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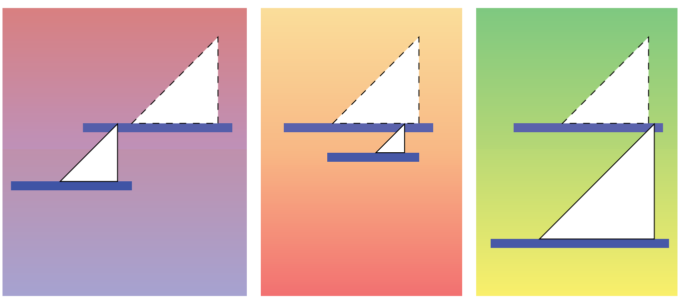

Impartial Games and Recursive Functions

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Abstract

Interest in 2-player impartial games often concerns the famous theory of Sprague-Grundy. In this thesis we study other aspects, bridging some gaps between combinatorial number theory, computer science and combinatorial games. The family of heap games is rewarding from the point of view of combinatorial number theory, partly because both the positions and the moves are represented simply by finite vectors of nonnegative integers. For example the famous game of Wythoff Nim on two heaps of tokens has a solution originating in Beatty sequences with modulus the Golden ratio. Sometimes generalizations of this game have similar properties, but mostly they are much harder to grasp fully. We study a spectrum of such variations, and our understanding of them ranges from being complete in the case of easier problems, to being very basic in the case of the harder ones. One of the most far reaching results concerns the convergence properties of a certain $\star\star$-operator for invariant subtraction games, introduced here to resolve an open problem in the area. The convergence holds for any game in any finite dimension. We also have a complete understanding of the reflexive properties of such games. Furthermore, interesting problems regarding computability can be formulated in this setting. In fact, we present two Turing complete families of impartial (heap) games. This implies that certain questions regarding their behavior are algorithmically undecidable, such as: Does a given finite sequence of move options alternate between N- and P-positions? Do two games have the same sets of P-positions? The notion of N- and P-positions is very central to the class of normal play impartial games. A position is in P if and only if it is safe to move there. This is virtually the only theory that we need. Therefore we hope that our material will inspire even advanced undergraduate students in future research projects. However we would not consider it impossible that the universality of our games will bridge even more gaps to other territories of mathematics and perhaps other sciences as well. In addition, some of our findings may apply as recreational games/mathematics.

Impartial Games and Recursive Functions

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Preface

When my daughter Hanna was little, we spent a lot of time playing silly games. One such game was an "imitation game" where we challenged each other by creating funny words to imitate. The lead of the game could alter at any point. Repetitions of imitations were allowed (it was often not even clear who was imitating whom). The game ended when either of us gave up imitating, sort of with a great laugh. There were a lot of details in the game, which of course altered over the years, many of which I have forgotten by now.

Some years later, I began studying combinatorial number theory, type Szemeredi's theorem, with Peter Hegarty. Quite soon, we discovered that certain permutations of the natural numbers mimicked the outcome functions of a famous combinatorial game, namely Wythoff Nim, together with a certain blocking maneuver (rediscovered by me). This was a good result and we published a paper. But, even so, I could not quite let go of it. For some reason I kept recalling the imitation game Hanna and I played so many times. And there it was, hiding in the same sequences of numbers as (the blocking variation of) Wythoff Nim, the game of *Imitation Nim*. This is how my Ph.D. position began, included as Paper 1 in this thesis.

At this point I was also invited to a conference, GONC, at the Banff centre in Canada by Professor Richard Nowakowski and met a community of researchers studying many aspects of combinatorial games, such as the fascinating theory of Berlekamp, Conway and Guy. By then, I had already become a regular participant at Professor Melvyn Nathanson's CANT conferences. Professor Bruce Landman's INTEGERS conferences in West Georgia has also been a big source for inspiration. Not to mention Professor Aviezri Fraenkel and his multitude of amazing papers incorporating computer science, combinatorial games and combinatorial number theory.

Figure 1: Chess or Xiangqi is a popular combinatorial game. Although it takes a lifetime to master, the rules are easy. By the age of five, I played the western variation with my grandfather.

A little later, my adviser Johan Wästlund introduced the concept of algorithmic undecidability and Turing completeness, to my combinatorial game studies. We discovered how standard heap games, also originating in very simple children's games, proved to be capable of emulating universal Turing machines. At this point however, I got somewhat bothered, since S. Wolphram's famous rule 110 cellular automaton, which was recently proved undecidable by M. Cook, did not appear to have a 2-player game equivalent, so I set out to construct two such games, together with a suitable generalization, the final paper in the thesis.

I would like to thank Chalmers and all my colleagues at the department of Mathematical Sciences for great times at the institution and the many opportunities it has brought. I would like to thank all great colleagues and friends that I met at the above mentioned conferences and workshops, including Games at Dal, the Weizmann Institute of Science and Ludus Recreational

Mathematics Colloquia, during the last few years. I would also like to thank my family and all friends for great support and much fun. This thesis is devoted to my mother who shares my passion for patterns, but I believe she found her expression in patchwork (see Figure 4), and also to my father, who shared my interest in mathematics and games, but sadly passed away a few days into the new year of 2013. Finally, I would like to thank my grandfather who taught me how to play chess. Without his early input, this thesis would not have come into play. Many thanks to the anonymous players in Figure 1.

To my dear parents

1 Papers

The thesis consists of 8 papers.

Paper 1: 2-pile Nim with a Restricted Number of Move-size Imitations (with an appendix by Peter Hegarty), also Section 4.1.

Paper 2: Blocking Wythoff Nim, also Section 4.2.

Paper 3: A Generalized Diagonal Wythoff Nim, also Section 4.3.

Paper 4: Maharaja Nim, Wythoff's Queen meets the Knight (with Johan Wästlund), also Section 4.4.

Paper 5: Invariant and dual subtraction games resolving the Duchêne-Rigo conjecture (with Peter Hegarty and Aviezri S. Fraenkel), also Section 4.5.

Paper 6: The $\star$-operator and invariant subtraction games, also Section 4.6.

Paper 7: From heaps of matches to the limits of computability (with Johan Wästlund), also Section 4.7.

Paper 8: Impartial games emulating one-dimensional cellular automata and undecidability, also Section 4.8.

Papers 1,2,3,5, 6 and 8 are published as described in the bibliography. The other two have been submitted to journals for peer reviewing. My coauthors are Johan Wästlund (Papers 4 and 7) and Peter Hegarty, Aviezri S. Fraenkel (Paper 5). P. Hegarty has also contributed an appendix for Paper 1.

One way to categorize the papers is as follows: Papers 1 and 2 concern blocking maneuvers and move-size dynamic variations of Wythoff Nim. Papers 3 and 4 concern natural extensions of Wythoff Nim that 'adjoin moves' to those of Wythoff Nim. Papers 5 and 6 concern a certain $\star$-operator of (vast generalizations of classical) invariant subtraction games. Papers 7 and 8 concern Turing completeness of two families of impartial (heap) games.

An informal discussion of patterns, and how they sometimes can be generated via combinatorial games, is the topic in Section 2. A formal introduction, to games and the problem of computability, is given in Section 3. After this, in Section 4, we give an overview of the included papers.

2 Patterns and their games

Patterns occur in nature in many different ways. They can be repetitive, creating, for example, larger periodic, self similar or fractal patterns, or otherwise they may appear chaotic in some sense, or a mixture of all.

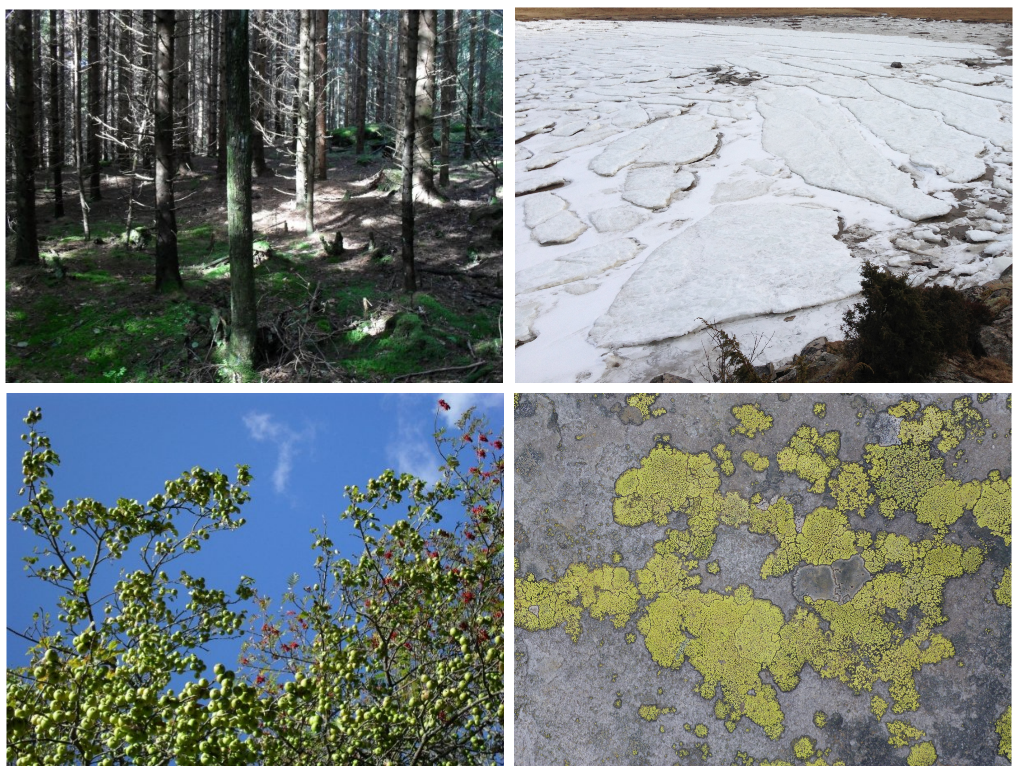

Figure 2: Nature is rich in fascinating patterns. In this thesis, we are interested in the combinatorial and computational aspects of a given patterns, in particular, if they can be produced by certain 2-player games.

If we know how a pattern is generated, by a computer say, we can claim to understand it fully. To know how a pattern is generated means that we can recreate it in detail whenever we wish.

But perhaps there is more than one algorithm that generates one and the same pattern? By studying a few, we can learn different aspects. In a

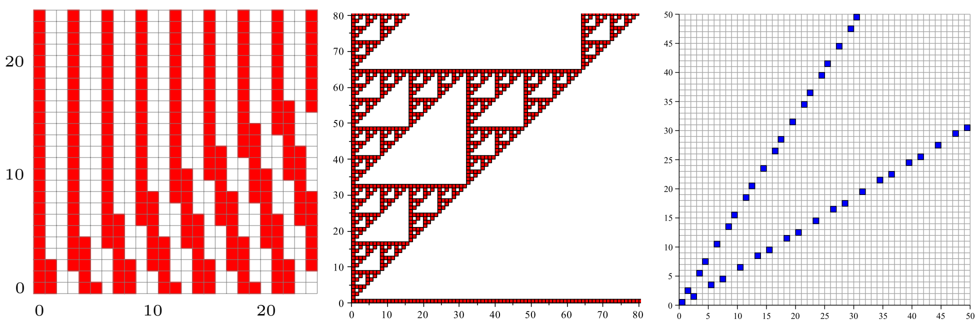

Figure 3: What do we see in a pattern? Can we imagine an infinite continuation? How would you generate these three 2-dimensional patterns? The leftmost contains two periodic sectors, the other two exhibit well known fractal or self-similar behavior. They can be generated in a few different ways. In this thesis we show that such patterns sometimes can be interpreted as solutions of combinatorial games on heaps of tokens and/or matches. There are two overarching research questions in this thesis. (1) What patterns do combinatorial games with given rules produce? (2) Given a pattern, can we find a combinatorial game that produces it? The leftmost pattern is described in Paper 7, the middle one is Pascal's triangle modulo 2 concerning Papers 7 and 8, whereas the rightmost pattern is explained in Example 4.

sense, a pattern would be richer if it had many interpretations. On the one hand, we will study patterns generated by certain 2-player games. The idea is very simple: the instructions for how to win will be coded via some two-state algorithm, to be explained in detail in Section 3. On the other hand, given some (not necessarily repetitive) pattern, we will ask if it is possible to generate it via some game. Sometimes the description of a game can be regarded simply as another pattern, which can be produced by another game, and so on. We will explain later what is meant by this. The point we wish to make here is that, whether we study mathematics or something else, as human beings we are good at sensing whether something appears repetitive or irregular. In this thesis we play around with distinguishing such areas in a simplified mathematical world. There is some murky territory between regularity and chaos, where it sometimes is possible to obtain some partial knowledge of what is going on. Such areas can be very interesting, as is discussed for example in Papers 3 and 4. If we think there is some kind of regularity, then it can be interesting to see if we can prove precisely what this counts of. If we see an overwhelming disorder, then it can be interesting to see if one can prove that given problems are very hard. A system is rich if we cannot determine what is going on for sure. But, on the other hand, beautiful patterns may not survive in absolute chaos. “Beauty”, of course, is a very subjective matter, but it is also an indispensable guide in finding interesting research areas.

- How can we generalize some known system? Interpretations as games can be fruitful, partly because games have natural variations, simply by adjoining or removing moves, or for example by applying move-size dynamic rules or blocking maneuvers; to be defined later. (See the Discussion in Paper 8 for example, but this is a recurring theme in the thesis.) Sometimes seemingly contra intuitive results shows that more complicated game rules produce simpler patterns (see for example the game of Blocking Wythoff Nim in Paper 2).

- Whenever we observe some regular behavior, we may wonder if it will continue forever (Figure 6). When should we expect a larger and more universal regularity from an observed smaller and local regularity? When can we predict “the future”, in this sense?

- On the other hand, apparent chaotic behavior may settle and become regular in the long run (compare with decidability problems later). Or, we may at least be able to bound its behavior to some partially intelligible structure, as for the game of Maharaja Nim in Paper 4.

- Even if we know the precise mechanisms that generate our sequences, we may sometimes not be able to determine whether certain patterns will be infinite or not (the so-called halting problem), or whether they will converge to certain accumulation points (see for example the game of GDWN in Paper 3).

- If we generate our patterns simply by the rules of our games, the procedure is often terribly slow (exponential in succinct input size). We would be looking for a closed formula or perhaps a polynomial time algorithm, to claim some better understanding. From a computational perspective it is certainly preferable.

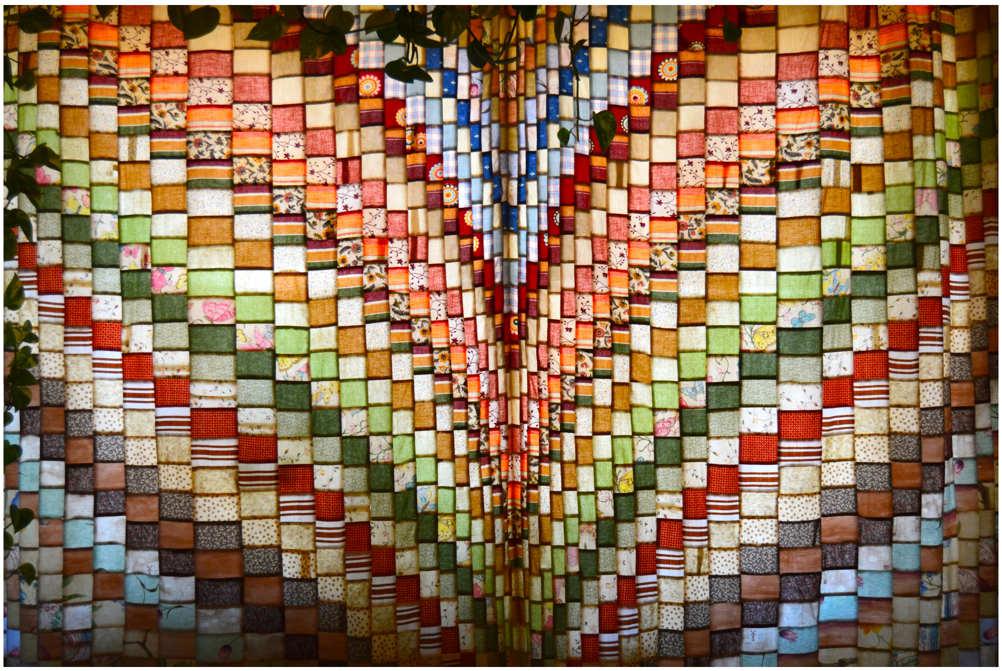

Figure 4: This is part of an impressive patchwork, created by my mother. Although the patterns appear regular, they cannot be simulated via the outcomes of a normal play impartial game (perhaps via Sprague and Grundy's famous "mex-function" though). Namely, as we will see, those are normally interpreted as two-state algorithms: 2 players alternate turns, given some impartial ruleset and according to standard normal play axioms, either the *First* or the *Second* player wins.

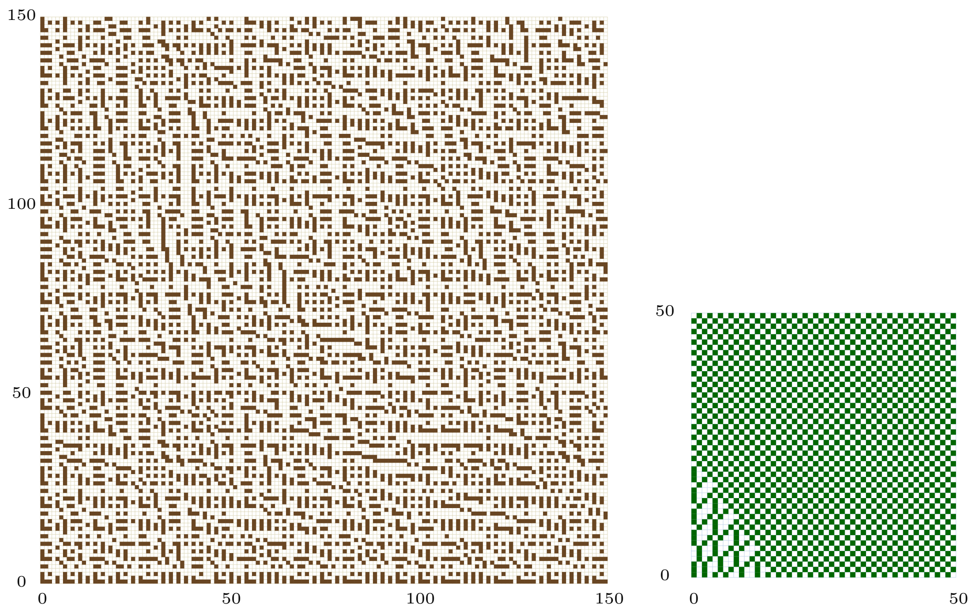

Figure 5: Is there any regularity (except NW-SE symmetry) in the leftmost picture? We generated the dark cells recursively via a very simple 2-player game, with rules: from a given cell (lattice point) in the first quadrant, a player in turn to move has three options, either (Option A) “move 3 steps down and 2 steps to the right”, or (Option B) “move 3 steps to the left and 2 steps up”, or (Option C) “move one step down and one step to the left”. The 2 players alternate moves, but can never move outside of the first quadrant. The picture shows that, in fact, if you are in a light cell when it is you turn to move, you will always be able to find a dark cell to move to. If, on the other hand, you move from a dark cell, then you are forced to move to a light cell. Thus, the dark cells are safe to move to. In particular $(0,0)$ is a final position from which no move is possible (the positions $(0,1),(0,2),(1,0),(2,0)$ are also final in this game). We don't yet have any other precise description of the visible patterns, then what has just been said. (Quite neat formulas are known for the corresponding patterns in Figure 3.) The rightmost figure has nearly the same generating options; only (Option C) is slightly different, it is exchanged for (Option C') “move one step to the left”. Using the notation from Paper 7, we get the ‘symmetric’ move options $\mathcal{M} = \{(2,-3),(-1,-1),(-3,2)\}$ for the ABC game and $\mathcal{M}' = \{(2,-3),(-1,0),(-3,2)\}$ for the ABC' game.

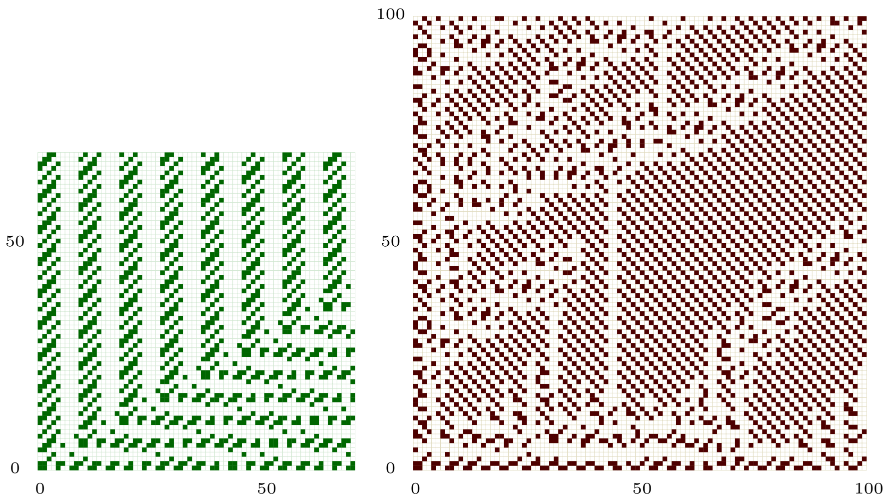

Figure 6: Similar to Figure 1, the patterns to the left describe the safe positions, called $\mathcal{P}(\mathcal{M})$, in a game $\mathcal{M}$ (containing 7 move options). Two periodic sectors appear, similar to the leftmost picture in Figure 3. To the right, precisely one move option is adjoined to the move set $\mathcal{M}$, producing the safe positions $\mathcal{P}(\mathcal{M}')$ of a new game $\mathcal{M}'$. In this way, it appears that we go from order to chaos. Does the rightmost figure eventually settle into patterns that can be fully understood, or is this a world of ever increasing complexity, where surprises will await us regardless of how far we take our computations? However, for the leftmost game $\mathcal{M}$ it is possible to adjoin another move option, obtaining a new game $\mathcal{M}''$ such that the patterns of safe positions remain the same, that is such that $\mathcal{P}(\mathcal{M}) = \mathcal{P}(\mathcal{M}'')$, but $\mathcal{M} \neq \mathcal{M}''$. Later we call such games P-equivalent. Sometimes a new move option does not change the safe positions of a game. Other times we go from order to chaos. In fact, in Paper 7 we will show that it is very hard to determine P-equivalence in general. The rightmost picture seems to be very sensitive to disturbances in the move set. We have not yet found any P-equivalent game to it. Perhaps the safe positions of this game are uniquely described by $\mathcal{M}'$ (in the sense of Paper 7)?

Rules of popular recreational games are supposedly easy to learn. It is sometimes argued that good games have simple rules, in a sense that an averaged intelligent five year old child can learn them. But even for very simple rules, it can be surprisingly hard to find a winning strategy, that is, to predict what type of patterns the game generate. This theme is central in the thesis. See also Figure 10 for an exception to the "simplicity rule". Perhaps some of the mathematical games discussed in this thesis apply as recreational mathematics, à la Martin Gardner [GB], see, for example, Figures 12, 13, 15 and 16.

3 Impartial games and their outcomes

We study interconnections between *combinatorial games* [C1976, BCG1982], *number theory* and *computer science*, with an emphasis on the former. A game consists of a finite set of *positions*, a *ruleset*, two *players* alternating moves and a declaration of who begins. Thus, given any position, the move options are also known. If no move is available, the game is *terminal* and the player who is *not* in turn to move is declared the *winner*.

Example 1. *The* game of "21" *is a popular children's game. Two players alternate in subtracting one of the numbers* 1 *or* 2 *from a given non-negative integer, starting with* 21 *until the position is* 0*. If both play optimally, will the* First *or the* Second *player win?*

We will return to this example in a moment. As we have mentioned in Section 2, sometimes very simple rules can make a game nearly trivial, other times interesting patterns emerge, yet other games can exhibit never ending complexities. Of course we all know that the game of Chess has a simple ruleset but is very hard to master. The famous game of Go is another such example of even greater complexity. But we will later show that even by restricting us to so-called *impartial heap games*, many interesting problems will withstand arbitrarily sophisticated computations.

In our setting, the ruleset is the same for both players. That is, given any position, the set of options is the same no matter whose turn it is to move. Thus we study the family of so-called *impartial games*. For example, the game of Chess is not impartial. In general terms, the rules are similar for both players, but if we look at a given position, the move options are usually very different depending on whether Black or White is the current player.

By our axioms, each game position has finitely many options (as in Chess) and terminates within a finite number of moves (Chess might not). There is no hidden information such as in Whist or Poker, neither is there any "random element" such as a dice (nor is there any influence of psychology or such unpredictable conditions). In games such as Prisoner's dilemma or Rock-paper-scissors, the players typically move simultaneously, so neither these games qualify. On the other hand, some of the readers may have played the popular children's game Geography, which usually follows normal play impartial rules.

There is some standard terminology for the family of impartial games, but not much is needed. For the *outcome* of a game, we denote by N any position from which the *next player* (the player in turn to move) wins. Any other position is denoted by P, the *previous player* wins. Since a game is finite, there will be no infinite loops or ties, hence the sets of N- and P-positions partition the set of positions of a given game. Clearly, any terminal position is in P. Then all positions that have moves to terminal positions can be listed. They will all be labeled N. Any position not yet listed, that has only listed N-positions as options, must belong to the set of P-positions, and so on. In general, any position that has a move to some P-position is in N, and otherwise, if no P-position is available, it is in P.

By this simple deterministic algorithm we see that our games belong to the class of so-called *perfect information* games: given any position and sufficient computational power it is always possible to compute a winning move, if there is one. In fact, all combinatorial games belong to this family where absolute knowledge is possible.

On the other hand, in a game of Yatzy, for example, the meaning of "optimal play" is completely different, since absolute knowledge is not possible. A reasonable guide would be to compute the expected score of all options and follow the maximal expectation. Even if you play at best from a favorable position in Yatzy there is in general a positive probability that you will lose. If you start with 2,2,3,6,6 and three- and four of a kind were the only remaining slots, then everybody would agree that we save 6,6 and throw three dices. But what if the two remaining throws both produce 2, 2, 1? You play optimally, but lose 8 points (if keeping the 2s would have resulted in the same throws). Games of perfect information don't exhibit such behavior.

We will often restrict our attention to a subfamily of all impartial games, the so-called *heap games*. A position consists of a finite number of *heaps* (or *piles*) each with a finite number of *tokens* (or *matches* etc). Various subclasses of such games are often called "take-away" games [G1966, S1970, Z1996], whereas games such as "21" are often called *subtraction games*.

Example 2. *Let us examine the winning strategy for the game of "21" in Example 1, interpreted as a heap game. Since the unique terminal position* *is* 0*, a player who can move to a pile with three tokens wins. This follows since* $3 - 1 = 2$ *and* $3 - 2 = 1$*. Now, by the same argument a pile with* 6 *tokens is losing and in general all heap sizes divisible by* 3*. Hence the game of "21" is losing for the First player, it is a P-position.*

For heap games, we often consider a game's enumerably infinite set of all possible starting positions. In particular it will allow us to explore the famous territory of the Church-Turing Thesis concerning recursive or computable functions.

In the 17-18th century W. G. Leibniz asked whether it is possible to build a mechanical device that can test any mathematical proposition's accuracy. This later became known as Hilbert's Entscheidungsproblem, which asks for an algorithm to decide whether a given statement is provable from the axioms of first order logic. A similar problem is to try to invent an algorithm that evaluates whether two given propositions are equivalent. In the 1930s Turing and Church independently proved that such algorithms do not exist. Turing's approach was to reduce the *halting problem* of his universal "machine" to the Entscheidungsproblem. He had already established that the halting problem is algorithmically undecidable: there is no Turing machine that can take as input the code of another Turing machine and decide whether it will halt or not for a given input. Both the code that describes the machine and its input are finite. Thus, if we want to reduce the halting problem to some problem in our setting of impartial games, we must ensure that the "code", whatever we use, is finitely described, and also by the Church-Turing thesis, if it produces an "output" in finite time, it must be correct. If it does not "halt" (which will have different meanings in our setting), it must not produce an incorrect result. Today there is a spectrum of abstract or even real machines capable of universal computation, such as all everyday computers, universal Turing machines [T1936], Posts Tag productions [M1961, P1943], some families of Cellular Automata [N1963, HU1979, W1984a, W1984b, W1984c, C2004] and many more. (In each case we assume that there is no limit in time or memory used.) We will return to these type of problems in Papers 7 and 8.

In the previous discussions many rulesets were 'finite'. Before we enter the next section with the summary of our papers, let us mention two more examples of classical impartial games with particularly simple, yet infinite, rulesets.

Example 3. *[B1902]* k-pile Nim*, for* k *a positive integer. Positions:* k *heaps of finite numbers of tokens. Moves: remove any positive number of tokens* *from precisely one of the piles, at most a whole pile. P-positions: The nim-sum of the heap sizes (addition modulo 2 of the binary expansions of the heap sizes) equals zero. In case $k = 2$, this means that the 2 piles have the same number of tokens.*

Example 4. *[W1907]* Wythoff Nim. *Positions: 2 heaps of a finite number of tokens. Moves: As 2-pile Nim but also a "diagonal" type of move, remove the same number of tokens from both piles as long as each remaining pile contains a non-negative number of tokens. Thus, the game can equivalently be played on a large Chess board with one single piece which moves as the Queen, but with the restriction that, by moving, the "Manhattan distance" to position $(0,0)$ must decrease. P-positions: Let $\phi = \frac{1+\sqrt{5}}{2}$ denote the golden ratio. The P-positions are of the forms $(\lfloor \phi n \rfloor, \lfloor n\phi^2 \rfloor)$ and $(\lfloor \phi^2 n \rfloor, \lfloor n\phi \rfloor)$ for all non-negative integers n. They are displayed in Figure 3 to the right.*

4 Overview of papers

Much inspiration for this thesis came from our three initial examples of impartial take-away games, with a certain emphasis on variations of Wythoff Nim. A major theme is to construct new games out of known ones: Papers 1 to 6.

Another theme is to emulate or mimic other interesting mathematical structures and sequences by the P-positions of a heap game: Papers 4, 5 and 8. In this context we also investigate whether apparently different games can have the same sets of P-positions: Papers 1, 5 and 7. In fact, this problem is demonstrated algorithmically undecidable in Paper 7.

Exploiting the similarities of the descriptions of positions and move options of our heap games, surprising properties arise. In this context, the $\star$-operator is introduced in Paper 5 to resolve a recent conjecture of Duchêne and Rigo [DR2010]. In Paper 6 we investigate further properties of this operator.

In Papers 7 and 8, we demonstrate that machines capable of universal computation, can be emulated, via sequences of P-positions, by certain impartial heap games. For the latter case a so-called *move-size dynamic* rule is used, where the options for the next move depend on the previous move. We introduce a variation of this already in our first paper.

4.1 Imitation Nim and Wythoff's sequences, Paper 1

This paper concerns a variation of the classical game of Nim on two piles as in Example 3. Suppose that the previous player removed x tokens from the smaller heap (any heap if they have equal size). Then the next player may not remove x tokens from the larger heap. We call this game *Imitation Nim* (although, as remarked by Aviezri Fraenkel the first time I met him, *L*imitation Nim would also have been an appropriate name). Notice that by this move restriction, the winning strategy of 2-pile Nim is altered. For example, the player who moves from the position $(1, 1)$ will lose in Nim, but win in Imitation Nim. It turns out that the P-positions correspond to those of Wythoff Nim, Example 4. The game generalizes nicely. Suppose that $m - 1$ consecutive imitations from one and the same player are allowed, but not the mth one. For example, with $m = 2$ and $0 < x \leq y$, suppose that the three most recent moves were $(x, y) \to (x - z, y) \to (x - z, y - z) \to (x - z - w, y - z)$, alternating between the two players. Then precisely the move to $(x - z - w, y - z - w)$ is prohibited. The P-positions of this generalization of Imitation Nim correspond to those of a variation of Wythoff Nim with a so-called *blocking maneuver* on the diagonal options, studied first in [HL2006], see also Section 4.2.

4.2 Blocking Wythoff Nim and the quest for possible Beatty sequences, Paper 2

Let us begin by giving some background to this paper. Let m be a positive integer. In the variation of Wythoff Nim with a *blocking maneuver* proposed in [HL2006], the previous player may, before the next player moves, block off $m - 1$ of the diagonal type options and declare them forbidden. After the next player has moved, any blocking maneuver is forgotten. It turns out that the P-positions of these generalizations have similar structures as those of Wythoff Nim (Example 4). The sequences of their coordinates approximate very closely straight lines with irrational slopes on the 2-dimensional integer lattice. Namely they can be approximated by so-called homogeneous Beatty sequences $(\lfloor n\alpha \rfloor)$, where α is a positive irrational and n ranges over the positive integers. (This was proved independently by Hegarty in [L2009, Appendix] and Fraenkel, Peled in [FP].)

Two sets of positive integers are *complementary* if each positive integer occurs in precisely one of them. It is a well known result [B1926] that the sets

$\{\lfloor n\alpha \rfloor\}$ and $\{\lfloor n\beta \rfloor\}$, where n ranges over the positive integers, are complementary if and only if α, β are positive irrationals satisfying $\alpha^{-1} + \beta^{-1} = 1$. A special case of this is given in Example 4.

Combinatorial games with a blocking maneuver, or so-called Muller Twist, were proposed via the game Quarto in "Mensa Best Mind games Award" in 1993. Later the idea appeared in the literature [HR2001, SS2002, GS2004].

Having observed, in [HL2006], that a blocking maneuver on the diagonal type moves gives rise to interesting sequences of integers, I set out to study two other natural variations of Wythoff Nim with a blocking maneuver, [L] and Paper 2, where at most a given finite number of options can be blocked, at each stage of the game. In [L] blocking is allowed exclusively on the Nim-type options, whereas in the included paper blocking is allowed on all options of Wythoff Nim.

Here, an exact formula for the P-positions is given, if at most one (non-restricted) option may be blocked. For this game, the *upper* P-positions have *split* into two sequences of P-positions, one with slope ϕ, similar to the Beatty type formula for Wythoff Nim, and the other with slope 2. A position (x, y) is *upper* if $y \geq x$. We also give a closed formula expression for the P-positions for the game with at most two blocked off options and state precise conjectures for some greater blocking parameters. The general problem seems very hard. (In contrast, the P-positions of the games in [L] can be described via Beatty sequences for all blocking parameters, generalizing A. S. Fraenkel's classical p-Wythoff Nim [F1982].)

A last remark regarding this paper is that a certain family of impartial *comply games* defined here, is not of the form considered in [S1981], where it is proved that for all impartial games in consideration, almost all positions are next player winning. In fact, our comply maneuver can be applied to any impartial game, labeling almost all positions as previous player winning. The reason for this is that, at each stage of game, the previous player has to present a non-empty set of options for the next player's consideration. In a sense, this turns the usual (blocking) rules inside-out.

4.3 A Generalized Diagonal Wythoff Nim and splitting beams of P-positions, Paper 3

A possible *splitting* of sequences of P-positions into two sequences of distinct slopes is discussed also in this paper. The P-positions of Nim lie on the single beam of slope 1, whereas those of Wythoff Nim lie on the beams of slopes ϕ and ϕ^{-1}. Therefore, going from Nim to Wythoff Nim has split the single *beam* of P-positions in Nim into two new P-beams for Wythoff Nim of distinct slopes. Let p, q be positive integers. If we adjoin, to the game of Wythoff Nim, new moves of the form (pt, qt) and (qt, pt), for all positive integers t, will the upper P-positions of the new game, denoted (p, q)-GDWN, split once again into two new distinct slopes? Here we prove that the ratio of the coordinates of the upper P-positions of this game do not have a unique accumulation point if $p = 1$ and $q = 2$. Via experimental results we conjecture that the upper P-positions of (p, q)-GDWN split if and only if (p, q) is either a Wythoff pair or a dual Wythoff pair, that is of the form $(p, q) = (\lfloor \phi n \rfloor, \lfloor n\phi^2 \rfloor)$ or $(\lceil \phi n \rceil, \lceil n\phi^2 \rceil)$, for n a positive integer. In a recent preprint [L1], which is not included in this thesis, I prove that $(1, 2)$-GDWN splits. Two new discoveries made this possible.

Lemma 5. *Let $\{(x_i, y_i)\}$ define the set of upper P-positions of some* extension *of Wythoff Nim (for all i, $x_i \leq y_i$ and $x_i < x_{i+1}$). Then, for all positive integers n,*

$$\liminf_{n\to\infty} \frac{\#\{i > 0 \mid x_i < n\}}{n} \geq \phi^{-1}.$$

Lemma 6. *If there is a positive lower asymptotic density of x-coordinates of P-positions above the line $y = 2x$, then the upper P-positions $\{(a_n, b_n)\}$ of* $(1, 2)$*-GDWN split.*

In [L1], we show that $(1, 2)$-GDWN satisfies the hypothesis of Lemma 6 and that the first result implies the second.

The conjecture is that there are precisely two accumulation points for the upper P-beams, namely to the ratio of coordinates $1.477\ldots$ and $2.247\ldots$ respectively, see Figure 8, the rightmost picture. Another related research project is [L3].

4.4 Maharaja Nim and a dictionary process, Paper 4

In this paper, coauthored with J. Wästlund, we study an *extension* of Wythoff Nim, where the Queen and Knight of Chess are combined in one and the same piece, the *Maharaja* (no coordinate increases by moving). The game is called *Maharaja Nim*. One can also view this game as a *restriction* of $(1, 2)$-GDWN.

It is clear that the P-positions of Wythoff Nim will be altered for this game. Namely, the “smallest” non-zero P-positions of Wythoff Nim are $(1, 2)$ and $(2, 1)$, corresponding precisely to the new move options introduced for Maharaja Nim. However, we have succeeded in proving that the P-positions remain within a bounded distance of the half-lines of slopes ϕ and ϕ^{-1} respectively. To obtain such a result we have used an unconventional method in this field, namely, relating the upper P-positions to a certain dictionary process on binary words, a process that we also prove is in general undecidable. We also give a short proof for a generalization of an already very nice result in [FP], concerning complementary sequences, to a “Central Lemma” in our paper.

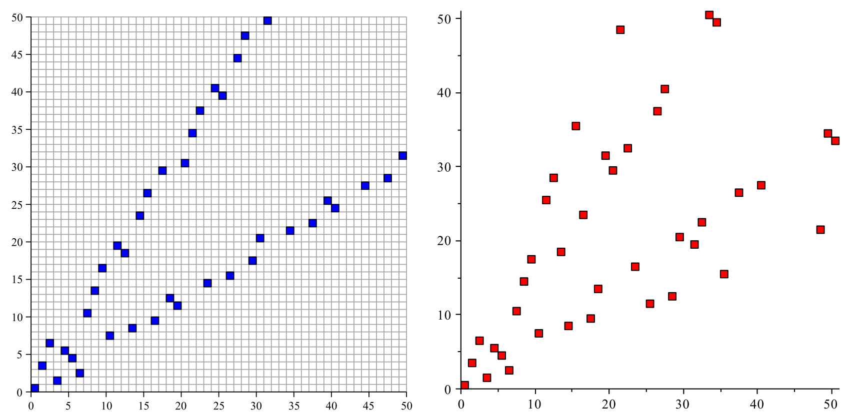

Figure 7: We have plotted the first few P-positions of Maharaja Nim and $(1, 2)$-GDWN respectively. How regular are these games?

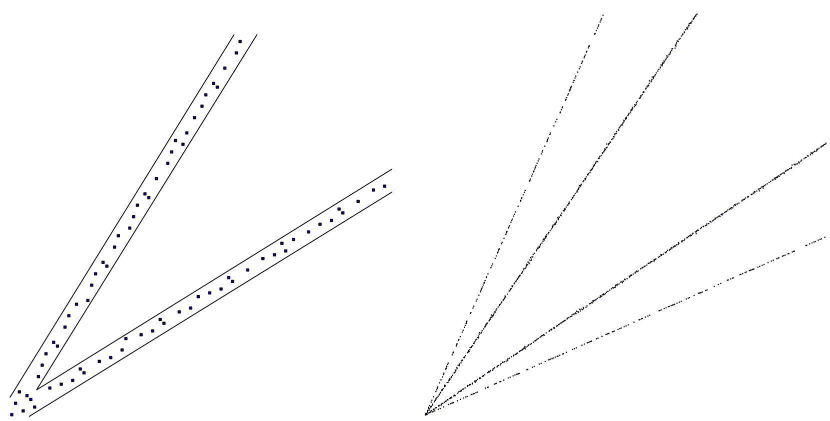

Figure 8: A few more P-positions have been computed for the respective games in Figure 7. The leftmost picture indicates that we are able to capture the behavior of Maharaja Nim's upper P-positions within a narrow stripe of slope ϕ. On the other hand, we have proved that the P-positions of GDWN to the right will eventually depart from any such stripe, however wide we make it. Extensive computations make us believe that perhaps the upper pair of P-beams' slopes will converge to the accumulation points $1.478\ldots$ and $2.248\ldots$ respectively. See also [L1] for a proof of an actual split of the upper P-beams, a result that we have chosen not to include in this thesis because it is in the process of being peer reviewed.

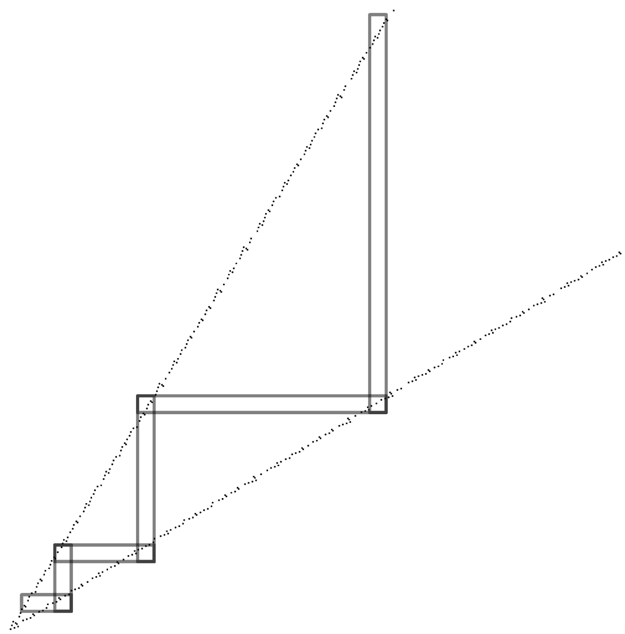

Figure 9: An open question: is it possible to decide in polynomial time, whether a given position is in P for Maharaja Nim? A "telescope" with focus $O(1)$ and reflectors along the lines ϕn and n/ϕ attempts to determine the outcome (P or N) of some position, (x, y) at the top of the picture. The method is successful for a similar game called $(2, 3)$-Maharaja Nim [L4]. (It gives the correct value for all extensions of Wythoff Nim with a finite non-terminating converging dictionary). The focus is kept sufficiently wide (a constant) to provide correct translations in each step. The number of steps is linear in $\log(xy)$.

4.5 Invariant games, the $\star$-operator and complementary Beatty sequences, Paper 5

This paper is joint with P. Hegarty and A.S. Fraenkel. An *invariant subtraction game* $G = G(\mathcal{M})$ is defined with a *move set* $\mathcal{M}$ of k-tuples of non-negative integers (not all zero). Given a position, that is another k-tuple of non-negative integers $\boldsymbol{x} = (x_1, \ldots, x_k)$ (possibly all zero), a player may use any vector $\boldsymbol{m} = (m_1, \ldots, m_k)$ from the move set and subtract it from $\boldsymbol{x}$ to obtain the next position $\boldsymbol{x} \ominus \boldsymbol{m} = (x_1 - m_1, \ldots, x_k - m_k)$, provided $\boldsymbol{x} \succeq \boldsymbol{m}$, that is $x_i \geq m_i$ for each i. As usual a player unable to move loses. Examples 1 to 4 belong to this class of games.

In this paper we resolve a conjecture from [DR2010]. They conjectured that, given a pair of complementary Beatty sequences (a_i) and (b_i) (as described in the second paragraph in Section 4.2), there is an invariant subtraction game for which the P-positions constitute precisely all the pairs (a_i, b_i) and (b_i, a_i), together with the terminal position $(0, 0)$.

We give a surprisingly simple solution to this problem. Namely take the description of the candidate P-positions (without $(0, 0)$) as *moves* in another invariant subtraction game G. Then the non-zero P-positions of the new game $\mathcal{P}(G) \setminus \{(0, 0)\}$ correspond precisely to the moves of another invariant subtraction game $G^\star$. This game has the original candidate set of P-positions as its set of P-positions, that is $G = (G^\star)^\star$.

In fact, we extend the result to a somewhat larger class of 'super-additive' sequences. See Figure 10, for the foremost example, a game which is P-equivalent to Wythoff Nim.

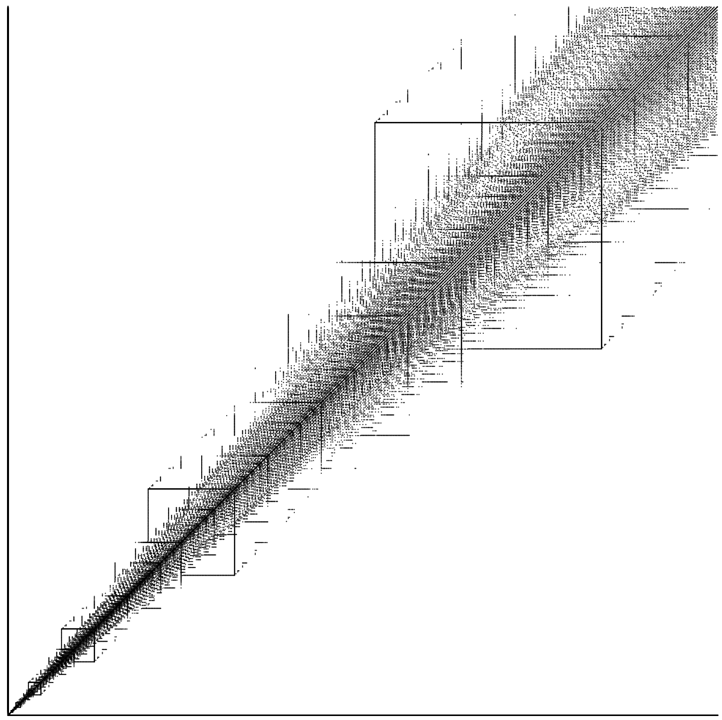

Figure 10: The picture illustrates the initial P-positions of the game (Wythoff Nim)*, or equivalently $(0, 0)$ (the lower left corner) together with the moves of the game (Wythoff Nim)$^{\star\star} \neq$ Wythoff Nim (!). The picture shows all coordinates less than 5000, but we have made computations to 12000 obtaining a similar behavior. We understand some of its behavior, but the overall pattern remains a mystery, although it is contained between half lines from the origin of slopes ϕ^{-1} and ϕ. In fact, a characterization of infinitely many log-periodic positions has been obtained in [L5], a result which is not included here. (Wythoff Nim)** is a very complicated game to play intelligently, although it has precisely the same set of P-positions as Wythoff Nim. But, on the other hand, the former game has a very nice property, which is absent in Wythoff Nim, namely it is reflexive, that is (Wythoff Nim)**=(Wythoff Nim)^{2k*} for all $k \geq 1$. Thus, simple rules do not always give the 'nicest' game properties.

4.6 Convergence of the $\star\star$-operator, Paper 6

Here we study some basic properties of the $\star$-operator from Paper 5. We identify a game with its move set and call $\mathcal{M}^\star$ the *dual* of $\mathcal{M}$. Whenever $\mathcal{M} = \mathcal{M}^{\star\star} = (\mathcal{M}^\star)^\star$ holds we say that $\mathcal{M}$ is *reflexive*. We prove that $\mathcal{M}$ is reflexive if and only if the difference set

$$\{\boldsymbol{m}_1 \ominus \boldsymbol{m}_2 \succ \boldsymbol{0} \mid \boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathcal{M}\}$$

is a subset of the set of N-positions, $\mathcal{N}(\mathcal{M})$, see Figure 11 for an example. We define the notion of *convergence* of a sequence of invariant subtraction games. Then, given an invariant subtraction game, we prove that the *limit game*, resulting from an infinite recursive application of the $\star\star$-operator, exists. Many problems remain to be resolved, such as: find an explicit formulation of some limit game, without using the notion of a sequence of invariant subtraction games.

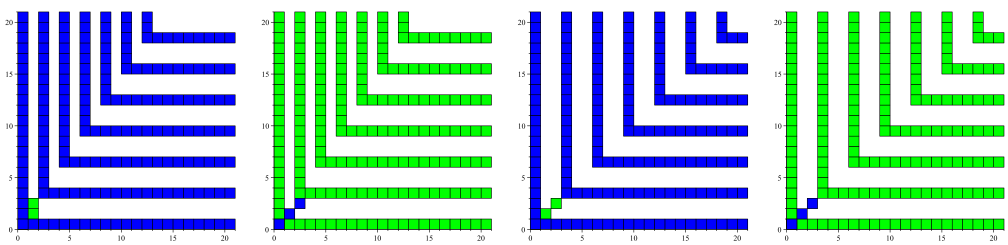

Figure 11: The figures illustrate three recursive applications of the $\star$-operator on $\mathcal{M} = \{(1,1),(1,2)\}$ for small positions. In the first figure the green (light) squares represent the two moves in $\mathcal{M}$ and the repetitive blue (dark) pattern its initial set of P-positions, $\mathcal{P}(\mathcal{M})$. The next figure illustrates the repetitive patterns in $\mathcal{M}^\star$ together with its three P-positions in $\mathcal{P}(\mathcal{M}^\star)$, and so on. The game $\mathcal{M}$ is non-reflexive since $(1,2) \ominus (1,1) = (0,1) \in \mathcal{P}(\mathcal{M})$. Neither is the dual, $\mathcal{M}^\star$, since $(1,0)$ and $(3,2)$ are moves, but $(3,2) \ominus (1,0) = (2,2) \in \mathcal{P}(\mathcal{M}^\star)$. On the other hand $\mathcal{M}^{\star\star} = \{(1,1),(2,2)\}$ is reflexive, since $(2,2) \ominus (1,1) = (1,1) \in \mathcal{M}^{\star\star} \subset \mathcal{N}(\mathcal{M}^{\star\star})$. Hence $\mathcal{M}^{n\star}$ is reflexive for all $n \geq 2$. We conjecture that this holds for any $\mathcal{M}$, with at most two moves, in any dimension.

4.7 Invariant heap games, cellular automata and undecidability, Paper 7

In this paper we discuss a family of heap games, played on k heaps of matches, with a finite number of invariant move options (generalizing Examples 1 and 2). Here the rules are relaxed so that, by moving, the *total* number of matches in all heaps must decrease, but the number may increase in individual heaps. (The move sets in Papers 5 and 6 have a different interpretation than those in this paper, although the notation will be the same.)

We prove, by relating the P-positions of a game to the updates of one-dimensional cellular automata (CA), that it is algorithmically undecidable whether two games have identical sets of P-positions. In fact, we reduce this problem from that of determining whether a finite binary string "101" occurs in the update of the CA, a problem which is known to be equivalent to the halting problem of a universal Turing machine. The construction uses an injective map from one-dimensional cellular automata to a class of (non-invariant) so-called *modular games* on two heaps of matches, a "tape heap" and a "time heap". A given n-ary update function for the CA is simulated via specific move options that are legal from distinct congruence classes modulo n, prescribed by the size of the time heap. The size of the tape heap simulates the position of the CA's tape whereas the size of the time heap, kn, simulates the kth update of the CA. The computation is carried out via the modular game's (binary) outcome function. Then, by introducing k more heaps, called the *gadget*, we emulate the modular games via a subset of invariant games on $k + 2$ heaps.

Remark 7. *Returning to the problem in Figure 6, we here show that, in general, such questions are undecidable. But the problem is wide open in two dimensions. The patterns in the rightmost picture resemble in a certain sense the updates of the rule 110 cellular automaton (Figure 14 and Paper 8), which has been demonstrated Turing complete. There are periodic "gliders" that seem to interact in strange ways, in a periodic background pattern (ether), beyond any reasonable guess from our part. We have taken the computations somewhat further, but the complexity does not seem to diminish. Another question comes to mind. If the patterns generated by any such game can be described via a finite covering of periodic rational polyhedra (all but one infinite), does this imply that it is possible to find some distinct move that, when adjoining it to the move set, will not change the pattern? (One should* *be able to find an explicit expression for the least such move, but it could be technical.) The games in the figure are*

$$\mathcal{M} = \{(0,-2),(-2,0),(2,-3),(-3,2),(-5,4),(-5,-2),(-4,-3)\},$$

$$\mathcal{M}' = \{(0,-2),(-2,0),(2,-3),(-3,2),(-5,4),(-5,-2),(-4,-3),(-1,-4)\},$$

$$\mathcal{M}'' = \{(0,-2),(-2,0),(2,-3),(-3,2),(-5,4),(-4,-2),(-5,-2),(-4,-3)\}.$$

4.8 An impartial game on two heaps emulating the rule 110 CA, Paper 8

Inspired by the discoveries in Paper 7, we ask the following question: is it possible for an impartial heap game to encompass universal computation using only two heaps? In particular can one emulate directly some one-dimensional cellular automata for which many questions are known to be algorithmically undecidable? In this paper we discuss two constructions which emulate the *rule 110* CA, Figure 14, which was proved undecidable by Mathew Cook [C2004] (resolving a conjecture by Stephen Wolfram). Our heap game variant in Figure 15 uses only two heaps and is similar to the game of Imitation Nim in that it takes advantage of a certain kind of move-size dynamics, which gives the heaps different meanings, simulating "time" and "space/tape" respectively. We prove that the patterns of P-positions of our game are equivalent to the patterns in the update of the rule 110 CA and thereby many questions regarding our heap game are undecidable. For this a creation black or white coloring of each token is required. See Figure 15.

In fact, in this paper we define an infinite family of move-size dynamic take-away games on two heaps, including also the well-known patterns of Pascal's triangle modulo 2, corresponding to the cellular automaton with update function the "Xor gate" (rule 60 in Wolfram's notation), see the middle picture in Figure 3.

The rules of this rule 60 game are particularly simple. There is one finite heap of matches and one finite heap of tokens. The current player removes at least one match (at most the whole heap) and at most as many tokens, as the number of matches removed by the previous player (possibly zero). It is not allowed to remove the remaining match(es) unless the tape-heap of tokens is empty. Hence, if a player cannot remove a match (from the time-heap), the game ends and the other player wins. See Figures 12 and 13.

We also show how these games have nice interpretations as board games, see Figure 16.

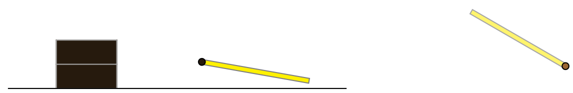

Figure 12: The previous player removed the rightmost match in the rule 60 game. Hence at most one token may be removed, which means that no move is possible and hence the previous player wins.

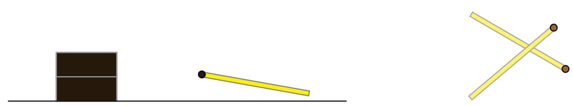

Figure 13: In this game, the next player wins by removing the last match together with both tokens.

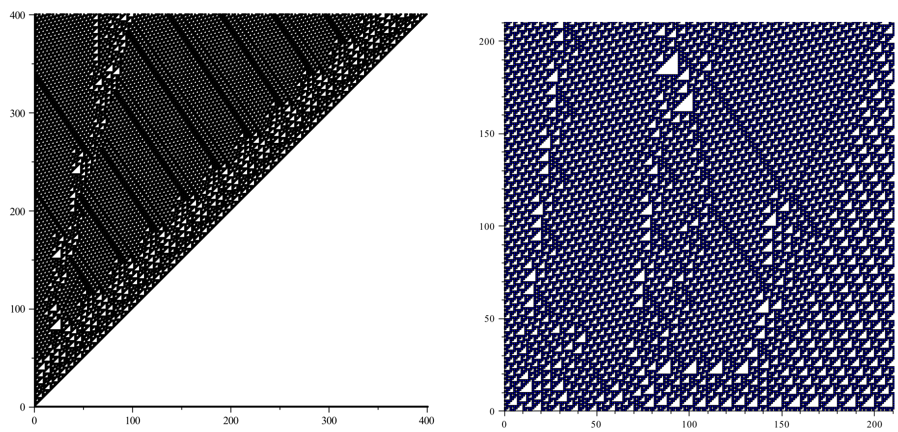

Figure 14: The pictures are produced by the updates of the elementary cellular automaton, rule 110. The initial one dimensional pattern, for the left most picture, is given by the doubly infinite initial string $\ldots 0011 \ldots$ (the black cells at the bottom are the 1s and time flows upwards). The central cell in each pattern is updated as follows: $000 \to 0, 001 \to 0, 010 \to 1, 011 \to 1, 100 \to 1, 101 \to 1, 110 \to 1, 111 \to 0$. The name originates in the number $110_{\text{dec}} = 1101110_{\text{bin}}$, obtained by letting time run downwards: the rule 124 CA is isomorphic to rule 110. We have omitted the negative part of the binary string since this area will be covered by 0s. (Compare with the patterns of the much simpler rule 60 CA, the middle picture in Figure 3.) The rightmost figure is produced by the same automaton with a somewhat more complicated initial string. Periodic "gliders" appear in a periodic background ether, in general interacting in unpredictable ways.

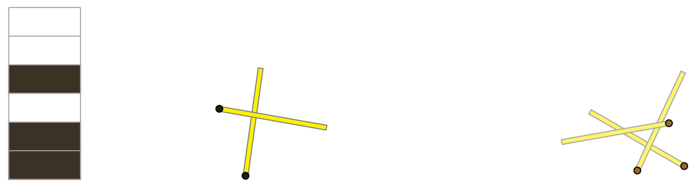

Figure 15: A rule 110 game variation on two heaps. The rightmost heap represents the previous player's removal of matches. The coloring of the tokens is essential: the final y matches can be removed if and only if the top y tokens are non-black. (Thus, if there are no tokens left, then the last match can always be removed.) The number of tokens a player can remove depends both on the current and the previous player's removal of matches. If the previous player removed m_p matches then $m - 1 \leq t \leq m_p + m$ tokens must be removed, together with $1 \leq m$ matches. Who wins the current game? Here $m_p = 3$ and $m \in \{1, 2\}$. If $m = 2$ then $1 \leq t \leq 5$, which violates the final condition. Hence only $m = 1$, $0 \leq t \leq 4$ is possible, which gives a win for the second player, by removing the last match together with $0, 1$ or 2 tokens as appropriate to avoid a top black token.

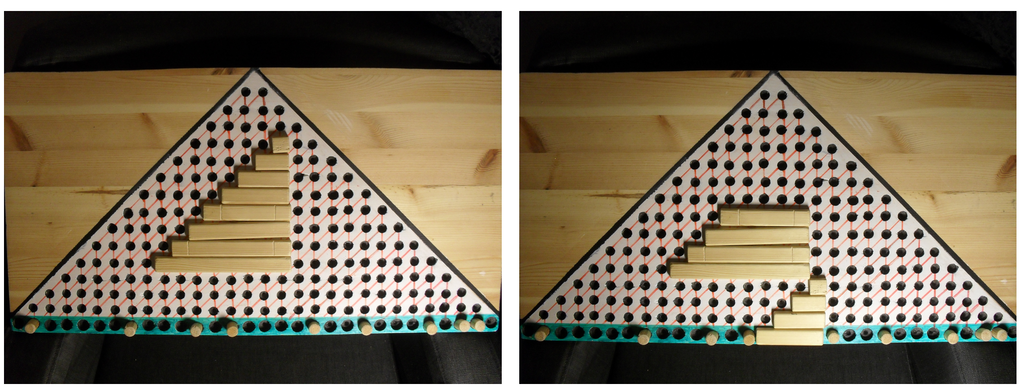

Figure 16: A board game variation of the rule 110 game, called the Triangle placing game, illustrating how the next player uses the only terminal move option from the given position. Rules: the top of the next right triangle must touch the base of the previous right triangle (strictly below). The right angle has to be to the right. The pegs at the bottom prevent certain moves. In the rightmost picture the top of the previous triangle has been removed, to build the next triangle.

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