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Zero-mode dynamics in supersymmetric Yang-Mills-Chern-Simons theory

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We consider minimally supersymmetric Yang-Mills theory with a Chern-Simons term on a flat spatial two-torus in the limit when the torus becomes small. The zero modes of the fields then decouple from the nonzero modes and give rise to a spectrum of states with energies that are given by multiples of the square of the coupling constant. We discuss the determination of this low-energy spectrum, both for simply connected gauge groups and for gauge groups of adjoint type, with a few examples worked out in detail.

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I. INTRODUCTION

In three space-time dimensions, the Yang-Mills coupling constant e has the dimension of $(\text{mass})^{1/2}$. Such theories are thus trivial in the ultraviolet but can be expected to have nontrivial infrared dynamics. In this paper, we will study them on a flat spatial two-torus

$$T^2 = \mathbb{C}/(L\mathbb{Z} + \tau L\mathbb{Z}), \tag{1.1}$$

where the modular parameter $\tau = \tau_1 + i\tau_2$ is valued in the complex upper half-plane and L is a length [of dimension $(\text{mass})^{-1}$]. Most of the quantum states involve the nonzero modes of the fields. Their energy eigenvalues depend on e, L and τ (presumably in a quite complicated way) and diverge in $L \to 0$ limit. However, some states involve only the zero modes of the fields, and their energies will be given by finite multiples of e^2 in this limit. Computing this weak coupling (or equivalently small volume) spectrum is the aim of the present paper.

More specifically, we will be considering a supersymmetric Yang-Mills-Chern-Simons theory with action

$$S = S_{\text{YM}} + S_{\text{CS}} + S_{\text{Fermion}}, \tag{1.2}$$

where the usual pure Yang-Mills action

$$S_{\text{YM}} = \frac{1}{4e^2} \int \text{Tr}(F \wedge *F) \tag{1.3}$$

is complemented by a Chern-Simons term [1]

$$S_{\text{CS}} = \frac{k}{4\pi} \int \text{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \tag{1.4}$$

and fermionic terms

$$S_{\text{Fermion}} = \frac{1}{4e^2} \int d^3x \text{Tr}(\bar{\lambda} \not{D} \lambda) + \frac{k}{4\pi} \int \text{Tr}(\bar{\lambda}\lambda). \tag{1.5}$$

Here the fermionic field λ is a Majorana spinor in the adjoint representation of the gauge group G. For topological reasons, the level k is quantized. This is best expressed in terms of a shifted level [2,3]

$$k' = k - h/2, \tag{1.6}$$

where h is the dual Coxeter number of G. For a simply connected gauge group

$$k' = 0 \mod \mathbb{Z}, \tag{1.7}$$

and for a gauge group of adjoint type (i.e., a simply connected group divided by its center)

$$k' = 0 \mod h\mathbb{Z}. \tag{1.8}$$

This theory has a mass gap which is visible already in perturbation theory, which makes many general problems of quantum field theory more accessible. By supersymmetry, the energy spectrum is non-negative, and a very basic question is to understand the structure of zero-energy states.

The number of such states (i.e., the Witten index of the theory [4]) was elegantly determined for simply connected gauge groups in [5] by algebro-geometric means. In short, the computation was done in the large volume limit, where the dynamics is dominated by the Chern-Simons term. The ground state wave functions localize on the moduli space of flat connections, the geometric structure of which has been described in the earlier mathematical literature in terms of (weighted) complex projective spaces. Similar results were obtained in [6] by rather different means. Here the computation was carried out in a Born-Oppenheimer approximation in the small volume limit, rather like in [4]. An important subtlety in this approach is how to correctly take bosonic and fermionic loop corrections into account.

In [7], the computation is redone in a third, arguably more pedestrian and straightforward fashion. This method has the advantage of also allowing for a determination of the actual wave functions of the states. In the cases with gauge groups of adjoint type, it allows for a refinement where the states can be further classified by their discrete electric and magnetic 't Hooft fluxes. For convenience, the main results of [7] will be reviewed in the present paper, but we will not repeat all the details. Our main focus here is instead to go further in this direction by determining the complete spectrum of low-energy states (i.e., states the energies of which remain finite in the $L \to 0$ limit).

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A natural next step would be to determine the complete spectrum in the weak coupling limit (i.e., the $e \longrightarrow 0$ limit with L fixed). We note that somewhat similar questions for Yang-Mills-Chern-Simons theory in noncompact space have been considered in e.g., [8].

In the next section, we will describe the low-energy theory. The determination of the low-energy spectrum for a simply connected gauge group is discussed in Sec. III and performed in full detail in the SU(2) and SU(3) cases. In Sec. IV, we consider the refinement with discrete electric and magnetic 't Hooft fluxes that is possible with a gauge group of adjoint form [9] and exemplify with $SU(2)/\mathbb{Z}_2$.

II. THE LOW-ENERGY THEORY

Starting with the bosonic degrees of freedom, we note that the Hamiltonian of the theory contains a magnetic contribution proportional to $\frac{1}{e^2}L^{-2}$ and the square of the magnetic field strength. This term diverges as $L \longrightarrow 0$ unless it vanishes, so in this limit, the low-energy theory localizes on gauge field configurations of zero magnetic field strength. Such spatially flat connections are completely determined by their commuting holonomies around the two cycles of T^2. By a gauge transformation (acting by conjugation) these can be taken to lie in a maximal torus subgroup of G. Furthermore, the complex structure of T^2 induces a complex structure on the space of such pairs of holonomies. They can thus be assembled into an element Z of the complex torus

$$X = \mathbb{C} \otimes V/(\Gamma \otimes \Lambda). \tag{2.1}$$

Here V is the root space and Λ the root lattice of the gauge group G so that V/Λ is a maximal torus, and Γ is the lattice $\mathbb{Z} + \tau\mathbb{Z}$ so that $\mathbb{C}/\Gamma$ is a torus "dual" to the spatial T^2. The bosonic low-energy degrees of freedom Z can be viewed as the zero modes of the gauge field A restricted from the non-Abelian Lie algebra of G to a Cartan subalgebra. Finally, we must identify elements of X that are related by the residual gauge symmetry, so the moduli space of spatially flat connections is really

$$\mathcal{M} = X/W, \tag{2.2}$$

where W is the Weyl group of G.

The vector space underlying the torus X has a metric $\mathcal{C}_{ab}$, $a, b = 1, \ldots,$ rank G given by the Killing form of G restricted to the maximal torus, and the bosonic variable Z should in more detail be written as a multicomponent vector Z^a. We will suppress this from the notation, though, and just denote the scalar product between such vectors with a raised dot. Hopefully, this will not cause any confusion.

We continue with the fermionic degrees of freedom. At a generic point of the moduli space $\mathcal{M}$, the gauge group G is spontaneously broken by the holonomies to the Abelian subgroup V. Some components of the spinor field λ are then massive and can be integrated out. This procedure is responsible for the shift of the Chern-Simons level from k to $k' = k - h/2$ as mentioned in the introduction [2,3]. We denote the remaining fermionic degrees of freedom, which consist of spatially constant V-valued modes, as η_+ and η_-.[1]

The dynamics of the low-energy degrees of freedom Z, $\bar{Z}$, η_+ and η_- is governed by the action

$$S = \frac{1}{4e^2}\int dt\left(\frac{dZ}{dt}\cdot\frac{d\bar{Z}}{dt} + \eta_+\cdot\frac{d\eta_-}{dt} + \eta_-\cdot\frac{d\eta_+}{dt}\right) + \frac{\pi k'}{\tau - \bar{\tau}}\int dt\left(Z\cdot\frac{d\bar{Z}}{dt} - \bar{Z}\cdot\frac{dZ}{dt} + \eta_+\cdot\eta_-\right). \tag{2.3}$$

The conserved supercharges are

$$Q_+ = \frac{1}{4e^2}\eta_-\cdot\frac{dZ}{dt} \qquad Q_- = \frac{1}{4e^2}\eta_+\cdot\frac{d\bar{Z}}{dt}. \tag{2.4}$$

By a standard canonical analysis and quantization in a Schrödinger representation, they correspond to operators

$$Q_+ = e\eta_-\cdot i\frac{D}{DZ} \qquad Q_- = e\eta_+\cdot i\frac{D}{D\bar{Z}}. \tag{2.5}$$

Here the covariant derivatives are given by

$$\frac{D}{DZ} = \frac{\partial}{\partial Z} - \frac{i\pi k'}{\tau - \bar{\tau}}\bar{Z} \qquad \frac{D}{D\bar{Z}} = \frac{\partial}{\partial \bar{Z}} + \frac{i\pi k'}{\tau - \bar{\tau}}Z \tag{2.6}$$

and fulfill the commutation relations

$$\left[\frac{D}{DZ}, \frac{D}{DZ}\right] = 0 \quad \left[\frac{D}{D\bar{Z}}, \frac{D}{D\bar{Z}}\right] = 0 \quad \left[\frac{D}{DZ}, \frac{D}{D\bar{Z}}\right] = \frac{\pi k'}{\tau_2}\mathcal{C}, \tag{2.7}$$

i.e., the different components of $\frac{D}{DZ}$ commute with one another, whereas the commutators between the components $\frac{D}{DZ}$ and $\frac{D}{D\bar{Z}}$ are given by a multiple of the Killing metric. Analogously, the fermionic operators fulfil the canonical anticommutation relations

$$\{\eta_+, \eta_+\} = 0 \qquad \{\eta_-, \eta_-\} = 0 \qquad \{\eta_+, \eta_-\} = \frac{\tau_2}{\pi}\mathcal{C}. \tag{2.8}$$

As usual in a supersymmetric theory, the Hamiltonian can be written in a manifestly positive definite way as

$$H = \{Q_+, Q_-\} = e^2\left(\frac{\tau_2}{\pi}i\frac{D}{DZ}\cdot i\frac{D}{D\bar{Z}} - \frac{\pi k'}{\tau_2}\eta_-\cdot\eta_+\right). \tag{2.9}$$

The operators $\frac{D}{DZ}$ and η_+ can now be interpreted as bosonic and fermionic creation operators respectively in the sense that

[1] The subscript indicates the spatial chirality.

$$\left[H, \frac{D}{DZ}\right] = e^2 k' \frac{D}{DZ} \qquad [H, \eta_+] = e^2 k' \eta_+; \tag{2.10}$$

i.e., they raise the energy by an amount $e^2 k'$. The corresponding annihilation operators $\frac{D}{D\bar{Z}}$ and η_- obey

$$\left[H, \frac{D}{D\bar{Z}}\right] = -e^2 k' \frac{D}{DZ} \qquad [H, \eta_-] = -e^2 k' \eta_- \tag{2.11}$$

and lower the energy by $e^2 k'$. The supercharges Q_+ and Q_- of course commute with the Hamiltonian H:

$$[H, Q_+] = [H, Q_-] = 0. \tag{2.12}$$

The bosonic Hilbert space consists of square integrable "wave sections" $\Psi(Z, \bar{Z})$ of a certain line bundle $\mathcal{L}^{k'}$ over the complex torus X [7]. The fermionic Hilbert space is spanned by states obtained by acting with components of the creation operators η_+ on a fermionic vacuum state $|0\rangle$ which is annihilated by all components of η_-. The ground states of the theory (which are annihilated by Q_+ and Q_- and thus have zero energy) are then precisely given by the states of the form

$$\Psi(Z, \bar{Z}) \otimes |0\rangle, \tag{2.13}$$

where Ψ obeys the holomorphicity condition

$$\frac{D}{D\bar{Z}} \Psi(Z, \bar{Z}) = 0. \tag{2.14}$$

A distinguished solution to this condition is given by

$$\Psi(Z, \bar{Z}) = \exp\left(\frac{i\pi k'}{\tau - \bar{\tau}} (Z - \bar{Z}) \cdot Z\right) \psi(Z) \tag{2.15}$$

with the holomorphic theta function

$$\psi(Z) = \sum_{\lambda \in \Lambda} \exp\left(i\pi k' \tau \lambda \cdot \lambda + 2\pi i k' \lambda \cdot Z\right). \tag{2.16}$$

A basis of solutions is then given by

$$(T_{\epsilon\tau}\Psi)(Z, \bar{Z}) = \exp\left(\frac{i\pi k'}{\tau - \bar{\tau}} (Z - \bar{Z}) \cdot Z\right) (T_{\epsilon\tau}\psi)(Z), \tag{2.17}$$

for $\epsilon \in \frac{1}{k'} \Lambda^* / \Lambda$. Here Λ^* is the weight lattice of G which is dual to the root lattice Λ, and $T_{\epsilon\tau}\psi$ is obtained by translating ψ on X:

$$\begin{aligned}(T_{\epsilon\tau}\psi)(Z) &= \pm \exp\left(i\pi k' \tau \epsilon \cdot \epsilon + 2\pi i k' \epsilon \cdot Z\right) \psi(Z + \epsilon\tau) \\ &= \sum_{\lambda \in \Lambda} \exp\left(i\pi k' \tau (\lambda + \epsilon) \cdot (\lambda + \epsilon)\right. \\ &\qquad \left. + 2\pi i k' (\lambda + \epsilon) \cdot Z\right).\end{aligned} \tag{2.18}$$

The prefactor is included to give the quasiperiodicity conditions appropriate for a section of $\mathcal{L}^{k'}$. The Weyl group W of G leaves ψ invariant and acts on the sections $T_{\epsilon\tau}\psi$ via its permutation action on $\epsilon \in \frac{1}{k'} \Lambda^* / \Lambda$. See [7] for more details on this construction.

It appears that the complete set of states can be obtained by acting on the ground states $(T_{\epsilon\tau}\Psi)(Z, \bar{Z}) \otimes |0\rangle$ with the bosonic and fermionic creation operators $\frac{D}{DZ}$ and η_+. The Weyl group W acts not only on the ground states as described in the previous paragraph, but also on these excitations. The space of gauge invariant physical states is obtained by projecting the complete set of states onto its Weyl invariant subspace.

III. SIMPLY CONNECTED GAUGE GROUPS

Rather than describing the procedure to determine the spectrum in general, we will discuss two concrete examples with gauge groups $G = \mathrm{SU}(2)$ and $G = \mathrm{SU}(3)$ in detail. Other groups should be amenable to an analogous analysis, but the computations would get more involved. We do not expect any striking new phenomena to occur, though.

A. $G = \mathrm{SU}(2)$

With gauge group $G = \mathrm{SU}(2)$, the Weyl group W is isomorphic to the permutation group S_2 on two objects. It has two irreducible unitary representations, both of dimension one: the trivial representation **1**, and another representation **1′** in which the nontrivial element acts as -1. The tensor products of these are of course given by the multiplication table

$$\begin{array}{c|cc} & \mathbf{1} & \mathbf{1}' \\ \hline \mathbf{1} & \mathbf{1} & \mathbf{1}' \\ \mathbf{1}' & \mathbf{1}' & \mathbf{1}. \end{array} \tag{3.1}$$

The ground state label ϵ takes its values in

$$\frac{1}{k'} \Lambda^* / \Lambda \simeq \left\{ (a, b) \in \left(\frac{1}{2k'} \mathbb{Z}/\mathbb{Z}\right) \times \left(\frac{1}{2k'} \mathbb{Z}/\mathbb{Z}\right) \middle| a + b = 0 \right\}. \tag{3.2}$$

The Weyl group W acts by permutation on (a, b), so there are two Weyl invariants

$$(0, 0) \qquad \left(\frac{1}{2}, \frac{1}{2}\right) \tag{3.3}$$

and $k' - 1$ Weyl pairs

$$\left(\frac{l}{2k'}, \frac{2k' - l}{2k'}\right), \qquad \left(\frac{2k' - l}{2k'}, \frac{l}{2k'}\right) \quad \text{for } l = 1, \ldots, k' - 1. \tag{3.4}$$

This gives $k' + 1$ ground states that transform in the **1** representation

$$\Psi_{\mathbf{1}} = \begin{cases} \Psi \otimes |0\rangle \\ T_{(\frac{1}{2},\frac{1}{2})\tau}\Psi \otimes |0\rangle \\ \frac{1}{\sqrt{2}}(T_{(\frac{l}{2k'},\frac{2k'-l}{2k'})\tau} + T_{(\frac{2k'-l}{2k'},\frac{l}{2k'})\tau})\Psi \otimes |0\rangle \quad \text{for } l = 1, \ldots, k'-1, \end{cases} \tag{3.5}$$

and $k' - 1$ ground states in the $\mathbf{1}'$ representation

$$\Psi_{\mathbf{1}'} = \frac{1}{\sqrt{2}}\left(T_{(\frac{l}{2k'},\frac{2k'-l}{2k'})\tau} - T_{(\frac{2k'-l}{2k'},\frac{l}{2k'})\tau}\right)\Psi \otimes |0\rangle$$

for $l = 1, \ldots, k' - 1$. (3.6)

[Here Ψ is the distinguished Weyl invariant solution (2.15).] All these ground states have bosonic statistics, and the gauge invariant states are those that transform in the $\mathbf{1}$ representation, so we get the physical ground state spectrum

$$k' + 1 \quad \text{bosonic states with} \quad E = 0. \tag{3.7}$$

The bosonic and fermionic creation operators $\frac{D}{DZ}$ and η_+ both transform in the $\mathbf{1}'$ representation under W. Gauge invariant excited states can now be obtained in two different ways: either by acting on a ground state in the $\mathbf{1}$ representation with an even number of creation operators

$$\begin{aligned} &\frac{D^{2n}}{DZ^{2n}}\Psi_{\mathbf{1}} \otimes |0\rangle \quad \text{bosonic} \\ &\frac{D^{2n-1}}{DZ^{2n-1}}\Psi_{\mathbf{1}} \otimes \eta_+|0\rangle \quad \text{fermionic,} \end{aligned} \tag{3.8}$$

or by acting on a ground state in the $\mathbf{1}'$ representation with an odd number of operators

$$\begin{aligned} &\frac{D^{2n+1}}{DZ^{2n+1}}\Psi_{\mathbf{1}'} \otimes |0\rangle \quad \text{bosonic} \\ &\frac{D^{2n}}{DZ^{2n}}\Psi_{\mathbf{1}'} \otimes \eta_+|0\rangle \quad \text{fermionic.} \end{aligned} \tag{3.9}$$

So the spectrum of excited states is

$$\begin{aligned} &k'+1 \ \ \text{bosonic and fermionic states with} \ \ E = 2ne^2k' \\ &k'-1 \ \ \text{bosonic and fermionic states with} \ \ E = (2n+1)e^2k'. \end{aligned} \tag{3.10}$$

The second formula is valid for $n = 0, 1, \ldots,$ and the first formula only for $n = 1, 2, \ldots$.

B. $G = \mathrm{SU}(3)$

With gauge group $G = \mathrm{SU}(3)$, the Weyl group W is isomorphic to the permutation group S_3 on three objects. It has three irreducible unitary representations $\mathbf{1}$, $\mathbf{1}'$, and $\mathbf{2}$ of the indicated dimensions. The tensor products of these are given by the multiplication table

$$\begin{array}{c|ccc} & \mathbf{1} & \mathbf{1}' & \mathbf{2} \\ \hline \mathbf{1} & \mathbf{1} & \mathbf{1}' & \mathbf{2} \\ \mathbf{1}' & \mathbf{1}' & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{2} & \mathbf{2} & [\mathbf{1} \oplus \mathbf{2}]_S \oplus [\mathbf{1}']_A. \end{array} \tag{3.11}$$

The ground state label ϵ takes its values in

$$\frac{1}{k'}\Lambda^*/\Lambda \simeq \left\{(a,b,c) \in \left(\frac{1}{3k'}\mathbb{Z}/\mathbb{Z}\right) \times \left(\frac{1}{3k'}\mathbb{Z}/\mathbb{Z}\right) \times \left(\frac{1}{3k'}\mathbb{Z}/\mathbb{Z}\right) \middle| a+b+c = 0\right\}. \tag{3.12}$$

The Weyl group acts by permutation on (a, b, c), so there are three Weyl invariants

$$\left(\frac{l}{3k'}, \frac{l}{3k'}, \frac{l}{3k'}\right) \quad \text{for } l = 0, k', 2k', \tag{3.13}$$

$3k' - 3$ Weyl triplets

$$\begin{aligned} &\left(\frac{l}{3k'}, \frac{l}{3k'}, \frac{3k'-2l}{3k'}\right), \qquad \left(\frac{l}{3k'}, \frac{3k'-2l}{3k'}, \frac{l}{3k'}\right), \\ &\left(\frac{3k'-2l}{3k'}, \frac{l}{3k'}, \frac{l}{3k'}\right) \quad \text{for } l \neq 0, k', 2k' \end{aligned} \tag{3.14}$$

and $\frac{1}{2}(k'-1)(k'-2)$ Weyl sextets

$$(a,b,c), (b,c,a), (c,a,b), (c,b,a), (b,a,c), (a,c,b) \tag{3.15}$$

with $a, b, c \in \frac{1}{3k'}\mathbb{Z}/\mathbb{Z}$ all different but $a = b = c \bmod \frac{1}{k'}$. The Weyl invariant elements give rise to ground states in the $\mathbf{1}$ representation

$$\Psi_{\mathbf{1}} = T_{(\frac{l}{3k'},\frac{l}{3k'},\frac{l}{3k'})\tau}\Psi \otimes |0\rangle. \tag{3.16}$$

Each Weyl triplet gives rise to a $\mathbf{1}$ state

$$\Psi_{\mathbf{1}} = \frac{1}{\sqrt{3}}\left(T_{(\frac{l}{3k'},\frac{l}{3k'},\frac{3k'-2l}{3k'})\tau} + T_{(\frac{l}{3k'},\frac{3k'-2l}{3k'},\frac{l}{3k'})\tau} + T_{(\frac{3k'-2l}{3k'},\frac{l}{3k'},\frac{l}{3k'})\tau}\right)\Psi \otimes |0\rangle \tag{3.17}$$

and two states in the $\mathbf{2}$ representation

$$\Psi_{\mathbf{2}} = \begin{cases} \frac{1}{\sqrt{3}}\left(T_{(\frac{l}{3k'},\frac{l}{3k'},\frac{3k'-2l}{3k'})\tau} + e^{2\pi i/3}T_{(\frac{l}{3k'},\frac{3k'-2l}{3k'},\frac{l}{3k'})\tau} + e^{4\pi i/3}T_{(\frac{3k'-2l}{3k'},\frac{l}{3k'},\frac{l}{3k'})\tau}\right)\Psi \otimes |0\rangle \\ \frac{1}{\sqrt{3}}\left(T_{(\frac{l}{3k'},\frac{l}{3k'},\frac{3k'-2l}{3k'})\tau} + e^{4\pi i/3}T_{(\frac{l}{3k'},\frac{3k'-2l}{3k'},\frac{l}{3k'})\tau} + e^{2\pi i/3}T_{(\frac{3k'-2l}{3k'},\frac{l}{3k'},\frac{l}{3k'})\tau}\right)\Psi \otimes |0\rangle. \end{cases} \tag{3.18}$$

Each Weyl sextet gives rise to a **1** state

$$\Psi_{\mathbf{1}} = \frac{1}{\sqrt{6}}(T_{(a,b,c)\tau} + T_{(b,c,a)\tau} + T_{(c,a,b)\tau} + T_{(c,b,a)\tau} + T_{(b,a,c)\tau} + T_{(a,c,b)\tau})\Psi \otimes |0\rangle, \tag{3.19}$$

a $\mathbf{1}'$ state

$$\Psi_{\mathbf{1}'} = \frac{1}{\sqrt{6}}(T_{(a,b,c)\tau} + T_{(b,c,a)\tau} + T_{(c,a,b)\tau} - T_{(c,b,a)\tau} - T_{(b,a,c)\tau} - T_{(a,c,b)\tau})\Psi \otimes |0\rangle, \tag{3.20}$$

and four states transforming as two copies of **2**

$$\Psi_{\mathbf{2}} = \begin{cases} \frac{1}{\sqrt{3}}(T_{(a,b,c)\tau} + e^{2\pi i/3}T_{(b,c,a)\tau} + e^{4\pi i/3}T_{(c,a,b)\tau})\Psi \otimes |0\rangle \\ \frac{1}{\sqrt{3}}(T_{(c,b,a)\tau} + e^{2\pi i/3}T_{(b,a,c)\tau} + e^{4\pi i/3}T_{(a,c,b)\tau})\Psi \otimes |0\rangle \end{cases} \tag{3.21}$$

and

$$\Psi_{\mathbf{2}} = \begin{cases} \frac{1}{\sqrt{3}}(T_{(a,b,c)\tau} + e^{4\pi i/3}T_{(b,c,a)\tau} + e^{2\pi i/3}T_{(c,a,b)\tau})\Psi \otimes |0\rangle \\ \frac{1}{\sqrt{3}}(T_{(c,b,a)\tau} + e^{4\pi i/3}T_{(b,a,c)\tau} + e^{2\pi i/3}T_{(a,c,b)\tau})\Psi \otimes |0\rangle \end{cases}. \tag{3.22}$$

Altogether, the ground states transform in the representation

$$R_{\text{ground states}} = \frac{1}{2}(k'+2)(k'+1) \times \mathbf{1} \oplus \frac{1}{2}(k'-1)(k'-2) \times \mathbf{1}' \oplus (k'^2-1) \times \mathbf{2}. \tag{3.23}$$

[By this we mean a direct sum of $\frac{1}{2}(k'+2)(k'+1)$ copies of **1**, $\frac{1}{2}(k'-1)(k'-2)$ copies of $\mathbf{1}'$, and k'^2-1 copies of **2**.] All these ground states have bosonic statistics, and the gauge invariant states are those in the **1** representation, so we get the physical ground state spectrum

$$\frac{1}{2}(k'+2)(k'+1) \quad \text{bosonic states with} \quad E = 0. \tag{3.24}$$

The bosonic and fermionic creation operators $\frac{D}{DZ}$ and η_+ both transform in the **2** representation under W. Multiple bosonic excitations give rise to the symmetric product representations

$$\begin{aligned}
[\mathbf{2}]_S^{\otimes 6n} &= (n+1) \times \mathbf{1} \oplus n \times \mathbf{1}' \oplus 2n \times \mathbf{2} \\
[\mathbf{2}]_S^{\otimes(6n+1)} &= n \times \mathbf{1} \oplus n \times \mathbf{1}' \oplus (2n+1) \times \mathbf{2} \\
[\mathbf{2}]_S^{\otimes(6n+2)} &= (n+1) \times \mathbf{1} \oplus n \times \mathbf{1}' \oplus (2n+1) \times \mathbf{2} \\
[\mathbf{2}]_S^{\otimes(6n+3)} &= (n+1) \times \mathbf{1} \oplus (n+1) \times \mathbf{1}' \oplus (2n+1) \times \mathbf{2} \\
[\mathbf{2}]_S^{\otimes(6n+4)} &= (n+1) \times \mathbf{1} \oplus n \times \mathbf{1}' \oplus (2n+2) \times \mathbf{2} \\
[\mathbf{2}]_S^{\otimes(6n+5)} &= (n+1) \times \mathbf{1} \oplus (n+1) \times \mathbf{1}' \oplus (2n+2) \times \mathbf{2}
\end{aligned} \tag{3.25}$$

whereas zero, one or two fermionic excitations give the antisymmetric products

$$[\mathbf{2}]_A^{\otimes 0} = \mathbf{1} \qquad [\mathbf{2}]_A^{\otimes 1} = \mathbf{2} \qquad [\mathbf{2}]_A^{\otimes 2} = \mathbf{1}'. \tag{3.26}$$

At energy level $E = me^2k'$ for $m = 1, 2, \ldots$, we have bosonic excitations transforming in the representation

$$R_m = [\mathbf{2}]_S^{\otimes m} \otimes [\mathbf{2}]_A^{\otimes 0} \oplus [\mathbf{2}]_S^{\otimes(m-2)} \otimes [\mathbf{2}]_A^{\otimes 2}. \tag{3.27}$$

By supersymmetry, this has to agree with the fermionic representation

$$R_m = [\mathbf{2}]_S^{\otimes(m-1)} \otimes [\mathbf{2}]_A^{\otimes 1}. \tag{3.28}$$

It is given by

$$\begin{aligned}
R_{3n} &= n \times \mathbf{1} \oplus n \times \mathbf{1}' \oplus 2n \times \mathbf{2} \\
R_{3n+1} &= n \times \mathbf{1} \oplus n \times \mathbf{1}' \oplus (2n+1) \times \mathbf{2} \\
R_{3n+2} &= (n+1) \times \mathbf{1} \oplus (n+1) \times \mathbf{1}' \oplus (2n+1) \times \mathbf{2}.
\end{aligned} \tag{3.29}$$

Gauge invariant excited states correspond to **1** terms in the tensor product of these excitation representations with the ground state representation $R_{\text{ground state}}$. The spectrum of excited states is thus

$$\begin{aligned}
3k'^2n &\quad \text{bosonic and fermionic states with} \quad E = 3ne^2k' \\
3k'^2n + k'^2 - 1 &\quad \text{bosonic and fermionic states with} \quad E = (3n+1)e^2k' \\
3k'^2n + 2k'^2 + 1 &\quad \text{bosonic and fermionic states with} \quad E = (3n+2)e^2k'.
\end{aligned} \tag{3.30}$$

The second and third formulas are valid for $n = 0, 1, \ldots$, and the first formula only for $n = 1, 2, \ldots$.

IV. GAUGE GROUPS OF ADJOINT TYPE

A gauge group of the form $G = \hat{G}/C$, where $\hat{G}$ is a simply connected group with center subgroup $C = \Lambda^*/\Lambda$, is said to be of adjoint type. It imposes the requirement $k' = 0 \bmod h\mathbb{Z}$ on the shifted level k', where h is the dual Coxeter number of G. The theory is then invariant under translations $T_{\mu+\nu\tau}$ on the complex torus X with $\mu, \nu \in \Lambda^*/\Lambda$. Physical states can be characterized by the phase factor $\exp 2\pi i(\mu \cdot e_1 + \nu \cdot e_2)$ picked up under such a transformation. This defines the discrete electric 't Hooft fluxes $e_1, e_2 \in \Lambda^*/\Lambda$. In our case, these transformations act on the ground states as

$$(T_{\mu+\nu\tau}T_{\epsilon\tau}\Psi)(Z) = \exp(i\pi k'\mu \cdot \mu + i\pi k'\nu \cdot \nu + 2\pi i k'\epsilon \cdot \mu)(T_{(\nu+\epsilon)\tau}\Psi)(Z) \tag{4.1}$$

and commute with the creation operators $\frac{D}{DZ}$ and η_+. The phase factor in the transformation law of the ground states thus determines e_1, whereas state of definite e_2 are obtained by taking appropriate linear combinations of the states $T_{(\nu+\epsilon)\tau}\Psi$ for $\nu \in \Lambda^*/\Lambda$. There is also a magnetic 't Hooft flux m_{12} which measures the obstruction to lifting the structure group of the gauge bundle from $G = \hat{G}/C$ to its universal covering group $\hat{G}$.

Here we will only consider the simplest case of $G = \mathrm{SU}(2)/\mathbb{Z}_2$. The case of $G = \mathrm{SU}(3)/\mathbb{Z}_3$ can be easily worked out by extending the ground state analysis of Sec. III B of the present paper and Sec. 3.2 of [7]. Other groups would involve more work.

A. $G = \mathrm{SU}(2)/\mathbb{Z}_2$

The nontrivial element $(\frac{1}{2}, \frac{1}{2})$ of the center subgroup $C = \Lambda^*/\Lambda \simeq \mathbb{Z}_2$ has norm $(\frac{1}{2}, \frac{1}{2}) \cdot (\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$. The dual Coxeter number of SU(2) is $h = 2$, so the shifted level k' has to be an even integer. In fact, it is convenient to consider the two cases when k' is divisible by four or not separately.

1. $k' = 0 \bmod 4$

The transformation law (4.1) now simplifies to

$$(T_{\mu+\nu\tau}T_{\epsilon\tau}\Psi)(Z) = \exp(2\pi i k' \epsilon \cdot \mu)(T_{(\nu+\epsilon)\tau}\Psi)(Z). \quad (4.2)$$

The ground states in the **1** representation can be further classified by their electric 't Hooft flux components e_1 and e_2 which determine the phases under translations by $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})\tau$ on X. There are $k'/4 + 1$ states with both e_1 and e_2 trivial:

$$\Psi_{\mathbf{1}} = \begin{cases} \frac{1}{\sqrt{2}}\Big(T_{(0,0)\tau} + T_{(\frac{1}{2},-\frac{1}{2})\tau}\Big)\Psi \\ \frac{1}{\sqrt{2}}\Big(T_{(\frac{1}{4},-\frac{1}{4})\tau} + T_{(\frac{3}{4},-\frac{3}{4})\tau}\Big)\Psi \\ \frac{1}{2}\Big(T_{(\frac{l}{2k'},-\frac{l}{2k'})\tau} + T_{(\frac{2k'-l}{2k'},-\frac{2k'-l}{2k'})\tau} + T_{(\frac{k'+l}{2k'},-\frac{k'+l}{2k'})\tau} + T_{(\frac{k'-l}{2k'},-\frac{k'-l}{2k'})\tau}\Big)\Psi, \end{cases} \quad (4.3)$$

where $l = 2, 4, \ldots, k'/2 - 2$ in the last line. There are also $k'/4$ states for each remaining combination of e_1 and e_2:

$$\Psi_{\mathbf{1}} = \begin{cases} \frac{1}{\sqrt{2}}\Big(T_{(0,0)\tau} - T_{(\frac{1}{2},-\frac{1}{2})\tau}\Big)\Psi \\ \frac{1}{2}\Big(T_{(\frac{l}{2k'},-\frac{l}{2k'})\tau} + T_{(\frac{2k'-l}{2k'},-\frac{2k'-l}{2k'})\tau} - T_{(\frac{k'+l}{2k'},-\frac{k'+l}{2k'})\tau} - T_{(\frac{k'-l}{2k'},-\frac{k'-l}{2k'})\tau}\Big)\Psi, \end{cases} \quad (4.4)$$

where $l = 2, 4, \ldots, k'/2 - 2$ in the last line, have e_1 trivial and e_2 nontrivial.

$$\Psi_{\mathbf{1}} = \frac{1}{2}\Big(T_{(\frac{l}{2k'},-\frac{l}{2k'})\tau} + T_{(\frac{2k'-l}{2k'},-\frac{2k'-l}{2k'})\tau} + T_{(\frac{k'+l}{2k'},-\frac{k'+l}{2k'})\tau} + T_{(\frac{k'-l}{2k'},-\frac{k'-l}{2k'})\tau}\Big)\Psi, \quad (4.5)$$

where $l = 1, 3, \ldots, k'/2 - 1$, have e_1 nontrivial and e_2 trivial.

$$\Psi_{\mathbf{1}} = \frac{1}{2}\Big(T_{(\frac{l}{2k'},-\frac{l}{2k'})\tau} + T_{(\frac{2k'-l}{2k'},-\frac{2k'-l}{2k'})\tau} - T_{(\frac{k'+l}{2k'},-\frac{k'+l}{2k'})\tau} - T_{(\frac{k'-l}{2k'},-\frac{k'-l}{2k'})\tau}\Big)\Psi, \quad (4.6)$$

where $l = 1, 3, \ldots, k'/2 - 1$, have both e_1 and e_2 nontrivial. Acting on these states with an even number of creation operators gives excited states as described in the previous section.

The ground states in the $\mathbf{1}'$ representation can be treated analogously. There are $k'/4 - 1$ states with both e_1 and e_2 trivial:

$$\Psi_{\mathbf{1}'} = \frac{1}{2}\Big(T_{(\frac{l}{2k'},-\frac{l}{2k'})\tau} - T_{(\frac{2k'-l}{2k'},-\frac{2k'-l}{2k'})\tau} + T_{(\frac{k'+l}{2k'},-\frac{k'+l}{2k'})\tau} - T_{(\frac{k'-l}{2k'},-\frac{k'-l}{2k'})\tau}\Big)\Psi, \quad (4.7)$$

where $l = 2, 4, \ldots, k'/2 - 2$. There are also $k'/4$ states for each remaining combination of e_1 and e_2:

$$\Psi_{\mathbf{1}'} = \begin{cases} \frac{1}{\sqrt{2}}\Big(T_{(\frac{1}{4},-\frac{1}{4})\tau} - T_{(\frac{3}{4},-\frac{3}{4})\tau}\Big)\Psi \\ \frac{1}{2}\Big(T_{(\frac{l}{2k'},-\frac{l}{2k'})\tau} - T_{(\frac{2k'-l}{2k'},-\frac{2k'-l}{2k'})\tau} - T_{(\frac{k'+l}{2k'},-\frac{k'+l}{2k'})\tau} + T_{(\frac{k'-l}{2k'},-\frac{k'-l}{2k'})\tau}\Big)\Psi \end{cases}, \quad (4.8)$$

where $l = 2, 4, \ldots, k'/2 - 2$ in the last line, have e_1 trivial and e_2 nontrivial.

$$\Psi_{\mathbf{1}'} = \frac{1}{2}\Big(T_{(\frac{l}{2k'},-\frac{l}{2k'})\tau} - T_{(\frac{2k'-l}{2k'},-\frac{2k'-l}{2k'})\tau} + T_{(\frac{k'+l}{2k'},-\frac{k'+l}{2k'})\tau} - T_{(\frac{k'-l}{2k'},-\frac{k'-l}{2k'})\tau}\Big)\Psi, \quad (4.9)$$

where $l = 1, 3, \ldots, k'/2 - 1$, have e_1 nontrivial and e_2 trivial.

$$\Psi_{\mathbf{1}'} = \frac{1}{2}\Big(T_{(\frac{l}{2k'}, -\frac{l}{2k'})\tau} - T_{(\frac{2k'-l}{2k'}, -\frac{2k'-l}{2k'})\tau} - T_{(\frac{k'+l}{2k'}, -\frac{k'+l}{2k'})\tau} + T_{(\frac{k'-l}{2k'}, -\frac{k'-l}{2k'})\tau}\Big)\Psi, \quad (4.10)$$

where $l = 1, 3, \ldots, k'/2 - 1$, have both e_1 and e_2 nontrivial. These states should be acted on by an odd number of creation operators to obtain physical excited states.

The states described above all have trivial magnetic 't Hooft flux; i.e., the gauge bundle can be lifted to an SU(2) bundle over T^2. There is also an additional gauge invariant ground state with nontrivial magnetic 't Hooft flux. Like the other ground states, this state has bosonic statistics [7]. There are however no excitations with energies of order e^2 above this ground state.

2. $k' = 2\,mod\,4$

The transformation law now acquires additional signs for nontrivial μ or ν as compared to (4.2). The analysis is otherwise precisely analogous, and we give only the results for the number of ground states of different quantum numbers:

There are $(k' + 2)/4 - 1$ ground states in the $\mathbf{1}$ representation with both e_1 and e_2 trivial, and $(k' + 2)/4$ ground states for each of the three remaining combinations of e_1 and e_2.

There are $(k' + 2)/4$ ground states in the $\mathbf{1}'$ representation with both e_1 and e_2 trivial, and $(k' + 2)/4 - 1$ ground states for each of the tree remaining combinations of e_1 and e_2.

Excited states are obtained by acting on these ground states with an even or odd number of creation operators respectively.

Finally, there is a single ground state with nontrivial magnetic 't Hooft flux. An interesting feature of this state is that it has fermionic statistics [7]. There are no low-energy excitations above this ground state either.

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[1] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982); **185**, 406(E) (1988); **185**, 406 (1988); **281**, 409 (2000).

[2] H.-C. Kao, K.-M. Lee, and T. Lee, Phys. Lett. B **373**, 94 (1996).

[3] G. Amelino-Camelia, I. I. Kogan, and R. J. Szabo, Nucl. Phys. **B480**, 413 (1996).

[4] E. Witten, Nucl. Phys. **B202**, 253 (1982).

[5] E. Witten, in *The Many Faces of the Superworld*, edited by M. A. Shifman (World Scientific, Singapore, 1999), pp. 156–184.

[6] A. V. Smilga, J. High Energy Phys. 05 (2012), 103; 01 (2010), 086.

[7] M. Henningson, J. High Energy Phys. 11 (2012), 013.

[8] D. Karabali, C.-j. Kim, and V. P. Nair, Nucl. Phys. **B566**, 331 (2000).

[9] G. 't Hooft, Nucl. Phys. **B138**, 1 (1978); **B153**, 141 (1979); **B190**, 455 (1981).