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***Abstract*—In this paper we propose the set of target trajectories as a state variable for target tracking. We argue that target tracking is fundamentally about computing the posterior distribution of the target trajectories, which all existing filtering solutions fail to produce. The new state variable enables us to solve the tracking problem in a principled and straightforward manner, without involving non-physical parameters such as an ordering or labels. We develop all the theoretical tools needed to use the set of trajectories as the state variable in a filtering framework; we adapt standard motion and measurement to random finite sets of trajectories and we also discuss general filtering recursions as well as the involved integrals. Another important component is that we present the exact filtering recursions using a conjugate family of distributions, which ensures that all future predicted and updated filtering distributions belong to the same family of distributions. The paper includes a small numerical example where we illustrate the properties of the proposed approach to trajectory estimation.**

## I. Introduction

The goal of tracking is to estimate the parameters, usually the kinematics, of objects over time, thus building a sequence of estimates for each object. Such sequences are called trajectories or tracks. We will use the two terms interchangeably. A focus on the filtering problem, i.e., estimating the parameters at the current time given the available measurements [1]–[3], has tended to somewhat obscure the goal of trajectory formation. The focus on filtering is understandable since smoothing, i.e, estimating the parameters at times preceding the current time, inevitably involves an increase in computational expense. However, as will be discussed below, computing only filtering densities creates difficulties when it is desired to form tracks. This is particularly so when targets move in close proximity and the available measurements are not informative about the identity of the targets under observation. Then, the usual approach is to apply a post-processing step in which estimates obtained at different times are linked over time to form trajectories. Such methods are generally *ad hoc* and provide no guarantee of satisfactory performance under all conditions.

Recently there has been increased interest in principled ways of forming trajectories. By far the most popular approach has been to add labels to the usual state vector [4]–[9]. Labels are unique and static quantities which identify a target over its life time. Tracks can then be formed simply by linking state estimates with the same label. While this seems like a reasonable idea, it becomes apparent on further contemplation that track formation using a sequence of labelled filtering posterior densities poses a number of practical and theoretical issues. The source of these difficulties can be traced to the use of a mathematical model which is not appropriate for the problem under consideration. As argued in [10, p. 405], the mathematical model used for tracking should match the physical reality as closely as possible. Attempting to perform track formation using labelled filtering posterior densities departs from this ideal in two ways. First, using filtering posterior densities effectively treats as the object of interest the individual target states, collected in a vector or a set, at a particular time. This is at odds with the actual aim of estimating sequences of states. Second, in order to compensate for the lack of trajectory information in the filtering densities, labels which have no physical significance are added to the state. The ramifications are explored in more detail in Section II.

We therefore require a different mathematical representation to perform track formation. In particular, if it is indeed desired to estimate target trajectories, we contend that the state variable used for tracking should 1) have a one-to-one correspondence with the physical state and 2) directly provide estimates of target trajectories. We discuss the construction of such a representation satisfying these two requirements below.

In the most common approach to multiple target tracking individual target states are concatenated in a multi-target state vector [1]. When this is done an arbitrary decision is made as to the ordering of the individual target states within the multi-target state vector. This is because there is no one-to-one relationship between the mathematical model and the physical reality: any ordering of target states would reflect the same physical reality. From this point-of-view the random finite set (RFS) approach proposed by Mahler is preferable as it avoids imposing an order on the target states [10]. Aside from being theoretically appealing, the RFS approach also removes uncertainty in the posterior induced by the ordering imposed in a vector-based approach. This has practical benefits, as can be seen by the set JPDAF which exploits the removal of ordering in a set representation to provide better Gaussian approximation of the posterior [11].

Tracking in the RFS framework has been performed only with a state variable which is a set containing individual target states at a given time. This applies to both RFS filters [12], [13] and smoothers [14], [15]. Computing filtering posterior densities in the RFS framework does not directly provide the information required for track formation, even if labels are added. This is because the dependencies between state estimates obtained at different times are not available and so

must be inferred using the target dynamics. However, track formation could be performed with the *joint* posterior density of the labelled target states at each time. This approach, which does not seem to have been pursued, would directly provide the desired target trajectory estimates, but the presence of labels means there is not a unique link between the mathematical representation and the physical state. As such, using the joint posterior of the RFSs of labelled states satisfies our second requirement but not our first. The problem is that, when the states of a particular target at different times belong to different sets, labels are needed to link estimates over time. This need can be avoided by using as a state variable a single set containing sequences of target states at all times. Note that this differs from a sequence of sets containing target states. The latter can include time links only through the addition of labels to the state vector. This is not necessary in the former. An RFS of trajectories therefore satisfies both of our requirements for a state variable suitable for trajectory estimation.

The purpose of this paper is to introduce the notion of tracking using an RFS of trajectories and to develop the necessary mathematical tools for defining and manipulating densities over the space of sets of trajectories. Our state variable is a set in which each element is a vector containing the start and end times of the trajectory and the trajectory of target states at each time. We define a set trajectory integral, analogous to Mahler's set integral, and develop motion and measurement models which extend those of Mahler. These developments are used to construct a conjugate filtering recursion, in the spirit of [8], for computing the posterior of the set of trajectories.

It is not intended to suggest that no techniques currently exist for trajectory estimation. For instance, the multiple hypothesis tracker (MHT) [3], [16] can provide trajectories of state estimates as can dynamic programming approaches [17], [18], although the latter are usually restricted to single targets. However, these do not represent formal approaches to performing inference over trajectories of target states. Invariably, there is no explicit definition of what the state variable is so that trajectory estimation is essentially an implementation step. Further, the notion of a distribution over collections of trajectories does not exist. The accumulated density approach of [19] computes the joint posterior density of the target state at each time, but has been developed only for a single target. In contrast, we provide a representation suitable for an unknown and varying number of targets.

The paper is organised as follows. In Section II we present an analysis of track formation using a sequence of labelled filtering posterior densities. This motivates an approach to tracking based on estimating an RFS of trajectories, described in Section III. Standard motion and measurement models which extend those of Mahler are given in Section IV and then used to establish a filtering recursion in Section V. In the final paper a numerical example will be given in Section VI.

## II. Motivation for sets of trajectories

As mentioned previously, the most popular approach to forming trajectories is based on computing, or approximating, the filtering posterior of a labelled target state [4]–[8]. This idea has been applied in a conventional vector-based framework as well as the RFS framework [8]. As shown in [4], labelled estimation in these frameworks is equivalent. Tracks are formed by linking filtered estimates with the same labels over time. In this section we will discuss the disadvantages of this methodology which lead us to propose a different approach to forming trajectories.

In a Bayesian context, labels are effectively additional parameters which are estimated along with the usual target states. To accommodate these parameters, the tracking model must be extended in such a way that the labels are unique and fixed over time. Aside from the inconvenience of having to specify an expanded tracking model, labelling introduces some undesirable side-effects. The underlying problem seems to be that labels have no direct and unambiguous connection to the physical phenomena under observation. A multitude of labelling schemes can be developed satisfying the basic requirements of fixity and distinctness. Although some such schemes would be obviously poor there is no general way of distinguishing the merits of a given scheme. This contravenes the usual notion of there being "a one-to-one correspondence between physical states and their mathematical representations" [10, p. 405]. We explore some of the ramifications of this via some simple examples.

*Example 1:* The addition of extra parameters to the state vector introduces artificial uncertainty into the estimation problem. Depending on how labels are defined, the amount of additional uncertainty can be significant. To see this consider a simple example in which two static targets located on the real line are observed. The targets appear at locations $1$ and $-1$ at time $k = 5$ and are detected with probability one with zero measurement noise. Thus, from $k = 5$ the posterior has its entire mass located at $[-1, 1]'$. According to the Poisson labelling scheme of [8], the targets would be assigned the labels (1,5) and (2,5) with equal probability. The labelled state posterior would then have half its mass located at $[-1, (1, 5), 1, (2, 5)]'$ and half at $[1, (1, 5), -1, (2, 5)]'$. In other words, labelling has produced a bi-modal posterior from a uni-modal posterior. In this simple example it is not difficult to envisage ways of handling this complication. However, such *ad hoc* operations are inelegant from a theoretical perspective and would conceivably fail in other, more complicated examples.

*Example 2:* Apart from post-processing, the concerns raised in Example 1 could be addressed by assigning labels in a different manner. Even then difficulties occur when labelling uncertainty arises, for example, after targets have moved in proximity. Consider an example in which two targets moving on the real line approach the same point, remain close for eight time steps and then separate. Assume that the targets are unambiguously labelled as 1 and 2 prior to their approach and, for convenience, that the measurement noise is zero. In this case the target identities become completely confused when the targets reach the meeting point and remain so afterwards. This means that, if the targets have position $x_1$ and $x_2$ at a particular time, the filtering posterior will have half its mass at $[[x_1, 1], [x_2, 2]]'$ and half at $[[x_1, 2], [x_2, 1]]'$. Two possible ways in which these filtering posteriors could be used to construct tracks are shown in Fig. 1. Each of these scenarios is *equally likely* given the filtering posterior densities. This situation can only be resolved by recourse to the dynamical equation which would rule out swapping once the targets are sufficiently well-separated. However, the need for a post-processing, using dynamical prior information which has already been used, is problematic and suggests a deficiency in the methodology.

In summary, we argue that the focus on computing filtering posteriors has led to the practice of adding labels to the state

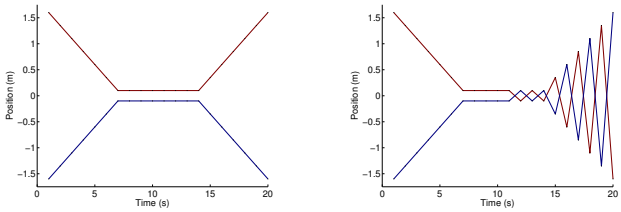


Fig. 1. Possible trajectories obtained from the sequence of filtering posteriors for Example 2.

vector in order to construct tracks. While this approach can yield acceptable results in many situations it is not entirely satisfactory because:

- a labelled state vector does not uniquely represent an underlying physical reality;
- adding labels to the state artificially increases the uncertainty in the tracking problem, significantly so in some cases (see Example 1);
- the sequence of filtering posterior densities does not contain sufficient information to construct tracks, even with labels appended to the state vector.

With these concerns in mind we advocate an approach in which trajectories are formed without the use of labels. We do this by considering the object of interest to be a set of sequences of target states, i.e, a random finite set (RFS) of trajectories. The reasoning behind this approach is as follows. In the single target case, it is clear that a sequence of target states directly gives the required target trajectory. If multiple targets can be present then no preference should be given to any particular ordering of the trajectories. Thus, in general, we should consider RFSs of trajectories. Such a quantity elegantly encapsulates what we would like to obtain from the measurements without the problematic aspects of labels.

The scenario of Example 2 involving two targets approaching on the real line, pausing and then separating illustrates the appeal of using a RFS of trajectories. We add measurement noise with standard deviation $1/100$ and compute the RFS trajectory posterior. This is a mixture with each component corresponding to a different association of measurements to targets. Fig. 2 shows the estimated sets of trajectories for the hypotheses with the four highest posterior probabilities. The first hypothesis contains the correct trajectories. The next three hypotheses have trajectories which include swaps when the targets meet or as they separate. These swaps have non-negligible posterior probability because of process noise in the transition model which is required to accommodate the sudden stopping and starting of the targets. Together, the top four sets of trajectories have a posterior probability of 0.991. Trajectories with multiple swaps more than one time step before or after the meeting point, as shown in Fig. 1(b), have negligible posterior probability. Here, we have obtained this information directly from the filter. This differs from the conventional approach of using a sequence of labelled filtering posterior densities where such trajectories are ruled out only through a post-processing step.

Realisation of our proposed representation requires a slight extension of the mathematical machinery developed for performing inference of RFSs of individual target states. This is the topic of the following sections.

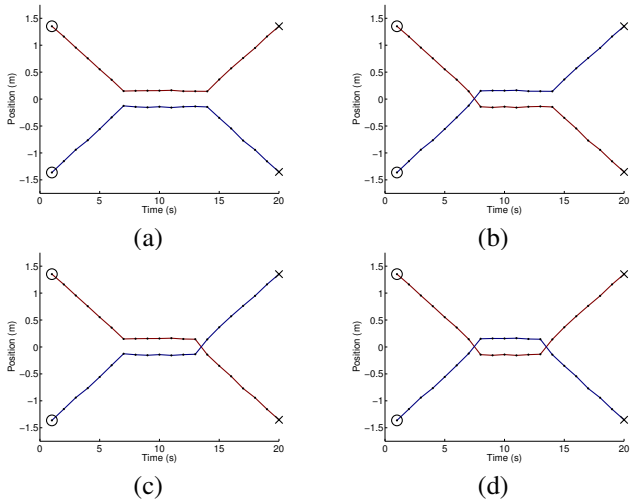


Fig. 2. The sets of estimated trajectories for the hypotheses with the four highest posterior probabilities for the scenario of Example 2. Subfigures (a)-(d) show the estimated trajectories in order of descending posterior probability.

## III. Random finite sets of trajectories

In this section we introduce a representation for the sets of trajectories and present fundamental theoretical tools such as general filtering recursions and define an integral over the space of finite sets of trajectories. The mathematical proofs, including the measure theoretic details that underlie the set trajectory integral, have been omitted for the sake of brevity.

### A. State space representation

We propose the *set of all trajectories* as state variable

$$\mathbf{X}_k = \{X_k^1, \ldots, X_k^{n_k}\}, \tag{1}$$

where $n_k$ is the number of trajectories that have existed until time $k$. The state variable is therefore a random finite set that contains all the variables of interest and we argue that this is the best available representation in order to compute the posterior distribution of the target trajectories, see Section II for a discussion on other alternatives.

In this paper, we represent the individual trajectories as

$$X_k^i = \begin{bmatrix} t_k^i \\ \tau_k^i \\ x_k^i \end{bmatrix}, \tag{2}$$

where $t_k^i$ and $\tau_k^i$ denote the start and minimum end times of trajectory $i$, whereas $x_k^i$ is a vector that contains the target states in the time interval $[t_k^i, \tau_k^i]$. It is assumed that $0 \leq t_k^i \leq \tau_k^i \leq k$ and that the time of birth, $t_k^i$, does not change over time. Both the minimum end time and the state sequence change whenever trajectory $i$ persists from time $k$ to $k+1$, and it then holds that $\tau_k^i = k$, $\tau_{k+1}^i = k+1$ and that $x_{k+1}^i$ contains one more state vector than $x_k^i$. A simple illustration of a set of trajectories is given in Fig. 3, where $n_5 = 2$, $\mathbf{X}_5 = \{X_5^1, X_5^2\}$ and the two trajectories are described by

$$\begin{aligned} X_5^1 &= [1,\, 3,\, [1,\, 1.5,\, 2]]^T \\ X_5^2 &= [2,\, 5,\, [2.4,\, 2.6,\, 2.8,\, 3]]^T, \end{aligned} \tag{3}$$

or vice versa. That is, since the trajectory numbers are arbitrary we could also have $X_5^1 = [2,\, 5,\, [2.4,\, 2.6,\, 2.8,\, 3]]^T$ and $X_5^2 =$

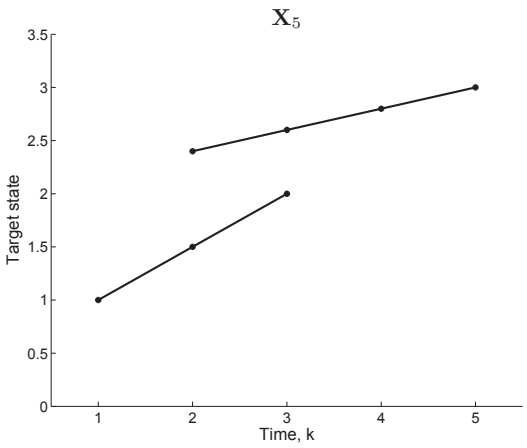


Fig. 3. An illustration of $\mathbf{X}_5$ in an example where $n_5 = 2$ and the single object states are scalar.

$[1,\, 3,\, [1,\, 1.5,\, 2]]^T$. It is clear that one of the trajectories ended at time 3 whereas the other trajectories is still present at time 5; whether or not it will persist until time 6 remains to be seen.

The remainder of this paper contains a description of how the set trajectory representation can be used to perform target tracking, including integration theory, models and a conjugate prior for the filtering recursions of the standard model. As a first step, we will look at the general filtering recursions.

### B. General filtering recursions

The objective in Bayesian filtering is to compute the posterior distribution, $p_{k|k}(\mathbf{X}_k|\mathbf{z}_{1:k}))$, of the state variable, $\mathbf{X}_k$, given all the measurements up to and including time $k$, $\mathbf{z}_{1:k} \triangleq \{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k\}$. We wish to perform filtering using the prediction equation

$$p_{k|k-1}(\mathbf{X}_k|\mathbf{z}_{1:k-1}) = \int f_{k|k-1}(\mathbf{X}_k|\mathbf{X}) p_{k-1|k-1}(\mathbf{X}|\mathbf{z}_{1:k-1})\, \delta\mathbf{X} \quad (4)$$

and the update equation

$$p_{k|k}(\mathbf{X}_k|\mathbf{z}_{1:k}) = \frac{g_k(\mathbf{z}_k|\mathbf{X}_k) p_{k|k-1}(\mathbf{X}_k|\mathbf{z}_{1:k-1})}{\int g_k(\mathbf{z}_k|\mathbf{X}) p_{k|k-1}(\mathbf{X}|\mathbf{z}_{1:k-1})\, \delta\mathbf{X}}, \quad (5)$$

which are analogous to the conventional filtering equations [10], [20] often known as the Chapman-Kolmogorov equation and Bayes' rule, respectively.

We assume that the posterior distribution is (a slightly generalized version of) a multiobject density in the FISST sense, and it is therefore equivalent to a probability density [10], [13]. To perform the prediction and update steps, we need a motion model $f_{k|k-1}(\mathbf{X}_k|\mathbf{X}_{k-1})$, a measurement model $g_k(\mathbf{z}_k|\mathbf{X}_k)$ and a suitable definition of the set integral $\int f(\mathbf{X})\, \delta\mathbf{X}$, when the elements of $\mathbf{X}$ are trajectories. In the remainder of this section we introduce the set integral for RFSs of trajectories and discuss the link between multi-object and ordered densities.

### C. Integrals

The most commonly used RFSs, $\mathbf{x}$, contain elements that are vectors in $\mathbb{R}^r$ and the *set integral* for such RFSs is given by

$$\int f(\mathbf{x})\delta\mathbf{x} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\substack{x^i \in \mathbb{R}^d \\ i=1,\ldots,n}} f(\{x^1, \ldots, x^n\}) \prod_{i=1}^{n} dx^i, \quad (6)$$

where $f(\mathbf{x})$ is a real-valued function of a finite-set variable $\mathbf{x}$, i.e., $f : \mathcal{F}(\mathbb{R}^r) \to \mathbb{R}$, where $\mathcal{F}(\mathbb{R}^r)$ denotes the collection of finite subsets of $\mathbb{R}^r$. The theory of FISST also describes set integrals for sets of hybrid vectors that contain both discrete and continuous elements [21]. What differentiates the RFSs considered here from those in previous work is that the dimensionality of the elements in the set is not fixed, but depends on the length of the trajectory. The two trajectories in Fig. 3 illustrate this since $X_5^1$ and $X_5^2$ have different lengths.

We assume that the single target states belong to $D = \mathbb{R}^r$, such that the RFS $\mathbf{X}$ contain elements (trajectories) $X$ in

$$E = \left\{ \begin{bmatrix} t \\ \tau \\ x \end{bmatrix} : \begin{bmatrix} t \\ \tau \\ x \end{bmatrix} \in \mathbb{N}_0^2 \times D^{\tau - t + 1}, 0 \leq \tau - t < \infty \right\} \quad (7)$$

i.e., the first two elements are natural numbers and the third element is a vector whose dimensionality matches the values of the two discrete elements. For technical reasons, we further assume that $\tau - t$ is finite; more specifically to ensure that $E$ is locally compact. Using this notation, the elements of $\mathbf{x}$ belong to $D$ whereas the elements of $\mathbf{X}$ belong to $E$.

The set integral in (6) can be extended to RFSs of trajectories in a natural fashion:

$$\int f(\mathbf{X})\delta\mathbf{X} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{\tau^i \geq t^i \geq 0 \\ i=1,\ldots,n}} \int_{\substack{x^i \in D^{\tau^i - t^i + 1} \\ i=1,\ldots,n}} f\left( \left\{ \begin{bmatrix} t^1 \\ \tau^1 \\ x^1 \end{bmatrix}, \ldots, \begin{bmatrix} t^n \\ \tau^n \\ x^n \end{bmatrix} \right\} \right) \prod_{i=1}^{n} dx^i, \quad (8)$$

where $f(\mathbf{X})$ now denotes a real-valued function of $\mathbf{X}$, i.e., $f : \mathcal{F}(E) \to \mathbb{R}$, where $\mathcal{F}(E)$ is the collection of finite subsets of $E$. For each cardinality $n$, the original set integral in (6) integrates over $D^n = \mathbb{R}^{rn}$ and in an analogous fashion, the integral in (8) computes the corresponding combined summation and integral over $E^n$. In Section V we use (4), (5) and (8) to derive exact filter recursions for a conjugate prior and the models described in Section IV.

### D. Multi-object densities vs ordered densities

The posterior densities of the set of trajectories are multi-object densities, but in order to express these using the conjugate family in Section V we make use of ordered densities. A multi-object density is a mapping $p : \mathcal{F}(E) \to \mathbb{R}$, where $\mathcal{F}(E)$ is the collection of finite subsets of $E$, whereas an ordered density is a mapping $p^o : \cup_{n=0}^{\infty} E^n \to \mathbb{R}$. That is, $p$ describes the distribution of a set of trajectories, whereas $p^o$ describes the distribution of an ordered list of trajectories. One can compute $p$ from $p^o$ using

$$p(\{X^1, \ldots, X^n\}) = \sum_{\sigma} p^o(X^{\sigma(1)}, \ldots, X^{\sigma(n)}), \quad (9)$$

where we sum over all permutations $\sigma$ of the numbers $\{1, 2, \ldots, n\}$.

The ordered list of trajectories, $(X^1, \ldots, X^n)$, is in fact another possible state representation which would be sufficiently informative in order for us to solve the complete tracking problem. However, similar to the labeled RFSs it has a non-physical part, namely the ordering, which is not generally given even if we know all the trajectories. Considering that we would like to have a one-to-one correspondence between physical states and their mathematical representations, the ordered list of trajectories is not entirely suitable.

A related aspect, which can be observed from (9), is that there are many ordered densities that represent the same multi-object density. In principle, any ordered density that represents the correct multi-object density is equally valid and we could therefore select the ordered density that best fits our purposes; such a strategy was explored in [11] to perform accurate merging, in a setting without birth and death events. Though this may be a useful property, we do not explore it in this paper.

## IV. Standard motion and measurement models

In this section we develop standard motion and measurement models for the case when the state variable is an RFS of trajectories. The intention is not to propose new model assumptions, but to express the standard model [10] in terms of sets of trajectories instead of sets of targets.

### A. The motion model

To perform the prediction step in (4), we express the standard motion model in target tracking in terms of the set trajectory density, $f_{k|k-1}(\mathbf{X}_+|\mathbf{X})$, where $\mathbf{X}_+ = \{X_+^1, \ldots, X_+^{n_+}\}$ and $\mathbf{X} = \{X^1, \ldots, X^n\}$ are sets of trajectories. The usual motion model in target tracking assumes that targets move independently, that a target with state $\tilde{x}$ survives until the next time instant with probability $p_s$ (or more generally $p_s(\tilde{x})$) and then moves to a new state $\tilde{x}_+ \sim \mathcal{N}(\phi(\tilde{x}), Q)$. Appearing targets are commonly modelled as a Poisson process with expected value $\mu_0$, and with a physical distribution $b(x)$.

By construction, trajectories are never removed from the set and it therefore holds that $f_{k|k-1}(\mathbf{X}_+ \big| \mathbf{X}) = 0$ when $n_+ < n$. For $n_+ \geq n$ we can express this model as

$$f_{k|k-1}(\mathbf{X}_+ \Big| \mathbf{X}) = \sum_{\sigma_+} \prod_{i=1}^{n} f^s_{k|k-1}(X_+^{\sigma_+(i)}|X^i) \frac{f_k^B\left(\left\{X_+^{\sigma_+(n+1:n_+)}\right\}\right)}{(n_+ - n)!} \quad (10)$$

where $f^s_{k|k-1}(X_+|X)$ denotes the single trajectory motion model and we sum over all permutations, $\sigma_+$, of the numbers $\{1, 2, \ldots, n_+\}$. The birth process is described by the Poisson model

$$f_k^B(\mathbf{B}) = e^{-\mu_0} \prod_{[t,\tau,x]^T \in \mathbf{B}} \mu_0 b(x) \delta[t-k]\delta[\tau-k], \quad (11)$$

under the additional assumption that new targets have the time indexes $t = \tau = k$. For brevity, we have introduced the notation $\{X_+^{\sigma(n+1:n_+)}\} \triangleq \{X_+^{\sigma(n+1)}, \ldots, X_+^{\sigma(n_+)}\}$. The factor $1/(n_+ - n)!$ appears because there are $(n_+ - n)!$ terms with the same set of new targets $\{X_+^{\sigma(n+1:n_+)}\}$.

Let us now work out the details for the single trajectory motion model, $f^s_{k|k-1}(X_+|X)$. The probability that the target that corresponds to a trajectory $X$ survives until time $k$ is

$$P_{s,k}(X) = p_s \delta[\tau - k + 1], \quad (12)$$

i.e., it can only survive if it was still present at time $k-1$. For notational convenience, we use $x_|$ to denote the last subvector and $x_{\backslash|}$ to denote all but the last subvector of a vector $x$, such that

$$x = \begin{bmatrix} x_{\backslash|} \\ x_| \end{bmatrix}.$$

Under the assumption that a trajectory $X_+$ survives, it must satisfy $t_+ = t$, $\tau_+ = \tau + 1 = k$, $x_{+,\backslash|} = x$ and $x_{+,|} \sim \mathcal{N}(\phi(x_|), Q)$. That is, the trajectory still starts at the same time, $t$, and the state sequence until time $k$, denoted $x$, has not changed, but the trajectory now contains one more element, $x_{+,|}$, at time $\tau_+ = k$. The transition density for trajectory $X$ therefore has two parts, related to the events that it has survived or died before time $k$:

$$f^s_{k|k-1}(X_+|X) = (1 - P_{s,k}(X))\, \delta[t_+ - t]\delta[\tau_+ - \tau]\delta(x_+ - x) + P_{s,k}(X)\delta[t_+ - t]\delta[\tau_+ - k]\delta(x_{+,\backslash|} - x)\mathcal{N}(x_{+,|}; \phi(x_|), Q). \quad (13)$$

The above equations jointly define the motion model.

*Remark 1:* One can view the summation in (10) as a version of the convolution formula in FISST.

*Remark 2:* The above transition density cannot be derived using the usual set derivatives of belief mass functions, since it contains Dirac delta functions. One can instead verify the expression using the closely related Choquet theorem [22, p. 30], [23, p. 10], [10, p. 713], but it is beyond the scope of this paper to present the details of that derivation.

### B. The measurement model

In target tracking, the most commonly used measurement model assumes that we observe a set of measurements

$$\mathbf{z}_k = \psi_k(\mathbf{X}_k) \cup \mathbf{c}_k, \quad (14)$$

where $\psi_k(\mathbf{X}_k)$ is the set of target detections and $\mathbf{c}_k$ is the set of false (clutter) detections, all at time $k$. The false detections are modeled by a Poisson RFS [10], with mean value $\lambda$ and spatial distribution $c(z)$, such that the multi-target density of $\mathbf{c}_k$ is

$$f_C(\mathbf{c}_k) = e^{-\lambda} \prod_{z \in \mathbf{c}_k} \lambda\, c(z). \quad (15)$$

It is important to note that $\mathbf{z}_k$, $\psi_k(\mathbf{X}_k)$ and $\mathbf{c}_k$ are all conventional RFSs, in the sense that the elements in these sets all belong to $\mathbb{R}^{r_z}$.

The set of target detections, $\psi_k(\mathbf{X}_k)$, is an RFS that only depends on the current set of targets, $\mathbf{x}_k = \{x_k^1, \ldots, x_k^{n'}\}$, since only the current targets can be detected. Every target $x \in \mathbf{x}_k$ is either detected with probability $p_D(x)$, and then generates a measurement $z \sim f(z|x)$, or missed with probability $1 - p_D(x)$; in this paper we consider the case when the probability of detection is constant, $p_D(x) = p_D$. It is further assumed that target detections from different targets make up RFSs which are independent of each other as well as of $\mathbf{c}_k$.

In terms of trajectories, this means that $X \in \mathbf{X}_k$ is detected with probability

$$P_{D,k}(X) = p_D\, \delta[\tau - k], \quad (16)$$

and then generates a measurement $z \sim f(z|x_|)$, or missed with probability $1 - P_{D,k}(X)$. Note that $x_|$ is the current single target state when $\tau = k$, and that $P_{D,k}(X) = 0$ when $\tau < k$ since an ended trajectory can no longer be detected. We can now decompose the target detections into independent RFSs

$$\psi_k(\{X_k^1, \ldots, X_k^n\}) = \psi_k(X_k^1) \cup \cdots \cup \psi_k(X_k^n), \quad (17)$$

where $\psi(X_k^i)$ is the detection set for trajectory $X_k^i$, and use this to derive the multitarget likelihood function by means of standard FISST techniques [10]. The resulting multi-target density of $\mathbf{z}_k$ is:

$$g_k(\mathbf{z}_k|\mathbf{X}_k) = f_C(\mathbf{z}_k) \prod_{X \in \mathbf{X}_k} (1 - P_{D,k}(X)) \sum_{\theta} \prod_{i:\theta(i)>0} \frac{p_D\, \delta[\tau^i - k] f(z_k^{\theta(i)}|x_{k,|}^i)}{(1 - p_D\, \delta[\tau^i - k])\, \lambda\, c(z_k^{\theta(i)})}, \quad (18)$$

where $\theta : \{1, 2, \ldots, n\} \to \{0, 1, \ldots, m_k\}$, defines the association between the $n$ trajectories and the $m_k$ detections[1]. The interpretation of $\theta(i) = j > 0$ is that $x_k^i$ generated $z_k^j$ and since a detection can not be generated by more than one target it must hold that $\theta(i) = \theta(i') > 0$ implies that $i = i'$.

## V. Exact filtering recursions using a conjugate prior

In order to perform filtering (tracking) using the state variable $\mathbf{X}_k$ and the models in Section IV, it is essential to have a representation of its distribution. In this section we present a family of conjugate distributions for which the prediction and update steps are easy to describe. The fact that it is conjugate means that as long as the prior distribution belongs to the conjugate family the predicted and updated distributions also belong to the same family.

Conjugacy is an important property and the literature contains examples like the Gaussian distribution in the linear-Gaussian filtering problem (computed using the Kalman filter), Gaussian mixture distributions in the single target MHT setting and the generalized labeled multi-Bernoulli [8] for general labeled target tracking. Note that we use the term conjugate in same relaxed sense as in [8]. It does not imply that a fixed and finite number of parameters can be used to describe the posterior at all times. Instead the number of hypotheses grows exponentially, a property inherent in problems with data association uncertainties, and approximations involving pruning and merging are required in practice.

### A. A conjugate family of distributions

We represent our posterior distribution at time $k$, given data up to time $l$, on the form

$$p_{k|l}(\{X^1, \ldots, X^n\}|\mathbf{z}_{1:l}) = \sum_{\sigma} p^o_{k|l}(X^{\sigma(1)}, \ldots, X^{\sigma(n)}|\mathbf{z}_{1:l}) \quad (19)$$

where

$$p^o_{k|l}(X^1, \ldots, X^n \Big| \mathbf{z}_{1:l}) = \sum_{\mathbf{t} \in \mathcal{T}_{n,k}} \sum_{\boldsymbol{\xi} \in \Xi_l(\mathbf{t})} w_{k|l}^{\mathbf{t},\boldsymbol{\xi}} \prod_{i=1}^{n} \tilde{p}(X^i \Big| a^i, b^i, \xi^i, \mathbf{z}_{1:l}), \quad (20)$$

which is a conjugate family for our filtering problem. Let us now introduce the components in (20). The variables $\mathbf{t}$ and $\boldsymbol{\xi}$ represent existence and data association hypotheses, respectively, and the weight $w_{k|l}^{\mathbf{t},\boldsymbol{\xi}}$ denotes the probability of hypothesis $(\mathbf{t}, \boldsymbol{\xi})$ at time $k$, conditioned on $\mathbf{z}_{1:l}$. The set $\mathcal{T}_{n,k}$ contains all matrices

$$\mathbf{t} = \begin{bmatrix} a^1 & a^2 & \ldots & a^n \\ b^1 & b^2 & \ldots & b^n \end{bmatrix}, \quad (21)$$

that satisfy $0 \leq a^i \leq b^i \leq k$ for $i = 1, \ldots, n$, and the variables $a^i$ and $b^i$ define the values of $t^i$ and $\tau^i$ according to hypothesis $(\mathbf{t}, \xi)$, see (22).

The variable $\boldsymbol{\xi} = (\xi^1, \ldots, \xi^n)$ contains discrete valued vectors $\xi^i$ that define a time sequence of associations between trajectory $i$ and detections at different times. Clearly, a target cannot be detected when it does not exist and we do not wish to define data associations at times when we have not observed any measurements, i.e., after time $l$. The vector $\xi^i$ is therefore empty if $l < a^i$ and otherwise it has the length $\min(b^i, l) - a^i + 1$ and defines the associations at times $a^i, a^i+1, \ldots, \min(b^i, l)$. The interpretation of $\xi^i(j) = s > 0$ is that trajectory $i$ generated detection $s$ at time $a^i + j - 1$. The set $\Xi_l(\mathbf{t})$ contains all variables $\boldsymbol{\xi}$ that match $l$ and $\mathbf{t}$ and that take possible values; $\xi^i(j)$ must take values in $\{0, 1, \ldots, m_{a^i+j-1}\}$ and $\xi^i(j) = \xi^{i'}(j) > 0$ must imply that $i = i'$.

Finally, $\tilde{p}$ is a distribution on the single trajectory space:

$$\tilde{p}(X|a, b, \xi, \mathbf{z}_{1:l}) = \delta[t - a]\delta[\tau - b]\breve{p}(x|a, b, \xi, \mathbf{z}_{1:l}) \quad (22)$$

where $x$ is a vector that contains the target states at times, $a, a+1, \ldots, b$, where it is assumed to exist. The density $\breve{p}(x|a, b, \xi, \mathbf{z}_{1:l})$ is the posterior distribution of $x$ given $\mathbf{z}_{1:l}$ and known data associations, which is straightforward to compute recursively using prediction and update steps, see e.g. [19] for a detailed description of such calculations.

### B. Chapman-Kolmogorov prediction

In the prediction step, we assume that we know $p_{k-1|k-1}(\{X^1, \ldots, X^n\} \Big| \mathbf{z}_{1:k-1})$, i.e., that we are given

$$p^o_{k-1|k-1}(X^1, \ldots, X^n \Big| \mathbf{z}_{1:k-1}) = \sum_{\mathbf{t} \in \mathcal{T}_{n,k-1}} \sum_{\xi \in \Xi_{k-1}(\mathbf{t})} w_{k-1|k-1}^{\mathbf{t},\xi} \prod_{i=1}^{n} \tilde{p}(X^i \Big| a^i, b^i, \xi^i, \mathbf{z}_{1:k-1}). \quad (23)$$

The objective is to find $p_{k|k-1}(\mathbf{X}_k|\mathbf{z}_{1:k-1})$ and express it using our conjugate family, i.e., by means of a density

$$p^o_{k|k-1}(X_k^1, \ldots, X_k^{n_k} \Big| \mathbf{z}_{1:k-1}) = \sum_{\mathbf{t}_+ \in \mathcal{T}_{n_k,k}} \sum_{\boldsymbol{\xi}_+ \in \Xi_{k-1}(\mathbf{t}_+)} w_{k|k-1}^{\mathbf{t}_+,\xi_+} \prod_{i=1}^{n_k} \tilde{p}(X_k^i \Big| a_+^i, b_+^i, \xi_+^i, \mathbf{z}_{1:k-1}). \quad (24)$$

By combining (23), with the Chapman-Kolmogorov equation (4), the definition of the integral (8) and the motion model (10), one can find the predicted density $p_{k|k-1}(\mathbf{X}_k|\mathbf{z}_{1:k-1})$.

One ordered density that represents $p_{k|k-1}(\mathbf{X}_k|\mathbf{z}_{1:k-1})$ exactly corresponds to what one may expect to obtain by performing prediction on the ordered density directly: the distribution of $X_+^i$ is obtained from the distribution of $X^i$, for $i = 1, \ldots, n$, and the new trajectories are given the indices

[1] We implicitly assume a certain ordering of the elements of $\mathbf{z}_k$, informally denoted $\mathbf{z}_k = \{z_k^1, \ldots, z_k^{m_k}\}$, such that $z_k^1, \ldots, z_k^{m_k}$ are uniquely defined by $\mathbf{z}_k$.

$n+1,\ldots,n_+$. To describe this ordered density in detail, we relate every hypothesis $\mathbf{t}_+ \in \mathcal{T}_{n_k,k}$ to $n$ and $\mathbf{t} \in \mathcal{T}_{n,k-1}$ as follows:

$$\begin{aligned} n &= \max_{i:a^i_+<k} i \\ a^i &= a^i_+ \qquad\qquad i=1,2,\ldots,n \\ b^i &= b^i_+ - \delta[b^i_+ - k] \qquad i=1,2,\ldots,n, \end{aligned} \tag{25}$$

which can be understood from the fact that $a^i_+ = k$ for new trajectories and $b^i_+ = k, a^i_+ < k$ for persisting trajectories.

It is possible to show that the weights in (24) can be expressed as

$$\begin{aligned} &w^{\mathbf{t}_+,\boldsymbol{\xi}_+}_{k|k-1} = \\ &w^{\mathbf{t},\boldsymbol{\xi}^{1:n}_+}_{k-1|k-1} \prod_{i=1}^{n}(1-p_s\delta[b^i_+-k+1]+(p_s-1)\delta[b^i_+-k]) \\ &\Delta(\xi^{n+1:n^+}_+)\frac{e^{-\mu_0}\mu_0}{(n_+-n)!}\prod_{i=n+1}^{n_+}\delta[a^i_+-k]\delta[b^i_+-k] \end{aligned} \tag{26}$$

where $n$ and $\mathbf{t}$ can be determined from $\mathbf{t}_+$, and where we have introduced the notation: $\boldsymbol{\xi}^{1:n}_+ \triangleq (\xi^1_+,\ldots,\xi^n_+)$, $\boldsymbol{\xi}^{n+1:n_+}_+ \triangleq (\xi^{n+1}_+,\ldots,\xi^{n_+}_+)$ and

$$\Delta(\xi^{n+1:n_+}_+) \triangleq \begin{cases} 1 & \text{if } \xi^{n+1}_+ = \cdots = \xi^{n_+}_+ = [\,] \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

The second line of (26) describes the probability of the events related to the trajectories that were present at time $k-1$ and the last line of (26) computes the probability of the independent events that trajectory number $n+1, n+2, \ldots, n^+$ are all new trajectories. With these equations we are able to compute the predicted distribution and represent it using the proposed conjugate family.

### C. Measurement update equations

In the update step, we combine the predicted density $p_{k|k-1}(\mathbf{X}\big|\mathbf{z}_{1:k-1})$, described using (24), with the likelihood, $g_k(\mathbf{z}_k|\mathbf{X})$, to obtain the posterior distribution $p_{k|k}(\mathbf{X}|\mathbf{z}_{1:k})$. A useful observation is that $g_k(\mathbf{z}_k|\mathbf{X}) = g_k(\mathbf{z}_k|\{X^{\sigma(1)},\ldots,X^{\sigma(n)}\})$, for all $\sigma$, which implies that an unnormalized version of (5) is

$$\begin{aligned} &p_{k|k}(\{X^1,\ldots,X^n\}|\mathbf{z}_{1:k}) \\ &\propto \sum_\sigma \Big(p^o_{k|k-1}(X^{\sigma(1)},\ldots,X^{\sigma(n)}|\mathbf{z}_{1:k-1}) \\ &\qquad g_k(\mathbf{z}_k|\{X^{\sigma(1)},\ldots,X^{\sigma(n)}\})\Big). \end{aligned} \tag{28}$$

One density $p^o_{k|k}$ that represents the multiobject posterior distribution correctly is hence given by

$$\begin{aligned} &p^o_{k|k}(X^1,\ldots,X^n|\mathbf{z}_{1:k}) \\ &\propto p^o_{k|k-1}(X^1,\ldots,X^n|\mathbf{z}_{1:k-1})g_k(\mathbf{z}_k|\{X^1,\ldots,X^n\}), \end{aligned} \tag{29}$$

and we would now like to express this density on the conjugate prior form

$$\begin{aligned} &p^o_{k|k}(X^1,\ldots,X^n\Big|\mathbf{z}_{1:k}) \\ &= \sum_{\mathbf{t}\in\mathcal{T}_{n,k}}\sum_{\boldsymbol{\xi}\in\Xi_k(\mathbf{t})} w^{\mathbf{t},\boldsymbol{\xi}}_{k|k}\prod_{i=1}^{n}\tilde{p}(X^i\Big|a^i,b^i,\xi^i,\mathbf{z}_{1:k}). \end{aligned} \tag{30}$$

To compute the weights $w^{\mathbf{t},\boldsymbol{\xi}}_{k|k}$ in $p^o_{k|k}$, we need to relate every hypothesis $\boldsymbol{\xi} \in \Xi_k(\mathbf{t})$ to the corresponding hypotheses $\boldsymbol{\xi}_+ \in \Xi_{k-1}(\mathbf{t})$, which appear in $p^o_{k|k-1}$, and $\theta$, which is part of the likelihood $g_k$. We can find $(\boldsymbol{\xi}_+,\theta)$ from $\boldsymbol{\xi}$ by setting

$$\begin{aligned} \xi^i_+ &= \xi^i, \quad \theta(i)=0 \quad \text{if } \tau^i < k \\ \xi^i_+ &= \xi^i_{|}, \quad \theta(i)=\xi^i_{|} \quad \text{if } \tau^i = k \end{aligned} \tag{31}$$

for $i=1,2,\ldots,n$. To express the updated weights, we make use of a function

$$w^{a,b,\xi}_k \triangleq \tag{32}$$

$$\begin{cases} 1 & \text{if } b<k \\ 1-p_D & \text{if } b=k, \xi_{|}=0 \\ \dfrac{p_D}{\lambda\, c\Big(z^{\xi_{|}}_k\Big)}\displaystyle\int f\Big(z^{\xi_{|}}_k|x_{|}\Big)\breve{p}(x\Big|a,b,\xi,\mathbf{z}_{1:k-1})dx & \text{if } b=k, \xi_{|}>0 \end{cases}$$

which is related to the probability that $\xi_{|}$ correctly describes which detection was generated by a trajectory. One can show that the unnormalized weights can be expressed as

$$w^{\mathbf{t},\boldsymbol{\xi}}_{k|k} \propto \tilde{w}^{\mathbf{t},\boldsymbol{\xi}}_{k|k} = w^{\mathbf{t},\boldsymbol{\xi}^+}_{k|k-1}\prod_{i=1}^{n} w^{a^i,b^i,\xi^i}_k, \tag{33}$$

and the normalized weights are thus given by

$$w^{\mathbf{t},\boldsymbol{\xi}}_{k|k} = \frac{\tilde{w}^{\mathbf{t},\boldsymbol{\xi}}_{k|k}}{\sum_{\mathbf{t}'\in\mathcal{T}_{n,k}}\sum_{\boldsymbol{\xi}'\in\Xi_k(\mathbf{t})}\tilde{w}^{\mathbf{t}',\boldsymbol{\xi}'}_{k|k}}. \tag{34}$$

We have now provided a complete description of how one may perform the filtering recursions.

## VI. A NUMERICAL EXAMPLE

We revisit Example 2 of Section II and extend it to allow for birth and death of targets using the model described in Section IV-A. Recall that Example 2 involves two targets approaching on the real line, pausing and then separating. Both targets are detected with probability one and no clutter is present, i.e., $p_D = 1$ and $\lambda = 0$. The target position is observed with measurement noise standard deviation $1/100$. Targets persist with probability $p_s = 0.95$ and the birth target density is $\mu_0 = 0.1$. The target state contains position and velocity and transitions according to $x_+ \sim \mathcal{N}(\phi(x), Q)$ where

$$\phi(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x, \qquad Q = 1/20 \begin{bmatrix} 1/3 & 1/2 \\ 1/2 & 1 \end{bmatrix}. \tag{35}$$

For this model the RFS trajectory posterior, found as described in Section V, is a mixture with each component hypothesising a different association between the measurements and the targets and different start and end times for the existing targets. Note that exactly two targets always exist because of the unity detection probability and zero clutter density.

We consider two scenarios. In the first scenario the targets approach and depart with velocities of $1/5$ and in the second the target velocities are 1. The aim is to see how the RFS trajectory posterior is affected by the target behaviour. A typical measurement realisation in the first scenario produces results like those shown in Fig. 2. Here, unbroken trajectories have the highest posterior probability. Typical results for the second scenario, in which the targets move with velocity 1, are given in Fig. 4. In this figure, circles and crosses indicate the

start and end of trajectories, respectively, and different colours are used for each hypothesised trajectory. Note that the $y$-axis is expanded in Fig. 4 compared to Fig. 2 because of the higher target velocities. The minimum target separation is the same in both scenarios. The faster target motion produces more severe manoeuvres which are not fully accommodated by the process noise. As a result hypotheses in which trajectories are broken at the starting and stopping times have non-negligible posterior probability. The increased speed of the targets also prevents the correct hypothesis having significant posterior probability as the targets tend to "overshoot" the stopping points.

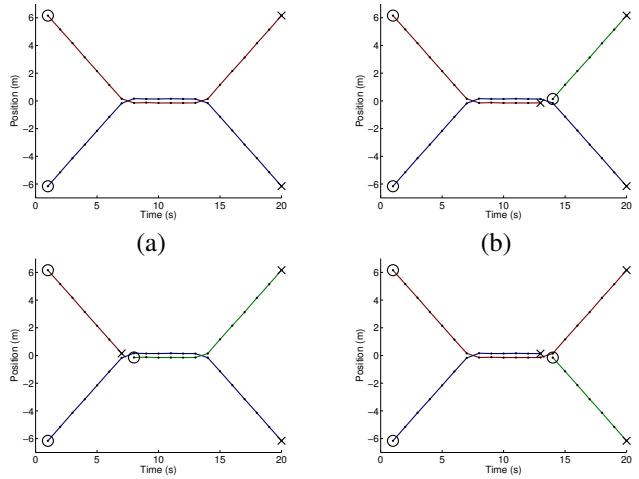


Fig. 4. The sets of estimated trajectories for the hypotheses with the four highest posterior probabilities for the scenario of Example 2 with target velocities of 1. Subfigures (a)-(d) show the estimated trajectories in order of descending posterior probability.

## VII. Conclusion

In this paper we have assumed that, as its name suggests, the aim of target tracking is to estimate the trajectories of target states across time. With this goal in mind we have proposed a new approach to target tracking in which the state variable is a random finite set (RFS) of trajectories. In order to perform inference on this state variable we have developed set trajectory analogues to the tools developed by Mahler. Our approach is general and can be applied in scenarios with an unknown and varying number of targets. It is expected that this novel approach to tracking proposed here will provide a number of avenues for future work. For example, the metrics usually adopted for assessing performance, such as the mean squared error, do not apply to a state variable which is an RFS of trajectories. One possibility is to modify the optimal subpattern assignment (OSPA) metric. Also, the in-principle solution developed here for computing the RFS trajectory posterior is not practically implementable so approximations are required. With slight modifications, we expect it to be possible to develop set trajectory versions of many well established tracking algorithms such as the MHT algorithm and the probability hypothesis density (PHD) filter.

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