Journal of Agricultural and Resource Economics 25(1):28–50

Modeling Technical Trade Barriers Under Uncertainty

Alan P. Ker

As traditional forms of agricultural protection continue to decline, agricultural interests will likely seek alternative protection in the form of technical barriers. A flexible framework for theoretically and empirically analyzing technical barriers under various sources of uncertainty is derived. Attention is focused on uncertainty arising from the variation in the product attribute levels, a source not yet considered by the literature. Ex ante and ex post densities of domestic and international quantities and prices as well as the densities of their respective extreme-order statistics are derived. An example is presented to illustrate the application of the developed framework.

Key words: nontariff barriers, trade modeling, uncertainty

Introduction

The competitiveness of a country's agricultural sector is influenced by its comparative advantage relative to its trading competitors. Although factor endowments and technological advantages influence comparative advantage, a country's competitive standing is also heavily influenced by trade policy. Currently, the policy environment under which many agricultural sectors operate is in a period of transition given the Uruguay Round Agreement (URA). As traditional forms of agricultural protection/support decline, agricultural interests will seek new forms. It seems reasonable to expect that nontariff barriers (NTBs), specifically technical barriers (TBs) such as sanitary and phytosanitary (SPS) measures, will become prominent. These measures are attractive because they can be both politically inexpensive for the domestic country and extremely difficult for a foreign country to prove illegal under World Trade Organization (WTO) agreements. Thus it is not surprising that in 1996 there existed more than 300 TBs in 63 foreign countries threatening $4.97 billion of U.S. exports (Roberts and DeRemer).

The primary objective of this article is to develop an analytical framework for evaluating, both theoretically and empirically, the effects of TBs under uncertainty. At first glance, analysis of the effects of trade intervention using standard trade models seems straightforward. Indeed, the empirical and theoretical literature contains an extensive evaluation and assessment of trade policies resulting from the URA. However, two important factors often overlooked in conventional analyses of agricultural trade policies are the role of TBs and uncertainty. As Sumner and Lee point out, "Incorporation of SPS

Alan P. Ker is assistant professor, Department of Agricultural and Resource Economics, University of Arizona, Tucson. The author would like to thank Barry Goodwin, Bruce Beattie, Rob Innes, and Jimmye Hillman for many helpful discussions and comments. Thanks also to two anonymous referees for their constructive comments. The financial support of the USDA-NRI grants program (No. 9801708) is gratefully acknowledged. Any remaining errors are the responsibility of the author.

and other technical barriers into empirical trade models remains a challenge" (p. 282). Although the academic literature dealing with TBs is growing (see, for example, Josling 1997; Sumner and Lee; Paarlberg and Lee; Calvin and Krissoff), it has largely ignored uncertainty despite demonstration by Young and Anderson, and by Pelcovits (among others) that uncertainty can significantly alter the effect of trade barriers. In light of the stochastic nature of agricultural production and possibly the effect of the TB itself, the influences of uncertainty are very important. Therefore, the development of a tractable framework for modeling the effects of TBs while taking into account various sources of uncertainty is particularly topical. The derived theoretical results and the illustrative example focus on uncertainty arising from the attribute levels, a source not yet considered in the literature.

Hooker and Caswell argue that trade agreements and trading blocs have adopted provisions in the URA on SPS regulations so that TBs are closely monitored to ensure gains from reduced tariffs are not eroded.[1] Although these provisions require TBs to be consistent with science, it is naive to believe that agricultural producers will not be successful, at least to some extent, in trading political support for protection. Recent history suggests protectionist agricultural interests will be successful in many countries.

First, consider Orden's exposition of rent-seeking behavior of U.S., Canadian, and Mexican agricultural interest groups and their successes as evidenced by the various concessions granted in NAFTA. Second, consider the following post-URA example from Beattie and Biggerstaff. In 1983, more than 10 years after the reported discovery of Karnal bunt in northwestern Mexico, the U.S. placed a quarantine on the import of all Mexican wheat and wheat seed. Subsequently, all major exporting countries placed a quarantine on Mexican wheat and wheat seed, increasing the number of countries with trade restrictions from four to 22. After the U.S. Department of Agriculture (USDA) quarantined the desert southwest in 1996, the number of countries banning the import of Karnal bunt contaminated wheat increased to more than 50. Surprisingly, Karnal bunt poses no known health risks to animals or humans.

The above example is not an anomaly, nor is it unique to U.S. agriculture. Trefler found when trade protection is modeled endogenously rather than exogenously as is commonly done, its restrictive impact on imports is roughly *ten times* as large. Obviously, rent-seekers have become very creative in developing alternatives to conventional trade barriers. With respect to the agricultural sector, TBs will likely comprise the majority of those alternate policies despite new constraints under various trade agreements (see Hillman 1990, 1996; Neary; Sumner; or Josling 1994). Thus, an analytically tractable framework to analyze TBs should become an important tool in considering future trade policy.

In contrast to standard approaches, the developed framework recognizes uncertainty at various levels and recovers the probability density functions associated with quantities and prices. To the best of the author's knowledge, this is the first such attempt at recovering the ex ante and ex post densities. By recovering the densities, many additional questions can be considered. For example, the effect of trade policy on higher moments is revealed. Similarly, estimates of the extreme import levels and their respective densities, particularly important for seasonal goods such as fruits and vegetables, are easily recovered. Welfare changes under a wider spectrum of scenarios,

[1] See the WTO "Agreement on the Application of Sanitary and Phytosanitary Measures" and "Agreement of Technical Barriers to Trade."

including deviations from risk neutrality, can be analyzed. The developed framework will permit the construction of a maximum Trade Restrictive Index (see Anderson, Bannister, and Neary; Anderson and Neary 1990, 1992). The ability to recover the maximum and minimum index levels, in addition to the expected index level, should be valuable to policy makers as future trade agreements are negotiated. Finally, not only does the developed framework lend itself to the derivation of theoretical results (see "Derived Results" section), it is straightforward to empirically implement (see "Illustrative Example" section).

A small perfectly competitive country, denoted the domestic country, imposing a TB is considered. The framework takes a partial equilibrium approach. Supply is stochastic in both the domestic country and the rest of the world (ROW). The innovation densities are assumed to be independent Gaussians with zero first moment and finite second moment. In rare circumstances, the Gaussian density may not be appropriate. However, given the size of the markets under consideration (national), Central Limit Theorems for either independent or spatially dependent processes suggest that the Gaussian density would be appropriate in most circumstances. Although supply is stochastic, demand is assumed to be deterministic given price. To keep the distributions tractable, both demand and supply are assumed to be linear in price. Note, it is sufficient to assume that the relationship is linear only over the space with positive probability measure under the TB environment and the free trade environment. If the demand and supply mappings are not approximately linear, then, depending on the nonlinearities, the relevant densities would need to be recovered numerically rather than analytically. Finally, the exchange rate is assumed to be deterministic.[2]

Structure of TBs

Many NTBs such as end-use certificates, customs regulations, and packaging and labeling requirements can be considered an implicit specific or ad valorem tariff. For example, labeling requirements in Canada require both English and French labeling. This is simply an additional marginal cost which may be considered an implicit specific tariff. These types of barriers are easily dealt with in the framework below as implicit ad valorem or specific tariffs. TBs such as SPS regulations present much greater modeling challenges and represent one of the major contributions of this article.

The nature of the TB is such that each unit i of the particular good of interest has an associated attribute level, denoted ω_i. That is, for unit j the attribute level is ω_j, and for unit k the attribute level is ω_k. If the attribute level is above a lower bound, say κ, then that unit is not importable. It is assumed that the sequence $\{\omega_1, \omega_2, \ldots, \omega_N\}$ is independent and identically distributed. Therefore, the probability that any unit has an attribute level below the SPS bound (κ) is ρ, where $\rho = \int_0^\kappa d\mathcal{P}$, and $\mathcal{P}$ is the probability measure associated with ω. Additionally, it is assumed that the number of units (N) is distributed Gaussian with mean λ and variance σ^2. Therefore, the quantity of importable product (denoted IP) is defined as $IP = \sum_{i=1}^{N} I_i$, where $I_i = 1$ if $\omega_i < \kappa$, and otherwise 0. Thus, the following hierarchical model results:

---

[2] An earlier version of the article allowed the exchange rate to be stochastic. The framework, albeit mildly more general, is significantly more complex as well as requiring additional assumptions to recover tractable densities.

(1)
$$IP|N \sim \text{Binomial}(N, \rho),$$
$$\text{where} \quad N \sim \text{Gaussian}(\lambda, \sigma^2).$$

Obviously, the marginal distribution of importable product (*IP*) is not recoverable given the above inconsistency. On one hand, it must be that $N \in I^+$ for the conditional distribution of *IP* given *N*. On the other hand, $N \in \Re$ since the marginal distribution for *N* is Gaussian. Although this represents an inconsistency, the first two moments of the marginal distribution of importable product can be recovered from the marginal and conditional moments: $E[IP] = E[E[IP|N]] = E[N\rho] = \lambda\rho$, and $\text{Var}[IP] = E[\text{Var}[IP|N]] + \text{Var}[E[IP|N]] = E[N\rho(1-\rho)] + \text{Var}[N\rho] = \lambda\rho(1-\rho) + \rho^2\sigma^2$. Assume that λ and σ^2 would represent the moments of ROW production, and that $\rho \in [0,1]$; the marginal density of *IP* may be approximated by a Gaussian with mean $\lambda\rho$ and variance $\lambda\rho(1-\rho)+\rho^2\sigma^2$.[3]

Note that this model structure for the TB is extremely flexible. First, the supply of importable product is explicitly derived from the bound on the SPS measure. Second, the supply of importable product considers two sources of uncertainty: total supply ($\rho^2\sigma^2$) and the attribute levels ($\lambda\rho(1-\rho)$). Note that under complete certainty with respect to supply, uncertainty remains because of the variability in the attribute levels ($\text{Var}[IP] = N\rho(1-\rho) \neq 0$). Third, the framework is not unduly restrictive in that one may consider TBs which result in a continuum of importable product levels, not simply the two extremes; importable product equals zero almost surely (a.s.) and importable product is greater than import demand a.s. Finally, although the hierarchical model appears to be unduly restrictive in that each unit must be sampled and tested for its associated attribute level, this is not the case. Rather than defining $\rho = \int_0^\kappa d\mathcal{P}$, one could define ρ as the integral over a product measure. The product measure, say π, would have two marginal probability measures: one depicting the attribute level, the other depicting the sampling scheme. Under this case, $\rho = 1 - \int_\Upsilon d\pi$, where Υ represents the set where both the attribute level is above the bound κ *and* the unit is sampled. Note, whether ρ is defined as the integral over a product measure (π) or over the initial probability measure ($\mathcal{P}$), the model remains unchanged given ρ.

The Small Perfectly Competitive Case

This section outlines the domestic and ROW market equilibriums under both the free trade environment and the TB environment. Recall, the domestic country is the small perfectly competitive importer.[4]

Free Trade Environment

The ROW market is illustrated as follows:

(2)
$$Q^s = \alpha + \beta P^e + \upsilon,$$
$$Q^d = \gamma + \delta P,$$
$$Q^s = Q^d,$$

[3] Alternatively, one could assume the marginal distribution for *N* is Poisson. In such a situation, the marginal distribution of importable product would also be Poisson. Although the Poisson can be approximated by a Gaussian, the mean and variance would necessarily be equal. It was felt that the inconsistency is justified given the relaxation of this constraint.

[4] The definition of a free trade environment allows a tariff rate. This can be set to zero if a tariff does not exist.

where Q^s represents ROW supply, P^e represents ROW market expected price, Q^d represents ROW demand, P represents ROW market clearing price, and υ represents an innovation following a Gaussian density with mean 0 and finite variance σ^2. Note, $\beta > 0$, $\delta < 0$, and $\gamma > \alpha$ ensures ROW production, an upward-sloping supply curve, and a downward-sloping demand curve. Note also that the ROW market-clearing condition does not consider exports to the domestic market. When the domestic market is small relative to the ROW market, it faces an infinitely elastic supply at the ROW price and thus cannot affect the ROW market. ROW producers target supply based on expected price. Given the model of the ROW market, $E[P] = P^e = (\alpha - \gamma)/(\delta - \beta)$. ROW producers target production based on P^e, and thus

(3) $$Q^s \sim N\left(\alpha + \beta\left(\frac{\alpha - \gamma}{\delta - \beta}\right), \sigma^2\right)$$

and

(4) $$P \sim N\left(\frac{\alpha - \gamma}{\delta - \beta}, \frac{\sigma^2}{\delta^2}\right).$$

The small perfectly competitive country situation is illustrated as follows:

(5) $$\begin{aligned} Q_d^s &= \alpha_d + \beta_d P_d^e + e, \\ Q_d^d &= \gamma_d + \delta_d P_d, \\ Q_d^s &= Q_d^d - I_s, \end{aligned}$$

where Q_d^s represents domestic supply, P_d^e represents the domestic market expected price, Q_d^d represents domestic demand, I_s represents import supply, P_d represents the domestic market-clearing price, and e represents an innovation density with mean 0 and variance σ_d^2. Note, $\beta_d > 0$, $\delta_d < 0$, and $\gamma_d > \alpha_d$ ensures domestic production, an upward-sloping supply curve, and a downward-sloping demand curve. Finally, we have the landed price equation:

(6) $$P_\ell = P \times ER(1 + \tau),$$

where P_ℓ is the landed price, ER is the exchange rate, and τ is the tariff should one exist.[5] For notational convenience, define $\mu_{e\tau} = ER(1 + \tau)$. Therefore, landed price is distributed as

(7) $$P_\ell \sim N\left(\left(\frac{\alpha - \gamma}{\delta - \beta}\right)\mu_{e\tau}, \mu_{e\tau}^2 \frac{\sigma^2}{\delta^2}\right).$$

Since the small country is a price taker, $P_d = P_\ell$, and thus

[5] There may exist a price premium on imported goods in light of the technical barrier. This would arise given the cost of testing the goods prior to exportation to the domestic country and/or given the probability that a fraction of those goods may not be allowed into the domestic country upon testing. It is sufficient to assume the tariff, τ, encapsulates the expected costs associated with those activities.

(8) $$P_d \sim N\left(\left(\frac{\alpha - \gamma}{\delta - \beta}\right)\mu_{e\tau},\ \mu_{e\tau}^2 \frac{\sigma^2}{\delta^2}\right).$$

Domestic producers take into account that imports are infinitely elastic at the realized ROW price. Therefore, they target production based on $P_d^e = P_\ell^e$, and hence

(9) $$Q_d^s \sim N\left(\alpha_d + \beta_d\left(\frac{\alpha - \gamma}{\delta - \beta}\right)\mu_{e\tau},\ \sigma_d^2\right).$$

One might anticipate that import supply, $I_s = Q_d^d - Q_d^s$, is a location transformation of the marginal density of Q_d^s because Q_d^d does not contain an innovation. However, this is not true since Q_d^d is stochastic because the landed price is stochastic. Therefore,

(10) $$I_s = (\gamma_d - \alpha_d) - \beta_d\left(\frac{\alpha - \gamma}{\delta - \beta}\right)\mu_{e\tau} + \delta_d P_\ell - e,$$

where $\delta_d P_\ell$ is independent of e. As a result,

(11) $$I_s \sim N\left((\gamma_d - \alpha_d) + \mu_{e\tau}\frac{(\alpha - \gamma)(\delta_d - \beta_d)}{\delta - \beta},\ \sigma_d^2 + \delta_d^2\mu_{e\tau}^2\frac{\sigma^2}{\delta^2}\right).$$

With respect to the domestic market, price is influenced by ROW variations and not by domestic variations. Conversely, supply is influenced by domestic variations and not by ROW variations. As a result, import supply must act as a compensating factor to achieve domestic equilibrium and is thus influenced by both domestic and ROW variations.

TB Environment

Intuitively, imports will equal import demand at the landed price unless there is insufficient importable product. Thus, $I_s = \min[IP, ID|_{P_d = P_\ell^{FT}}]$, where IP is the supply of importable product while $ID|_{P_d = P_\ell^{FT}}$ is the import demand at the landed price under free trade. Recall, $IP \sim N(\lambda\rho, \lambda\rho(1-\rho) + \rho^2\sigma^2) = N(\rho(\alpha + \beta P^e), (\alpha + \beta P^e)\rho(1-\rho) + \rho^2\sigma^2)$, and $ID|_{P_d = P_\ell^{FT}}$ may be derived as

(12) $$ID|_{P_d = P_\ell^{FT}} = \gamma_d + \delta_d P_\ell - \alpha_d - \beta_d P_d^e - e,$$

and thus

(13) $$ID|_{P_d = P_\ell^{FT}} \sim N\left((\gamma_d - \alpha_d) + \mu_{e\tau}\delta_d\left(\frac{\gamma - \alpha}{\delta - \beta}\right) - \beta_d P_d^e,\ \sigma_d^2 + \delta_d^2\mu_{e\tau}^2\frac{\sigma^2}{\delta^2}\right).$$

Note, IP and $ID|_{P_d = P_\ell^{FT}}$ are dependent (directly correlated) and jointly Gaussian. This makes intuitive sense because when ROW supply increases, the available importable product will increase but ROW price will decrease, causing greater domestic import demand. Therefore, IP and $ID|_{P_d = P_\ell^{FT}}$ are directly correlated. This makes the density of I_s less tractable, and so the derivation is placed in appendix A. The density is:

$$(14) \qquad f_{I_s}(i_s) = \frac{1}{\sigma_1}\phi(A_1(i_s)) \times \left[1 - \Phi\left(\frac{A_2(i_s) - \eta A_1(i_s)}{\sqrt{1-\eta^2}}\right)\right] + \frac{1}{\sigma_2}\phi(A_2(i_s)) \times \left[1 - \Phi\left(\frac{A_1(i_s) - \eta A_2(i_s)}{\sqrt{1-\eta^2}}\right)\right],$$

where

$$A_1(i_s) = \frac{i_s - \left((\gamma_d - \alpha_d) + \mu_{e\tau}\delta_d\left(\frac{\gamma - \alpha}{\delta - \beta}\right) - \beta_d P_d^e\right)}{\sigma_1}, \quad A_2(i_s) = \frac{i_s - \rho(\alpha + \beta P^e)}{\sigma_2},$$

$$\sigma_1 = \left(\sigma_d^2 + \frac{\delta_d^2\mu_{e\tau}^2\sigma^2}{\delta^2}\right)^{1/2}, \quad \sigma_2 = \left((\alpha + \beta P^e)\rho(1-\rho) + \rho^2\sigma^2\right)^{1/2},$$

and η is the correlation between IP and $ID|_{P_d = P_\ell^{FT}}$ with covariance $(\sigma^2\rho\delta_d\mu_{e\tau})/(\delta - \beta) \geq 0$. Recovery of the domestic price distribution is also not easily tractable. If $IP < ID|_{P_d = P_\ell^{FT}}$, then $P_d = P_d^*$, where P_d^* is the market-clearing domestic price given $IP < ID|_{P_d = P_\ell^{FT}}$. However, if $IP \geq ID|_{P_d = P_\ell^{FT}}$, then $P_d = P_\ell$. Intuitively, imports will increase up to the point where importable product is exhausted or the domestic price decreases to the ROW price at which point imports will no longer increase. Therefore, $P_d = \max(P_d^*, P_\ell)$, where P_d^* is recovered from the market-clearing condition $Q_d^d = Q_d^s + IP$. Therefore,

$$(15) \qquad P_d^* = \frac{\alpha_d - \gamma_d + \rho(\alpha + \beta P^e) + \beta_d P_d^e}{\delta_d} + \frac{e + \rho\upsilon}{\delta_d},$$

and thus

$$(16) \qquad P_d^* \sim N\left(\frac{\alpha_d - \gamma_d + \rho(\alpha + \beta P^e) + \beta_d P_d^e}{\delta_d}, \frac{\sigma_d^2 + (\alpha + \beta P^e)\rho(1-\rho) + \rho^2\sigma^2}{\delta_d^2}\right).$$

Note that P_d^* and P_ℓ are also directly correlated and jointly Gaussian. Again, this makes intuitive sense because when ROW supply increases, the ROW price decreases while the importable product increases. Increases in importable product cause the market-clearing domestic price (given $IP < ID|_{P_d = P_\ell^{FT}}$) to decrease. Following the same derivation as with I_s, the density of P_d is recovered:

$$(17) \qquad f_{P_d}(p_d) = \frac{1}{\sigma_1}\phi(A_1(p_d)) \times \left[\Phi\left(\frac{A_2(p_d) - \eta A_1(p_d)}{\sqrt{1-\eta^2}}\right)\right] + \frac{1}{\sigma_2}\phi(A_2(p_d)) \times \left[\Phi\left(\frac{A_1(p_d) - \eta A_2(p_d)}{\sqrt{1-\eta^2}}\right)\right],$$

where

$$A_1(p_d) = \frac{p_d - \left(\dfrac{\alpha_d - \gamma_d + \rho(\alpha + \beta P^e) + \beta_d P_d^e}{\delta_d}\right)}{\sigma_1}, \quad A_2(p_d) = \frac{p_d - \left(\dfrac{\alpha - \gamma}{\delta - \beta}\right)\mu_{e\tau}}{\sigma_2},$$

$$\sigma_1 = \left(\frac{\sigma_d^2 + (\alpha + \beta P^e)\rho(1 - \rho) + \rho^2\sigma^2}{\delta_d^2}\right)^{1/2}, \quad \sigma_2 = \left(\frac{\mu_{e\tau}^2\sigma^2}{\delta^2}\right)^{1/2},$$

and η is the correlation between P_d^* and P_ℓ with covariance $(\sigma^2\rho\mu_{e\tau})/(\delta_d(\delta - \beta)) \geq 0$. Since we consider the maximum of a bivariate Gaussian in the above situation, the cumulative probability rather than one minus the cumulative probability materializes.

In the above derivation, $f_{P_d}(p_d)$ depends on an unknown quantity P_d^e through $A_1(p_d)$. A simple one-dimensional grid search can be employed to find P_d^e. By lemma 1 in appendix B, it must be the case that $P_d^e > P_\ell^e$, and thus P_ℓ^e is an ideal starting point for a search in the upward direction. Given a particular search value, say $\tilde{P}_d^e$, an estimate for P_d^e, say $\hat{P}_d^e$, is easily recovered from the moment equation in appendix B. P_d^e is found, where $\hat{P}_d^e = \tilde{P}_d^e$. Given P_d^e, $f_{P_d}(p_d)$ and $f_{I_s}(i_s)$ are recoverable, and thus the domestic supply distribution is

$$Q_d^s \sim N\left(\alpha_d + \beta_d P_d^e,\ \sigma_d^2\right). \tag{18}$$

Both domestic price and quantity supplied are bounded from below by the free trade environment. Therefore, the marginal distributions of domestic price and quantity under the free trade (FT) environment dominate, pointwise, their respective marginal distributions under the TB environment. That is,

$$F_{P_d}^{FT}(p_d) \geq F_{P_d}^{TB}(p_d)\ \forall\, p_d \in \Re, \text{ and } F_{Q_d^s}^{FT}(q_d^s) \geq F_{Q_d^s}^{TB}(q_d^s)\ \forall\, q_d^s \in \Re. \tag{19}$$

Similarly, $E[P_d^{FT}] \leq E[P_d^{TB}]$, and $E[Q_d^s]^{FT} \leq E[Q_d^s]^{TB}$.

Previously, it was mentioned that the framework allowed a continuum of importable product levels, not simply the two extremes: importable product equals zero a.s., and importable product is greater than import demand a.s. However, for the small country, the two extreme cases are very relevant. The latter situation is equivalent to the free trade environment since the TB is of no effect. The former situation results in a closed economy which is considered below as a special case.

SPECIAL CASE: Closed Economy (CE), $IP = 0$ a.s.

The case assumes that $IP = 0$ a.s., or $\rho = \int_0^\kappa d\mathcal{P} = 0$, and hence $I_s = 0$ a.s. As a result, the domestic market is closed from the ROW market and achieves equilibrium at $Q_d^s = Q_d^d$. Therefore, supply is targeted based on $E[P_d] = P_d^e = (\alpha_d - \delta_d)/(\delta_d - \beta_d)$, and hence

$$Q_d^s \sim N\left(\alpha_d + \beta_d\left(\frac{\alpha_d - \gamma_d}{\delta_d - \beta_d}\right),\ \sigma_d^2\right). \tag{20}$$

Therefore, the marginal density of domestic price is

(21) $$P_d \sim N\left(\frac{\alpha_d - \gamma_d}{\delta_d - \beta_d}, \frac{\sigma_d^2}{\delta_d^2}\right).$$

Given that the domestic country is an importer, it must be the case that domestic price, and thus domestic supply, increases. That is,

(22) $$\left(\frac{\alpha - \gamma}{\delta - \beta}\right)\mu_{e\tau} < \left(\frac{\alpha_d - \gamma_d}{\delta_d - \beta_d}\right),$$

or $E[P_d^{FT}] < E[P_d^{CE}]$ and $E[Q_d^s]^{FT} < E[Q_d^s]^{CE}$. The marginal distributions of domestic price and quantity under the free trade environment dominate, pointwise, their respective marginal distributions under the TB environment, which in turn dominate this closed economy (CE) case. That is,

(23) $$F_{P_d}^{FT}(p_d) \geq F_{P_d}^{TB}(p_d) \geq F_{P_d}^{CE}(p_d)\ \forall\, p_d \in \Re, \text{ and}$$
$$F_{Q_d^s}^{FT}(q_d^s) \geq F_{Q_d^s}^{TB}(q_d^s) \geq F_{Q_d^s}^{CE}(q_d^s)\ \forall\, q_d^s \in \Re.$$

Derived Results

This section derives some theoretical results based on the developed model. The first three results are of a general nature, while the final three focus on the uncertainty arising from the variation in the product attribute levels, a source not yet considered by the literature. To prove these results, four lemmas are required (stated below) whose proofs are located in appendix B given their surprisingly technical nature. Also somewhat surprising is the nonexistence of these proofs in the probability and mathematical statistics literature. It is necessary to introduce some notation. Let X and Y be two real-valued random variables which are bivariate Gaussian, and define $R = \max[X, Y]$ and $Z = \min[X, Y]$. The following four lemmas are proved:

1. $E[R] \geq \max[\mu_x, \mu_y]$, $E[Z] \leq \min[\mu_x, \mu_y]$;
2. $\frac{\partial E[R]}{\partial \mu_x} > 0$, $\frac{\partial E[R]}{\partial \mu_y} > 0$, $\frac{\partial E[Z]}{\partial \mu_x} > 0$, $\frac{\partial E[Z]}{\partial \mu_y} > 0$;
3. $\frac{\partial E[R]}{\partial \sigma_x^2} > 0$, $\frac{\partial E[R]}{\partial \sigma_y^2} > 0$, $\frac{\partial E[Z]}{\partial \sigma_x^2} < 0$, $\frac{\partial E[Z]}{\partial \sigma_y^2} < 0$; and
4. $\frac{\partial \mathrm{Var}[R]}{\partial \sigma_x^2} [>, =, <]\, 0$, $\frac{\partial \mathrm{Var}[R]}{\partial \sigma_y^2} [>, =, <]\, 0$, $\frac{\partial \mathrm{Var}[Z]}{\partial \sigma_x^2} [>, =, <]\, 0$, $\frac{\partial \mathrm{Var}[Z]}{\partial \sigma_y^2} [>, =, <]\, 0$.

■ RESULT 1. *When supply uncertainty is ignored, expected producer surplus may be underestimated while expected consumer surplus may be overestimated.*

When uncertainty is ignored, it is common to assume which state of nature is realized: importable product is sufficient to satisfy import demand at landed ROW prices a.s.; or, importable product is insufficient to satisfy import demand at landed ROW prices a.s. In the former situation, $P_d^e = P_\ell^e$, whereas in the latter situation, $P_d^e = P_d^{*e}$. However, by lemma 1, if $P_d = \max[P_d^*, P_\ell] \rightarrow P_d^e \geq \max[P_d^{*e}, P_\ell^e]$. As a result, expected producer surplus may be underestimated while expected consumer surplus may be overestimated if $P_d^e = P_d^{*e}$ or $P_d^e = P_\ell^e$ is assumed, because supply uncertainty is neglected.

- RESULT 2. *If the mean of importable product (IP) or import demand (given domestic price equals the free trade price,* $ID|_{P_d=P_\ell^{FT}}$*) decreases (increases), then the mean of* I_s *decreases (increases).*

Recall, under the TB environment, $I_s = \min[IP, ID|_{P_d=P_\ell^{FT}}]$. Then, by lemma 2,

$$\frac{\partial E[I_s]}{\partial E[ID|_{P_d=P_\ell^{FT}}]} > 0 \rightarrow E[I_s] \downarrow \text{ when } E[ID|_{P_d=P_\ell^{FT}}] \downarrow.$$

Similarly,

$$\frac{\partial E[I_s]}{\partial E[IP]} > 0 \rightarrow E[I_s] \downarrow \text{ when } E[IP] \downarrow,$$

again by lemma 2. This lemma states a rather intuitive and obvious result that if the mean of importable product decreases or the import demand at ROW prices decreases, the mean of imports will likewise decrease.

- RESULT 3. *If the mean of market-clearing domestic price given* $IP < ID|_{P_d=P_\ell^{FT}}$ (P_d^*) *or the landed price* (P_ℓ) *decreases (increases), the mean of* P_d *decreases (increases).*

Recall, under the TB environment, $P_d = \max(P_d^*, P_\ell)$. Then, by lemma 2,

$$\frac{\partial E[P_d]}{\partial E[P_\ell]} > 0 \rightarrow E[P_d] \downarrow \text{ when } E[P_\ell] \downarrow.$$

Similarly,

$$\frac{\partial E[P_d]}{\partial E[P_d^*]} > 0 \rightarrow E[P_d] \downarrow \text{ when } E[P_d^*] \downarrow,$$

again by lemma 2. This lemma states a second obvious result that if the mean of landed price decreases or the mean of the domestic price (given importable product exceeds import demand) decreases, then the mean of the domestic price likewise decreases.

- RESULT 4. *If uncertainty resulting from the variability in the attribute levels* $\{\omega_1, \omega_2, \ldots, \omega_N\}$ *is neglected, the expected level of imports is biased upward, whereas the variance is not necessarily biased downward.*

Recall, under the TB environment, $I_s = \min[IP, ID|_{P_d=P_\ell^{FT}}]$, where $IP \sim N(\rho(\alpha + \beta P^e), (\alpha + \beta P^e)\rho(1-\rho) + \rho^2\sigma^2)$. Also recall that $(\alpha + \beta P^e)\rho(1-\rho)$, which is strictly positive, is the variability in IP arising from the variability in the attribute levels. By lemma 3,

$$\frac{\partial E[I_s]}{\partial \text{Var}[IP]} < 0 \rightarrow E[I_s] \downarrow \text{ when Var}[IP] \uparrow.$$

Therefore, if the uncertainty arising from the attribute levels is neglected, then expected imports will be overestimated. By lemma 4,

$$\frac{\partial \text{Var}[I_s]}{\partial \text{Var}[IP]} \; [<, =, >] \; 0,$$

which implies the somewhat surprising result that the variance of the imports does not necessarily increase when the variance of importable product increases. Thus, if uncertainty arising from the attribute levels is neglected, then the variance of imports will not necessarily be biased downward.

■ RESULT 5. *If uncertainty resulting from the variability in the attribute levels* $\{\omega_1, \omega_2, \ldots, \omega_N\}$ *is neglected, the expected domestic price is biased downward, whereas the variance is not necessarily biased downward.*

Recall, under the TB environment, $P_d = \max(P_d^*, P_\ell)$, where

$$P_d^* \sim N\left(\frac{\alpha_d - \gamma_d + \rho(\alpha + \beta P^e) + \beta_d P_d^e}{\delta_d}, \; \frac{\sigma_d^2 + (\alpha + \beta P^e)\rho(1 - \rho) + \rho^2\sigma^2}{\delta_d^2}\right).$$

Note that $(\alpha + \beta P^e)\rho(1 - \rho)$, which is strictly positive, is the variability in P_d^* arising from the variability in the attribute levels. By lemma 3,

$$\frac{\partial E[P_d]}{\partial \text{Var}[P_d^*]} > 0 \rightarrow E[P_d] \uparrow \text{ when } \text{Var}[P_d^*] \uparrow .$$

Therefore, if the uncertainty arising from the attribute levels is neglected, then expected domestic price will be underestimated. By lemma 4,

$$\frac{\partial \text{Var}[P_d]}{\partial \text{Var}[P_d^*]} \; [<, =, >] \; 0,$$

which implies the somewhat surprising result that the variance of the domestic price does not necessarily increase when the variance of domestic price (given $IP < ID\,|_{P_d = P_\ell^{FT}}$) increases. Thus, if uncertainty arising from the attribute levels is neglected, then the variance of domestic price will not necessarily be biased downward.

■ RESULT 6. *When uncertainty resulting from the variability in the attribute levels* $\{\omega_1, \omega_2, \ldots, \omega_N\}$ *is ignored, expected producer surplus will be underestimated while expected consumer surplus will be overestimated.*

This follows directly from result 5. When uncertainty arising from the attribute levels is neglected, P_d^e will be biased downward (lemma 3). Consequently, expected producer surplus will be underestimated while expected consumer surplus will be overestimated.

Illustrative Example

The purpose of this section is to illustrate how the derived distributions are empirically recovered. In doing so, results 4 and 5 from the previous section are illustrated as well as the exact densities for the extreme-order statistics. Such densities, unrecoverable without the underlying densities, are of particular importance to the domestic producers in, for example, a highly volatile industry like the fruit and vegetable sector.

The arbitrarily assumed ROW demand and supply equations are, respectively,

$$Q^s = 96 + 0.48P^e + \upsilon, \quad Q^d = 156 - 0.72P, \tag{24}$$

while the respective arbitrarily assumed domestic demand and supply equations are

$$Q_d^s = 18 + 0.12P_d^e + e, \quad Q_d^d = 48 - 0.18P_d. \tag{25}$$

Note that the chosen parameters ensure both positive domestic and ROW production, upward-sloping supply curves, and downward-sloping demand curves. These parameters may be recovered from domestic and ROW demand and supply elasticities. Additionally, it was assumed that $\sigma^2 = 40$, $\sigma_d^2 = 4$, $\mu_{e\tau} = 1$, and $\rho = 0.1$. These parameters may be recovered from domestic and ROW production data, exchange rate data, and information on the technical barrier. Given the above parameters, the import densities (figure 1) and the domestic price densities (figure 3) are recovered using the expressions for their respective densities.[6] For comparison purposes, the densities are recovered under three environments: the free trade environment, the TB environment, and the TB environment neglecting the uncertainty resulting from the variability of the attribute levels.

The import densities for the three environments are located in figure 1. Consistent with lemma 1 and result 1, the introduction of the TB reduces the mean level of imports. Consistent with lemma 3 and result 4, neglecting the uncertainty from the attribute levels increases the mean of imports, but only negligibly. However, the effect of neglecting the uncertainty in the attribute levels is most pronounced in the variance of imports. When the uncertainty with respect to the attribute levels is neglected, the variance of the import density is dramatically reduced. This result arises because the $\mathrm{Var}(IP) = ((\alpha + \beta P^e)\rho(1 - \rho) + \rho^2\sigma^2)$, where $(\alpha + \beta P^e)\rho(1 - \rho)$ is a relatively large fraction *and* $E[IP] \ll E[ID|_{P_d = P_\ell^{FT}}]$ in the example. Not surprisingly, the increased variance has a pronounced effect on the densities of the extreme-order statistics. Figure 2 illustrates the densities for the maximum imports given $N = 10$ independent draws from the import densities. For $N = 10$, the density is defined as

$$f_{I_{s10}}(i_s) = 10 f_{I_s}(i_s) \times F_{I_s}(i_s)^9, \tag{26}$$

[6] For this case, the densities do not appear to differ significantly from the Gaussian. However, without the derivation of the densities for the minimum or maximum and their subsequent moment-generating functions, the true mean and variance would not have been recoverable.

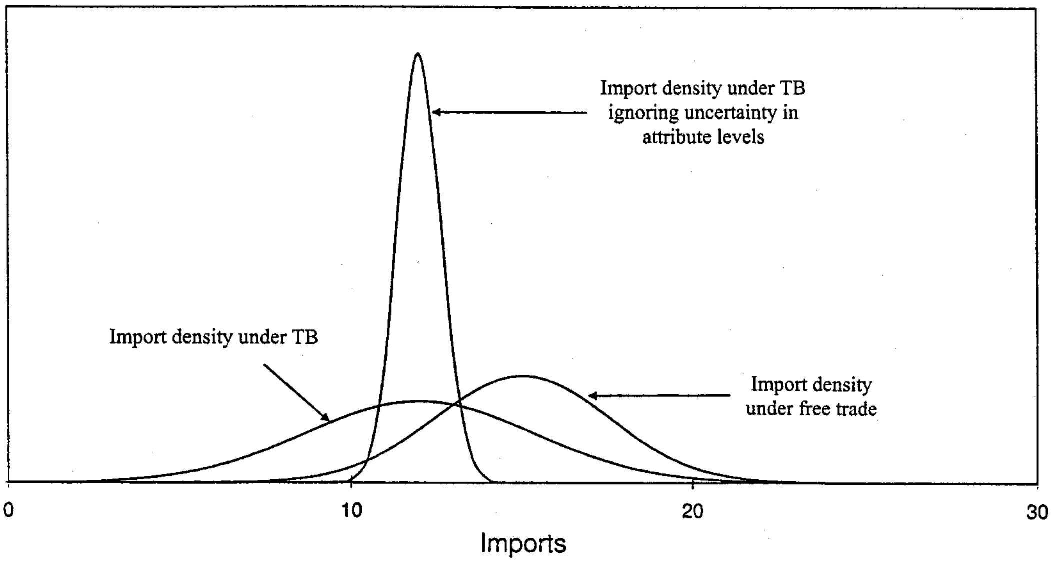

Figure 1. Illustrative example: Import densities

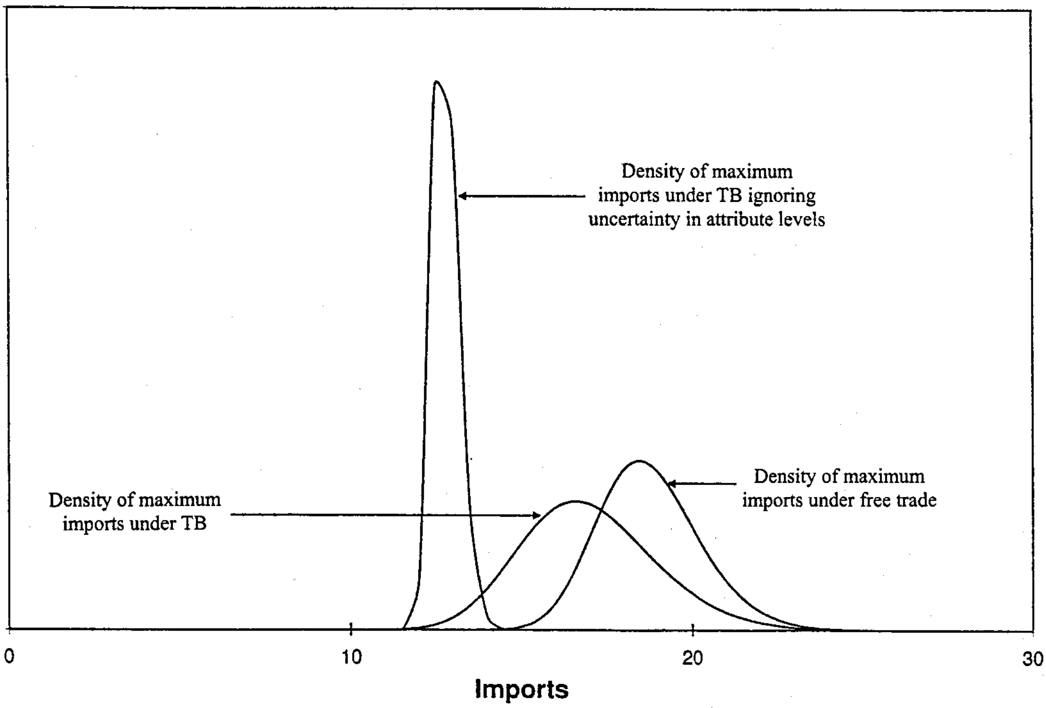

Figure 2. Illustrative example: Densities of maximum imports (N = 10)

where $f(\cdot)$ and $F(\cdot)$ are the density and distribution functions, respectively. Not surprisingly, when the TB is introduced, the mean of the maximum imports decreases. Most notably, though, when the uncertainty in the attribute levels is neglected, the density of the maximum imports is very poorly estimated; both the mean and variance are severely biased downward.

The domestic price densities for the three environments are located in figure 3. Consistent with lemma 1, the introduction of the TB increases the mean domestic price. Consistent with lemma 2 and result 5, neglecting uncertainty from the attribute levels decreases the mean of domestic prices, but only negligibly. Unlike the import density, the effect of neglecting uncertainty in the attribute levels is not very significant in the variance of the domestic price density. This result is also not surprising since $(\alpha + \beta P^e)\rho(1 - \rho)$ is a relatively smaller portion of the total variance, and the $E[P_d^*]$ and $E[P_\ell]$ are not significantly different. Therefore, neglecting the uncertainty with respect to the attribute levels does not appear to have a pronounced effect on the domestic price, and thus should not have a pronounced effect on the densities of the extreme-order statistics. Figure 4 illustrates the density for the minimum domestic price for $N = 10$ independent draws from the domestic price densities. For $N = 10$, the density is defined as follows:

$$f_{P_{d_{10}}}(p_d) = 10 f_{P_d}(p_d) \times [1 - F_{P_d}(p_d)]^9, \tag{27}$$

where $f(\cdot)$ and $F(\cdot)$, respectively, are the density and distribution functions. Not surprisingly, when the TB is introduced, the mean of the minimum domestic price increases. When the uncertainty in the attribute levels is ignored, the density of the minimum domestic price is not severely biased as is the case for the imports.

Conclusions

The competitiveness of a country's agricultural sector is influenced not only by its comparative advantage relative to its trading competitors, but also by the domestic and foreign policy environments. As traditional forms of agricultural protection decline in light of recent trade agreements, agricultural interests will likely seek protection in the form of TBs. Thus, it is of great interest that a theoretical framework which models the effects of TBs under uncertainty be developed.

The main objective of this article has been to develop an analytical framework that may be employed to recover the effect of TBs on domestic and international quantities and prices of traded goods. In contrast to standard approaches, this framework explicitly recognizes the effects of uncertainty at various levels and recovers ex ante and ex post probability density functions of relevant economic variables. The framework is not unduly restrictive in that one may consider SPS measures which result in a continuum of importable product levels, not simply the two extremes: importable product equals zero a.s., and importable product is greater than import demand a.s.

The flexibility of the framework yields a very strong analytical tool. In particular, derivation of the densities allows for the consideration of numerous questions. For example, densities for the extreme-order statistics, which are of great importance to highly volatile sectors like fruits and vegetables, were recovered. Other questions such

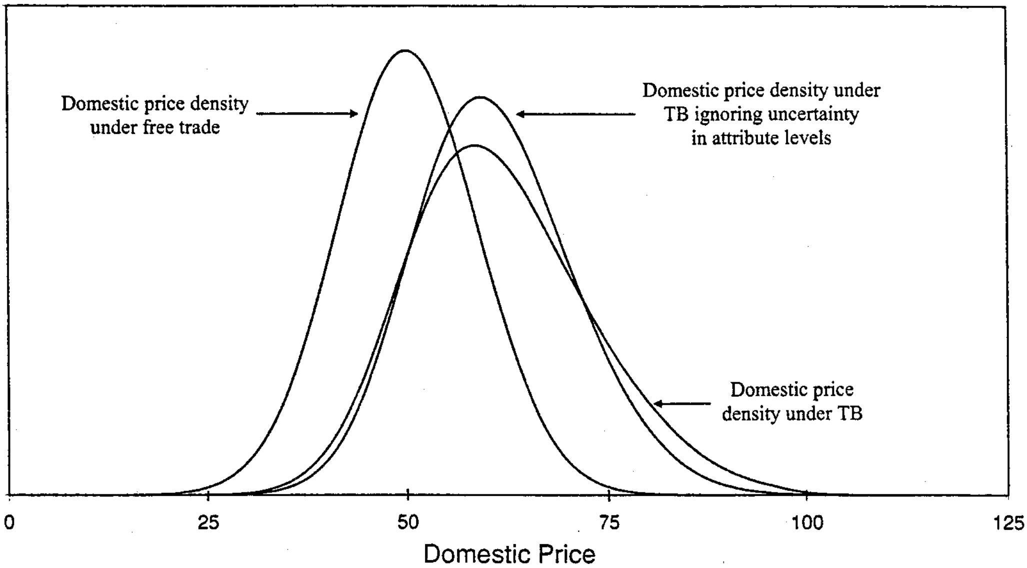

Figure 3. Illustrative example: Domestic price densities

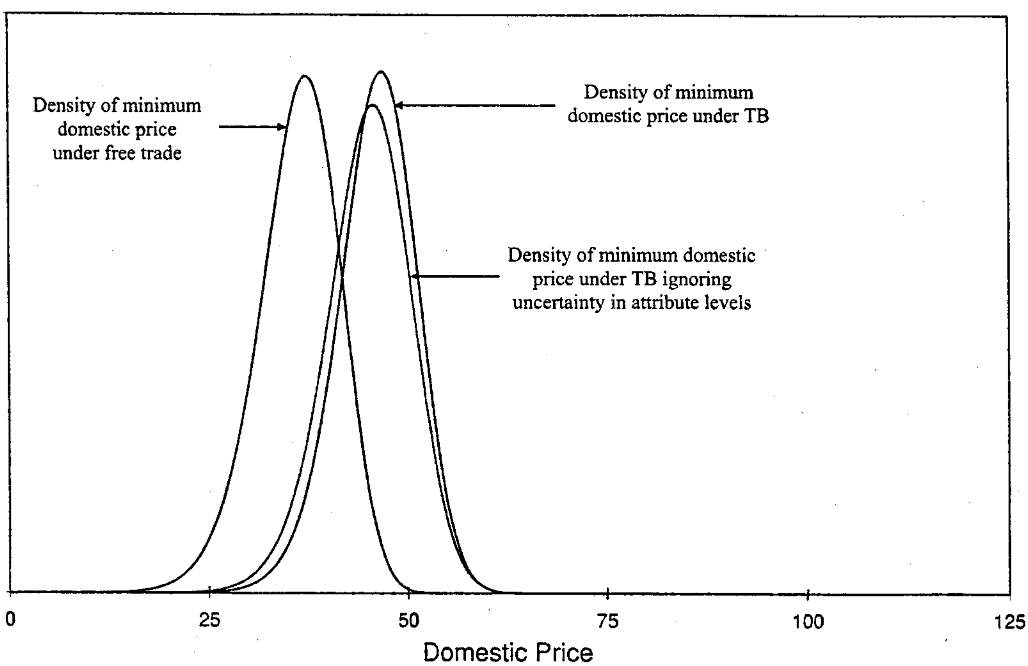

Figure 4. Illustrative example: Densities of minimum domestic prices (N = 10)

as those involving deviations from risk neutrality can be considered. To the best of the author's knowledge, this article represents the first attempt to explicitly model TBs and derive the ex ante and ex post probability density functions. Admittedly, it was necessary to make some minor sacrifices (importable product is Gaussian and demand and supply are linear in price) to keep the densities tractable. Despite this caveat, the developed model is simply implemented (as shown in the "Illustrative Example" section) with estimates of domestic and ROW supply elasticities, domestic and ROW demand elasticities, variation in domestic and ROW supply, variation in the exchange rate, and the probability of attribute level greater than the bound. This article considers a relatively simple case—the small perfectly competitive country. Further research should focus on the less simplistic scenarios, i.e., the small monopolistic country, the large perfectly competitive country, and the large monopolistic country.

In particular, focus was given to the uncertainty resulting from the attribute levels, a source not yet considered in the literature. Both the theoretical results and illustrative example indicated that neglecting uncertainty will down-weight the impact of the TB in that both the decrease in expected imports and the increase in the expected domestic price will be underestimated. Although it could not be proved in general, the illustrative example indicated the variance of both imports and domestic prices will be underestimated. This bias becomes apparent in the densities of the extreme-order statistics.

[*Received April 1999; final revision received December 1999.*]

References

Anderson, J. E., G. E. Bannister, and J. P. Neary. "Domestic Distortions and International Trade." *Internat. Econ. Rev.* 36(1995):139–57.

Anderson, J. E., and J. P. Neary. "The Coefficient of Trade Utilization: Back to the Baldwin Envelope." In *The Political Economy of International Trade: Essays in Honor of Robert E. Baldwin*, eds., R. W. Jones and A. O. Krueger, pp. 49–72. Oxford: Basil Blackwell, 1990.

———. "A New Approach to Evaluating Trade Policy." Research Work. Pap. in International Trade No. WPS1022, The World Bank, Washington DC, 1992.

Beattie, B. R., and D. R. Biggerstaff. "Karnal Bunt: A Wimp of a Disease . . . But an Irresistible Political Opportunity." *Choices* (2nd Quarter 1999):4–7.

Cain, M. "The Moment-Generating Function of Bivariate Normal Random Variables." *Amer. Statistician* 48,2(1994):124–25.

Calvin, L., and B. Krissoff. "Technical Barriers to Trade: A Case Study of Phytosanitary Barriers and U.S.-Japanese Apple Trade." *J. Agr. and Resour. Econ.* 23(1998):351–66.

Casella, G., and R. L. Berger. *Statistical Inference.* Belmont CA: Brooks/Cole Pub. Co., 1990.

Hillman, J. S. "The Neoprotectionist Challenge for the WTO." *Policy* (Autumn 1996):16–18.

———. *Technical Barriers to Trade.* New York: Westview Pub. Co., 1990.

Hooker, N. H., and J. A. Caswell. "Regulatory Targets and Regimes for Food Safety: A Comparison of North American and European Approaches." Unnumbered NBER rep., National Bureau of Economic Research, Cambridge MA, 1995.

Josling, T. E. "An Analytical Framework for Assessing the Trade Impact of SPS and TBT Regulations." Paper presented at the Economic Research Service's Technical Barriers to Trade Workshop, Washington DC, 8–9 October 1997.

———. "The Uruguay Round Agreement on Agriculture: An Evaluation." IATRC Commissioned Paper No. 9, International Agricultural Trade Research Consortium, University of Minnesota, St. Paul, 1994.

Neary, J. P. "Trade Liberalization and Shadow Prices in the Presence of Tariffs and Quotas." *Internat. Econ. Rev.* 36,3(1995):521–55.

Orden, D. "Agricultural Interest Groups and the North American Free Trade Agreement." In *The Political Economy of American Trade Policy*, ed., A. O. Krueger. Chicago: The University of Chicago Press, 1996.

Paarlberg, P., and J. Lee. "Import Restrictions in the Presence of a Health Risk: An Illustration Using FMD." *Amer. J. Agr. Econ.* 80(1998):175–83.

Pelcovits, S. "Quotas versus Tariffs." *J. Internat. Econ.* 5(1976):363–70.

Roberts, D., and K. DeRemer. "Overview of Foreign Technical Barriers to U.S. Agricultural Exports." Staff Pap. No. 9075, USDA/Economic Research Service, Washington DC, 1996.

Sumner, D. A. *Agricultural Trade Policy: Letting Markets Work.* Washington DC: American Enterprise Institute Press, 1995.

Sumner, D. A., and H. Lee. "Sanitary and Phytosanitary Trade Barriers and Empirical Trade Modeling." In *Understanding Technical Barriers to Agricultural Trade*, Proceedings of a Conference of the IATRC, eds., D. Orden and D. Roberts, pp. 273–85. St. Paul MN: International Agricultural Trade Research Consortium, University of Minnesota, 1997.

Trefler, D. "Trade Liberalization and the Theory of Endogenous Protection: An Econometric Study of U.S. Import Policy." *J. Polit. Econ.* 101(1993):138–60.

Young, L., and J. E. Anderson. "The Optimal Policies for Restricting Trade Under Uncertainty." *Rev. Econ. Stud.* 47,5(October 1980):927–32.

Appendix A: Derivation of Density of Minimum of Bivariate Normal Random Variables

Consider two real-valued random variables, denoted X and Y, which have joint density $f_{XY}(x, y)$. Define a third real-valued random variable as $Z = \min[X, Y]$. Let K be the random variable that counts the number of X, Y less than z.[7] Therefore,

$$F_Z(z) = 1 - \mathcal{P}(k = 0) = 1 - \int_z^\infty \int_z^\infty f_{XY}(x, y)\, dx\, dy.$$

Hence, the density

$$f_Z(z) = \frac{-d\left(\int_z^\infty \int_z^\infty f_{XY}(x, y)\, dx\, dy \right)}{dz}.$$

Because X and Y are not independent, the joint cannot be marginalized, and as such, the marginal distributions are not sufficient to recover the minimum distribution. We assume that e and υ are independent Gaussians with zero means and finite variances σ_d^2 and σ^2, respectively. If we define

$$X = a_x e + b_x \upsilon + c_x,$$
$$Y = a_y e + b_y \upsilon + c_y,$$

where e and υ are independent Gaussians, then the joint distribution of X and Y is a bivariate Gaussian with parameters

$$\mu_x = E[X] = c_x,$$
$$\mu_y = E[Y] = c_y,$$
$$\eta = \mathrm{Cor}[X, Y] = \frac{a_x a_y \sigma_d^2 + b_x b_y \sigma^2}{\sigma_x \sigma_y},$$
$$\sigma_x^2 = \mathrm{Var}[X] = a_x^2 \sigma_d^2 + b_x^2 \sigma^2,$$
$$\sigma_y^2 = \mathrm{Var}[Y] = a_y^2 \sigma_d^2 + b_y^2 \sigma^2.$$

[7] If X and Y were independent and identically distributed random variables, then K would be binomial.

The joint density $f_{XY}(x, y)$, which is bivariate Gaussian, is of the form

$$\text{(A1)}\quad f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\eta^2}} \exp\left[-\frac{1}{2(1-\eta^2)} \left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\eta\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 \right]\right].$$

By making the transformation

$$u = \frac{x-\mu_x}{\sigma_x} \quad \text{and} \quad \upsilon = \frac{y-\mu_y}{\sigma_y},$$

then

$$\text{(A2)}\quad \int_z^\infty \int_z^\infty f_{XY}(x,y)\,dx\,dy = \int_{A_y(z)}^\infty \int_{A_x(z)}^\infty \frac{1}{2\pi\sqrt{1-\eta^2}} \exp\left[-\frac{1}{2(1-\eta^2)} (u^2 - 2\eta u\upsilon + \upsilon^2)\right] du\,d\upsilon,$$

where

$$A_x(z) = \frac{z-\mu_x}{\sigma_x} \quad \text{and} \quad A_y(z) = \frac{z-\mu_y}{\sigma_y}.$$

By rearranging terms, we get

$$\text{(A3)}\quad \int_z^\infty \int_z^\infty f_{XY}(x,y)\,dx\,dy = \int_{A_y(z)}^\infty \int_{A_x(z)}^\infty \frac{1}{2\pi\sqrt{1-\eta^2}} \exp\left[-\frac{1}{2}\left(\frac{(u-\eta\upsilon)^2}{1-\eta^2} + \upsilon^2\right)\right] du\,d\upsilon.$$

Making the transformation

$$w = \frac{u-\eta\upsilon}{\sqrt{1-\eta^2}}$$

yields

$$\text{(A4)}\quad \int_z^\infty \int_z^\infty f_{XY}(x,y)\,dx\,dy = \int_{A_y(z)}^\infty \left[\int_{A_w(z,\upsilon)}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right) dw \right] \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\upsilon^2}{2}\right) d\upsilon$$
$$= \int_{A_y(z)}^\infty \left[\left(1 - \Phi\left(A_w(z,\upsilon)\right)\right)\phi(\upsilon)\right] d\upsilon,$$

where

$$A_w(z, \upsilon) = \frac{A_x - \eta\upsilon}{\sqrt{1-\eta^2}},$$

$\Phi(\cdot)$ is the standard Gaussian distribution, and $\phi(\cdot)$ is the standard Gaussian density. Recall that

$$\text{(A5)}\quad f_Z(z) = -\frac{d\left(\int_z^\infty \int_z^\infty f_{XY}(x,y)\,dx\,dy\right)}{dz}$$
$$= -\frac{d\left(\int_{A_y(z)}^\infty \left[\left(1 - \Phi\left(A_w(z,\upsilon)\right)\right)\phi(\upsilon)\right] d\upsilon\right)}{dz}.$$

By using Leibnitz' rule and the Lebesgue Dominating Convergence Theorem,

(A6)
$$f_Z(z) = -\left[-\int_{A_y(z)}^{\infty} \phi\left(A_w(z,\upsilon)\right) \frac{dA_w(z,\upsilon)}{dz} \phi(\upsilon)\, d\upsilon - \left[1 - \Phi\left(A_w\left(z, A_y(z)\right)\right)\right] \phi\left(A_y(z)\right) \frac{dA_y(z)}{dz} \right].$$

Consider the first term of the above equation:

(A7)
$$\begin{aligned}
&\int_{A_y(z)}^{\infty} \phi\left(A_w(z,\upsilon)\right) \frac{dA_w(z,\upsilon)}{dz} \phi(\upsilon)\, d\upsilon \\
&= \int_{A_y(z)}^{\infty} \frac{1}{2\pi\sigma_x\sqrt{1-\eta^2}} \exp\left[-\frac{A_x(z)^2 - 2A_x(z)\eta\upsilon + \eta^2\upsilon^2 + \upsilon^2 - \eta^2\upsilon^2}{2(1-\eta^2)} \right] d\upsilon \\
&= \int_{A_y(z)}^{\infty} \frac{1}{2\pi\sigma_x\sqrt{1-\eta^2}} \exp\left[-\frac{(\upsilon - \eta A_x(z))^2}{2(1-\eta^2)} \right] \exp\left[-\frac{A_x(z)^2}{2} \right] d\upsilon \\
&= \frac{1}{\sigma_x} \phi\left(A_x(z)\right) \int_{A_y(z)}^{\infty} \frac{1}{2\pi\sqrt{1-\eta^2}} \exp\left[-\frac{(\upsilon - \eta A_x(z))^2}{2(1-\eta^2)} \right] d\upsilon \\
&= \frac{1}{\sigma_x} \phi\left(A_x(z)\right) \times \left[1 - \Phi\left(\frac{A_y(z) - \eta A_x(z)}{\sqrt{1-\eta^2}} \right) \right].
\end{aligned}$$

Now consider the second term in the above equation:

(A8)
$$\left[1 - \Phi\left(A_w\left(z, A_y(z)\right)\right)\right] \phi\left(A_y(z)\right) \frac{dA_y(z)}{dz} = \frac{1}{\sigma_y} \phi\left(A_y(z)\right) \times \left[1 - \Phi\left(\frac{A_x(z) - \eta A_y(z)}{\sqrt{1-\eta^2}} \right) \right].$$

Therefore,

(A9)
$$\begin{aligned}
f_Z(z) &= \frac{1}{\sigma_x} \phi\left(A_x(z)\right) \times \left[1 - \Phi\left(\frac{A_y(z) - \eta A_x(z)}{\sqrt{1-\eta^2}} \right) \right] \\
&\quad + \frac{1}{\sigma_y} \phi\left(A_y(z)\right) \times \left[1 - \Phi\left(\frac{A_x(z) - \eta A_y(z)}{\sqrt{1-\eta^2}} \right) \right].
\end{aligned}$$

Notice, if the random variables were independent, i.e., $\eta = 0$, then the usual result would hold:

$$f_Z(z)\big|_{\eta=0} = \frac{1}{\sigma_x} \phi\left(A_x(z)\right)\left[1 - \Phi\left(A_y(z)\right)\right] + \frac{1}{\sigma_y} \phi\left(A_y(z)\right)\left[1 - \Phi\left(A_x(z)\right)\right].$$

Appendix B: Proof of Lemmas

The purpose of this appendix is to prove the lemmas employed in the "Derived Results" section of the main text. Somewhat surprisingly, these lemmas are not in either the probability or mathematical statistics literature. Using the same notation as in appendix A, consider two real-valued random variables, denoted X and Y, which are bivariate Gaussian. Define two real-valued random variables, $R = \max[X, Y]$ and $Z = \min[X, Y]$. Fortunately, Cain derived the moment-generating function for Z as well as the first two raw moments:

(A10) $$E[Z] = \mu_x \Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) + \mu_y \Phi\left(\frac{\mu_x - \mu_y}{\theta}\right) - \theta\phi\left(\frac{\mu_y - \mu_x}{\theta}\right)$$

and

(A11) $$E[Z^2] = \left(\sigma_x^2 + \mu_x^2\right)\Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) + \left(\sigma_y^2 + \mu_y^2\right)\Phi\left(\frac{\mu_x - \mu_y}{\theta}\right) - \left(\mu_x + \mu_y\right)\theta\phi\left(\frac{\mu_y - \mu_x}{\theta}\right),$$

where $\theta = (\sigma_x^2 - 2\eta + \sigma_y^2)^{½}$, and η is the COV(X,Y). Following the same derivation, the moment-generating function for R is also recoverable. From this, the first two raw moments are:

(A12) $$E[R] = \mu_x \Phi\left(\frac{\mu_x - \mu_y}{\theta}\right) + \mu_y \Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) + \theta\phi\left(\frac{\mu_y - \mu_x}{\theta}\right)$$

and

(A13) $$E[R^2] = \left(\sigma_x^2 + \mu_x^2\right)\Phi\left(\frac{\mu_x - \mu_y}{\theta}\right) + \left(\sigma_y^2 + \mu_y^2\right)\Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) + \left(\mu_x + \mu_y\right)\theta\phi\left(\frac{\mu_y - \mu_x}{\theta}\right),$$

where again, $\theta = (\sigma_x^2 - 2\eta + \sigma_y^2)^{½}$, and η is the COV(X,Y).

■ LEMMA 1. $E[Z] \le \min[\mu_x, \mu_y]$.

First, note that μ_x and μ_y are completely interchangeable, and thus it is sufficient to prove $E[Z] \le \mu_y$ while assuming $\mu_y \le \mu_x$. Hence,

(A14) $$\begin{aligned} E[Z] - \mu_y &= \mu_x \Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) + \mu_y\left(1 - \Phi\left(\frac{\mu_y - \mu_x}{\theta}\right)\right) - \theta\phi\left(\frac{\mu_x - \mu_y}{\theta}\right) - \mu_y \\ &= (\mu_x - \mu_y)\Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) - \theta\phi\left(\frac{\mu_x - \mu_y}{\theta}\right) \\ &= \theta\left[\left(\frac{\mu_x - \mu_y}{\theta}\right)\left(1 - \Phi\left(\frac{\mu_x - \mu_y}{\theta}\right)\right) - \phi\left(\frac{\mu_x - \mu_y}{\theta}\right)\right] \\ &= -\theta[\phi(\Lambda) - \Lambda(1 - \Phi(\Lambda))], \end{aligned}$$

where

$$\Lambda = \frac{\mu_x - \mu_y}{\theta}.$$

Although Chebychev's inequality is widely applicable, tighter bounds can often be found for specific distributions. One such inequality for the Gaussian distribution is that

$$1 - \Phi(\Lambda) \le \frac{\phi(\Lambda)}{\Lambda}$$

(see Casella and Berger, p. 186), which implies $\Lambda(1 - \Phi(\Lambda)) \le \phi(\Lambda)$ for $\Lambda > 0$. Therefore, since

$$\theta \ge 0 \rightarrow -\theta[\phi(\Lambda) - \Lambda(1 - \Phi(\Lambda))] \le 0 \rightarrow E[Z] \le \mu_y \rightarrow E[Z] \le \min[\mu_x, \mu_y].$$

Using the same approach, the converse can be shown: $E[R] \ge \max[\mu_x, \mu_y]$.

This lemma suggests that the mean of the minimum of two random variables which are bivariate Gaussian must be less than or equal to the means of either of the two random variables. Similarly, the mean of the maximum of two random variables which are bivariate Gaussian must be greater than or equal to the means of either of the two random variables.

■ LEMMA 2. $\dfrac{\partial E[Z]}{\partial \mu_x} > 0,\ \dfrac{\partial E[Z]}{\partial \mu_y} > 0.$

Again, it is sufficient to prove

$$\frac{\partial E[Z]}{\partial \mu_x} \geq 0$$

because μ_x and μ_y are interchangeable:

$$\text{(A15)} \qquad \frac{\partial E[Z]}{\partial \mu_x} = \Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) - \frac{\mu_x}{\theta}\phi\left(\frac{\mu_y - \mu_x}{\theta}\right) + \frac{\mu_y}{\theta}\phi\left(\frac{\mu_x - \mu_y}{\theta}\right) + \phi'\left(\frac{\mu_y - \mu_x}{\theta}\right),$$

where

$$\phi'(t) = \frac{\partial \phi(t)}{\partial t}.$$

Note, however, that $\phi'(t) = -t\phi(t)$, and thus

$$\text{(A16)} \qquad \begin{aligned} \frac{\partial E[Z]}{\partial \mu_x} &= \Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) + \left(\frac{\mu_y - \mu_x}{\theta}\right)\phi\left(\frac{\mu_y - \mu_x}{\theta}\right) - \left(\frac{\mu_y - \mu_x}{\theta}\right)\phi\left(\frac{\mu_y - \mu_x}{\theta}\right) \\ &= \Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) \\ &> 0. \end{aligned}$$

Using the same approach, the converse can also be shown:

$$\frac{\partial E[R]}{\partial \mu_x} > 0 \quad \text{and} \quad \frac{\partial E[R]}{\partial \mu_y} > 0.$$

This lemma indicates that the mean of the minimum of two random variables which are bivariate Gaussian must increase (decrease) if the mean of either of the two random variables increases (decreases) while everything else remains constant. Similarly, the mean of the maximum of two random variables which are bivariate Gaussian must increase (decrease) if the mean of either of the two random variables increases (decreases) while everything else remains constant.

■ LEMMA 3. $\dfrac{\partial E[Z]}{\partial \sigma_x^2} < 0,\ \dfrac{\partial E[Z]}{\partial \sigma_y^2} < 0.$

Note that

$$\frac{\partial E[Z]}{\partial \sigma_x^2} = \frac{\partial E[Z]}{\partial \theta}\frac{\partial \theta}{\partial \sigma_x^2}.$$

It is easily seen that

$$\frac{\partial \theta}{\partial \sigma_x^2} > 0 \quad \text{and} \quad \frac{\partial \theta}{\partial \sigma_y^2} > 0.$$

(Both inequalities assume X and Y have nonzero variance.) What is less obvious is that

$$\frac{\partial E[Z]}{\partial \theta} < 0:$$

(A17)
$$\frac{\partial E[Z]}{\partial\theta} = \mu_x\phi\left(\frac{\mu_y-\mu_x}{\theta}\right)\left(\frac{\mu_x-\mu_y}{\theta^2}\right)\frac{\partial\theta}{\partial\sigma_x^2} + \mu_y\phi\left(\frac{\mu_x-\mu_y}{\theta}\right)\left(\frac{\mu_y-\mu_x}{\theta^2}\right)\frac{\partial\theta}{\partial\sigma_x^2}$$
$$- \phi\left(\frac{\mu_y-\mu_x}{\theta}\right)\frac{\partial\theta}{\partial\sigma_x^2} - \theta\phi'\left(\frac{\mu_y-\mu_x}{\theta}\right)\frac{\partial\theta}{\partial\sigma_x^2}\left(\frac{\mu_x-\mu_y}{\theta^2}\right)$$
$$= -\phi\left(\frac{\mu_y-\mu_x}{\theta}\right)\frac{\partial\theta}{\partial\sigma_x^2} + \phi\left(\frac{\mu_y-\mu_x}{\theta}\right)\left(\frac{\mu_x-\mu_y}{\theta^2}\right)(\mu_x-\mu_y+\mu_y-\mu_x)\frac{\partial\theta}{\partial\sigma_x^2}$$
$$= -\phi\left(\frac{\mu_y-\mu_x}{\theta}\right)\frac{\partial\theta}{\partial\sigma_x^2}$$
$$< 0.$$

Given

$$\frac{\partial\theta}{\partial\sigma_x^2} > 0 \text{ and } \frac{\partial E[Z]}{\partial\theta} < 0 \rightarrow \frac{\partial E[Z]}{\partial\sigma_x^2} < 0.$$

Similarly, given

$$\frac{\partial\theta}{\partial\sigma_y^2} > 0 \text{ and } \frac{\partial E[Z]}{\partial\theta} < 0 \rightarrow \frac{\partial E[Z]}{\partial\sigma_y^2} < 0.$$

Using the same approach, it can be shown that

$$\frac{\partial E[R]}{\partial\theta} > 0,$$

and thus

$$\frac{\partial E[R]}{\partial\sigma_x^2} > 0 \text{ and } \frac{\partial E[R]}{\partial\sigma_y^2} > 0.$$

This lemma suggests that the mean of the minimum of two random variables which are bivariate Gaussian must increase (decrease) if the variance of either of the two decreases (increases) while everything else remains constant. Similarly, the mean of the maximum of two random variables which are bivariate Gaussian must increase (decrease) if the variance of either of the two increases (decreases) while everything else remains constant.

■ LEMMA 4. $\frac{\partial \mathrm{Var}[Z]}{\partial\sigma_x^2}$ [>, =, <] 0, $\frac{\partial \mathrm{Var}[Z]}{\partial\sigma_y^2}$ [>, =, <] 0.

Note that the $\mathrm{Var}[Z] = E[Z^2] - E[Z]^2$, and thus

$$\frac{\partial \mathrm{Var}[Z]}{\partial\sigma_x^2} = \frac{\partial E[Z^2]}{\partial\sigma_x^2} - 2E[Z]\frac{\partial E[Z]}{\partial\sigma_x^2}.$$

Recall from lemma 3 that

$$\frac{\partial E[Z]}{\partial\sigma_x^2} = -\phi\left(\frac{\mu_y-\mu_x}{\theta}\right)\frac{\partial\theta}{\partial\sigma_x^2}.$$

Therefore, concentrating on

$$\frac{\partial E[Z^2]}{\partial\sigma_x^2}$$

yields

(A18)
$$\frac{\partial E[Z^2]}{\partial \sigma_x^2} = \Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) + \left(\sigma_x^2 + \mu_x^2\right)\phi\left(\frac{\mu_y - \mu_x}{\theta}\right)\left(\frac{\mu_x - \mu_y}{\theta^2}\right)\frac{\partial \theta}{\partial \sigma_x^2}$$
$$+ \left(\sigma_y^2 + \mu_y^2\right)\phi\left(\frac{\mu_y - \mu_x}{\theta}\right)\left(\frac{\mu_y - \mu_x}{\theta^2}\right)\frac{\partial \theta}{\partial \sigma_x^2}$$
$$- (\mu_x + \mu_y)\phi\left(\frac{\mu_y - \mu_x}{\theta}\right)\frac{\partial \theta}{\partial \sigma_x^2}$$
$$- (\mu_x + \mu_y)\theta\phi'\left(\frac{\mu_x - \mu_y}{\theta}\right)\left(\frac{\mu_y - \mu_x}{\theta^2}\right)\frac{\partial \theta}{\partial \sigma_x^2}.$$

Noting again that $\phi'(t) = -t\phi(t)$ and simplifying yields

(A19)
$$\frac{\partial E[Z^2]}{\partial \sigma_x^2} = \Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) + \phi\left(\frac{\mu_y - \mu_x}{\theta}\right)\frac{\partial \theta}{\partial \sigma_x^2}\left(\frac{\mu_y - \mu_x}{\theta^2}\right)\left(\sigma_y^2 - \sigma_x^2 - \theta^2\frac{\mu_x + \mu_y}{\mu_y - \mu_x}\right)$$
$$= \Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) + \phi\left(\frac{\mu_y - \mu_x}{\theta}\right)\frac{\partial \theta}{\partial \sigma_x^2}\frac{2}{\theta^2}\left(\mu_x\left(\eta - \sigma_y^2\right) + \mu_y\left(\eta - \sigma_x^2\right)\right).$$

Therefore,

(A20)
$$\frac{\partial \mathrm{Var}[Z]}{\partial \sigma_x^2} = \Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) + \phi\left(\frac{\mu_y - \mu_x}{\theta}\right)\frac{\partial \theta}{\partial \sigma_x^2}\frac{2}{\theta^2}\left(\mu_x\left(\eta - \sigma_y^2\right) + \mu_y\left(\eta - \sigma_x^2\right)\right)$$
$$+ \phi\left(\frac{\mu_y - \mu_x}{\theta}\right)\frac{\partial \theta}{\partial \sigma_x^2}\frac{2}{\theta^2}\left(\mu_x\theta^2\Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) + \mu_y\theta^2\Phi\left(\frac{\mu_x - \mu_y}{\theta}\right)\right.$$
$$\left. - \theta^3\phi\left(\frac{\mu_y - \mu_x}{\theta}\right)\right).$$

Noting that
$$\frac{\partial \theta}{\partial \sigma_x^2} = \frac{1}{2\theta}$$

and $\Phi(t) = 1 - \Phi(-t)$, then by gathering terms,

(A21)
$$\frac{\partial \mathrm{Var}[Z]}{\partial \sigma_x^2} = \phi\left(\frac{\mu_y - \mu_x}{\theta}\right)\frac{1}{\theta^3}\left((\mu_x - \mu_y)\left(\eta - \sigma_y^2 + \theta^2\Phi\left(\frac{\mu_y - \mu_x}{\theta}\right)\right)\right)$$
$$+ \Phi\left(\frac{\mu_y - \mu_x}{\theta}\right) - \phi\left(\frac{\mu_y - \mu_x}{\theta}\right)^2 [>, =, <]\ 0,$$

shown by counter-example. This lemma suggests that the variance of the minimum of two random variables which are bivariate Gaussian does not necessarily increase (decrease) if the variance of either of the two increases (decreases) while everything else remains constant. Similarly, the variance of the maximum of two random variables which are bivariate Gaussian does not necessarily decrease (increase) if the variance of either of the two increases (decreases) while everything else remains constant.