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When Flexible Forms Are Asked to Flex Too Much

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Taylor series-based flexible forms cannot be interpreted as Taylor series approximations unless all data used in estimation lie in a region of convergence. When flexible forms lose their Taylor series interpretation, elasticity estimates will be biased. When the flexible form is a translog, Rotterdam, or AIDS model, the region of convergence is shown to be the entire positive orthant. Regions of convergence associated with quadratic, Leontief, and any flexible form that does not employ logged arguments are smaller and may not encompass the entire data set. Implications for production and demand analyses and experimental design are discussed.

Key words: flexible forms, Leontief, quadratic, selecting functional forms, translog.

Introduction

Today, production and demand analysts have a variety of flexible forms from which to choose. These include the translog, generalized Leontief, and quadratic forms (collectively referred to as TSFFs), all of which have been characterized as second-order Taylor series (TS) approximations by Blackorby, Primont, and Russell. A more general form that includes the above forms as a special or limiting case is the generalized Box–Cox (Berndt and Khaled). Recently, flexible forms based on the Fourier series (Gallant; Chalfant and Gallant) have been added to the growing list.

However, what form is most appropriate for empirical work is still debated. Monte Carlo studies have been undertaken to determine what flexible form profit and cost functions provide the best approximations, but the results have never conclusively favored one form or another. Summarizing work in the area, Judge et al. concluded that although efforts to develop new flexible forms enrich the family of specification alternatives, the question of how to choose among them remains. In this article, a criterion is proposed for narrowing the range of sensible choices within the TSFF group.

A functional form that may be interpreted as a TS approximation has some nice properties. In a neighborhood of the point of approximation, the error of approximation is bounded and the approximation converges to the underlying function as higher-order terms are added. In this neighborhood, derivatives of the approximation converge to derivatives of the underlying function as higher-order terms are added. The convergence properties of TS approximations become more attractive the larger the neighborhood in which they hold. For a TS approximation in the variable u, the size of this neighborhood is influenced by three factors: (a) the function being approximated, (b) the point of approximation, and (c) whether $u = \ln(x)$, $u = x$, $u = x^{1/2}$, etc.

The purpose of this study is twofold. First, find the TS approximation with the largest neighborhood of convergence when the underlying function is a production, profit, cost, or indirect utility function, and argue that it makes sense to select a TSFF based on this particular approximation. Second, demonstrate that when the neighborhood of conver-

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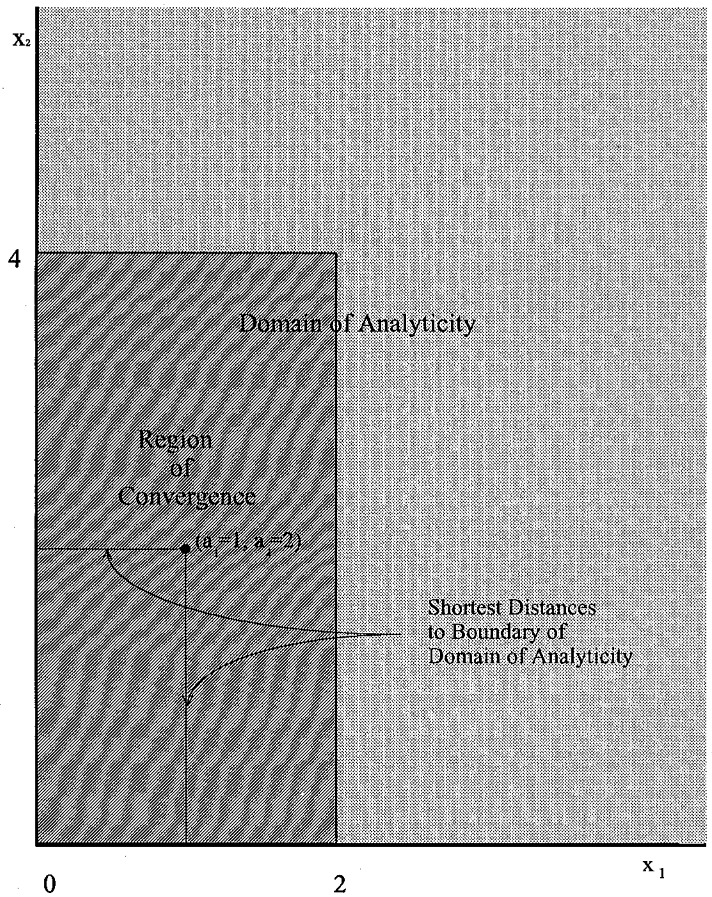

Figure 1. Domain of analyticity and region of convergence [Note: Domain of analyticity is all of R^n_+; region of convergence is ($0 < x_1 < 2$, $0 < x_2 < 4$) when point of approximation is $\boldsymbol{a}_1 = 1$, $\boldsymbol{a}_2 = 2$.]

gence is small and does not encompass all data points in the sample, the TSFF is, loosely speaking, asked to flex too much. As a result, estimated coefficients of TSFFs diverge from TS coefficients as higher-order terms are added to the approximation and the estimated functional form loses its interpretation as a TS approximation.

The remainder of the article is partitioned into three sections. In the first section, conditions under which a production, profit, or indirect utility function has a TS representation are reviewed. The region over which the TS approximation converges to the underlying function is established. The second section is devoted to finding regions of convergence associated with TSFFs. The effects of data scaling and transposing the axes are discussed. In the last section, the Monte Carlo experiment and results are presented.

Approximating Direct and Indirect Functions with Taylor Series

The Region of Convergence of a Taylor Series Approximation

Production, cost, profit, utility, and indirect utility functions (hereafter collectively referred to as the functions of interest) are all defined on the domain, R^n_+. Outside this domain, function values and derivatives are not defined. This common property is of major importance in determining the region over which a TS approximation and derivatives of the TS approximation will converge to the function value and its derivatives.

If $f(x)$, a real function of a single real variable having derivatives of all orders at the point $\boldsymbol{a}$, can be represented by the power series

$$(1) \qquad f(x) = \sum_{n=0}^{\infty} c_n(x - \boldsymbol{a})^n \qquad \text{for } |x - \boldsymbol{a}| < R, \text{ with } R > 0,$$

then the c_n are given by

$$(2) \qquad c_n = \frac{f^{(n)}(\boldsymbol{a})}{n!},$$

where $f^{(n)}(\boldsymbol{a})$ is the nth partial derivative of $f(x)$ evaluated at $\boldsymbol{a}$ (Protter and Morey, p. 233). The power series in (1) is called a Taylor series expansion.

If $f(x)$ has a power series representation over the interval $|x - \boldsymbol{a}| < R$, $R > 0$, then f is said to be analytic at point $\boldsymbol{a}$. A function f is analytic on a domain if and only if it is analytic at each point in the domain (Protter and Morey, p. 233).[1] Conversely, if f is analytic in some domain G, $\boldsymbol{a}$ is an arbitrary point in G, **and** R is the distance between the point $\boldsymbol{a}$ and the nearest boundary of G, then there exists a power series converging to f on the interval $|x - \boldsymbol{a}| < R$ (Silverman, p. 204; also see Fleming, pp. 94 and 97 for the n-variable theorems). The interval $|x - \boldsymbol{a}| < R$ is known as the region of convergence. Rudin (p. 172) notes that R may be $+\infty$. If $f(\boldsymbol{x})$ is a function of n variables, the region of convergence can be thought of as a cube given by $|x_i - \boldsymbol{a}_i| < R_i$ for $i = 1, n$, where $(\boldsymbol{a}_1, \ldots, \mathbf{a}_n)$ is the point of expansion (Protter and Morey, pp. 137 and 185).

If a function of interest has a power series representation over some domain G, it must be analytic on that domain. A necessary condition for the function to be analytic is that both the function value and function derivatives of all orders exist at every point in G.[2] Since values and derivatives of the functions of interest are not defined for $x_i < 0$ (and sometimes for $x_i \leq 0$), $R_i < \boldsymbol{a}_i$, where $\boldsymbol{a}_i > 0$. Assuming that the function of interest is analytic on R^n_+, the region of convergence of the power series in (1) is given by $0 < x_i < 2\boldsymbol{a}_i$. For a function of interest in two variables, the region of convergence is depicted in figure 1 as $(0 < x_i < 2, 0 < x_2 < 4)$ when the point of approximation is $(\boldsymbol{a}_1 = 1, \boldsymbol{a}_2 = 2)$.

Example 1: Consider the region of convergence for the real function:

$$f(x_1, x_2) = \frac{x_1}{(1 - x_1)(1 - x_2)}.$$

A TS expansion in powers of x about the point (0, 0) has as a region of convergence the square $\{|x_1| < 1, |x_2| < 1\}$. The function value and its derivatives are undefined in the plane $x_1 = 1$ and in the plane $x_2 = 1$; therefore, the function is not analytic at these points. The Taylor series will not converge at these points or at any point (x_1, x_2) where $|x_1| > 1$ or $|x_2| > 1$. A TS expansion about (0, 0) converges for $\{(x_1, x_2): -1 < x_1 < 1, -1 < x_2 < 1\}$.

Example 2: Consider the region of convergence for the function:

$$f(x_1, x_2) = \frac{x_1}{(1 - x_2)}.$$

A TS expansion in powers of $\boldsymbol{x}$ about (0, 0) converges everywhere that $|x_2| < 1$. Because the variable x_1 cannot cause the function value and its derivatives to be unbounded, the Taylor series converges for all values of x_1, provided $|x_2| < 1$.

Properties of a Taylor Series Approximation

Uniform Convergence. Suppose that the series in (1) converges for $x = x_1$, with $x_1 \neq \boldsymbol{a}$. Then the series converges uniformly on the interval $I = \{x: \boldsymbol{a} - h \leq x \leq \boldsymbol{a} + h\}$ for each $h < |x_1 - \boldsymbol{a}|$ (Protter and Morey, p. 231). *Uniform convergence* implies that for any $\epsilon > 0$, an N can be found such that the absolute value of the difference between an nth-order approximation evaluated at arbitrary x in I and the function evaluated at x is less than ϵ for all $n > N$, where N depends only on ϵ and not on the particular x in I (Protter and Morey, p. 222). This implies that for some error ϵ (*possibly* small), a second-order TS approximation will track $f(x)$ with error less than ϵ over the entire region of convergence.

Derivatives of f(x). Let $f(x)$ be a real analytic function defined on an open interval I. Then $f(x)$ is continuous, and has real analytic derivatives of all orders. For each positive integer m, the derivatives $f^{(m)}(x)$ are given by term-by-term differentiation of (1) m times (Krantz, p. 224). The radius of convergence of the derived series is identical to that of the original power series (Krantz, p. 223). Therefore, TS approximations to gradient and hessian terms of $f(\boldsymbol{x})$ may be obtained by differentiating the TS approximation to $f(\boldsymbol{x})$ and these derived approximations converge to the derivatives of $f(x)$ over the same region as the original series. **This derivative property does not apply in general to polynomial approximations.**

Uniqueness. The representation of $f(\boldsymbol{x})$ by the series (1) is unique (Krantz, p. 224). This has implications for the Monte Carlo experiment reported below. Since TS coefficients are unique, if estimated flexible form coefficients do not converge to TS coefficients at the point of approximation, then the estimated flexible form cannot be interpreted as a TS approximation.

Divergence Outside the Region of Convergence. If the TS approximation is not convergent at a point, it is divergent (Rudin, p. 69). If at the point (x_1) the TS approximation diverges, then it diverges for $|x_1^* - \boldsymbol{a}_1| > |x_1 - \boldsymbol{a}_1|$. Outside the region of convergence, the TS approximation diverges from $f(x)$.

Functions of n *Variables.* All of the properties discussed above apply to TS approximations to functions of n variables (Gunning and Rossi, pp. 1–4).

Approximations in Powers of Functions of x. Suppose that $f(x_1, \ldots, x_n)$ is ultimately a function of variables y_i, $i = 1, n$; for instance, $f(\boldsymbol{x}) = f(x_1(y_1), \ldots, x_n(y_n))$ and $x_i(y_i) = \ln(y_i)$. All properties of the Taylor series still apply and the expansion may be made either in powers of $x_i = \ln(y_i)$ or y_i. If the expansion is in powers of $\ln(y_i)$, the TS coefficients involve derivatives of f with respect to $\ln(y_i)$. As demonstrated below, the region of convergence can be affected by the function of y_i employed in the TS expansion.

Influencing the Region of Convergence of Approximations to Direct and Indirect Functions

Profit, cost, and production functions are defined only in R^n_+. Since their function values and derivatives are not defined outside of R^n_+, the domain over which they can be analytic is confined to R^n_+. For a production function, the boundaries of this domain are given by the planes in input space $x_1 = 0, \ldots, x_n = 0$. For profit functions, the boundaries in relative price space are $p_1/p_y = 0, \ldots, p_n/p_y = 0$. For the production function, *if* the point of expansion is the mean of the data $(\mu_1, \ldots, \mu_n)$, a TS expansion in powers of x will converge in the cube $\{0 < x_1 < 2\mu_1, \ldots, 0 < x_n < 2\mu_n\}$.

Choosing Series Expansions to Enlarge the Region of Convergence

In what follows, it is assumed that only at and beyond the boundary points of R^n_+ do the functions of interest fail to be analytic. They are assumed to be analytic at all interior points. There is really no point in discussing TS approximations to these functions in the absence of this assumption (functions that are not analytic do not have power series representations). If the function of interest is analytic everywhere in R^n_+, then it can be approximated by a Taylor series and the region of convergence made large by *moving the boundaries* arbitrarily far away from the point of expansion. One way of shifting the boundaries is by expanding in powers of $u = \ln(x)$. By doing this, the boundaries given by the planes $x_i = 0$ essentially are moved to $u_i = -\infty$. A TS expansion in powers of ln(x) will converge in the cube $\{-\infty < \ln(x_1) < \infty, \ldots, -\infty < \ln(x_n) < \infty\}$ or $\{0 < x_1 < \infty, \ldots, 0 < x_n < \infty\}$.[3]

The following example illustrates the difference between expanding in powers of x and powers of ln(x). The example illustrates the differences between using the quadratic, generalized Leontief, and translog flexible forms. Consider the function $f(x) = x\ln(x)$. Neither the function value nor its derivatives are defined at $x = 0$; therefore, the function is not analytic there. Since the function is not analytic at $x = 0$, the region of convergence of a TS approximation expanded about $\boldsymbol{a} = 1$ is given by $|x - \boldsymbol{a}| < R = 1$, or $0 < x < 2$. This is verified using the ratio test (Krantz, p. 94). First, expand $f(x) = x\ln(x)$ in powers of x about $\boldsymbol{a} = 1$:

$$x\ln(x) = x\ln(x)|_{x=1} + (\ln(x) + 1)|_{x=1}(x - 1) + 1/2(1/x)|_{x=1}(x - 1)^2 + \ldots$$
$$= 0 + (x - 1) + 1/(2x)|_{x=1}(x - 1)^2 - 1/3!(1/x^2)|_{x=1}(x - 1)^3$$
$$+ 1/4!(2/x^3)|_{x=1}(x - 1)^4 - 1/5!(3!/x^4)|_{x=1}(x - 1)^5 + \ldots$$

To see over what x this series converges, use the ratio test

$$\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}(n-1)!}{(n+1)!}\cdot\frac{n!}{(n-2)!(x-1)n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(x-1)(n-1)}{(n+1)}\right| = |x - 1|\lim_{n\to\infty}[(n-1)/(n+1)]$$
$$= |x - 1|.$$

For convergence, $|x - 1| < 1$. As expected, this series converges for $0 < x < 2$.

Now expand $f(x) = x\ln(x)$ in powers of ln(x) about $\boldsymbol{a}^* = 1$, or $\ln(\boldsymbol{a}^*) = 0$. This is equivalent to employing the change of variables technique where $u = \ln(x)$. In other words, the expansion about to be made is equivalent to taking a Taylor series expansion in u about 0 of the function $f(u) = ue^u$. Both expansions have the same region of convergence:

$$x\ln(x) = x\ln(x)|_{\ln(x)=0} + d(x\ln(x))/d\ln(x)|_{\ln(x)=0}(\ln(x) - 0)$$
$$+ 1/2[d^2(x\ln(x))/d\ln(x)^2]|_{\ln(x)=0}(\ln(x) - 0)^2 + \ldots$$
$$= 0 + (x + x\ln(x))|_{\ln(x)=0}(\ln(x)) + 1/2(2x + x\ln(x))|_{\ln(x)=0}(\ln(x))^2$$
$$+ (1/3!)(3x + x\ln(x))|_{\ln(x)=0}(\ln(x))^3 + \ldots$$

Determine the region of convergence using the ratio test:

$$\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| = \lim_{n\to\infty}\left|\frac{(\ln(x))^{n+1}(n+1)}{(n+1)!}\cdot\frac{n!}{n(\ln(x))^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{\ln(x)}{n}\right| = |\ln(x)|\lim_{n\to\infty}(1/n).$$

Convergence requires $|\ln(x)| \cdot 0 < 1$. This series converges for all $x > 0$.

By choosing a Taylor series expansion in powers of ln(x) and any strictly nonzero point of expansion, in essence, a change of variables is undertaken and the expansion is made in $u_i = \ln(x_i)$. Boundaries of the domain of analyticity in the planes $x_i = 0$ become boundaries at $u_i = -\infty$ and can only limit the region of convergence to $\boldsymbol{x} \in R^n_+$ or $\boldsymbol{u} \in R^n$, a much larger region than that associated with the expansions in powers of x or $x^{1/2}$.[4]

As Blackorby, Primont, and Russell have shown (below), the quadratic, generalized Leontief, and translog forms may be interpreted as a Taylor series expansion in powers of x, $x^{1/2}$, and $\ln(x)$, respectively.

Flexible Form Parameter Estimates: Series Expansion Coefficients or What?

Blackorby, Primont, and Russell developed a generalized Taylor series expansion in powers of $h_i(x_i)$ and interpreted a number of flexible form production and utility functions (including the quadratic, translog, and Leontief) as TS approximations. In subsequent papers, econometricians have debated whether it is appropriate to interpret TSFF coefficient estimates as TS coefficients. For instance, using a Cobb–Douglas example, White argues that OLS coefficient estimates need not represent TS coefficients at any expansion point. Byron and Bera highlight errors in White's example and argue that researchers seldom restrict themselves to first-order approximations like a Cobb–Douglas. They claim that estimated coefficients may be interpreted as coefficients of a TS expansion as long as sufficient higher-order terms are included in the flexible form specification.

If estimated TSFFs can be interpreted as TS approximations, then the properties of TS approximations apply as well to the estimated TSFF. If they cannot be interpreted as series expansions, there is no guarantee that estimates of function value, gradients, and hessian terms (elasticities) converge to their function counterparts as higher-order terms are added.

Blackorby, Primont, and Russell [p. 293, equations (8)–(17)] give a general representation of the quadratic, generalized Leontief, and translog forms as

$$\psi[z] = \psi[f(\mathbf{x})] = \alpha_0 + \sum_i \alpha_i f_i(x_i) + \frac{1}{2}\sum_i \sum_j \beta_{ij} f_i(x_i) f_j(x_j), \qquad (3)$$
$$\beta_{ij} = \beta_{ji}.$$

If $f_i(x_i) = x_i$ and $\psi(z) = z$, then (3) reduces to the quadratic. If $f_i(x_i) = x_i^{1/2}$ and $\psi(z) = z$, then (3) becomes the generalized Leontief. If $f_i(x_i) = \ln(x_i)$ and $\psi(z) = \ln(z)$, then (3) is translog. The authors then show that

$$\alpha_0 = \tilde{\alpha}_0 - \sum_i \tilde{\alpha}_i f_i(\tilde{x}_i) + \frac{1}{2}\sum_i \sum_j \tilde{\beta}_{ij} f_i(\tilde{x}_i) f_j(\tilde{x}_j) \qquad (4)$$
$$\alpha_i = \tilde{\alpha}_i - \sum_j \tilde{\beta}_{ij} f_j(\tilde{x}_j)$$
$$\beta_{ij} = \tilde{\beta}_{ij},$$

where $\tilde{\alpha}_0$, $\tilde{\alpha}_i$, and $\tilde{\beta}_{ij}$ may be interpreted as TS coefficients evaluated at the point $\tilde{\mathbf{x}}$. With estimates of the α_is and β_{ij}s, the "Taylor series" coefficients (which may be interpreted as gradient and hessian terms) can be computed at $\tilde{\mathbf{x}}$. Under appropriate conditions, this is true of all points $\tilde{\mathbf{x}}$ in the region of convergence and does not indicate that $\tilde{\mathbf{x}}$ is the (implicit) point of expansion in estimation.

Above, it was established that the underlying function can be approximated by a Taylor series only in the region of convergence. Since the domain of analyticity of the functions of interest is R^n_+, the region of convergence for the estimated TSFF can be determined once the point of expansion is identified. Unfortunately, when estimating a flexible form, the point of expansion is not explicitly specified and cannot be controlled. Cramer gives an analytical proof that the point of expansion is the mean of the data when the functional form is a first-order approximation and an OLS estimator is employed.[5]

In order to establish that OLS and ITSUR estimates of TSFF coefficients can be interpreted as series coefficients evaluated at $\bar{\mathbf{x}}$, the underlying function is assumed to have an *identical* TS representation at all sample points. As a consequence, all sample points ($t = 1, T$) must lie in the region of convergence of this TS approximation. If the data have been transformed, the expansion point is the mean of the transformed data $(\overline{f_1(x_1)}, \ldots, \overline{f_n(x_n)})$ where $\overline{f_i(x_i)} = \Sigma_t f_i(x_{it})/T$. Therefore, at all T observations and for all explanatory variables x_i, data points must satisfy $|f_i(x_{it}) - \overline{f_i(x_i)}| < R_i = \overline{f_i(x_i)} - f_i(0)$, where, for instance, $f_i(x_i) = \ln(x_i)$ in the case of the translog. Equivalently, all data must lie in the cube $\{f_i(0) < f_i(x_{it}) < 2\,\overline{f_i(x_i)} - f_i(0)\}$. If this assumption is violated, OLS and ITSUR coefficient estimates need not converge to their Taylor series counterparts at $\bar{\boldsymbol{x}}$. Outside the region of convergence, the Taylor series is divergent and remainders can be large. The divergence property of the TS expansion and the minimization criterion of estimators like OLS or ITSUR are incompatible. For instance, the OLS estimator seeks to reduce squared prediction errors even at those points at which the TS expansion is divergent and the remainder term is large.

To be able to interpret estimated TSFFs as TS approximations, all data must lie in the region of convergence. As the regions of convergence associated with Taylor series expansions in powers of x_i and $x_i^{1/2}$ can be small, estimated quadratic and generalized Leontief TSFFs may not qualify as TS approximations to direct and indirect functions for some data sets.

Data Scaling and Transposition

The region of convergence cannot be influenced by scaling data by a factor λ (as claimed by Thursby and Lovell) or by data transposition (moving the origin). Scaling data by a factor affects the magnitude of OLS or ITSUR coefficient estimates (by an inverse scaling) but cannot render a divergent series convergent because there is no net effect from the scaling. Transposing the data cannot help, since the boundaries of R^n_+ are moved as well and the region of convergence is not altered. Further, flexible form coefficient estimates using the transposed data can be computed from estimates obtained with the original data by (4); therefore, no consequential effect is achieved.

When Sample Data Lie Outside the Region of Convergence: A Monte Carlo Example

In this section, a Monte Carlo experiment is undertaken to explore the extent to which TSFF coefficient estimates differ from TS coefficients when the sample includes data points that lie outside the region of convergence. The investigations require two experiments. In the first experiment, data are generated so that all sample points lie in the region of convergence for the quadratic where the mean of the data is the point of expansion. Translog and quadratic models (of orders 2–5) are estimated using OLS and ITSUR estimators. For both OLS and ITSUR estimators, coefficient estimates of both TSFFs are expected to converge to TS coefficients at the mean of the transformed data as higher-order terms are added to the expansion.

In the second experiment, 10% of the sample data are large in magnitude relative to remaining observations. All of these points lie outside the region of convergence of the quadratic (that is, $|x_{it} - \bar{x}_i| > \bar{x}_i$ for some i). For both OLS and ITSUR estimators, estimates of quadratic coefficients are expected to diverge from TS coefficients, but those of the translog models are expected to converge.

Although in practice, econometricians rarely have the luxury of employing approximations of order larger than three, the higher-order models are included here to demonstrate that, when all data lie inside the region of convergence, coefficient bias observed for low-order expansions is caused solely by omitted higher-order terms. When some data lie outside the region of convergence, the results from higher-order models help demonstrate that the bias will not disappear and, in fact, bias increases as higher-order terms are added.

For the first experiment, 200 observations of $\boldsymbol{x}_t = (x_{1t}, x_{2t}, x_{3t})$ are drawn randomly from the following distribution:

Table 1. Generalized CES Parameters for Nine Functions

Function	Parameters				Allen Partial Elasticities		
	ρ	ρ_1	ρ_2	ρ_3	*AES*12	*AES*13	*AES*23
1	−.67	−.67	−.67	−.67	3.030	3.030	3.030
2	2.00	2.00	2.00	2.00	.333	.333	.333
3	−.40	−.67	−.25	−.25	1.775	1.775	.780
4	−.55	−.30	−.50	−.70	1.079	1.790	2.510
5	2.00	.20	2.00	3.00	.840	.630	.252
6	.40	.67	.25	.25	.687	.687	.919
7	.55	.30	.50	.70	.793	.700	.607
8	−1.10	−1.50	−.50	−.30	5.109	3.649	−3.649
9	−.80	−1.10	−.50	−.40	4.383	3.652	−.730

Note: AES_{ij} are evaluated at the means of the data.

(5)
$$\ln(\boldsymbol{x}) \sim N\left[\begin{pmatrix} .239 \\ .582 \\ .391 \end{pmatrix}, \begin{pmatrix} .026 & .000 & .000 \\ .000 & .066 & .000 \\ .000 & .000 & .065 \end{pmatrix}\right].$$

Data cannot take on negative values.[6] Since the covariance matrix of $\ln(\boldsymbol{x})$ has zero off-diagonal elements, collinearity is minimized. Given all the squared and interactive terms in the higher-order forms, fairly severe collinearity ultimately does arise. This does not measurably affect results.

Two distinct data sets are created from the raw data. For the first data set, to be used when estimating quadratic flexible forms, $x^*_{it} = x_{it} - \bar{x}_i$; for the second data set, to be used when estimating translog flexible forms, $\ln(x^*_{it}) = \ln(x_{it}) - \overline{\ln(x_i)}$. By centering the data about (0, 0, 0), TSFF coefficient estimates may be compared directly to Taylor series coefficient estimates evaluated at the mean of the transformed data without using (4). Next, $Y_t = f(\boldsymbol{x}_t)$ is calculated from the raw data using the generalized CES function,

(6)
$$Y_t = \left(\sum_i \delta_i x_{it}^{-\rho_i}\right)^{-1/\rho}.$$

This is an attractive function since the function derivatives are undefined for $\boldsymbol{x}_t \notin R^3_+$ or, when $\rho_i > 0$, $\rho > 0$, for $x_{it} = 0$. In this respect, the function exhibits properties identical to those of direct and indirect functions. Last, to permit the use of an ITSUR estimator, all first partial derivatives and first partial log derivatives are computed at all sample points and added to the data set.

For both experiments, nine different functions are created from (6). In table 1, parameter values are given for each of the nine functions as well as Allen partials evaluated at the mean of the data. The nine functions cover a wide range of possibilities.

Results of First Experiment

In the first experiment, all data lie in regions of convergence of the Taylor series expansions in powers of x and powers of $\ln(x)$. Given the functional form in (6), the fact that data are restricted to R^3_+ guarantees that the region of convergence of a Taylor series in powers of $\ln(x)$ about the mean of the data is all of R^3_+. Using the generated data, coefficient estimates are obtained for second-, third-, fourth-, and fifth-order quadratic and translog forms.

Due to space limitations, only the results for function 5 using ITSUR estimators are presented in the top half of table 2. The results using OLS estimators are similar.[7] In the tables, first- and second-order Taylor series coefficients evaluated at the mean of the data are recorded along with the standard deviation of these coefficients over the sample.[8] First

Table 2. Function 5, Taylor Series Coefficient Bias Associated with Flexible Form Coefficient Estimates at Data Means Using ITSUR

	Taylor Series Coefficients								
	*D*1	*D*2	*D*3	*D*11	*D*12	*D*13	*D*22	*D*23	*D*33
	First Experiment[a]								
Quadratic:									
Taylor Coef.	.05978	.13584	.36339	−.04515	.01711	.04577	−.18092	.10402	−.70559
Std. Dev.	.02525	.07369	.29825	.03449	.01301	.02395	.11758	.07175	.43396
Bias-order 2	−.00303	−.00464	.07760	−.00981	−.02839	.02165	.04564	−.03223	.14252
Bias-order 3	.00019	−.00131	.02359	−.00250	.00118	.00659	−.00928	.00593	−.05709
Bias-order 4	−.00230	−.00087	.00538	−.00321	−.00292	.00389	−.00228	.00410	−.00842
Bias-order 5	−.00014	−.00136	.00279	−.00206	−.00339	.00313	.00446	.00177	−.02188
Translog:									
Taylor Coef.	.05545	.17570	.40461	−.00494	.01948	.04487	−.28966	.14218	−.88642
Std. Dev.	.01246	.06272	.23919	.00037	.00856	.01281	.07924	.05328	.24093
Bias-order 2	−.00189	−.00042	.02898	.00014	−.00175	.00053	.01688	.00100	−.00945
Bias-order 3	−.00028	.00045	.00358	.00018	−.00012	−.00251	.00654	−.00802	.04974
Bias-order 4	−.00022	−.00044	.00392	.00017	−.00020	−.00227	.00357	−.00234	.03762
Bias-order 5	−.00002	−.00014	.00039	.00022	−.00027	−.00009	.00164	−.00097	.00122
	Second Experiment[b]								
Quadratic:									
Taylor Coef.	.09163	.20821	.00195	−.06541	.03486	.00032	−.25768	.00074	−.00128
Std. Dev.	.02819	.09733	.32350	.03784	.01801	.02897	.14294	.06903	.49854
Bias-order 2	−.10513	−.53846	−.11067	.25315	.27315	.00428	.12247	.00686	.01341
Bias-order 3	−.06190	−.41207	−.05564	.29256	.05364	−.00762	−.48772	−.04784	.02365
Bias-order 4	−.08002	−.33174	.18497	.26782	−.12806	−.00987	.03232	−.05296	.02942
Bias-order 5	−.11556	−.68905	.37402	.62552	−.05510	−.03363	.59130	−.10657	.11519
Translog:									
Taylor Coef.	.07106	.22516	.09630	−.00411	.03200	.01368	−.34893	.04336	−.27036
Std. Dev.	.01352	.07699	.27665	.00049	.01144	.02027	.08892	.06663	.38976
Bias-order 2	−.00900	−.02060	.16595	.00075	−.00557	−.00293	.06065	−.00744	.05514
Bias-order 3	.00263	.01102	−.05597	−.00006	−.00871	.01406	.05580	.04693	−.28129
Bias-order 4	.00053	.00223	−.00934	−.00093	.00029	.00469	−.00167	.01284	−.08367
Bias-order 5	After 2,000 iterations, the ITSUR estimator had not converged.								

Note: First and second partial derivatives evaluated at data means.
[a] In the first experiment, all data lie inside region of convergence of the quadratic.
[b] In the second experiment, some data lie outside region of convergence of the quadratic.

partial derivatives $\partial y/\partial x_1$, $\partial y/\partial x_2$, and $\partial y/\partial x_3$ are denoted $D1$, $D2$, and $D3$. Second own-partials, $\partial^2 y/\partial x_1^2$, and cross-partials, $\partial^2 y/\partial x_1 \partial x_2$, are denoted $D11$, $D12$, etc. For the quadratic, derivatives are $\partial y/\partial x_i$, etc., and for the translog, the derivatives are $\partial \ln(y)/\partial \ln(x_i)$. The deviation from TS coefficients associated with (nth-order) flexible form coefficient estimates of first- and second-order Taylor series coefficients at the mean of the data are then listed across from the rows labeled Bias-order 2, Bias-order 3, etc. For all functions in the experiment, the TS coefficient bias associated with quadratic and translog coefficient estimates approaches zero as higher-order terms are added to the model. For most functions, estimates obtained from the second-order models are not biased very much. Coefficient bias associated with low-order approximations can be considered omitted variable bias.

The results of the first experiment suggest that for some data sets, estimated TSFFs may be interpreted as TS approximations. Coefficients of second-order approximations may exhibit some omitted variable bias but, for the functions examined here, this bias is relatively small (compared to the bias that results when some data lie outside the region of convergence) and may be reduced by adopting a higher-order form. This guarantees that estimates of gradient and hessian terms also converge to gradient and hessian values of the underlying function everywhere in the region of convergence. This follows from the derivative property of Taylor series approximations.

Results of Second Experiment

In the second experiment, x_3 is changed to 25 in each of the last 20 observations. The value 25 is about eight times $\max\{x_{3t}, t = 1, 180\}$ for the first experiment. The data are then centered as in the first experiment. The large outliers are chosen (*a*) to illustrate the consequences of including data in the sample that lie outside the region of convergence of the quadratic, and (*b*) to illustrate the large region over which the translog converges.

The results of the second experiment, function 5, appear in the bottom half of table 2. The translog coefficient estimates converge to TS coefficients evaluated at the mean of the data for function 5 and every other function. For all functions examined, quadratic flexible form coefficient estimates diverge from the TS coefficients as higher-order terms are added to the model. For many functions, the "best" estimates among the quadratic models are obtained from the second-order quadratic, but even these estimates can be 100% or more off the mark.[9] Since TSFF coefficients will be interpreted as derivatives of the function at the mean of the data, elasticity estimates at the mean of the data (and other points as well) will be biased.

Implications for Empirical Work

A Consumption Model/Profit Model Example. One additional experiment is conducted to mimic plausible modeling situations. Data are generated as in the first experiment, except that 10 is added to x_{3i}, $i = 141, 200$. Values of x_3 for the last 60 observations are less than three times the mean of x_3 and about four times the largest value that x_3 takes on in the first 140 observations.

This data set has characteristics similar to a household consumption data set where the distribution of individual incomes (or group expenditures) are skewed to the right and 30% of individual incomes take on values about three times the mean. The data set also has characteristics similar to a set of price data that might be used to estimate a profit function. For instance, let x_3 be the normalized price of energy, which took a fourfold leap in the mid-1970s. If an annual data set included the years 1960–82, the properties of the data would be similar to those of the current example.

For this final experiment, data are generated using function 2, a CES function. Quadratic and translog models are estimated using the ITSUR estimator. The results in table 3 illustrate the danger of employing a quadratic model when one or more regressors have a distribution that is skewed right. The coefficient bias associated with the second-order

Table 3. Function 2, Flexible Form Coefficient Bias at Data Means Using ITSUR

	Taylor Series Coefficients								
	*D*1	*D*2	*D*3	*D*11	*D*12	*D*13	*D*22	*D*23	*D*33
	Third Experiment								
Quadratic:									
Taylor Coef.	.39597	.39044	.14389	−.27199	.10460	.03855	−.21274	.03801	−.04062
Std. Dev.	.18987	.18743	.25455	.19606	.07144	.05476	.11681	.04741	.21306
Bias-order 2	−.46043	−.46508	−.21796	.11730	−.13124	.02817	.20656	.01109	.09600
Bias-order 3	−.29366	−.28426	−.07309	.14858	.01712	−.03250	.15971	.01458	.12083
Bias-order 4	−.20088	−.75943	−.45400	1.29098	−.34453	−.09186	1.40120	.00015	.09889
Bias-order 5	−.18799	−.19190	−.08731	1.07121	−.70716	−.06411	.88897	.05178	.16115
Translog:									
Taylor Coef.	.21548	.28855	.36594	−.03910	.03419	.04337	−.09848	.05807	−.18250
Std. Dev.	.03799	.05729	.16208	.00231	.01251	.01282	.01003	.01649	.05215
Bias-order 2	−.00348	−.00345	.01295	.00167	−.00023	−.00357	.00358	−.00449	.01461
Bias-order 3	−.00007	−.00009	.00029	.00105	−.00132	−.00061	.00273	−.00074	.00246
Bias-order 4	−.00041	.00017	.00072	.00101	−.00092	−.00106	.00074	.00110	.00092
Bias-order 5	.00002	.00002	−.00004	.00124	−.00045	.00044	.00017	−.00011	.00015

Notes: First and second partial derivatives evaluated at data means. In this experiment, some data lie outside region of convergence of the quadratic.

quadratic exceeds 100% of the coefficient values $D1$, $D2$, $D3$, $D12$, and $D33$. The bias generally increases for higher-order quadratics. The most severe bias associated with second-order translog coefficients is the 7% bias of $D23$. The coefficient bias gradually disappears as higher-order terms are added.

Estimating Production Functions Using Field Trial Data. If the production function is to be approximated with a quadratic form, then the sample design should prohibit the use of treatments that grow exponentially or are drawn from distributions that are skewed right (such as the exponential or log normal) unless the mean is sufficiently large. Conversely, if the sample design includes treatments drawn from a skewed distribution, then the production function should be estimated using a translog functional form. Alternatively, several different quadratic models may be estimated using data sets that have been partitioned so that within each set, all data lie in a common region of convergence.

When estimating production functions using field trial data, an OLS estimator typically is employed. When sample data lie outside the region of convergence, bias typically is much more severe when estimating single equations via OLS than systems of equations using ITSUR estimators.

Degrees of Freedom Limitations and the Order of Approximation. For the class of functions defined only on R^n_+, if the order of expansion is permitted to grow with sample size, two asymptotic properties of the translog follow. First, if elasticity functions are continuous at any point in R^n_+, consistent estimates of elasticities can be obtained from the translog at this point. Since elasticities are functions of the derivative estimates, and these estimates converge to their true values, elasticity estimates converge to their true values. Second, asymptotically size α tests can be achieved since the translog converges to the underlying function everywhere, i.e., no functional misspecification arises.

When sample size is small and degrees of freedom are insufficient to use more than a second-order approximation, a marked preference still is found for the translog in situations where one or more explanatory variables are drawn from a distribution that is skewed right. At the bottom of table 2, note how large the TS coefficient bias (in absolute and percentage terms) associated with second-order quadratic coefficient estimates is relative to the TS coefficient bias associated with second-order translog coefficients.

Summary and Conclusion

Taylor series approximations have the desirable property that the error associated with approximations of both function value and derivatives is bounded in some neighborhood of the point of approximation. Under appropriate conditions, TSFFs may be interpreted as TS approximations in a neighborhood called the region of convergence. For TS approximations in a variable u (a function of x), the size of the neighborhood is governed by domain of analyticity of the function being approximated, the point of expansion, and whether $u = x$, $u = \ln(x)$, $u = x^{1/2}$, etc. For production, cost, profit, utility, and indirect utility functions, the domain of analyticity is limited to R^n_+, and this places limitations on the region of convergence.

When a TSFF is specified to represent a direct or indirect function, and estimated by OLS or ITSUR, it is established that for all t observations and for all explanatory variables x_i, data points must lie in the cube $\{f_i(0) < f_i(x_{it}) < 2\,\overline{f_i(x_i)} - f_i(0)\}$, where, for instance, $f_i(x_i) = \ln(x_i)$ in the case of the translog. If this condition is not satisfied, the estimated TSFF does not have a TS interpretation. Among popular TSFFs, the translog is preferred for approximating production, profit, cost, and utility functions. Its coefficients can be estimated without concern over data dispersion, and the estimates of function value, gradient, and hessian terms can be evaluated anywhere in R^n_+. Although this study does not deal explicitly with the Rotterdam or AIDS models, the regions of convergence associated with these models will be infinite as well, since both models employ logged arguments.

Estimated quadratic flexible forms may not be interpreted as TS approximations when some sample data lie outside the region of convergence. This implies that first and second derivatives of the estimated quadratic will not provide good approximations to gradient and hessian terms of the underlying function. Although experiments did not include the generalized Leontief, its region of convergence may be small and therefore will be subject to the same convergence problems as the quadratic.

The results have implications for empirical work. Whenever explanatory variables have a distribution that is skewed to the right (e.g., income in indirect utility functions; prices in profit or cost functions), a quadratic flexible form may be inappropriate. When designing treatments for field trials, the sample design must not include treatments that are outside the region of convergence of the approximating function. One way to forego such restrictions on treatment design is to employ the translog as an approximating function.

If a quadratic or Leontief flexible form must be employed, all data should fall within the region of convergence. When observations fall outside the region of convergence, the data must be partitioned and two or more distinct forms estimated. A distinct region of convergence applies to each estimated form, and these forms may not be used to estimate elasticities at any point outside the associated region of convergence.

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Notes

[1] An infinitely differentiable function, whose power series representation at a point $\boldsymbol{a}$ does not converge to f, is not analytic at that point. (See Protter and Morey, p. 234, for an example.)

[2] This follows from equations (1) and (2). It is assumed that the sufficient conditions for analyticity are met. If they are not, there is no purpose in discussing power series representations.

[3] As argued below, when estimating flexible forms, the point of expansion cannot be controlled, so the region of convergence cannot be made large by choosing an appropriate expansion point.

[4] A Taylor series expansion of $x\ln(x)$ about $\boldsymbol{a} = 1$ in powers of $x^{1/2}$ converges for $0 < x < 2$.

[5] A proof that this result holds (under appropriate conditions) for higher-order approximations and for both OLS and ITSUR estimators is available on request from the author.

[6] Once 200 observations are generated, data means are calculated, and all sample points are checked to ensure that they lie in the region of convergence.

[7] The results from all experiments are available from the author on request.

[8] Notice that for both Taylor series, these coefficients vary quite a bit over the sample, and the variance is similar.

[9] Bias associated with OLS estimators using single-equation models is far more dramatic.

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