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WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR A SPACE-DEPENDENT ANYON BOLTZMANN EQUATION*

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Abstract. A fully nonlinear kinetic Boltzmann equation for anyons is studied in a periodic one-dimensional setting with large initial data. Strong L^1 solutions are obtained for the Cauchy problem. The main results concern global existence, uniqueness, and stability. We use the Bony functional, the two-dimensional velocity frame specific for anyons, and an initial layer analysis that moves the solution away from a critical value.

Key words. anyon, Haldane statistics, low temperature kinetic theory, quantum Boltzmann equation

AMS subject classifications. 82C10, 82C22, 82C40

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1. Anyons and the Boltzmann equation. Let us first recall the definition of an anyon. Consider the wave function $\psi(R, \theta, r, \varphi)$ for two identical particles with center of mass coordinates (R, θ) and relative coordinates (r, φ). Exchanging them, $\varphi \to \varphi + \pi$, gives a phase factor $e^{2\pi i}$ for bosons and $e^{\pi i}$ for fermions. In three or more dimensions those are all possibilities. Leinaas and Myrheim proved in 1977 [10] that in one and two dimensions any phase factor is possible in the particle exchange. This became an important topic after the first experimental confirmations in the early 1980s, and Wilczek [18] in analogy with the terms bos(e)-ons and fermi-ons coined the name any-ons for the new quasi particles with any phase. Anyon quasi particles with, e.g., fractional electric charge have since been observed in various types of experiments.

By moving to a definition in terms of a generalized Pauli exclusion principle, Haldane [9] extended the above concept to a fractional exclusion statistics valid for any dimension and coinciding with the anyon definition in the one- and two-dimensional cases. Haldane statistics has also been realized for neutral fermionic atoms at ultralow temperatures in three dimensions [3]. Wu later derived [19] occupation-number distributions for ideal gases under Haldane statistics by counting states under the new fractional exclusion principle. From the number of quantum states of N identical particles occupying G states being

$$\frac{(G+N-1)!}{N!(G-1)!} \quad \text{and} \quad \frac{G!}{N!(G-N)!}$$

in the boson (resp., fermion) cases, he derived the interpolated number of quantum states for the fractional exclusions to be

$$\frac{(G+(N-1)(1-\alpha))!}{N!(G-\alpha N-(1-\alpha))!}, \qquad 0<\alpha<1. \tag{1.1}$$

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He then obtained for ideal gases the equilibrium statistical distribution

$$\frac{1}{w(e^{(\epsilon-\mu)/T})+\alpha}\,, \tag{1.2}$$

where ϵ denotes particle energy, μ chemical potential, T temperature, and the function $w(\zeta)$ satisfies

$$w(\zeta)^{\alpha}(1+w(\zeta))^{1-\alpha}=\zeta\equiv e^{(\epsilon-\mu)/T}.$$

In particular, $w(\zeta)=\zeta-1$ for $\alpha=0$ (bosons) and $w(\zeta)=\zeta$ for $\alpha=1$ (fermions).

In elastic pair collisions, the velocities (v,v_*) before and (v',v'_*) after a collision are related by

$$v'=v-n[(v-v_*)\cdot n],\quad v'_*=v_*+n[(v-v_*)\cdot n],\quad n\in S^{d-1}.$$

This preserves mass, linear momentum, and energy in Boltzmann-type collision operators. We shall write $f=f(v),\ f_*=f(v_*),\ f'=f(v'),\ f'_*=f(v'_*)$. An important question for gases with fractional exclusion statistics is how to calculate their transport properties, particularly how the Boltzmann equation

$$\partial_t f+v\cdot\bigtriangledown_x f=Q(f)$$

gets modified. An answer was given by Bhaduri, Bhalerao, and Murthy [2] by generalizing to anyons the filling factors $F(f)$ from the fermion and boson cases, $F(f)=(1+\eta f)$, $\eta=\mp1$, and by inductive reasoning obtaining as anyon filling factors

$$F(f)=(1-\alpha f)^{\alpha}(1+(1-\alpha)f)^{1-\alpha},\quad 0<\alpha<1.$$

Namely, with a filling factor $F(f)$ in the collision operator Q, the entropy production term becomes

$$\int Q(f)\log\frac{f}{F(f)}dv,$$

which for equilibrium implies

$$\frac{f'}{F(f')}\frac{f'_*}{F(f'_*)}=\frac{f}{F(f)}\frac{f_*}{F(f_*)}.$$

Using conservation laws and properties of the Cauchy equation, one concludes that in equilibrium $\frac{f}{F(f)}$ is a Maxwellian. Inserting Wu's equilibrium (1.2) for f and taking the quotient Maxwellian as $e^{-(\epsilon-\mu)/T}$ with $\epsilon=|v-v_0|^2$ when the bulk velocity is v_0, this gives

$$f=\frac{1}{w(e^{(\epsilon-\mu)/T})+\alpha}\,,\qquad F(f)=fe^{(\epsilon-\mu)/T}=\frac{e^{(\epsilon-\mu)/T}}{w(e^{(\epsilon-\mu)/T})+\alpha}.$$

In particular in the fermion and boson cases,

$$f=\frac{1}{e^{(\epsilon-\mu)/T}-\eta},\qquad F(f)=\frac{e^{(\epsilon-\mu)/T}}{e^{(\epsilon-\mu)/T}-\eta},\quad \eta=\mp1.$$

This is consistent with taking an interpolation between the fermion and boson factors as general filling factor, $F(f)=(1-\alpha f)^{\alpha}(1+(1-\alpha)f)^{1-\alpha}$, $0<\alpha<1$. It gives the collision operator Q of [2] for Haldane statistics,

$$(1.3)\quad Q(f)(v) = \int_{\mathbb{R}^d \times S^{d-1}} B(|v-v_*|,\omega)\big[f'f'_*F(f)F(f_*) - ff_*F(f')F(f'_*)\big]dv_*d\omega.$$

Here $d\omega$ corresponds to the Lebesgue probability measure on the $(d-1)$-sphere. The collision kernel $B(z,\omega)$ in the variables $(z,\omega) \in \mathbb{R}^d \times \mathbb{S}^{d-1}$ is positive, locally integrable, and depends only on $|z|$ and $|(z,\omega)|$. The collision operator is discussed in [2], but, as common in quantum kinetic theory, without explicit bounds on the kernel. We restrict to a bounded collision kernel truncated for small relative velocities and grazing collisions. The precise assumptions on B are given at the beginning of section 2.

The anyon Boltzmann equation for $0 < \alpha < 1$ retains important properties from the Fermi–Dirac case. In the filling factor $F(f) = (1-\alpha f)^\alpha(1+(1-\alpha)f)^{1-\alpha}$, $0 < \alpha < 1$, the factor $(1-\alpha f)^\alpha$ requires that the value of f not exceed $\frac{1}{\alpha}$. This is formally preserved by the equation itself, since the gain term vanishes for $f = \frac{1}{\alpha}$, making the Q-term (1.3) and the derivative left-hand side of the Boltzmann equation negative there. Positivity is formally preserved, since the derivative equals the positive gain term for $f = 0$, where the loss term vanishes. F is concave with maximum value one at $f = 0$ for $\alpha \geq \frac{1}{2}$, and maximum value $(\frac{1}{\alpha}-1)^{1-2\alpha} > 1$ at $f = \frac{1-2\alpha}{\alpha(1-\alpha)}$ for $\alpha < \frac{1}{2}$. The collision operator vanishes identically for the equilibrium distribution functions obtained by Wu.

The Boltzmann equation for the limiting cases, representing boson statistics ($\alpha = 0$) and fermion statistics ($\alpha = 1$), was introduced by Nordheim [16] in 1928. Here the quartic terms in the collision integral cancel, which is used in the analysis. General existence results for the space-homogeneous isotropic boson large data case were obtained in [13], followed by a number of other papers (e.g., [7], [14], [8], [15]), and for the space-dependent case near equilibrium in [17]. In the space-dependent fermion case general existence results were obtained in [6], [11], [12], and [15].

For $0 < \alpha < 1$ there are no cancellations in the collision term. Moreover, the Lipschitz continuity of the collision term for $\alpha \in \{0,1\}$ is for $0 < \alpha < 1$ replaced by a weaker Hölder continuity near $f = \frac{1}{\alpha}$. The space-homogeneous initial value problem for the Boltzmann equation with Haldane statistics is

$$(1.4)\qquad \frac{df}{dt} = Q(f), \qquad f(0,v) = f_0(v).$$

Because of the filling factor F, the range for the initial value f_0 should belong to $[0, \frac{1}{\alpha}]$, which is also formally preserved by the equation. A good control of $\int f(t,x,v)dv$, which in the space-homogeneous case is given by the mass conservation, can be used to keep f uniformly away from $\frac{1}{\alpha}$, and $F(f)$ Lipschitz continuous. That was a basic observation behind the existence result for the space-homogeneous anyon Boltzmann equation.

PROPOSITION 1.1 (see [1]). *Consider the space-homogeneous equation* (1.4) *with velocities in* $\mathbb{R}^d$, $d \geq 2$, *and for hard potential kernels with*

$$(1.5)\qquad 0 < B(z,\theta) \leq C|z|^\beta|\sin\theta\cos\theta|^{d-1}, \quad (z,\theta) \in \mathbb{R}_+ \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

where $0 < \beta \leq 1$, $d > 2$ *or* $0 < \beta < 1$, $d = 2$. *Let the initial value* f_0 *have finite mass and energy. If* $0 < f_0 \leq \frac{1}{\alpha}$ *and* $\operatorname{ess\,sup}(1+|v|^s)f_0 < \infty$ *for* $s = d-1+\beta$, *then the initial value problem for* (1.4) *has a strong solution in the space of functions continuous from* $t \geq 0$ *into* $L^1 \cap L^\infty$, *which conserves mass and energy, and for* $t_0 > 0$

given has

$$\operatorname{ess}\sup_{v\in\mathbb{R}^d,\, t\leq t_0} |v|^{s'} f(t,v) \quad \textit{bounded, where} \quad s' = \min\left\{s, \frac{2\beta(d+1)+2}{d}\right\}.$$

The proof implies stability; given a sequence of positive initial values $(f_{0n})_{n\in\mathbb{N}}$ with

$$\sup_n \operatorname{ess}\sup\ f_{0n}(v) < \frac{1}{\alpha},$$

and converging in L^1 to f_0, there is a subsequence of the solutions converging in L^1 to a solution with initial value f_0.

2. The main results. The present paper considers the space-dependent anyon Boltzmann equation in a slab. With

$$\cos\theta = n \cdot \frac{v-v_*}{|v-v_*|},$$

the kernel $B(|v-v_*|,\omega)$ will from now on be written $B(|v-v_*|,\theta)$ and be assumed measurable with

$$0 \leq B \leq B_0 \tag{2.1}$$

for some $B_0 > 0$. It is also assumed for some $\gamma, \gamma', c_B > 0$ that

$$B(|v-v_*|,\theta) = 0 \quad \text{for } |\cos\theta| < \gamma', \text{ for } 1-|\cos\theta| < \gamma', \text{ and for } |v-v_*| < \gamma, \tag{2.2}$$

and that

$$\int B(|v-v_*|,\theta)d\theta \geq c_B > 0 \qquad \text{for } |v-v_*| \geq \gamma. \tag{2.3}$$

The initial datum $f_0(x,v)$, periodic in x, is assumed to be a measurable function with values in $]0,\frac{1}{\alpha}]$ and such that

$$(1+|v|^2)f_0(x,v) \in L^1([0,1]\times\mathbb{R}^2), \qquad \int \sup_{x\in[0,1]} f_0(x,v)dv = c_0 < \infty, \tag{2.4}$$
$$\inf_{x\in[0,1]} f_0(x,v) > 0, \qquad \text{a.a. } v \in \mathbb{R}^2.$$

With v_1 denoting the component of v in the x-direction, consider for functions periodic in x the initial value problem

$$\partial_t f(t,x,v) + v_1\partial_x f(t,x,v) = Q(f)(t,x,v), \qquad f(0,x,v) = f_0(x,v), \tag{2.5}$$
$$(t,x,v) \in \mathbb{R}_+ \times [0,1] \times \mathbb{R}^2.$$

The main results of the present paper are given in the following theorem.

THEOREM 2.1. *Assume* (2.1)–(2.3). *There exists a strong solution* $f \in \mathcal{C}([0,\infty[; L^1([0,1]\times\mathbb{R}^2))$ *of* (2.5) *with* $0 < f(t,\cdot,\cdot) < \frac{1}{\alpha}$ *for* $t > 0$. *There is* $t_m > 0$ *such that for any* $T > t_m$ *there is* $\eta_T > 0$ *so that*

$$f(t,\cdot,\cdot) \leq \frac{1}{\alpha} - \eta_T, \quad t \in [t_m, T].$$

The solution is unique and depends continuously in $\mathcal{C}([0,T]; L^1([0,1]\times\mathbb{R}^2))$ *on the initial* L^1*-datum. It conserves the mass, momentum, and energy.*

Remarks. The above results seem to be new also in the fermion case where $\alpha = 1$. Indeed, whereas global existence of weak solutions in the three-dimensional fermionic case is proved in [11], [12], and [15], we here prove the global existence and the uniqueness of strong solutions in the two-dimensional case.

This paper is restricted to the slab case, since the proof below uses an estimate for the Bony functional only valid in one space dimension.

Due to the filling factor $F(f)$, the proof in an essential way depends on the two-dimensional velocity frame, which corresponds to the anyon context. It does not extend to Haldane statistics in three or higher velocity dimensions. The approach in the paper can also be used to obtain regularity results. The control of $\int f(t,x,v)dv$ in the present space-dependent setting is nontrivial.

An entropy for (2.5) is

$$\int \left(f \log f + \left(\frac{1}{\alpha} - f\right) \log(1-\alpha f)^{\alpha} - \left(\frac{1}{1-\alpha} + f\right) \log(1+(1-\alpha)f)^{1-\alpha} \right) dxdv.$$

The asymptotic behavior of the solution when $t \to \infty$ is an interesting still open problem, as is the behavior of (2.5) beyond the anyon frame, i.e., for higher v-dimensions under Haldane statistics. It seems likely that a close to equilibrium approach, as in the classical case, could work with fairly general kernels B for close to equilibrium initial values f_0 with some regularity and strong decay conditions for large velocities. Any progress on the large data case in several space dimensions under Haldane statistics would be quite interesting.

The paper is organized as follows. The lack of Lipschitz continuity of $F(f)$ when f is in a neighborhood of $\frac{1}{\alpha}$ requires some care. Since the gain term vanishes when $f = \frac{1}{\alpha}$ and the derivative becomes negative there, f should start decreasing before reaching this value. The proof that this takes place uniformly over phase-space and approximations is based on a good control of $\int f(t,x,v)dv$ in the integration of the gain and loss parts of Q. That is a main topic in section 3, together with the study of a family of approximating equations with large velocity cut-off. Section 4 starts with an initial value analysis, which shows that $f(t,\cdot,\cdot) < \frac{1}{\alpha} - b_1 t$ for some constant $b_1 > 0$ on an initial layer and that f remains far from $\frac{1}{\alpha}$ afterwards. This is crucial for handling the Hölder continuity of $F(f)$ for values of f close to $\frac{1}{\alpha}$, $F(f)$ being Lipschitz continuous away from $\frac{1}{\alpha}$. Based on this control of the values of f, the well-posedness of the problem and the conservation properties of the solution are proven.

3. Approximations and control of mass density. The conditions (2.1)–(2.3) for the kernel B and (2.4) are assumed throughout this section. For any $j \in \mathbb{N}$ denote by ψ_j the cut-off function with

$$\psi_j(r) = 0 \text{ if } r > j \quad \text{and} \quad \psi_j(r) = 1 \text{ if } r \leq j,$$

and set

$$\chi_j(v, v_*, v', v'_*) = \psi_j(|v|)\psi_j(|v_*|)\psi_j(|v'|)\psi_j(|v'_*|).$$

Let the uniformly bounded function F_j be defined on $[0, \frac{1}{\alpha}]$ by

$$F_j(y) = \frac{1-\alpha y}{(\frac{1}{j} + 1 - \alpha y)^{1-\alpha}} (1+(1-\alpha)y)^{1-\alpha}.$$

Denote by Q_j (resp., Q_j^+, and Q_j^- to be used later) the operator

$$Q_j(f)(v) \\ := \frac{1}{\pi}\int B(|v-v_*|,\theta)\chi_j(v,v_*,v',v'_*)\Big(f'f'_*F_j(f)F_j(f_*) - ff_*F_j(f')F_j(f'_*)\Big)dv_*d\theta$$

(resp., its gain part

$$Q_j^+(f)(v) := \frac{1}{\pi}\int B(|v-v_*|,\theta)\chi_j(v,v_*,v',v'_*)f'f'_*F_j(f)F_j(f_*)dv_*d\theta$$

and its loss part

$$\left. Q_j^-(f)(v) := \frac{1}{\pi}\int B(|v-v_*|,\theta)\chi_j(v,v_*,v',v'_*)ff_*F_j(f')F_j(f'_*)dv_*d\theta \right).$$

For $j \in \mathbb{N}$, let a mollifier φ_j be defined by $\varphi_j(x,v) = j^3\varphi(jx,jv)$, where

$$\varphi \in C_0^\infty(\mathbb{R}^3), \quad supp(\varphi) \subset [0,1]\times\{v\in\mathbb{R}^2; |v|\le 1\},\ \varphi\ge 0, \quad \int\varphi(x,v)dxdv = 1.$$

Let $f_{0,j}$ be the restriction to $[0,1]\times\{v;|v|\le j\}$ of $\big(\min\{f_0, \frac{1}{\alpha}-\frac{1}{j}\}\big)*\varphi_j$. The following lemma concerns a corresponding approximation of (2.5).

LEMMA 3.1. *For $T>0$ there is a unique solution $f_j \in C([0,T]\times[0,1]; L^1(\{v;|v|\le j\}))$ to*

$$\partial_t f_j + v_1\partial_x f_j = Q_j(f_j), \qquad f_j(0,\cdot,\cdot) = f_{0,j}. \tag{3.1}$$

There exists some $\eta_j > 0$ such that f_j takes its values in $]0,\frac{1}{\alpha}-\eta_j]$. The solution conserves mass, first v-moment, and energy.

Proof. Let $T>0$ be given. We shall first prove by contraction that for $T_1>0$ and small enough there is a unique solution

$$f_j \in C\big([0,T_1]\times[0,1]; L^1(\{v;|v|\le j\})\big) \cap \left\{f; f\in\left[0,\frac{1}{\alpha}\right]\right\}$$

to (3.1). Let the map $\mathcal{C}$ be defined on periodic in x functions in

$$C\big([0,T]\times[0,1]; L^1(\{v;|v|\le j\})\big) \cap \left\{f; f\in\left[0,\frac{1}{\alpha}\right]\right\}$$

by $\mathcal{C}(f) = g$, where g is the unique solution of the following linear differential equation:

$$\begin{aligned}\partial_t g + v_1\partial_x g = {}& \frac{1}{\pi}(1-\alpha g)\left(\frac{1+(1-\alpha)f}{\frac{1}{j}+1-\alpha f}\right)^{1-\alpha}\int B\chi_j f'f'_*F_j(f_*)dv_*d\theta \\ & - \frac{g}{\pi}\int B\chi_j f_*F_j(f')F_j(f'_*)dv_*d\theta, \\ g(0,\cdot,\cdot) = {}& f_{0,j}.\end{aligned}$$

It follows from the linearity of the previous partial differential equation that it has a unique periodic solution g in $C([0,T]\times[0,1]; L^1(\{v;|v|\le j\}))$. For f with values in $[0,\frac{1}{\alpha}]$, g takes its values in $]0,\frac{1}{\alpha}]$. Indeed, defining

$$g^\sharp(t,x,v) = g(t,x+tv_1,v),$$

it holds that

$$\begin{aligned} g^\sharp(t,x,v) &= f_{0,j}(x,v)e^{-\int_0^t \bar{\sigma}^\sharp_f(r,x,v)dr} \\ &\quad + \frac{1}{\pi}\int_0^t ds \left((1-\alpha g)\left(\frac{1+(1-\alpha)f}{\frac{1}{j}+1-\alpha f}\right)^{1-\alpha} \right. \\ &\qquad \left. \cdot \int B\chi_j f' f'_* F_j(f_*) dv_* d\theta \right)^\sharp (s,x,v) e^{-\int_s^t \bar{\sigma}^\sharp_f(r,x,v)dr} \\ &\geq f_{0,j}(x,v)e^{-\int_0^t \bar{\sigma}^\sharp_f(r,x,v)dr} > 0 \end{aligned}$$

and

$$\begin{aligned} (1-\alpha g)^\sharp(t,x,v) &= (1-\alpha f_{0,j})(x,v)e^{-\int_0^t \tilde{\sigma}^\sharp_f(r,x,v)dr} \\ &\quad + \frac{\alpha}{\pi}\int_0^t \left(g\int B\chi_j f_* F_j(f') F_j(f'_*) dv_* d\theta \right)^\sharp (s,x,v) e^{-\int_s^t \tilde{\sigma}^\sharp_f(r,x,v)dr} ds \\ &\geq (1-\alpha f_{0,j})(x,v)e^{-\int_0^t \tilde{\sigma}^\sharp_f(r,x,v)dr} > 0. \end{aligned}$$

Here,

$$\begin{aligned} \bar{\sigma}_f &:= \frac{1}{\pi}\int B\chi_j f_* F_j(f') F_j(f'_*) dv_* d\theta, \\ \tilde{\sigma}_f &:= \frac{\alpha}{\pi}\left(\frac{1+(1-\alpha)f)}{\frac{1}{j}+1-\alpha f}\right)^{1-\alpha} \int B\chi_j f' f'_* F_j(f_*) dv_* d\theta. \end{aligned}$$

$\mathcal{C}$ is a contraction on $C([0,T_1]\times[0,1];L^1(\{v;|v|\leq j\}))\cap\{f; f\in[0,\frac{1}{\alpha}]\}$ for $T_1>0$ small enough depending only on j, since the derivative of the map F_j is bounded on $[0,\frac{1}{\alpha}]$. Let f_j be its fixed point, i.e., the solution of (3.1) on $[0,T_1]$. The argument can be repeated and the solution can be continued up to $t=T$. By Duhamel's form for f_j (resp., $1-\alpha f_j$),

$$f_j^\sharp(t,x,v) \geq f_{0,j}(x,v)e^{-\int_0^t \bar{\sigma}^\sharp_{f_j}(r,x,v)dr} > 0, \qquad t\in[0,T],\ x\in[0,1],\ |v|\leq j$$

(resp.,

$$\begin{aligned} (1-\alpha f_j)^\sharp(t,x,v) &\geq (1-\alpha f_{0,j})(x,v)e^{-\int_0^t \tilde{\sigma}^\sharp_{f_j}(r,x,v)dr} \\ &\geq \frac{1}{je^{cj^3T}}, \qquad t\in[0,T],\ x\in[0,1],\ |v|\leq j \Bigg). \end{aligned}$$

Consequently, for some $\eta_j>0$, there is a periodic in x solution $f_j\in C([0,T]\times[0,1];L^1(\{v;|v|\leq j\}))$ to (3.1) with values in $]0,\frac{1}{\alpha}-\eta_j]$. If there were another nonnegative local solution $\tilde{f}_j$ to (3.1), defined on $[0,T']$ for some $T'\in\,]0,T]$, then by the exponential form it would stay below $\frac{1}{\alpha}$. The difference $f_j-\tilde{f}_j$ would for some constant $c_{T'}$ satisfy

$$\begin{aligned} \int |(f_j-\tilde{f}_j)^\sharp(t,x,v)|dxdv &\leq c_{T'}\int_0^t |(f_j-\tilde{f}_j)^\sharp(s,x,v)|dsdxdv, \quad t\in[0,T'], \\ &\qquad (f_j-\tilde{f}_j)^\sharp(0,x,v)=0, \end{aligned}$$

implying that the difference would be identically zero on $[0, T']$. Thus f_j is the unique solution on $[0, T]$ to (3.1) and has its range contained in $]0, \frac{1}{\alpha} - \eta_j]$. ∎

The remaining part of this section is devoted to obtaining a uniform control with respect to $j \in \mathbb{N}$ of

$$\int \sup_{t\in[0,T],\, x\in[0,1]} f_j^\sharp(t, x, v)dv.$$

It relies on the following four lemmas, where the first is an estimate of the Bony functionals:

$$\bar{B}_j(t) := \int_0^1 \int |v - v_*|^2 B\chi_j f_j f_{j*} F_j(f_j') F_j(f_{j*}') dv dv_* d\theta dx, \quad t \geq 0.$$

LEMMA 3.2. *For $T > 0$ it holds that*

$$\int_0^T \bar{B}_j(t)dt \leq c_0'(1 + T), \quad j \in \mathbb{N},$$

with c_0' depending only on $\int f_0(x, v)dxdv$ and $\int |v|^2 f_0(x, v)dxdv$.

Proof. Denote f_j by f for simplicity. The proof is an extension of the classical one (cf. [4], [5]), together with the control of the filling factor when $v \in \mathbb{R}^2$, as follows.

The integral over time of the momentum $\int v_1 f(t, 0, v)dv$ (resp., the momentum flux $\int v_1^2 f(t, 0, v)dv$) is first controlled. Let $\beta \in C^1([0, 1])$ be such that $\beta(0) = -1$ and $\beta(1) = 1$. Multiply (3.1) by $\beta(x)$ (resp., $v_1\beta(x)$), and integrate over $[0, t] \times [0, 1] \times \mathbb{R}^2$. This gives

$$\int_0^t \int v_1 f(\tau, 0, v)dvd\tau = \frac{1}{2}\Big(\int \beta(x)f_0(x, v)dxdv - \int \beta(x)f(t, x, v)dxdv + \int_0^t \int \beta'(x)v_1 f(\tau, x, v)dxdvd\tau\Big)$$

(resp.,

$$\int_0^t \int v_1^2 f(\tau, 0, v)dvd\tau = \frac{1}{2}\Big(\int \beta(x)v_1 f_0(x, v)dxdv - \int \beta(x)v_1 f(t, x, v)dxdv + \int_0^t \int \beta'(x)v_1^2 f(\tau, x, v)dxdvd\tau\Big)\Big).$$

Consequently, using the conservation of mass and energy of f,

$$\left|\int_0^t \int v_1 f(\tau, 0, v)dvd\tau\right| + \int_0^t \int v_1^2 f(\tau, 0, v)dvd\tau \leq c(1 + t). \tag{3.2}$$

Let

$$\mathcal{I}(t) = \int_{x<y} (v_1 - v_{*1})f(t, x, v)f(t, y, v_*)dxdydvdv_*.$$

We find from

$$\mathcal{I}'(t) = -\int (v_1 - v_{*1})^2 f(t, x, v)f(t, x, v_*)dxdvdv_* + 2\int v_{*1}(v_{*1} - v_1)f(t, 0, v_*)f(t, x, v)dxdvdv_*$$

and the conservations of the mass, momentum, and energy of f that

$$\begin{aligned}
&\int_0^t \int_0^1 \int (v_1 - v_{*1})^2 f(s,x,v) f(s,x,v_*) dv dv_* dx ds \\
&\leq 2 \int f_0(x,v) dx dv \int |v_1| f_0(x,v) dv + 2 \int f(t,x,v) dx dv \int |v_1| f(t,x,v) dx dv \\
&\quad + 2 \int_0^t \int v_{*1}(v_{*1} - v_1) f(\tau, 0, v_*) f(\tau, x, v) dx dv dv_* d\tau \\
&\leq 2 \int f_0(x,v) dx dv \int (1+|v|^2) f_0(x,v) dv + 2 \int f(t,x,v) dx dv \int (1+|v|^2) f(t,x,v) dx dv \\
&\quad + 2 \int_0^t \left(\int v_{*1}^2 f(\tau, 0, v_*) dv_* \right) d\tau \int f_0(x,v) dx dv \\
&\quad - 2 \int_0^t \left(\int v_{*1} f(\tau, 0, v_*) dv_* \right) d\tau \int v_1 f_0(x,v) dx dv \\
&\leq c \left(1 + \int_0^t \int v_1^2 f(\tau, 0, v) dv d\tau + \left| \int_0^t \int v_1 f(\tau, 0, v) dv d\tau \right| \right).
\end{aligned}$$

And so, by (3.2),

$$\int_0^t \int_0^1 \int (v_1 - v_{*1})^2 f(\tau,x,v) f(\tau,x,v_*) dx dv dv_* d\tau \leq c(1+t). \tag{3.3}$$

Here, c is a constant depending only on $\int f_0(x,v)dxdv$ and $\int |v|^2 f_0(x,v)dxdv$. Define $u_1 = \frac{\int v_1 f dv}{\int f dv}$. Recalling (2.1), we have

$$\begin{aligned}
&\int_0^t \int_0^1 \int (v_1 - u_1)^2 B \chi_j f f_* F_j(f') F_j(f'_*)(s,x,v,v_*,\theta) dv dv_* d\theta dx ds \\
&\qquad \leq c \int_0^t \int_0^1 \int (v_1 - u_1)^2 f f_*(s,x,v,v_*) dv dv_* dx ds \\
&\qquad = \frac{c}{2} \int_0^t \int_0^1 \int (v_1 - v_{*1})^2 f f_*(s,x,v,v_*) dv dv_* dx ds \\
&\qquad \leq c(1+t). \qquad (3.4)
\end{aligned}$$

Multiply (3.1) for f by v_1^2, integrate, and use that $\int v_1^2 Q_j(f) dv = \int (v_1 - u_1)^2 Q_j(f) dv$ and (3.4). Then we have

$$\begin{aligned}
&\frac{1}{\pi} \int_0^t \int (v_1 - u_1)^2 B \chi_j f' f'_* F_j(f) F_j(f_*) dv dv_* d\theta dx ds \\
&= \int v_1^2 f(t,x,v) dx dv - \int v_1^2 f_0(x,v) dx dv \\
&\quad + \frac{1}{\pi} \int_0^t \int (v_1 - u_1)^2 B \chi_j f f_* F_j(f') F_j(f'_*) dx dv dv_* d\theta ds \\
&< c_0(1+t),
\end{aligned}$$

where c_0 is a constant depending only on $\int f_0(x,v)dxdv$ and $\int |v|^2 f_0(x,v)dxdv$.

After a change of variables, the left-hand side can be written as

$$\begin{aligned}&\frac{1}{\pi}\int_0^t\int(v_1'-u_1)^2B\chi_j ff_*F_j(f')F_j(f_*')dvdv_*d\theta dxds\\&=\frac{1}{\pi}\int_0^t\int(c_1-n_1[(v-v_*)\cdot n])^2B\chi_j ff_*F_j(f')F_j(f_*')dvdv_*d\theta dxds,\end{aligned}$$

where $c_1=v_1-u_1$. And so,

$$\begin{aligned}&\int_0^t\int n_1^2[(v-v_*)\cdot n]^2B\chi_j ff_*F_j(f')F_j(f_*')dvdv_*d\theta dxds\\&\leq\pi c_0(1+t)+2\int_0^t\int c_1n_1[(v-v_*)\cdot n]B\chi_j ff_*F_j(f')F_j(f_*')dvdv_*d\theta dxds.\end{aligned}$$

The term containing $n_1^2[(v-v_*)\cdot n]^2$ is estimated from below. When n is replaced by an orthogonal (direct) unit vector $n_\perp$, v' and v_*' are shifted, and the product $ff_*F_j(f')F_j(f_*')$ is unchanged. In $\mathbb{R}^2$ the ratio between the sum of the integrand factors $n_1^2[(v-v_*)\cdot n]^2+n_{\perp 1}^2[(v-v_*)\cdot n_\perp]^2$ and $|v-v_*|^2$ is, outside of the angular cut-off (2.2), uniformly bounded from below by γ'^2. Indeed, if θ_1 denotes the angle between $\frac{v-v_*}{|v-v_*|}$ and n,

$$\begin{aligned}n_1^2\left[\frac{v-v_*}{|v-v_*|}\cdot n\right]^2+n_{\perp 1}^2\left[\frac{v-v_*}{|v-v_*|}\cdot n_\perp\right]^2&=\cos^2\theta\cos^2\theta_1+\sin^2\theta\sin^2\theta_1\\&\geq\gamma'^2\cos^2\theta_1+\gamma'(2-\gamma')\sin^2\theta_1\\&\geq\gamma'^2,\quad\gamma'<|\cos\theta|<1-\gamma',\ \theta_1\in[0,2\pi].\end{aligned}$$

This is where the condition $v\in\mathbb{R}^2$ is used.

That leads to the lower bound

$$\begin{aligned}&\int_0^t\int n_1^2[(v-v_*)\cdot n]^2B\chi_j ff_*F_j(f')F_j(f_*')dvdv_*d\theta dxds\\&\geq\frac{\gamma'^2}{2}\int_0^t\int|v-v_*|^2B\chi_j ff_*F_j(f')F_j(f_*')dvdv_*d\theta dxds.\end{aligned}$$

And so,

$$\begin{aligned}&\gamma'^2\int_0^t\int|v-v_*|^2B\chi_j ff_*F_j(f')F_j(f_*')dvdv_*d\theta dxds\\&\leq2\pi c_0(1+t)+4\int_0^t\int(v_1-u_1)n_1[(v-v_*)\cdot n]B\chi_j ff_*F_j(f')F_j(f_*')dvdv_*d\theta dxds\\&\leq2\pi c_0(1+t)+4\int_0^t\int\Big(v_1(v_2-v_{*2})n_1n_2\Big)B\chi_j ff_*F_j(f')F_j(f_*')dvdv_*d\theta dxds,\end{aligned}$$

since

$$\begin{aligned}&\int u_1(v_1-v_{*1})n_1^2B\chi_j ff_*F_j(f')F_j(f_*')dvdv_*d\theta dx\\&=\int u_1(v_2-v_{*2})n_1n_2\chi_jBff_*F_j(f')F_j(f_*')dvdv_*d\theta dx=0,\end{aligned}$$

by an exchange of the variables v and v_*. Moreover, exchanging first the variables v and v_*,

$$
\begin{aligned}
&2\int_0^t \int v_1(v_2-v_{*2})n_1n_2B\chi_j ff_*F_j(f')F_j(f'_*)dvdv_*d\theta dxds \\
&\quad=\int_0^t \int (v_1-v_{*1})(v_2-v_{*2})n_1n_2B\chi_j ff_*F_j(f')F_j(f'_*)dvdv_*d\theta dxds \\
&\quad\le \frac{8}{\gamma'^2}\int_0^t \int (v_1-v_{*1})^2n_1^2B\chi_j ff_*F_j(f')F_j(f'_*)dvdv_*d\theta dxds \\
&\qquad+\frac{\gamma'^2}{8}\int_0^t \int (v_2-v_{*2})^2n_2^2B\chi_j ff_*F_j(f')F_j(f'_*)dvdv_*d\theta dxds \\
&\quad\le \frac{8\pi c_0}{\gamma'^2}(1+t)+\frac{\gamma'^2}{8}\int_0^t \int (v_2-v_{*2})^2n_2^2B\chi_j ff_*F_j(f')F_j(f'_*)dvdv_*d\theta dxds.
\end{aligned}
$$

It follows that

$$\int_0^t \int |v-v_*|^2B\chi_j ff_*F_j(f')F_j(f'_*)dvdv_*d\theta dxds \le c'_0(1+t),$$

with c'_0 depending only on $\int f_0(x,v)dxdv$ and $\int |v|^2f_0(x,v)dxdv$. This completes the proof of the lemma. ∎

Lemma 3.3. *Given $T>0$, the solution f_j of* (3.1) *satisfies*

$$\int \sup_{t\in[0,T]} f_j^\sharp(t,x,v)dxdv < c'_1+c'_2T, \quad j\in\mathbb{N},$$

where c'_1 and c'_2 depend only on T, $\int f_0(x,v)dxdv$, and $\int |v|^2f_0(x,v)dxdv$.

Proof. Denote f_j by f for simplicity. Since

$$f^\sharp(t,x,v)=f_0(x,v)+\int_0^t Q_j(f)(s,x+sv_1,v)ds,$$

it holds that

$$\sup_{t\in[0,T]} f^\sharp(t,x,v)\le f_0(x,v)+\int_0^T Q_j^+(f)(t,x+tv_1,v)dt. \tag{3.5}$$

Integrating (3.5) with respect to (x,v) and using Lemma 3.2 gives

$$
\begin{aligned}
&\int \sup_{0\le t\le T} f^\sharp(t,x,v)dxdv \le \int f_0(x,v)dxdv+\frac{1}{\pi}\int_0^T\int B\chi_j, \\
&f(t,x+tv_1,v')f(t,x+tv_1,v'_*)F_j(f)(t,x+tv_1,v)F_j(f)(t,x+tv_1,v_*)dvdv_*d\theta dxdt \\
&\qquad\le \int f_0(x,v)dxdv+\frac{1}{\gamma^2}\int_0^T\int B\chi_j|v-v_*|^2, \\
&f(t,x,v')f(t,x,v'_*)F_j(f)(t,x,v)F_j(f)(t,x,v_*)dvdv_*d\theta dxdt \\
&\qquad\le \int f_0(x,v)dxdv+\frac{C_1+C_2T}{\gamma^2}. \qquad ∎
\end{aligned}
$$

Lemma 3.4. *Given $T>0$ and $\delta_1>0$, there exist $\delta_2>0$ and $t_0>0$, depending only on $\int f_0(x,v)dxdv$ and $\int |v|^2f_0(x,v)dxdv$, such that for $t\le T$*

$$\sup_{x_0\in[0,1]}\int_{|x-x_0|<\delta_2}\sup_{t\le s\le t+t_0} f_j^\sharp(s,x,v)dxdv<\delta_1, \quad j\in\mathbb{N}.$$

Proof. Denote f_j by f for simplicity. For $s \in [t, t+t_0]$ it holds that

$$f^\sharp(s,x,v) = f^\sharp(t+t_0,x,v) - \int_s^{t+t_0} Q_j(f)(\tau, x+\tau v_1, v)d\tau$$
$$\leq f^\sharp(t+t_0,x,v) + \int_s^{t+t_0} Q_j^-(f)(\tau, x+\tau v_1, v)d\tau.$$

Thus

$$\sup_{t\leq s\leq t+t_0} f^\sharp(s,x,v) \leq f^\sharp(t+t_0,x,v) + \int_t^{t+t_0} Q_j^-(f)(s, x+sv_1, v)ds.$$

Integrating with respect to (x,v) and using Lemma 3.2 and the bound $\frac{1}{\alpha}$ of f from above gives

$$\int_{|x-x_0|<\delta_2} \sup_{t\leq s\leq t+t_0} f^\sharp(s,x,v)dxdv$$
$$\leq \int_{|x-x_0|<\delta_2} f^\sharp(t+t_0,x,v)dxdv$$
$$+ \frac{1}{\pi}\int_t^{t+t_0}\int B\chi_j f^\sharp(s,x,v) f(s,x+sv_1,v_*) F_j(f)(s,x+sv_1,v')$$
$$\cdot F_j(f)(s,x+sv_1,v'_*)dvdv_*d\theta dxds$$
$$\leq \int_{|x-x_0|<\delta_2} f^\sharp(t+t_0,x,v)dxdv$$
$$+ \frac{1}{\lambda^2}\int_t^{t+t_0}\int_{|v-v_*|\geq\lambda} B\chi_j |v-v_*|^2 f^\sharp(s,x,v) f(s,x+sv_1,v_*),$$
$$F_j(f)(s,x+sv_1,v')F_j(f)(s,x+sv_1,v'_*)dvdv_*d\theta dxds$$
$$+ c\int_t^{t+t_0}\int_{|v-v_*|<\lambda} B\chi_j f^\sharp(s,x,v) f(s,x+sv_1,v_*)dvdv_*d\theta dxds$$
$$\leq \int_{|x-x_0|<\delta_2} f^\sharp(t+t_0,x,v)dxdv + \frac{C_1+C_2T}{\lambda^2} + ct_0\lambda^2\int f_0(x,v)dxdv$$
$$\leq \frac{1}{\Lambda^2}\int v^2 f_0 dxdv + c\delta_2\Lambda^2 + \frac{C_1+C_2T}{\lambda^2} + ct_0\lambda^2\int f_0(x,v)dxdv.$$

Depending on δ_1, suitably choosing Λ and then δ_2, λ and then t_0, the lemma follows. ∎

The previous lemmas imply a t-dependent bound for the v-integral of $f_j^\#$ depending only on $\int f_0(x,v)dxdv$ and on $\int |v|^2 f_0(x,v)dxdv$, as will now be proved.

Lemma 3.5. *Given $T>0$, the solution f_j of* (3.1) *satisfies*

$$\int \sup_{(t,x)\in[0,T]\times[0,1]} f_j^\sharp(t,x,v)dv \leq c_1(T), \quad j\in\mathbb{N},$$

where $c_1(T)$ depends only on T, $\int f_0(x,v)dxdv$, and $\int |v|^2 f_0(x,v)dxdv$.

Proof. Denote by $E(x)$ the integer part of $x\in\mathbb{R}$, $E(x)\leq x < E(x)+1$. As in the proof of Lemma 3.3,

$$
\begin{aligned}
\sup_{s\le t} f^{\sharp}(s,x,v) &\le f_0(x,v) + \int_0^t Q_j^{+}(f)(s,x+sv_1,v)ds \\
&= f_0(x,v) + \frac{1}{\pi}\int_0^t \int B\chi_j f(s,x+sv_1,v')f(s,x+sv_1,v'_*)F_j(f)(s,x+sv_1,v) \\
&\qquad \cdot F_j(f)(s,x+sv_1,v_*)dv_* d\theta ds \qquad (3.6)\\
&\le f_0(x,v) + cA,
\end{aligned}
$$

where

$$A = \int_0^t \int B\chi_j \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-v'_1), v') \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-{v'}_{*1}), v'_*)dv_* d\theta ds.$$

For θ outside of the angular cut-off (2.2), let n be the unit vector in the direction $v-v'$, and $n_\perp$ the orthogonal unit vector in the direction $v-v'_*$. With e_1 a unit vector in the x-direction,

$$\max(|n\cdot e_1|, |n_\perp \cdot e_1|) \ge \frac{1}{\sqrt{2}}.$$

For $\delta_2 > 0$ that will be fixed later, split A into $A_1 + A_2 + A_3 + A_4$, where

$$
\begin{aligned}
A_1 &= \int_0^t \int_{|n\cdot e_1|\ge \frac{1}{\sqrt{2}},\, t|v_1-v'_1|>\delta_2} B\chi_j \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-v'_1), v') \\
&\qquad \cdot \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-{v'}_{*1}), v'_*)dv_* d\theta ds,\\
A_2 &= \int_0^t \int_{|n\cdot e_1|\ge \frac{1}{\sqrt{2}},\, t|v_1-v'_1|<\delta_2} B\chi_j \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-v'_1), v') \\
&\qquad \cdot \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-{v'}_{*1}), v'_*)dv_* d\theta ds,\\
A_3 &= \int_0^t \int_{|n_\perp\cdot e_1|\ge \frac{1}{\sqrt{2}},\, t|v_1-v'_1|>\delta_2} B\chi_j \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-v'_1), v') \\
&\qquad \cdot \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-{v'}_{*1}), v'_*)dv_* d\theta ds,\\
A_4 &= \int_0^t \int_{|n_\perp\cdot e_1|\ge \frac{1}{\sqrt{2}},\, t|v_1-v'_1|<\delta_2} B\chi_j \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-v'_1), v') \\
&\qquad \cdot \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-{v'}_{*1}), v'_*)dv_* d\theta ds.
\end{aligned}
$$

In A_1 and A_2, bound the factor $\sup_{\tau\in[0,t]} f^{\sharp}(\tau, x+s(v_1-v'_{*1}), v'_*)$ by its supremum over $x\in[0,1]$, and make the change of variables

$$s \to y = x + s(v_1 - v'_1)$$

with Jacobian

$$\frac{Ds}{Dy} = \frac{1}{|v_1-v'_1|} = \frac{1}{|v-v_*|\,\left|\left(n, \frac{v-v_*}{|v-v_*|}\right)\right|\,|n_1|} \le \frac{\sqrt{2}}{\gamma\gamma'}.$$

It holds that

$$A_1 \le \int_{t|v_1-v_1'|>\delta_2} \frac{B\chi_j}{|v_1-v_1'|} \left(\int_{y\in(x,x+t(v_1-v_1'))} \sup_{\tau\in[0,t]} f^\#(\tau,y,v')dy \right) \cdot \sup_{(\tau,X)\in[0,t]\times[0,1]} f^\#(\tau,X,v_*')dv_*d\theta$$

and

$$A_2 \le \frac{\sqrt{2}}{\gamma\gamma'} \int_{|n\cdot e_1|\ge\frac{1}{\sqrt{2}},\, t|v_1-v_1'|<\delta_2} B\chi_j \left(\int_{|y-x|<\delta_2} \sup_{\tau\in[0,t]} f^\#(\tau,y,v')dy \right) \cdot \sup_{(\tau,X)\in[0,t]\times[0,1]} f^\#(\tau,X,v_*')dv_*d\theta.$$

Then, performing the change of variables $(v,v_*,n)\to(v',v_*',-n)$,

$$\int \sup_{x\in[0,1]} A_1dv \le \int_{t|v_1-v_1'|>\delta_2} \frac{B\chi_j}{|v_1-v_1'|} \sup_{x\in[0,1]} \left(\int_{y\in(x,x+t(v_1'-v_1))} \sup_{\tau\in[0,t]} f^\#(\tau,y,v)dy \right) \cdot \sup_{(\tau,X)\in[0,t]\times[0,1]} f^\#(\tau,X,v_*)dvdv_*d\theta,$$

so that

$$\begin{aligned}
&\int \sup_{x\in[0,1]} A_1dv \\
&\le \int_{t|v_1-v_1'|>\delta_2} \frac{B\chi_j}{|v_1-v_1'|} \sup_{x\in[0,1]} \left(\int_{y\in(x,x+E(t(v_1'-v_1)+1))} \sup_{\tau\in[0,t]} f^\#(\tau,y,v)dy \right) \cdot \sup_{(\tau,X)\in[0,t]\times[0,1]} f^\#(\tau,X,v_*)dvdv_*d\theta \\
&= \int_{t|v_1-v_1'|>\delta_2} \frac{B\chi_j}{|v_1-v_1'|} |E(t(v_1'-v_1)+1)| \left(\int_0^1 \sup_{\tau\in[0,t]} f^\#(\tau,y,v)dy \right) \cdot \sup_{(\tau,X)\in[0,t]\times[0,1]} f^\#(\tau,X,v_*)dvdv_*d\theta \\
&\le t\left(1+\frac{1}{\delta_2}\right) \int B\chi_j \left(\int_0^1 \sup_{\tau\in[0,t]} f^\#(\tau,y,v)dy \right) \sup_{(\tau,X)\in[0,t]\times[0,1]} f^\#(\tau,X,v_*)dvdv_*d\theta \\
&\le B_0\pi t\left(1+\frac{1}{\delta_2}\right) \int \sup_{\tau\in[0,t]} f^\#(\tau,y,v)dydv \int \sup_{(\tau,X)\in[0,t]\times[0,1]} f^\#(\tau,X,v_*)dv_*.
\end{aligned}$$

Apply Lemma 3.3, so that

$$\int \sup_{x\in[0,1]} A_1dv \le B_0\pi t\left(1+\frac{1}{\delta_2}\right)(c_1'+c_2'T) \int \sup_{(\tau,X)\in[0,t]\times[0,1]} f^\#(\tau,X,v_*)dv_*. \tag{3.7}$$

Moreover, performing the change of variables $(v,v_*,n)\to(v_*',v',-n)$,

$$\int \sup_{x\in[0,1]} A_2dv \le \frac{B_0\pi\sqrt{2}}{\gamma\gamma'} \sup_{x\in[0,1]} \left(\int_{|y-x|<\delta_2} \sup_{\tau\in[0,t]} f^\#(\tau,y,v_*)dydv_* \right) \cdot \int \sup_{(\tau,X)\in[0,t]\times[0,1]} f^\#(\tau,X,v)dv.$$

Given $\delta_1 = \frac{\gamma\gamma'}{4B_0\pi\sqrt{2}}$, apply Lemma 3.4 with the corresponding δ_2 and t_0, so that for $t \leq t_0$,

$$\int \sup_{x\in[0,1]} A_2 dv \leq \frac{1}{4} \int \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v)dv. \tag{3.8}$$

The terms A_3 and A_4 are treated similarly, with the change of variables $s \to y = x + s(v_1 - v'_{*1})$. Using (3.7)–(3.8) and the corresponding bounds obtained for A_3 and A_4 leads to

$$\int \sup_{(s,x)\in[0,t]\times[0,1]} f^{\#}(s,x,v)dv \leq 2\int \sup_{x\in[0,1]} f_0(x,v)dv + 4B_0\pi t\left(1+\frac{1}{\delta_2}\right)(c'_1 + c'_2 T)\int \sup_{(s,x)\in[0,t]\times[0,1]} f^{\#}(s,x,v)dv, \quad t \leq t_0.$$

Hence

$$\int \sup_{(s,x)\in[0,t]\times[0,1]} f^{\#}(s,x,v)dv \leq 4\int \sup_{x\in[0,1]} f_0(x,v)dv, \quad t \leq \min\left\{t_0, \frac{\delta_2}{8B_0\pi(\delta_2+1)(c'_1+c'_2T)}\right\}.$$

Since t_0, c'_1, and c'_2 depend only on T, $\int f_0(x,v)dxdv$, and $\int |v|^2 f_0(x,v)dxdv$, it follows that the argument can be repeated up to $t = T$. This completes the proof of the lemma. ☐

Remark. Lemmas 3.2–3.5 also hold with essentially the same proofs, for strong solutions of (2.5) with locally bounded energy.

The following two preliminary lemmas are needed for the control of large velocities.

LEMMA 3.6. *Given $T > 0$, the solution f_j of* (3.1) *satisfies*

$$\int_0^1 \int_{|v|>\lambda} |v| \sup_{t\in[0,T]} f_j^{\sharp}(t,x,v)dvdx \leq \frac{c_T}{\lambda}, \quad j \in \mathbb{N},$$

where c_T depends only on T, $\int f_0(x,v)dxdv$, and $\int |v|^2 f_0(x,v)dxdv$.

Proof. For convenience j is dropped from the notation f_j. As in (3.5),

$$\sup_{t\in[0,T]} f^{\sharp}(t,x,v) \leq f_0(x,v) + \int_0^T Q_j^+(f)(s,x+sv_1,v)ds.$$

Integration with respect to (x,v) for $|v| > \lambda$ gives

$$\int_0^1 \int_{|v|>\lambda} |v| \sup_{t\in[0,T]} f^{\sharp}(t,x,v)dvdx \leq \int\int_{|v|>\lambda} |v| f_0(x,v)dvdx + \frac{1}{\pi}\int_0^T \int_{|v|>\lambda} B\chi_j \cdot |v| f(s,x+sv_1,v')f(s,x+sv_1,v'_*)F(f)(s,x+sv_1,v)F(f)(s,x+sv_1,v_*)dvdv_*d\theta dxds.$$

Here in the last integral, either $|v'|$ or $|v'_*|$ is the largest and larger than $\frac{\lambda}{\sqrt{2}}$. The two cases are symmetric, and we discuss the case $|v'| \geq |v'_*|$. After a translation in x, the integrand is estimated from above by

$$c|v'|f^{\#}(s,x,v') \sup_{(t,x)\in[0,T]\times[0,1]} f^{\#}(t,x,v'_*).$$

The change of variables $(v, v_*, n) \to (v', v'_*, -n)$, the integration over

$$(s, x, v, v_*, \omega) \in [0, T] \times [0, 1] \times \left\{ v \in \mathbb{R}^2; |v| > \frac{\lambda}{\sqrt{2}} \right\} \times \mathbb{R}^2 \times \left[-\frac{\pi}{2}, \frac{\pi}{2} \right],$$

and Lemma 3.5 give the bound

$$\frac{c}{\lambda} \left(\int_0^T \int |v|^2 f^{\#}(s, x, v) dx dv ds \right) \left(\int \sup_{(t,x) \in [0,T] \times [0,1]} f^{\#}(t, x, v_*) dv_* \right) \\ \leq \frac{cTc_1(T)}{\lambda} \int |v|^2 f_0(x, v) dx dv.$$

The lemma follows. ◻

LEMMA 3.7. *Given $T > 0$, the solution f_j of* (3.1) *satisfies*

$$\int_{|v| > \lambda} \sup_{(t,x) \in [0,T] \times [0,1]} f_j^{\sharp}(t, x, v) dv \leq \frac{c'_T}{\sqrt{\lambda}}, \quad j \in \mathbb{N},$$

where c'_T depends only on T, $\int f_0(x, v) dx dv$, and $\int |v|^2 f_0(x, v) dx dv$.

Proof. Take $\lambda > 2$. As above,

$$\int_{|v| > \lambda} \sup_{(t,x) \in [0,T] \times [0,1]} f^{\sharp}(t, x, v) dv \leq \int_{|v| > \lambda} \sup_{x \in [0,1]} f_0(x, v) dv + cC, \tag{3.9}$$

where

$$C = \int_{|v| > \lambda} \sup_{x \in [0,1]} \int_0^T \int B\chi_j f^{\#}(s, x + s(v_1 - v'_1), v') f^{\#}(s, x + s(v_1 - v'_{*1}), v'_*) dv dv_* d\theta ds.$$

For v', v'_* outside of the angular cut-off (2.2), let n be the unit vector in the direction $v - v'$, and $n_\perp$ the orthogonal unit vector in the direction $v - v'_*$. Let e_1 be a unit vector in the x-direction.

Split C as $C = \sum_{1 \leq i \leq 6} C_i$, where C_1 (resp., C_2, C_3) refers to integration with respect to (v_*, θ) on

$$\left\{ (v_*, \theta); \quad n \cdot e_1 \geq \frac{1}{\sqrt{2}}, \quad |v'| \geq |v'_*| \right\}$$

$$\left(\text{resp., } \left\{ (v_*, \theta); n \cdot e_1 \geq \sqrt{1 - \frac{1}{\lambda}}, \ |v'| \leq |v'_*| \right\}, \right. \\ \left. \left\{ (v_*, \theta); n \cdot e_1 \in \left[\frac{1}{\sqrt{2}}, \sqrt{1 - \frac{1}{\lambda}} \right], \ |v'| \leq |v'_*| \right\} \right),$$

and analogously for C_i, $4 \leq i \leq 6$, with n replaced by $n_\perp$. By symmetry, C_i, $4 \leq i \leq 6$, can be treated as C_i, $1 \leq i \leq 3$, so we discuss only the control of C_i, $1 \leq i \leq 3$. By the change of variables $(v, v_*, n) \to (v', v'_*, -n)$, and noticing that $|v'| \geq \frac{\lambda}{\sqrt{2}}$ in the domain of integration of C_1, it holds that

$$C_1 \leq \int_{|v|>\frac{\lambda}{\sqrt{2}}} \sup_{x\in[0,1]} \int_0^T \int_{n\cdot e_1 \geq \frac{1}{\sqrt{2}}} B\chi_j f^\#(s, x+s(v_1'-v_1), v) \cdot f^\#(s, x+s(v_1'-v_{*1}), v_*) dv_* d\theta ds dv$$
$$\leq \int_{|v|>\frac{\lambda}{\sqrt{2}}} \sup_{x\in[0,1]} \int_0^T \int_{n\cdot e_1 \geq \frac{1}{\sqrt{2}}} B\chi_j \sup_{\tau\in[0,T]} f^\#(\tau, x+s(v_1'-v_1), v) \cdot \sup_{(\tau,X)\in[0,T]\times[0,1]} f^\#(\tau, X, v_*) dv_* d\theta ds dv.$$

With the change of variables $s \to y = x + s(v_1' - v_1)$,

$$C_1 \leq \int_{|v|>\frac{\lambda}{\sqrt{2}}} \sup_{x\in[0,1]} \int_{n\cdot e_1 \geq \frac{1}{\sqrt{2}}} \int_{y\in(x,x+T(v_1'-v_1))} \frac{B\chi_j}{|v_1'-v_1|} \sup_{\tau\in[0,T]} f^\#(\tau, y, v) \cdot \sup_{(\tau,X)\in[0,T]\times[0,1]} f^\#(\tau, X, v_*) dy dv_* d\theta dv$$
$$\leq \int_{|v|>\frac{\lambda}{\sqrt{2}}} \int_{n\cdot e_1 \geq \frac{1}{\sqrt{2}}} \frac{|E(T(v_1'-v_1))+1)|}{|v_1'-v_1|} \int_0^1 B\chi_j \sup_{\tau\in[0,T]} f^\#(\tau, y, v) \cdot \sup_{(\tau,X)\in[0,T]\times[0,1]} f^\#(\tau, X, v_*) dy dv_* d\theta dv.$$

Moreover,

$$|E(T(v_1'-v_1))+1)| \leq T|v_1'-v_1| + 1 \leq \left(T + \frac{\sqrt{2}}{\gamma\gamma'}\right)|v_1'-v_1|,$$

where γ and γ' were defined in (2.2). Consequently,

$$C_1 \leq c(T+1) \int_0^1 \int_{|v|>\frac{\lambda}{\sqrt{2}}} \sup_{\tau\in[0,T]} f^\#(\tau, y, v) dy dv \int \sup_{(\tau,X)\in[0,T]\times[0,1]} f^\#(\tau, X, v_*) dv_*$$
$$\leq \frac{c(T+1)}{\lambda} \int_0^1 \int_{|v|>\frac{\lambda}{\sqrt{2}}} |v| \sup_{\tau\in[0,T]} f^\#(\tau, y, v) dy dv \int \sup_{(\tau,X)\in[0,T]\times[0,1]} f^\#(\tau, X, v_*) dv_*.$$

By Lemmas 3.5 and 3.6,

$$C_1 \leq \frac{c}{\lambda^2}(T+1) c_T c_1(T).$$

Moreover,

$$C_2 \leq \int_{|v'|>\lambda, |v_*|>|v|, n\cdot e_1 \geq \sqrt{1-\frac{1}{\lambda}}} \frac{B\chi_j}{|v_1'-v_1|} \cdot \sup_{x\in[0,1]} \int_{y\in(x,x+T(v_1'-v_1))} \sup_{\tau\in[0,T]} f^\#(\tau, y, v) \sup_{(\tau,X)\in[0,T]\times[0,1]} f^\#(\tau, X, v_*) dy dv dv_* d\theta$$
$$\leq c(T+1) \int_{n\cdot e_1 \geq \sqrt{1-\frac{1}{\lambda}}} d\theta \int \sup_{\tau\in[0,T]} f^\#(\tau, y, v) dy dv \int \sup_{(\tau,X)\in[0,T]\times[0,1]} f^\#(\tau, X, v_*) dv_*$$
$$\leq \frac{c}{\sqrt{\lambda}}(T+1)^2 c_1(T),$$

by Lemmas 3.3 and 3.5. Finally,

$$C_3 \leq \int_{|v_*|>\frac{\lambda}{\sqrt{2}}, \frac{1}{\sqrt{\lambda}} \leq n_\perp \cdot e_1 \leq \frac{1}{\sqrt{2}}} \sup_{(\tau,X)\in[0,T]\times[0,1]} f^{\#}(\tau, X, v) \frac{B\chi_j}{|v'_1 - v_{*1}|}$$

$$\cdot \sup_{x\in[0,1]} \left(\int_{y\in(x, x+T(v'_1 - v_{*1}))} \sup_{\tau\in[0,T]} f^{\#}(\tau, y, v_*) dy \right) dv dv_* d\theta$$

$$\leq c(T+1)\sqrt{\lambda} \left(\int \sup_{(\tau,X)\in[0,T]\times[0,1]} f^{\#}(\tau, X, v) dv \right) \left(\int_{|v_*|>\frac{\lambda}{\sqrt{2}}} \sup_{\tau\in[0,T]} f^{\#}(\tau, y, v_*) dy dv_* \right).$$

By Lemmas 3.5 and 3.6,

$$C_3 \leq \frac{c}{\sqrt{\lambda}} (T+1) c_1(T) c_T.$$

The lemma follows. □

4. Proof of the main theorem. This section is devoted to the proof of Theorem 2.1. It consists of four steps. In the first step, we prove the existence of an initial layer $[0, t_m]$, with t_m independent on j, where $f_j^\sharp(t,\cdot,\cdot) < \frac{1}{\alpha} - b_1 t$. In a second step, we prove the existence of a solution f to (2.5). In the third step, we prove its uniqueness and the stability result stated in Theorem 2.1. Finally, the fourth step proves the conservations of mass, momentum, and energy of the solution.

First step: Analysis of an initial layer. Define

$$\tilde{\nu}_j(f) := \frac{1}{\pi} \int B\chi_j f' f'_* F_j(f_*) dv_* d\theta, \qquad \nu_j(f) := \frac{1}{\pi} \int B\chi_j f_* F_j(f') F_j(f'_*) dv_* d\theta,$$

so that

$$Q_j(f) = F_j(f) \tilde{\nu}_j(f) - f \nu_j(f).$$

Consider

$$\nu_j(f_j)^\sharp(t,x,v)$$
$$= \frac{1}{\pi} \int B\chi_j f_j(t, x+tv_1, v_*) F_j(f_j(t, x+tv_1, v')) F_j(f_j(t, x+tv_1, v'_*)) dv_* d\theta.$$

With the angular cut-off (2.2), $v_* \to v'$ and $v_* \to v'_*$ are changes of variables. Indeed, if the polar coordinates of $v_* - v$ are (r_*, φ) and θ is the angle between $v_* - v$ and n, then the polar coordinates of $v' - v$ (resp., $v'_* - v$) are $(|r_* \cos\theta|, \varphi + \theta)$ (resp., $(|r_* \sin\theta|, \varphi + \theta + \frac{\pi}{2})$). It follows from the angular cut-off (2.2) that the Jacobians $\frac{Dv_*}{Dv'} = \frac{1}{|\cos\theta|}$ (resp., $\frac{Dv_*}{Dv'_*} = \frac{1}{|\sin\theta|}$) are bounded. Using these changes of variables and Lemma 3.5, for ω outside the integration cut-off, the measure of the set

$$Z_{(j,t,x,v,\omega)} := \left\{ v_*; f(t, x+tv_1, v') > \frac{1}{2} \quad \text{or} \quad f(t, x+tv_1, v'_*) > \frac{1}{2} \right\} \tag{4.1}$$

is uniformly bounded with respect to (x, v, ω), $t \leq T$, and $j \in \mathbb{N}$. Take j_T so large that πj_T^2 is at least eight times this uniform bound. Notice that here j_T depends only on T and $\int (1+|v|^2) f_0(x,v) dx dv$. Using Duhamel's form for the solution, (2.3), and Lemma 3.5, one gets that

$$f_j^\sharp(t, x, v_*) \geq c_{1T} f_0(x, v_*) > 0, \qquad j \geq j_T,\ t \leq T, \tag{4.2}$$

with c_{1T} independent of $j \geq j_T$. It follows from (4.2) and the third assumption in (2.4) that

$$\nu_j(f_j)^\sharp(t,x,v) > c_{2T} > 0, \quad (t,x,v) \in [0,T] \times [0,1] \times \{v \in \mathbb{R}^2; |v| \leq j\}, \tag{4.3}$$

uniformly with respect to $j \geq j_T$ and with c_{2T} depending only on T and f_0.

Using again the $v_* \to v'$ change of variables together with Lemma 3.5, one obtains that for some constant $c_{3T} > 0$,

$$\tilde{\nu}_j^\sharp(f_j)(t,x,v) \leq c_{3T}, \quad j \geq j_T, \quad (t,x,v) \in [0,T] \times [0,1] \times \{v \in \mathbb{R}^2; |v| \leq j\}.$$

The functions defined on $]0, \frac{1}{\alpha}]$ by $x \to \frac{F_j(x)}{x}$ are uniformly bounded from above with respect to j by

$$x \to c\alpha^{\alpha-1}\frac{(1-\alpha x)^\alpha}{x},$$

which is continuous and decreasing to zero at $x = \frac{1}{\alpha}$. Hence there is $\mu \in]0, \frac{1}{\alpha}[$ such that

$$x \in \left[\frac{1}{\alpha} - \mu, \frac{1}{\alpha}\right] \quad \text{implies} \quad \frac{F_j(x)}{x} \leq \frac{c_{2T}}{4c_{3T}}, \quad j \geq j_T.$$

Consequently, for $j \geq j_T$,

$$\begin{aligned} f_j^\sharp(t,x,v) \in \left[\frac{1}{\alpha} - \mu, \frac{1}{\alpha}\right] \Rightarrow D_t f_j^\sharp(t,x,v) &= \left(F_j(f_j^\sharp)\tilde{\nu}_j^\sharp - \frac{1}{2}f_j^\sharp\nu_j^\sharp\right)(t,x,v) - \frac{1}{2}f_j^\sharp\nu_j^\sharp(t,x,v) \\ &< -\frac{1}{2}f_j^\sharp\nu_j^\sharp(t,x,v) \\ &< -\frac{1}{2}\left(\frac{1}{\alpha} - \mu\right)c_{2T} := -b_1. \end{aligned} \tag{4.4}$$

This gives a maximum time $t_1 = \frac{\mu}{b_1}$ for $f_j^\#$ to reach $\frac{1}{\alpha} - \mu$ from an initial value $f_0(x,v) \in]\frac{1}{\alpha} - \mu, \frac{1}{\alpha}]$. On this time interval $D_t f_j^\sharp \leq -b_1$. If $t_1 \geq T$, then at $t = T$ the value of $f_j^\#$ is bounded from above by $\frac{1}{\alpha} - b_1 T := \frac{1}{\alpha} - \mu'$ with $0 < \mu' \leq \mu$. Take $t_m = \min(t_1, T)$, and from now on $\mu = t_m b_1$. For any (x,v), if $f_j(0,x,v) < \frac{1}{\alpha} - \mu$ were to reach $\frac{1}{\alpha} - \mu$ at (t,x,v) with $t \leq t_m$, then $D_t f_j^\#(t,x,v) \leq -b_1$, which excludes such a possibility. It follows that $f_j \leq \frac{1}{\alpha} - \mu$ everywhere for $t \in [t_m, T]$, and that

$$f_j^\sharp(t,x,v) \leq \frac{1}{\alpha} - b_1 t \quad \text{for } t \in [0, t_m]. \tag{4.5}$$

The previous estimates leading to the definition of t_m are independent of $j \geq j_T$.

Second step: Existence of a solution f to (2.5). Using the initial layer and the results in section 3, we shall prove for any $T > 0$ the convergence in $C([0,T]; L^1([0,T] \times \mathbb{R}^2))$ of the sequence (f_j) to a solution f of (2.5).

Let us prove that (f_j) is a Cauchy sequence in $L^1([0,T] \times [0,1] \times \mathbb{R}^2)$ when $j \to \infty$. We shall prove that, given $\beta > 0$, there exists $b \geq \max\{1, j_T\}$ such that

$$\sup_{t \in [0,T]} \int |g_j(t,x,v)| dx dv < \beta, \quad j > b, \tag{4.6}$$

where $g_j = f_j - f_b$. The function g_j satisfies the equation
(4.7)

$$\begin{aligned}
\partial_t g_j + v_1 \partial_x g_j = &\frac{1}{\pi}\int (\chi_j - \chi_b) B\Big(f'_j f'_{j*} F_j(f_j) F_j(f_{j*}) - f_j f_{j*} F_j(f'_j) F_j(f'_{j*})\Big) dv_* d\theta \\
&+ \frac{1}{\pi}\int \chi_b B (f'_j f'_{j*} - f'_b f'_{b*}) F_j(f_j) F_j(f_{j*}) dv_* d\theta \\
&- \frac{1}{\pi}\int \chi_b B (f_j f_{j*} - f_b f_{b*}) F_j(f'_j) F_j(f'_{j*}) dv_* d\theta \\
&+ \frac{1}{\pi}\int \chi_b B f'_b f'_{b*} \Big(F_j(f_{j*})\big(F_j(f_j) - F_j(f_b)\big) + F_b(f_b)\big(F_j(f_{j*}) - F_j(f_{b*})\big)\Big) dv_* d\theta \\
&+ \frac{1}{\pi}\int \chi_b B f'_b f'_{b*} \Big(F_j(f_{j*})\big(F_j(f_b) - F_b(f_b)\big) + F_b(f_b)\big(F_j(f_{b*}) - F_b(f_{b*})\big)\Big) dv_* d\theta \\
&- \frac{1}{\pi}\int \chi_b B f_b f_{b*} \Big(F_j(f'_{j*})\big(F_j(f'_j) - F_j(f'_b)\big) + F_b(f'_b)\big(F_j(f'_{j*}) - F_j(f'_{b*})\big)\Big) dv_* d\theta \\
&- \frac{1}{\pi}\int \chi_b B f_b f_{b*} \Big(F_j(f'_{j*})\big(F_j(f'_b) - F_b(f'_b)\big) + F_b(f'_b)\big(F_j(f'_{b*}) - F_b(f'_{b*})\big)\Big) dv_* d\theta.
\end{aligned}$$

Moreover, using Lemma 3.5,

$$\begin{aligned}
\int (\chi_j - \chi_b) B\Big(f'_j f'_{j*} F_j(f_j) F_j(f_{j*}) + f_j f_{j*} F_j(f'_j) F_j(f'_{j*})\Big) dx dv dv_* d\theta& \\
\leq c \int_{|v| > \frac{b}{\sqrt{2}}} f_j(t,x,v) dx dv& \\
\leq \frac{c}{b^2}, \quad \text{by the conservation of energy of } f_j;&
\end{aligned}$$

$$\begin{aligned}
&\int \chi_b B |f_j f_{j*} - f_b f_{b*}| F_j(f'_j) F_j(f'_{j*}) dx dv dv_* d\omega \\
&\quad \leq c \left(\int \sup_{(t,x)\in[0,T]\times[0,1]} f_j^\sharp(t,x,v) dv + \int \sup_{(t,x)\in[0,T]\times[0,1]} f_b^\sharp(t,x,v) dv \right) \\
&\qquad\qquad \cdot \int |(f_j^\sharp - f_b^\sharp)(t,x,v)| dx dv \\
&\quad \leq c \int |(f_j^\sharp - f_b^\sharp)(t,x,v)| dx dv.
\end{aligned}$$

Next,

$$\begin{aligned}
&\int \chi_b B \Big(f'_b f'_{b*} F_j(f_{j*}) |F_j(f_b) - F_b(f_b)|\Big)^\sharp dx dv dv_* d\theta \\
&= \int \chi_b B f'_b f'_{b*} F_j(f_{j*}) (1 - \alpha f_b)(1 + (1-\alpha) f_b)^{1-\alpha} \\
&\qquad \cdot \left| \left(\frac{1}{j} + 1 - \alpha f_b\right)^{\alpha - 1} - \left(\frac{1}{b} + 1 - \alpha f_b\right)^{\alpha - 1} \right| dx dv dv_* d\theta.
\end{aligned}$$

By Lemmas 3.3 and 3.5, this integral restricted to the set where $1 - \alpha f_b(t,x,v)) \leq \frac{2}{b}$, hence where

$$(1 - \alpha f_b) \left| \left(\frac{1}{j} + 1 - \alpha f_b\right)^{\alpha - 1} - \left(\frac{1}{b} + 1 - \alpha f_b\right)^{\alpha - 1} \right| \leq \frac{2^{\alpha+1}}{b^\alpha}$$

is bounded by $\frac{c}{b^\alpha}$ for some constant $c > 0$.

For the remaining domain of integration where $1 - \alpha f_b(t,x,v)) \geq \frac{2}{b}$,

$$\begin{aligned}
|F_j(f_b) - F_b(f_b)| &\leq c(1-\alpha f_b)^\alpha \left| \left(\frac{1}{j(1-\alpha f_b)} + 1\right)^{\alpha-1} - \left(\frac{1}{b(1-\alpha f_b)} + 1\right)^{\alpha-1} \right| \\
&= c\left(\frac{1}{j} - \frac{1}{b}\right)(1-\alpha f_b)^{\alpha-1}\lambda^{\alpha-2}, \quad \text{where } \lambda \in \left[1, \frac{3}{2}\right], \\
&\leq \frac{2^{\alpha-1}c}{b^\alpha}.
\end{aligned}$$

And so,

$$\int \chi_b B\Big(f'_b f'_{b*} F_j(f_{j*})|F_j(f_b) - F_b(f_b)|\Big)^\sharp dx dv dv_* d\theta \leq \frac{c}{b^\alpha}.$$

Finally,

$$\begin{aligned}
&\int \chi_b B\Big(f'_b f'_{b*} F_j(f_{j*})|F_j(f_j) - F_j(f_b)|\Big)^\sharp (t,x,v) dx dv dv_* d\theta \\
&\quad \leq c \int |F_j(f_j) - F_j(f_b)|^\sharp (t,x,v) dx dv.
\end{aligned}$$

Split the (x,v)-domain of integration of the latest integral into

$$\begin{aligned}
D_1 &:= \left\{ (x,v); (f_j^\sharp(t,x,v), f_b^\sharp(t,x,v)) \in \left[0, \frac{1}{\alpha} - \mu\right]^2 \right\}, \\
D_2 &:= \left\{ (x,v); (f_j^\sharp(t,x,v), f_b^\sharp(t,x,v)) \in \left[\frac{1}{\alpha} - \mu, \frac{1}{\alpha}\right]^2 \right\}, \\
D_3 &:= \left\{ (x,v); (f_j^\sharp, f_b^\sharp)(t,x,v) \in \left[\frac{1}{\alpha} - \mu, \frac{1}{\alpha}\right] \times \left[0, \frac{1}{\alpha} - \mu\right] \right. \\
&\qquad\qquad \left. \text{or } (f_j^\sharp, f_b^\sharp)(t,x,v)) \in \left[0, \frac{1}{\alpha} - \mu\right] \times \left[\frac{1}{\alpha} - \mu, \frac{1}{\alpha}\right] \right\}.
\end{aligned}$$

It holds that

$$\begin{aligned}
&\int_{D_1} |F_j(f_j) - F_j(f_b)|^\sharp (t,x,v) dx dv \leq c(\alpha\mu)^{\alpha-1} \int_{D_1} |g_j^\sharp(t,x,v)| dx dv, \\
&\int_{D_2} |F_j(f_j) - F_j(f_b)|^\sharp (t,x,v) dx dv \leq c t^{\alpha-1} \int_{D_2} |g_j^\sharp(t,x,v)| dx dv, \quad \text{by (4.5)}, \\
&\int_{D_3} |F_j(f_j) - F_j(f_b)|^\sharp (t,x,v) dx dv \leq c\big((\alpha\mu)^{\alpha-1} + t^{\alpha-1}\big) \int_{D_3} |g_j^\sharp(t,x,v)| dx dv.
\end{aligned}$$

The remaining terms to the right in (4.7) are of the same types as the ones just estimated. Consequently,

$$\frac{d}{dt} \int |g_j^\sharp(t,x,v)| dx dv \leq \frac{c}{b^\alpha} + c(1 + t^{\alpha-1}) \left(\int |g_j^\sharp(t,x,v)| dx dv \right). \tag{4.8}$$

And so,

$$\int |g_j^\sharp(t,x,v)|dxdv \leq \left(\int_{|v|>b} f_0(x,v)dxdv + \frac{cT}{b^\alpha}\right) e^{c(T+\frac{T^\alpha}{\alpha})},$$

which tends to zero when $b \to +\infty$, uniformly w.r.t. $j \geq b$. This proves that $(f_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^1([0,T]\times[0,1]\times\mathbb{R}^2)$ and ends the proof of the existence of a solution f to (2.5).

Third step: Uniqueness of the solution to (2.5) *and stability results.* The previous line of arguments can be followed to obtain that the solution is unique. Namely, assuming the existence of two solutions f_1 and f_2 to (2.5) with locally bounded energy, (4.5) holds for both solutions. The difference $g = f_1 - f_2$ satisfies

(4.9)

$$\begin{aligned}&\partial_t g + v_1\partial_x g\\ &= \frac{1}{\pi}\int B(f_1'f_{1*}' - f_2'f_{2*}')F(f_1)F(f_{1*})dv_*d\theta - \frac{1}{\pi}\int B(f_1f_{1*} - f_2f_{2*})F(f_1')F(f_{1*}')dv_*d\theta\\ &+ \frac{1}{\pi}\int Bf_2'f_{2*}'\Big(F(f_{1*})\big(F(f_1)-F(f_2)\big) + F(f_2)\big(F(f_{1*})-F(f_{2*})\big)\Big)dv_*d\theta\\ &- \frac{1}{\pi}\int Bf_2f_{2*}\Big(F(f_{1*}')\big(F(f_1')-F(f_2')\big) + F(f_2')\big(F(f_{1*}')-F(f_{2*}')\big)\Big)dv_*d\theta.\end{aligned}$$

The first line on the right-hand side of (4.9) gives rise to $c\int|g^\sharp(t,x,v)|dxdv$ in the bound from above of $\frac{d}{dt}|g^\sharp(t,x,v)|dxdv$, whereas the two last lines on the right-hand side of (4.9) give rise to the bound $c(1+t^{\alpha-1})\int|g^\sharp(t,x,v)|dxdv$. Consequently,

$$\frac{d}{dt}\int|g^\sharp(t,x,v)|dxdv \leq c(1+t^{\alpha-1})\int|g^\sharp(t,x,v)|dxdv.$$

This implies that $\int|g^\sharp(t,x,v)|dxdv$ is identically zero, since it is zero initially.

The proof of stability is similar.

Fourth step: Conservations of mass, momentum, and energy. The conservation of mass and first momentum of f follows from the boundedness of the total energy. The energy is nonincreasing by the construction of f. Energy conservation will follow if the energy is nondecreasing. Taking $\psi_\epsilon = \frac{|v^2|}{1+\epsilon|v|^2}$ as an approximation for $|v|^2$, it is enough to bound

$$\int Q(f,f)(t,x,v)\psi_\epsilon(v)dxdv = \frac{1}{\pi}\int B\psi_\epsilon\Big(f'f_*'F(f)F(f_*) - ff_*F(f')F(f_*')\Big)dxdvdv_*d\theta$$

from below by zero in the limit $\epsilon \to 0$. Similarly to [14],

$$\begin{aligned}\int Q(f,f)\psi_\epsilon dxdv &= \frac{1}{2\pi}\int Bff_*F(f')F(f_*')\Big(\psi_\epsilon(v') + \psi_\epsilon(v_*') - \psi_\epsilon(v) - \psi_\epsilon(v_*)\Big)dxdvdv_*d\theta\\ &\geq -\frac{1}{\pi}\int Bff_*F(f')F(f_*')\frac{\epsilon|v|^2|v_*|^2}{(1+\epsilon|v|^2)(1+\epsilon|v_*|^2)}dxdvdv_*d\theta.\end{aligned}$$

The previous line, with the integral taken over a bounded set in (v,v_*), converges to zero when $\epsilon \to 0$. In integrating over $|v|^2 + |v_*|^2 \geq 2\lambda^2$, there is symmetry between the subset of the domain with $|v|^2 > \lambda^2$ and that with $|v_*|^2 > \lambda^2$. We discuss the first subdomain, for which the integral in the last line is bounded from below by

$$-c\int |v_*|^2 f(t,x,v_*)dxdv_* \int_{|v|\geq\lambda} B \sup_{(s,x)\in[0,t]\times[0,1]} f^{\#}(s,x,v)dvd\theta$$
$$\geq -c\int_{|v|\geq\lambda}\sup_{0\leq(s,x)\in[0,t]\times[0,1]} f^{\#}(s,x,v)dv.$$

It follows from Lemma 3.7 that the right-hand side tends to zero when $\lambda\to\infty$. This implies that the energy is nondecreasing and bounded from below by its initial value. That completes the proof of the theorem. ∎

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