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# Cooperative Wireless Sensor Network Positioning via Implicit Convex Feasibility

Mohammad Reza Gholami, Luba Tetruashvili, Erik G. Ström, and Yair Censor

*Abstract*—We propose a distributed positioning algorithm to estimate the unknown positions of a number of target nodes, given distance measurements between target nodes and between target nodes and a number of reference nodes at known positions. Based on a geometric interpretation, we formulate the positioning problem as an implicit convex feasibility problem in which some of the sets depend on the unknown target positions, and apply a parallel projection onto convex sets approach to estimate the unknown target node positions. The proposed technique is suitable for parallel implementation in which every target node in parallel can update its position and share the estimate of its location with other targets. We mathematically prove convergence of the proposed algorithm. Simulation results reveal enhanced performance for the proposed approach compared to available techniques based on projections, especially for sparse networks.

*Index Terms*– Positioning, Cooperative wireless sensor network, Parallel projections onto convex sets, Convex feasibility problem, Implicit convex feasibility.

## I. INTRODUCTION

For many applications in wireless sensor networks (WSNs), position information is a vital requirement for the network to function as intended. Due to drawbacks of using global positioning system (GPS) receivers in sensor nodes, mainly due to limited access to GPS satellites, e.g., in an indoor scenario, the position recovery from the network, called positioning or localization, has been extensively studied in the literature [1]–[4]. It is usually assumed that there are a number of reference sensor nodes at known positions that can be used to estimate the location of a number of sensor nodes at unknown positions, henceforth called target nodes. To estimate the position of target nodes, some types of measurements are taken between different nodes such as received-signal-strength, angle-of-arrival, or time-of-arrival [1], [2], [5]. A popular technique in the positioning literature is to estimate distances between sensor nodes from measurements and then to apply a suitable positioning algorithm [1], and this is also the approach taken in this study.

From one point of view, positioning algorithms can be categorized into two groups: cooperative and noncooperative [5]. In a noncooperative network, measurements taken between a target and reference nodes are used to estimate the position of the target node, while in a cooperative network,

M. R. Gholami and E. G. Ström are with the Division of Communication Systems, Information Theory, and Antennas, Department of Signals and Systems, Chalmers University of Technology, SE-412 96 Gothenburg, Sweden. L. Tetruashvili and Y. Censor are with the Department of Mathematics, University of Haifa, Mt. Carmel, Haifa 3190501, Israel. The contributions of the first two authors to this work are of equal shares.

besides the measurements made between target nodes and reference nodes, measurements collected between target nodes are also used in the positioning process [5]. For low density networks, the cooperation technique can effectively improve the performance of the position estimate [5]. Compared to noncooperative scenarios, the positioning problem in a cooperative network is more challenging and it is not clear how to effectively use the distance measurements between target nodes for the positioning process. During the last few years, various cooperative positioning algorithms have been proposed in the literature. Classic estimators such as the maximum likelihood (ML) estimator and nonlinear least squares (NLS) estimator derived for the positioning problem are often too complex and pose difficult global optimization problems [2], [5], [6]. In the literature, a number of suboptimal techniques have been proposed to avoid the difficulty in solving the ML or the NLS problems, such as convex relaxation techniques, e.g., based on second order cone programming [7], [8], sum of squares [9], and semidefinite programming [10]–[12], and linearization techniques [6], [13]–[15].

From an implementation point of view, positioning algorithms can be categorized into two classes: centralized and distributed. In a centralized approach, all measurements gathered in the sensor nodes are transferred to a central unit, and a positioning algorithm is applied to find the locations of the target nodes. In a distributed approach, however, every target is allowed to locally process the measurements to obtain an estimate of its own position. To get benefits from cooperation, target nodes can share their estimates with other targets, e.g., by broadcasting the estimates. For example, algorithms based on weighted-multidimensional scaling [16] and distributed belief propagation [17] have been proposed to solve the positioning problem in distributed fashion. In addition to the need for a distributed implementation, another important factor in designing a positioning algorithm is the robustness against non-line-of-sight (NLOS) conditions, in which distances are measured with a large, normally positive, errors.

It is well-known that in the noncooperative case, a particular target node can be found in the intersection of a number of balls centered around the reference nodes, if the measurement errors are positive. In this case, it makes sense to formulate the positioning problem as a convex feasibility problem (CFP), i.e., the target position is estimated as a point inside the intersection of the appropriate balls. We can approach this by finding a point that minimizes the sum of squared distances to the balls. In case the intersection is nonempty, a minimizing point is obviously a solution to the CFP. In case the intersection is empty (i.e., when the CFP is inconsistent), the minimizing points are still reasonable estimates of the target

node positions. There exist algorithms based on projection methods that are proven to find a minimizer to the above-mentioned objective function, see, e.g., [5], [18] and references therein.

The class of *projection methods* is understood here as the class of methods that have the property that they can reach an aim related to the family of sets $\{C_1, C_2, \ldots, C_m\}$, such as, but not only, solving the CFP, or solving an optimization problem with these sets as constraints, by performing projections (orthogonal, i.e., least Euclidean distance, or others) onto the individual sets $C_i$. The advantage of such methods occurs in situations where projections onto the individual sets are computationally simple to perform. Such methods have been in recent decades extensively investigated mathematically and used experimentally with great success on some huge and sparse real-wold applications, consult, e.g., [19], [20] and the books [18], [21]–[27]. For further applications of projection methods, see, e.g., [28].

In this paper, we extend the projection ideas to the cooperative case. This leads to a new type of CFP, which we henceforth call an *implicit* CFP (ICFP), since some of the convex sets (balls) actually depend on the target positions. We formulate an algorithm that minimizes the sum of the square distances to a number of convex sets. We also present a mathematical analysis to support the validity of our approach in terms of convergence. Simulation results show an improved performance for the proposed algorithm compared to available techniques based on projection approaches.

It should be noted that projection techniques have been used in the past for cooperative positioning, see, e.g., [29], [30]. In fact, our proposed scheme can be viewed as an extension of the parallel projection method (PPM) proposed in [30]. However, our method is different from the one in [30] in terms of convergence. The approach proposed in [30] needs good initial points to provide good estimates of the targets positions, while for the proposed technique in this paper, convergence to an optimal solution is always guaranteed for any arbitrary initialization point. There are no convergence proofs for the algorithms proposed in [29], [30] and one aim of this study is to provide a framework for studying the convergence of a class of optimization problems with applications in wireless sensor network positioning.

The paper is organized as follows. In Section II, we describe the positioning problem based on a geometric notion. The proposed algorithm is defined in Section III, and a mathematical model of the ICFP problem is provided in Section IV. Some properties of the objective function and the convergence proof are found in Sections V and VI, respectively. In Section VII, the performance of the proposed algorithm is evaluated by computer simulations.

## II. PROBLEM STATEMENT

### A. System model

We consider a $d$-dimensional cooperative network, $d = 2$ or $d = 3$, consisting of a number of sensor nodes at known positions, called as reference nodes, and a number of target nodes at unknown positions. In this study, we assume that every target node can estimate the distance to nearby sensor nodes, e.g., using a two-way time-of-arrival technique. We also assume that sensor nodes are perfectly synchronized, meaning that measurements are no longer affected by clock imperfections. It is also assumed that target nodes can share the estimates of the positions by broadcasting the estimates to neighboring target nodes (without any round-off errors).

Let us consider a WSN consisting of $n+m$ sensor nodes distributed in a space. Let $x_i \in \mathbb{R}^d$ for $i \in I := \{1, 2, \ldots, n\}$ be the position of target nodes whose locations $x_i$ are unknown, and let $a_j \in \mathbb{R}^d$ for $j \in J := \{n+1, n+2, \ldots, n+m\}$ be the position of reference nodes whose locations $a_j$ are *a priori* known. To formulate a geometric positioning problem, we further define the following sets. For every $i \in I$, let

$$\mathcal{A}_i = \{j \in J \mid \text{target node } i \text{ can communicate with} \\ \text{reference node } j\} \tag{1}$$

and let

$$\mathcal{B}_i = \{q \in I \mid \text{target node } i \text{ can communicate with target} \\ \text{node } q\}. \tag{2}$$

The distance measurement between a pair of nodes (target, reference) or (target, target) is modeled as in [3], [31],

$$\hat{d}_{ij} = d_{ij} + \varepsilon_{ij}, \ j \in \mathcal{A}_i, \ i \in I \tag{3a}$$

$$\hat{l}_{iq} = l_{iq} + \varepsilon_{iq}, \ q \in \mathcal{B}_i, \ i \in I \tag{3b}$$

where $\varepsilon_{ij}$ and $\varepsilon_{iq}$ are measurement errors drawn from some distributions, and $d_{ij} = \|x_i - a_j\|$ and $l_{iq} = \|x_i - x_q\|$ are the actual Euclidean distances between reference node $j$ and target node $i$ and between target node $q$ and target node $i$, respectively. We also define the following convex sets. For $i \in I$, define for each $j \in \mathcal{A}_i$,

$$\mathcal{C}_{ij} = \{z \in \mathbb{R}^d \mid \|z - a_j\| \le \hat{d}_{ij}\} \tag{4}$$

and for $i \in I$, define for each $q \in \mathcal{B}_i$,

$$\mathcal{X}_{iq} = \{z \in \mathbb{R}^d \mid \|z - x_q\| \le \hat{l}_{iq}\}. \tag{5}$$

For convenience, we assume that the measurements are symmetric in the sense that if $q \in \mathcal{B}_i$ then $i \in \mathcal{B}_q$ and furthermore that $\hat{l}_{iq} = \hat{l}_{qi}$. That is, if target $i$ can measure the distance to target $q$ then the opposite is also true. This simplifying assumption can be motivated by extending the traditional two-way time-of-arrival (TOA) ranging procedure to a four-way TOA process in which

1) Node $i$ sends a message to node $q$
2) Node $q$ immediately returns the message to node $i$
3) Node $i$ can now compute the initial estimate $\hat{l}'_{iq}$ from the TOA measurements recorded from the two first transmissions
4) Node $i$ transmits $\hat{l}'_{iq}$ to node $q$
5) Node $q$ can now compute the initial estimate $\hat{l}'_{qi}$ from the TOA measurements from the second and third transmissions, and the final estimates are computed as $\hat{l}_{qi} = \hat{l}_{iq} = (\hat{l}'_{iq} + \hat{l}'_{qi})/2$
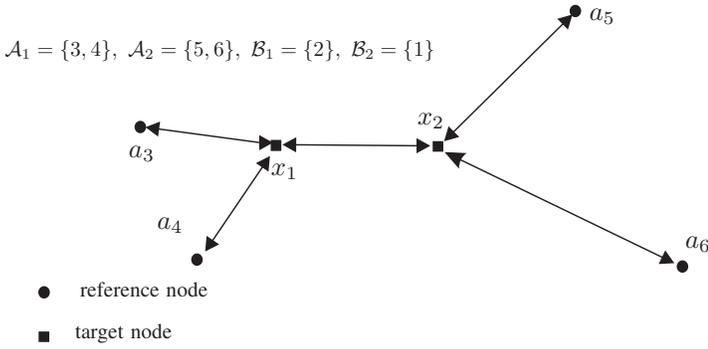6) Node $q$ transmits the final estimate $\hat{l}_{iq}$ to node $i$.

$\mathcal{A}_1 = \{3,4\}, \ \mathcal{A}_2 = \{5,6\}, \ \mathcal{B}_1 = \{2\}, \ \mathcal{B}_2 = \{1\}$



Fig. 1. An example of a cooperative network consisting of two target nodes and four reference nodes.
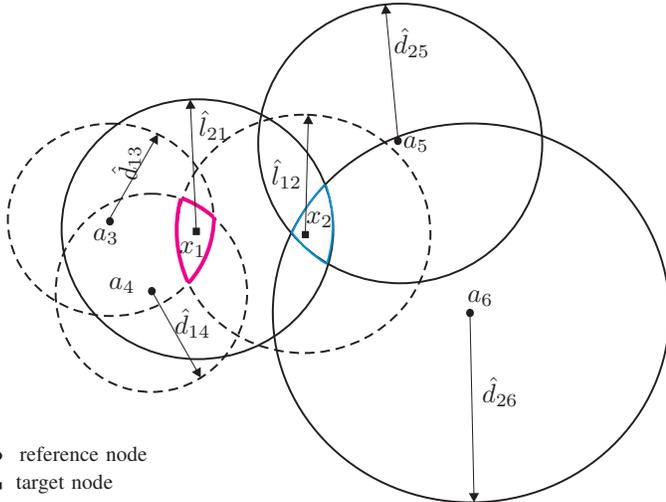


- reference node
- target node

Fig. 2. Deriving the intersections for target node one and target node two. Balls with dashed boundaries are used to derive the intersection for target node one and balls with solid boundaries are used to determine the intersection for target node two. The boundary of intersections for target node one and target node two are red and blue colored, respectively. Here, we assume that $\mathcal{X}_{12}$ and $\mathcal{X}_{21}$ are available for finding the red and blue areas. In a practical setting, however, $\mathcal{X}_{12}$ and $\mathcal{X}_{21}$ are not available from the start.

From Eq. (3) it is clear that in the absence of measurement errors, i.e., when $\varepsilon_{ij} = 0$ and $\varepsilon_{iq} = 0$, the position of target node $i$ belongs to the intersection of a number of closed balls centered at $a_j$ and $x_q$ with radii $d_{ij}$ and $l_{iq}$, respectively. Assuming positive measurement errors, we use the balls defined in Eqs. (4) and (5), and formulate a feasibility problem (CFP) aimed at solving the cooperative sensor network positioning problem as follows.

*Problem 1:* Find a set of vectors $x_i \in \mathbb{R}^d$, $i \in I$, such that

$$x_i \in \left( \bigcap_{j \in \mathcal{A}_i} \mathcal{C}_{ij} \right) \bigcap \left( \bigcap_{q \in \mathcal{B}_i} \mathcal{X}_{iq} \right). \qquad (6)$$

Note that when the problem is formulated, the target nodes $x_i \in \mathbb{R}^d$ for $i \in I$, are not known, and thus the sets $\mathcal{X}_{iq}$ are also unknown. For more details and other examples of geometric positioning problems, see [5], [29].

We clarify the formulation of Problem 1 with the following example. Consider Fig. 1 in which a 2-dimensional cooperative network consists of four reference nodes, i.e.,

$a_i \in \mathbb{R}^2$, $i \in \{3,4,5,6\}$, and two target nodes, $x_i \in \mathbb{R}^2$, $i \in \{1,2\}$. The sets $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$, and $\mathcal{B}_2$ for this network are shown in the figure. The sets $\mathcal{C}_{ij}$ and $\mathcal{X}_{iq}$ are defined as

$$\mathcal{C}_{13} = \{z \in \mathbb{R}^2 \mid \|z - a_3\| \le \hat{d}_{13}\},$$
$$\mathcal{C}_{14} = \{z \in \mathbb{R}^2 \mid \|z - a_4\| \le \hat{d}_{14}\},$$
$$\mathcal{C}_{25} = \{z \in \mathbb{R}^2 \mid \|z - a_5\| \le \hat{d}_{25}\},$$
$$\mathcal{C}_{26} = \{z \in \mathbb{R}^2 \mid \|z - a_6\| \le \hat{d}_{26}\},$$
$$\mathcal{X}_{12} = \{z \in \mathbb{R}^2 \mid \|z - x_2\| \le \hat{l}_{12}\},$$
$$\mathcal{X}_{21} = \{z \in \mathbb{R}^2 \mid \|z - x_1\| \le \hat{l}_{21}\}. \qquad (7)$$

Suppose that distance estimates between different nodes are greater than the actual distances, i.e., $\hat{d}_{ij} \ge d_{ij}$ for $i \in \{1,2\}$ and $j \in \{3,4,5,6\}$, and $\hat{l}_{iq} \ge l_{iq}$ for $i \in \{1,2\}$ and $q \in \{1,2\}$. Such a situation happens when the measurement errors are positive [32], [33]. The intersection involving target node one and the intersection involving target node two are shown in Fig. 2. Hence, we can write

$$x_1 \in \left( \mathcal{C}_{13} \bigcap \mathcal{C}_{14} \right) \bigcap \mathcal{X}_{12}, \qquad (8)$$
$$x_2 \in \left( \mathcal{C}_{25} \bigcap \mathcal{C}_{26} \right) \bigcap \mathcal{X}_{21}. \qquad (9)$$

*Remark 2:* In practical applications, some of the measurement errors $\epsilon_{ij}$ and $\epsilon_{iq}$ in (3) can be negative. In this case, the intersection (6) might not contain the target node locations, even if the intersection is nonempty. However, there are also practical situations when the measurement errors are positive. For example, errors in two-way time-of-arrival (TW-TOA) measurements tend to be positive in practice, even for line-of-sight conditions (as clarified in recent work on localization based on practical Ultra-wideband (UWB) measurements, please see [33]). Time-of-arrival is typically measured by computing the cross-correlation of the received signal with a copy of the transmitted signal and finding the first peak of the cross-correlation that exceeds a certain threshold. Even if the threshold is carefully adjusted, the detected peak rarely occurs before the true arrival time, especially for UWB signals. For more details of this phenomenon, see [33]. Finally, note that we have here considered positive errors to justify the ICFP as a means to estimate the target node positions. However, the algorithm proposed later in this paper does not rely on this assumption. It will function well even if errors are allowed to be negative.

### B. On the "implicit convex feasibility problem"

Problem 1 is not a standard CFP because, as can be seen from (6), the sets $\mathcal{X}_{iq}$ are not determined only by the data, but also on the, yet unknown, target node positions.

For CFPs with fixed feasible sets that are not dependent on the solution there exits a broad literature, see, e.g., [18] or [34].

This leads us to formulate a new class of CFPs that we call implicit CFP (ICFP). To start, let $Q_t$ for $t = 1, 2, \ldots, T$ be subsets of $\mathbb{R}^p$ defined for given convex functions $f_t : \mathbb{R}^p \to \mathbb{R}$ by their zero-level-sets:

$$Q_t := \{z \in \mathbb{R}^p \mid f_t(z) \le 0\}. \qquad (10)$$

*The convex feasibility problem* (CFP) in the finite-dimensional Euclidean space $\mathbb{R}^p$ is defined in the next Problem 2:

*Problem 2*: Find a point $z^*$ in the set $Q := \cap_{t=1}^{T} Q_t$.

Now let us consider the implicit convex feasibility problem (ICFP), that generalizes the CFP, and which can be used to describe the cooperative positioning problem.

Assume that $h_t : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ $(t = 1, 2, \ldots, T)$ are some given functions, such that for any point $u \in \mathbb{R}^p$, the function $h_t(z, u)$ is convex in the variable $z$. For any given point $u \in \mathbb{R}^p$ and $t = 1, 2, \ldots, T$ we define the sets

$$\hat{Q}_t(u) := \{z \in \mathbb{R}^p | h_t(z, u) \leq 0\}, \tag{11}$$

and look at the following ICFP:

*Problem 3:* Find a point $z^*$ in the set $\hat{Q}(z^*) := \cap_{t=1}^{T} \hat{Q}_t(z^*)$.

In the ICFP problem we are seeking a common point $z^*$ of sets which are defined by the unknown point $z^*$. Choosing $h_{ij}(z, a_j) = \|z - a_j\| - \hat{d}_{ij}$, for all $i \in I$ and $j \in \mathcal{A}_i$, and $h_{iq}(z, x_q) = \|z - x_q\| - \hat{l}_{iq}$, for all $i \in I$ and $q \in \mathcal{B}_i$, the ICFP translates the cooperative positioning problem into the following problem.

*Problem 4:* Find a point $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^{dn}$ such that it satisfies following relations for all $i = 1, 2, \ldots, n$:

$$x_i \in \left( \bigcap_{j \in \mathcal{A}_i} \{z \in \mathbb{R}^d \mid \|z - a_j\| - \hat{d}_{ij} \leq 0\} \right) \bigcap$$
$$\left( \bigcap_{q \in \mathcal{B}_i} \{z \in \mathbb{R}^d \mid \|z - x_q\| - \hat{l}_{iq} \leq 0\} \right). \tag{12}$$

### III. THE ALGORITHM

To solve Problem 4, as described in the previous section, we propose a distributed algorithm based on projections onto sets $\mathcal{C}_{ij}$ and $\mathcal{X}_{iq}$, defined in (4) and (5). The notation $P_\Omega(u)$ stands for the orthogonal (least Euclidean distance) projection of the point $u$ onto the set $\Omega$. We use boldface to denote vectors in the product space $\mathbb{R}^{dn}$ ($d = 2$ or $d = 3$), and denote the number of elements in a set $D$ by $|D|$.

*Algorithm 1:*

**Initialization:** Choose an arbitrary initial point $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in \mathbb{R}^{dn}$.

**General Iterative Step:** Given a current vector $\mathbf{x}^k = (x_1^k, x_2^k, \ldots, x_n^k) \in \mathbb{R}^{dn}$,

1) for every $i = 1, 2, \ldots, n$, define

$$\mathcal{X}_{iq}^k := \begin{cases} \{z \mid \|z - x_q^k\| \leq \hat{l}_{iq}\}, & \text{for } i < q, \\ \{z \mid \|z - x_q^{k+1}\| \leq \hat{l}_{iq}\}, & \text{for } i > q, \end{cases} \tag{13}$$

and calculate

$$x_i^{k+1} = \frac{1}{2}x_i^k + \frac{1}{2(|\mathcal{A}_i| + |\mathcal{B}_i|)} \left( \sum_{j \in \mathcal{A}_i} P_{\mathcal{C}_{ij}}(x_i^k) \right.$$
$$\left. + \sum_{q \in \mathcal{B}_i} P_{\mathcal{X}_{iq}^k}(x_i^k) \right), \tag{14}$$

2) set

$$\mathbf{x}^{k+1} = (x_1^{k+1}, x_2^{k+1}, \ldots, x_n^{k+1}). \tag{15}$$

**Stopping criterion:** If $\|\mathbf{x}^{k+1} - \mathbf{x}^k\|$ is small enough then stop.

Note that in the general iterative step there are $n$ update-steps (14) inside the $k$-th iteration, leading from $\mathbf{x}^k$ to $\mathbf{x}^{k+1}$. In the $i$th update step, we determine a new approximate location $x_i^{k+1}$ of the unknown target $x_i$ by computing projections onto balls around reference points and around target points for $q \neq i$ and then taking the mid-point on the line segment between the average of projections and the old approximate location $x_i^k$ of the target $x_i$.

### IV. THE MATHEMATICAL MODEL OF THE PROBLEM

Let us consider the mathematical model of the problem that is obtained by unconstrained minimization

$$\begin{cases} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in \mathbb{R}^{dn} \end{cases} \tag{16}$$

of the objective function $f : \mathbb{R}^{dn} \to \mathbb{R}$ given by:

$$f(\mathbf{x}) := \sum_{i=1}^{n} \sum_{j \in \mathcal{A}_i} (\max\{\|x_i - a_j\| - \hat{d}_{ij}, 0\})^2$$
$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{q \in \mathcal{B}_i} (\max\{\|x_i - x_q\| - \hat{l}_{iq}, 0\})^2 \tag{17}$$

for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $x_i \in \mathbb{R}^d$ for all $i = 1, 2, \ldots, n$. The factor $1/2$ in (17) is motivated by the fact that each term in the sum is repeated twice, due to the symmetry of the measurements ($q \in \mathcal{B}_i \Rightarrow i \in \mathcal{B}_q$ and $\hat{l}_{iq} = \hat{l}_{qi}$). The objective function $f$ is the sum of convex functions and is therefore convex. Obviously, the optimal value of this optimization problem is zero if we assume that there exists a point which satisfies all distance requirements. Under the last mentioned existence assumption, the motivation for introducing the optimization problem (16) is that any optimal solution of the problem (16)-(17) is a solution of the ICFP Problem 4.

Note that the objective function can also be rewritten equivalently in the following form:

$$f(\mathbf{x})$$
$$= \sum_{i=1}^{n} \sum_{j \in \mathcal{A}_i} \|x_i - P_{\mathcal{C}_{ij}}(x_i)\|^2 + \frac{1}{2} \sum_{i=1}^{n} \sum_{q \in \mathcal{B}_i} \|x_i - P_{\mathcal{X}_{iq}}(x_i)\|^2, \tag{18}$$

where $P$ is the orthogonal projection operator and the sets $\mathcal{C}_{ij}$ and $\mathcal{X}_{iq}$ are defined in (4) and (5), respectively.

We will later show that any sequence $\{\mathbf{x}^k\}_{k=0}^{\infty}$ generated by Algorithm 1 is such that its accumulation points solve (16) with the objective function (18). This convergence property holds regardless if the ICFP in (12) is consistent or not.

*Remark 3:* There are related approaches in the literature to solve the positioning problem. For example, the authors in [35] consider a mini-max approach and obtain a positioning problem based on convex relaxation. The nonconvex problem originating from the least squares criterion can be transformed to a convex problem by adopting, e.g., a convex relaxation [12]. For sparse networks, the author of [9] formulated a positioning algorithm based on sum of squares and derived a convex

optimization problem. The main difference between most of the available methods in the literature and the algorithm developed in this paper, is that the algorithm proposed here is a projection method that has several advantages as discussed, e.g., in [28, Subsection 3.2].

## V. SOME PROPERTIES OF THE OBJECTIVE FUNCTION

The following proposition contains some important properties of the objective function $f$ that are required for the convergence analysis in the sequel.

*Proposition 4:* The function $f$ of (18) is convex, continuously differentiable, and its gradient is block-coordinate-wise Lipschitz continuous with constants $L_i \triangleq 4(|\mathcal{A}_i| + |\mathcal{B}_i|)$ for all $i = 1, 2, \ldots, n$.

*Proof:* Convexity of the function $f$ follows from its representation in (17) where each summand is a composition of a square function and a nonnegative convex max-type function and thus $f$ is convex. Let us consider the following equivalent representation of (18):

$$f(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j \in \mathcal{A}_i} \|x_i - P_{\mathcal{C}_{ij}}(x_i)\|^2 + \sum_{q \in \mathcal{B}_s} \|x_s - P_{\mathcal{X}_{sq}}(x_s)\|^2$$

$$-\frac{1}{2} \sum_{q \in \mathcal{B}_s} \|x_s - P_{\mathcal{X}_{sq}}(x_s)\|^2 + \frac{1}{2} \sum_{i=1, i \neq s}^{n} \sum_{q \in \mathcal{B}_i} \|x_i - P_{\mathcal{X}_{iq}}(x_i)\|^2.$$
(19)

Note that for each pair of target neighbours $i, q$ the following identity is satisfied:

$$\|x_i - P_{\mathcal{X}_{iq}}(x_i)\| = \|x_q - P_{\mathcal{X}_{qi}}(x_q)\|. \tag{20}$$

Substituting (20) into (19) we get:

$$f(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j \in \mathcal{A}_i} \|x_i - P_{\mathcal{C}_{ij}}(x_i)\|^2 + \sum_{q \in \mathcal{B}_s} \|x_s - P_{\mathcal{X}_{sq}}(x_s)\|^2$$

$$+ \frac{1}{2} \sum_{i=1, i \neq s}^{n} \sum_{q \in \mathcal{B}_i, q \neq s} \|x_i - P_{\mathcal{X}_{iq}}(x_i)\|^2. \tag{21}$$

Thus, the partial derivative of $f$ at $(x_1, x_2, \ldots, x_n)$ with respect to the variables in the subvector $x_s$ is given by:

$$\nabla_s f(x_1, x_2, \ldots, x_n) = \sum_{j \in \mathcal{A}_s} 2(x_s - P_{\mathcal{C}_{sj}}(x_s))$$

$$+ \sum_{q \in \mathcal{B}_s} 2(x_s - P_{\mathcal{X}_{sq}}(x_s)), \tag{22}$$

and the gradient of $f$ is then given by

$$\nabla f(x_1, x_2, \ldots, x_n)$$
$$= \Big( \nabla_1 f(x_1, x_2, \ldots, x_n), \ldots, \nabla_n f(x_1, x_2, \ldots, x_n) \Big). \tag{23}$$

From continuity of $\nabla_s f(x_1, x_2, \ldots, x_n)$ for all $s = 1, 2, \ldots, n$, we conclude that the gradient of $f$ is continuously differentiable.

It remains to show that gradient is block-coordinate-wise Lipschitz continuous. Let us take two vectors $(x_1, x_2, \ldots, y_i, \ldots, x_n)$ and $(x_1, x_2, \ldots, x_i, \ldots, x_n)$. These two vectors are identical except for elements which correspond to the $i$-th subvector and we have

$$\|\nabla_i f(x_1, x_2, \ldots, y_i, \ldots, x_n) - \nabla_i f(x_1, x_2, \ldots, x_i, \ldots, x_n)\|$$
$$\leq 4(|\mathcal{A}_i| + |\mathcal{B}_i|)\|x_i - y_i\|,$$

where in the last inequality, we used the nonexpansivity of the projection operator $P$, i.e.,

$$\|P(x) - P(y)\| \leq \|x - y\|.$$

Thus, we have shown that the gradient of $f$ is block-coordinate-wise Lipschitz continuous with constants $4(|\mathcal{A}_i| + |\mathcal{B}_i|)$ for each block. ∎

## VI. CONVERGENCE ANALYSIS

In Proposition 4 we have shown that many, but not all, assumptions required for the convergence analysis of the Block Coordinate Gradient Descent (BCGD) algorithm in [36] are satisfied. In what follows, we present an alternative and self-contained proof of convergence of Algorithm 1, which does not require Lipschitz continuity of the gradients of the function $f$, as required in the proof of the BCGD algorithm in [36]. In our analysis we will make use of following results.

*Lemma 5:* (**Descent Lemma**, Appendix A in [37]) If $g : \mathbb{R}^p \to \mathbb{R}$ is a continuously differentiable function whose gradient $\nabla g$ is Lipschitz continuous with constant $L$ then

$$g(y) \leq g(x) + \langle \nabla g(x), y - x \rangle + (L/2)\|x - y\|^2 \text{ for all } x, y \in \mathbb{R}^p. \tag{24}$$

*Proposition 6:* For any $\mathbf{x}^0 \in \mathbb{R}^{dn}$ and for the function $f$ of (18) the level set

$$S = \{\mathbf{x} \mid f(\mathbf{x}) \leq f(\mathbf{x}^0)\} \tag{25}$$

is bounded.

*Proof:* The function $f$ of (18) is coercive, i.e., for all $\mathbf{x}$ such that $\|\mathbf{x}\| \to \infty$ we have $f(\mathbf{x}) \to \infty$. Thus, all level sets of $f$ are bounded. ∎

In the next proposition we show that any sequence generated by Algorithm 1 entails a non-increasing sequence of function values $\{f(\mathbf{x}^k)\}_{k=0}^{\infty}$.

*Proposition 7:* Let $\{x_i^k\}_{k=0}^{\infty}$ for all $i = 1, 2, \ldots, n$, be sequences generated by Algorithm 1 and let $\{f(\mathbf{x}^k)\}_{k=0}^{\infty}$ be the associated sequences of function values. Denote, for $0 \leq i \leq n$,

$$\mathbf{x}^{k,i} := (x_1^k, x_2^k, \ldots, x_i^k, x_{i+1}^{k-1}, \ldots, x_n^{k-1}). \tag{26}$$

Then, for every $k = 1, 2, \ldots$, we have

$$f(\mathbf{x}^k) \leq f(\mathbf{x}^{k-1}) \tag{27}$$

and

$$\lim_{k \to \infty} \|\nabla_i f(\mathbf{x}^{k,i-1})\| = 0, \text{ for all } i = 1, 2, \ldots, n, \tag{28}$$

and

$$\lim_{k \to \infty} \|\mathbf{x}^{k,i} - \mathbf{x}^{k,i-1}\| = 0, \text{ for all } i = 1, 2, \ldots, n. \tag{29}$$

*Proof:* Obviously, we have $\mathbf{x}^k = \mathbf{x}^{k,n}$ and $\mathbf{x}^{k-1} = \mathbf{x}^{k,0}$. In Proposition 4, we have shown that the gradient of $f$

is block-coordinate-wise Lipschitz continuous with constants $L_i = 4(|\mathcal{A}_i| + |\mathcal{B}_i|)$ for all $i = 1, 2, \ldots, n$. Thus, by Lemma 5, we have for all $i = 1, 2, \ldots, n$,

$$f(\mathbf{x}^{k,i}) \leq f(\mathbf{x}^{k,i-1}) + \langle \nabla_i f(\mathbf{x}^{k,i-1}), x_i^k - x_i^{k-1} \rangle \\ + (L_i/2)\|x_i^k - x_i^{k-1}\|^2, \tag{30}$$

where we used the equality

$$\|\mathbf{x}^{k,i} - \mathbf{x}^{k,i-1}\| = \|x_i^k - x_i^{k-1}\| \tag{31}$$

and

$$\langle \nabla f(\mathbf{x}^{k,i-1}), \mathbf{x}^{k,i} - \mathbf{x}^{k,i-1} \rangle = \langle \nabla_i f(\mathbf{x}^{k,i-1}), x_i^k - x_i^k \rangle. \tag{32}$$

From the construction of the iterative sequence in Algorithm 1 we have, for all $i = 1, 2, \ldots, n$ and for all $k > 0$,

$$x_i^k - x_i^{k-1} = -(1/L_i)\nabla_i f(\mathbf{x}^{k,i-1}). \tag{33}$$

Plugging (33) into (30) yields

$$f(\mathbf{x}^{k,i-1}) - f(\mathbf{x}^{k,i}) \geq (1/2L_i)\|\nabla_i f(\mathbf{x}^{k,i-1})\|^2, \ \ i = 1, \ldots, n. \tag{34}$$

Summing over all inequalities and denoting $\tau := \max\{L_i \mid i = 1, 2, \ldots, n\}$ we get

$$f(\mathbf{x}^{k-1}) - f(\mathbf{x}^k) \geq (1/2\tau)\sum_{i=1}^{n} \|\nabla_i f(\mathbf{x}^{k,i-1})\|^2, \tag{35}$$

thus, proving (27). From (35) it follows that

$$f(\mathbf{x}^0) \geq (1/2\tau)\sum_{k=0}^{\infty}\sum_{i=1}^{n} \|\nabla_i f(\mathbf{x}^{k,i-1})\|^2, \tag{36}$$

which yields (28), since by interchanging the summation, an infinite sum of positive values in the right-hand side is bounded from above by $f(\mathbf{x}^0)$. Note that (33) can be equivalently rewritten in the form

$$\|\mathbf{x}^{k,i} - \mathbf{x}^{k,i-1}\| = \|(1/L_i)\nabla_i f(\mathbf{x}^{k,i-1})\| \tag{37}$$

which can be combined with (28) to yield the desired result (29). ■

*Corollary 8:* Any sequence $\{\mathbf{x}^k\}_{k=0}^{\infty}$ generated by Algorithm 1 is bounded.

*Proof:* From Proposition 7, it follows that the sequence $\{\mathbf{x}^k\}_{k=0}^{\infty}$ belongs to the bounded level set $S = \{\mathbf{x} \mid f(\mathbf{x}) \leq f(\mathbf{x}^0)\}$, thus, by Proposition 6, it is bounded. ■

Now we are ready to present and prove the main result.

*Theorem 9:* Let $\{\mathbf{x}^k\}_{k=0}^{\infty}$ be any sequence generated by Algorithm 1. Then every limit point of $\{\mathbf{x}^k\}_{k=0}^{\infty}$ is an optimal solution of the problem (16) with the objective function (18) and the sequence $\{f(\mathbf{x}^k)\}_{k=0}^{\infty}$ converges to the optimal value $f^*$ of (16).

*Proof:* From Corollary 8, it follows that there exist converging subsequence of the sequences $\{\mathbf{x}^k\}_{k=0}^{\infty}$. Let $\{\mathbf{x}^{k_j}\}_{j=0}^{\infty}$ be such a convergent subsequence and denote by $\mathbf{x}^*$ its limit point. We show that $\nabla f(\mathbf{x}^*) = 0$, thus proving the optimality of the point $\mathbf{x}^*$.

Consider the sequence $\{\mathbf{x}^{k_j}\}_{j=0}^{\infty} \cup \{\{\mathbf{x}^{k_j,i}\}_{i=1}^{n}\}_{j=0}^{\infty}$ obtained by combining these two sequences and denote it by $\{\mathbf{y}^\ell\}_{\ell=0}^{\infty}$. From (29) in Proposition 7, it follows that the sequence

$\{\mathbf{y}^\ell\}_{\ell=0}^{\infty}$ also converges to the point $\mathbf{x}^*$. The sequence of gradients $\{\nabla f(\mathbf{y}^\ell)\}_{\ell=0}^{\infty}$ then converges to $\nabla f(\mathbf{x}^*)$ and so does every subsequence of it.

For the subsequence $\{\nabla f(\mathbf{x}^{k_j,1})\}_{j=0}^{\infty}$, we conclude, from (28) of Proposition 7, that it converges to a vector that has zeros in all its components that refer to the part that comes from the partial derivatives with respect to components of $x_1$. Thus, $\nabla f(\mathbf{x}^*)$ contains zeros in the same components. Repeating this argument we reach the conclusion that all elements of $\nabla f(\mathbf{x}^*)$ are zeros.

From (27) it follows that the sequence $\{f(\mathbf{x}^k)\}_{k=0}^{\infty}$ is non-increasing and, since it is also bounded from below by $f^*$, it converges. Since we showed that any converging subsequence $\{\mathbf{x}^{k_j}\}_{j=0}^{\infty}$ converges to the optimal solution, the subsequence $\{f(\mathbf{x}^{k_j})\}_{j=0}^{\infty}$ converges to the optimal value $f^*$, thus the entire sequence $\{f(\mathbf{x}^k)\}_{k=0}^{\infty}$ converges to $f^*$. ■

*Remark 10:* It is noted that Algorithm 1 converges even if the intersection in (6) does not contain the target nodes or even if it is empty–which could happen for negative measurement errors. In fact, the proposed algorithm solves the optimization problem in (16), regardless of the extent of the intersection in (1).

## VII. NUMERICAL RESULTS

In this section, we evaluate the performance of Algorithm 1 and two methods proposed in [29] and in [30] through computer simulations. To evaluate the different methods, we consider a 2-dimensional network consisting of a number of reference and target nodes. We will evaluate different algorithms ability to localize target nodes under line-of-sight (LOS) and non-line-of-sight (NLOS) conditions.

The two algorithms of [29] and [30] as well as Algorithm 1 cycle through the networks in an ordered fashion. Although it is possible to update the location of target nodes in an arbitrarily ordered fashion, which may be more suitable for a practical implementation, algorithms exchange the updated position estimates between nodes in the order $1, 2, \ldots, n$, where $n$ is the number of target nodes. In [30] the authors used a method based on projections (in parallel) onto spheres (projections onto the boundary of each individual set). This method is sensitive to the choice of the initial points and it may converge to local minima resulting in large errors. To avoid converging to local minima the algorithm of [30] is initialized at a target node with the position of the closest reference node connected to that target. However, in general, finding good initial estimates for all target nodes is a challenging task in positioning problems. In [29], a method based on projections onto convex sets (POCS) was considered to localize target nodes. Note that contrary to the algorithm proposed in this paper, there are no formal convergence proofs for the algorithms introduced in [29], [30].

### A. Simulation setup

The methods proposed in [29], [30], and Algorithm 1, are henceforth called cooperative parallel projection onto boundary (Coop. PPB), cooperative projection onto convex sets
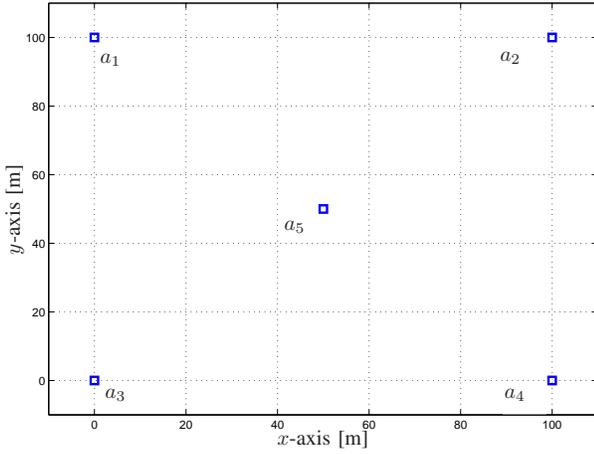
Fig. 3. Layout of the network considered in simulations, where squares denote the positions of reference nodes and target nodes are randomly distributed inside the convex hull of the reference nodes.

(Coop. POCS), and cooperative parallel projection method (Coop. PPM), respectively. We have conducted computer simulations for different scenarios and evaluated the algorithms based on two metrics: convergence and accuracy. We have considered a $100 \times 100$ m$^2$ square area shown in Fig. 3 with a number of reference nodes at fixed positions. In the simulation, we randomly distributed a number of target nodes inside the area[1].

The measurement errors in (3) are modeled as

$$\varepsilon_{mn} = \begin{cases} \varepsilon_{G,mn}, & \text{LOS conditions,} \\ \varepsilon_{G,mn} + b_{mn}\varepsilon_{U,mn}, & \text{NLOS conditions,} \end{cases}$$

where $\varepsilon_{G,mn}$ are iid zero-mean Gaussian random variables with variance $\sigma^2$, $b_{mn} \in \{0,1\}$ are iid Bernoulli random variables with parameter $p_{\text{NLOS}} = \Pr\{b_{mn} = 1\}$, and $\varepsilon_{U,mn} \in [0, L]$ are iid uniformly distributed random variables. The validity of uniform distribution to model NLOS noise has been considered in various works, e.g., [29], [38], [39]. In our simulations, we used $\sigma = 1$ m, $p_{\text{NLOS}} = 0.2$, and $L = 20$ m. The connectivity is defined based on the actual distances between sensor nodes. Namely, if the distance between two nodes is 40 m or less, we assume that the nodes are connected. To assess the positioning algorithms, we consider the cumulative distribution function (CDF) of the position error for all target nodes. That is, we define the position error for target node $i$ as

$$E_{i,m} = \|x_{i,m}^K - x_{i,m}\|, \quad i = 1, 2, \ldots, n, \quad m = 1, 2, \ldots, N \tag{38}$$

where $x_{i,m}^K$ is an estimate of target node $i$ position, $x_{i,m}$, after $K$ iterations for the $m$th network realization, and $N$ is the total number of network realizations. In the results below, we have used $K = 300$, which is large enough to ensure convergence and then compute the empirical CDF of the position error for

[1]We have also assessed the algorithms for 3-dimensional networks. The results show similar behavior as for 2-dimensional networks.

all target nodes collected in $\mathbf{E}$ as

$$\text{CDF}(\alpha) = \Pr(\text{position error} \leq \alpha)$$
$$= \frac{\sum_{i=1}^{n} \sum_{m=1}^{N} I(E_{i,m} - \alpha)}{nN}, \tag{39}$$

where the indicator function $I(t)$ is defined as

$$I(t) = \begin{cases} 1, & \text{if } t \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{40}$$

To study the convergence rate, we consider the average residual defined as

$$\bar{r}_k = 1/(nN) \sum_{m=1}^{N} \|\mathbf{x}_m^k - \mathbf{x}_m^{k-1}\|, \tag{41}$$

where

$$\mathbf{x}_m^k = \left(x_{1,m}^k, x_{2,m}^k \ldots, x_{n,m}^k\right). \tag{42}$$

To initialize the algorithm of [30], we set the initial estimate for a target node as the position of the closest reference node connected to the target. For a target $i$ which is not connected to any reference node, we pick the position of the closest target (already initialized) connected to the target node $i$.

### B. Performance of algorithms

In this section, we assess the above mentioned algorithms for LOS and NLOS scenarios. In Fig. 4 and Fig. 5, we plot the CDF of the position error of the algorithms for various numbers of reference and target nodes in the LOS scenario. The figures show a superior performance for the proposed approach Coop. PPM compared with the others, especially with respect to Coop. PPB. Coop. PPB has poor performance for both small number of reference nodes and large number of target nodes. The reason is that finding good initial points is difficult when the number of reference nodes decreases or the number target nodes increases. From the plots, we observe that Coop. POCS shows good performance compared with Coop. PPB.

To further investigate the algorithms, we assume 20% of distance measurements are NLOS. The position error CDFs are plotted in Fig. 6 for different numbers of target nodes in NLOS conditions. In this simulation, we used four reference nodes $a_1, \ldots, a_4$. From the plots, we observe that Coop. POCS and Coop. PPM show better performance compared with Coop. PPB and that they are robust against NLOS measurements. The poor performance of Coop. PPB in the considered scenario, is mainly due to poor initialization.

Based on the results presented here and on additional simulation results that we performed, we can conclude that the method proposed in this paper has superior performance compared with other projection approaches available in the positioning literature for sparse networks in which target nodes are connected to a few other nodes.
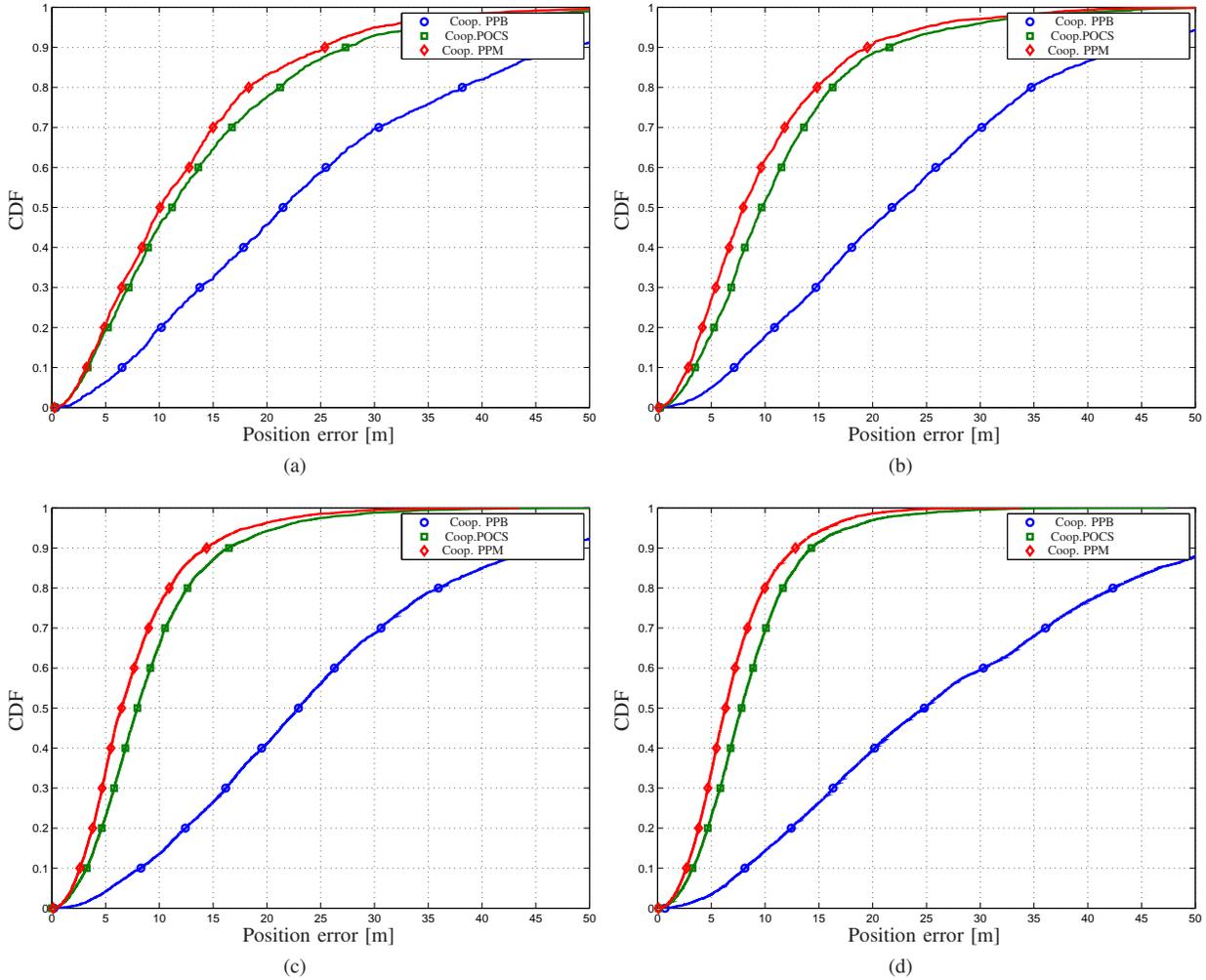
Fig. 4. The CDF of the position error of different algorithms in LOS scenarios for four reference nodes $a_1, a_2, a_3, a_4$ and (a) 20 target nodes, (b) 30 target nodes, (c) 60 target nodes, and (d) 100 target nodes.

## C. Convergence speed

In this section, we evaluate the convergence speed of the above mentioned algorithms through simulations. We consider the LOS scenario for the network shown in Fig. 3. The convergence speed is plotted in Fig. 7 for different numbers of target nodes. For Coop. POCS, we use $5D_i$ iterations for locally updating target node $i$, where $D_i$ is the number of sensor nodes (reference or localized target) with known or estimated location connected to target $i$. That is, for target $i$, we continue sequential projection onto $D_i$ balls corresponding to the nodes connected to target $i$. We first set the relaxation parameters equal to one and then decrease them to the value used in [29] after $3D_i$. In the figure, the $x$-axis shows the number of iterations for updating the vector $\mathbf{x}^k = \left(x_1^k, x_2^k, \ldots, x_n^k\right)$. It is observed that all algorithms converge after a few iterations. The Coop. POCS uses more local updatings, and it shows faster global convergence compared to the two other methods. Finally, from this figure we also see that the convergence rate of Coop. PPB may not be monotone, while Coop. PPM and Coop. POCS show monotonic convergence. In fact, the Coop. PPB, objective function is nonconvex, and we need to ensure that the starting point is sufficently close to the global

solution, since the algorithm might otherwise converge to a local minimum. We also note that the proposed algorithm has relatively slow convergence after, say, 40–50 iterations. However, monotonic convergence is guaranteed by Theorem 9.

## VIII. CONCLUSIONS

In this paper, we have considered a geometric interpretation of the cooperative positioning problem in wireless sensor networks and formulated the position problem as an implicit convex feasibility problem. We have then proposed a distributed algorithm based on projections approach to solve the problem. The proposed algorithm enjoys parallel implementation capabilities and is suitable for practical scenarios. We have also proven that the algorithm converges to the desired minimizer of an intuitively pleasing cost function, (18), regardless if the implicit convex feasibility problem is consistent or not. Simulation results show an enhanced performance of the proposed approach compared with available algorithms for sparse networks. Since the proposed algorithm has low convergence speed, one possible open problem for future studies is to improve the convergence speed of the algorithm, while maintaining convergence.
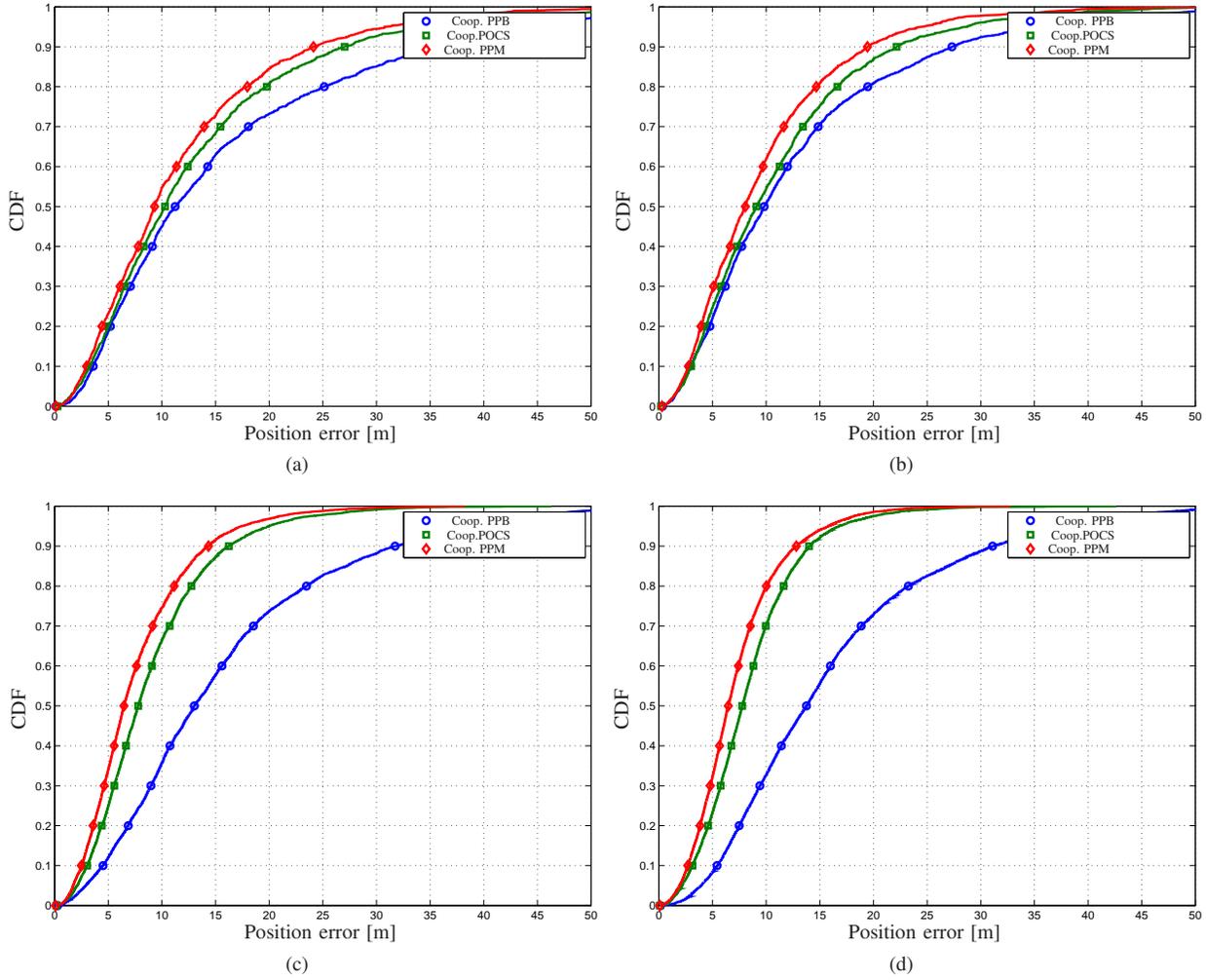
Fig. 5. The CDF of the position error of different algorithms in LOS scenario for five reference nodes $a_1, a_2, a_3, a_4, a_5$ and (a) 20 target nodes, (b) 30 target nodes, (c) 60 target nodes, and (d) 100 target nodes.
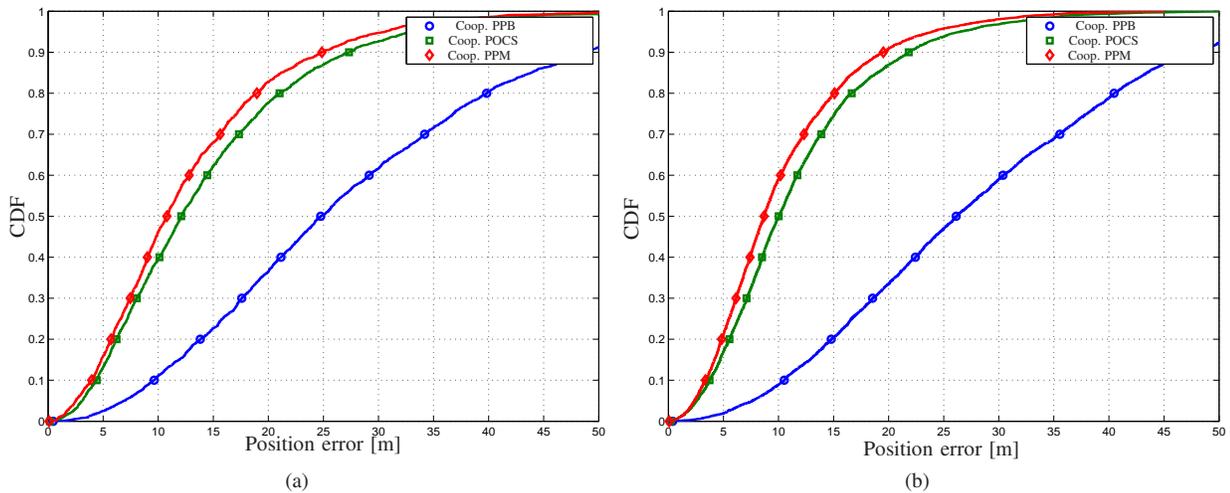


Fig. 6. The CDF of the position error in NLOS scenario for four reference nodes and (a) 30 target nodes and (b) 60 target nodes.
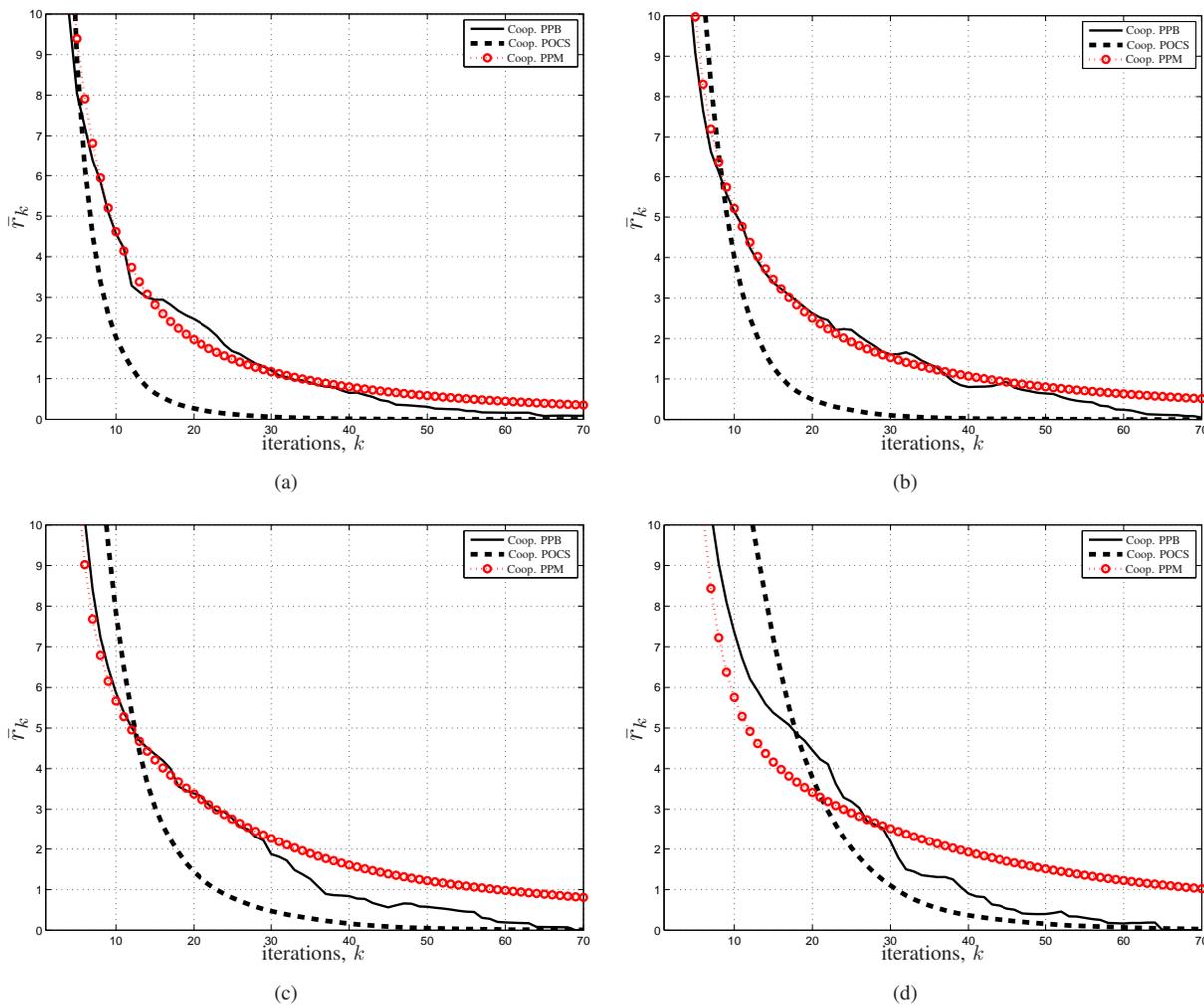
Fig. 7. Convergence speed of average residuals $\bar{r}_k = 1/(nN)\sum_{m=1}^{N} \|\mathbf{x}_m^k - \mathbf{x}_m^{k-1}\|$ versus the number of iterations, $k$, for different algorithms in LOS conditions for five reference nodes and (a) 20 target nodes, (b) 30 target nodes, (c) 60 target nodes, and (d) 100 target nodes.

## IX. Acknowledgments

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