by

$$\Delta\lambda_{\mathrm{PF}} \leq \frac{1}{2}(-1 + \sqrt{1 + 8(m+1)}) - \frac{2m}{n}. \tag{98}$$

The above inequality can be obtained as follows. The average degree of the graph is $\bar{d} = \frac{2m}{n}$. A lower bound on the largest real eigenvalue is given by $\lambda_{\mathrm{PF}} \geq \bar{d}$ (Cvetkovic and Rowlinson, 1990). An upper bound on the largest real eigenvalue is given by $\lambda_{\mathrm{PF}} \leq \frac{1}{2}(-1 + \sqrt{1 + 8m})$ (Stanley, 1987). Combining the two bounds yields the inequality in (98).

We apply the bound of Equation (98) on the change in the largest real eigenvalue, $\Delta\lambda_{\mathrm{PF}}$, by adding an edge to the graph G with m edges. Solving the equation $\Delta\lambda_{\mathrm{PF}} = c$ for m yields the maximal number m^* of edges that can be added to a graph of n nodes when the cost is c, $m^*(n,c) = \frac{n}{4}(-1-2c+n+\sqrt{n^2+9-2n(1+2c)})$. Notice that $m^*(n,c)$ decreases with increasing cost c. Imposing now this expression to be equal to one edge less than the number of edges in a complete graph K_n with n nodes, $\binom{n}{2} - 1 = \frac{n(n-1)}{2} - 1$, we get $c^* = \frac{2}{n}$. Thus, if costs exceed this value then the increase in eigenvalue corresponding to the creation of the link that would make the graph complete, is smaller than the cost. Notice that c^* decreases with n and tends to 0 for large n, as plotted in Figure (21), and therefore for any given c there is an n large enough such that the complete graph cannot be reached.

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