RSS-Based Sensor Localization in the Presence of Unknown Channel Parameters

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***Abstract*—This paper studies the received signal strength based localization problem when the transmit power or path-loss exponent is unknown. The corresponding maximum likelihood estimator (MLE) poses a difficult nonconvex optimization problem. To avoid the difficulty in solving the MLE, we use suitable approximations and formulate the localization problem as a general trust region subproblem, which can be solved exactly under mild conditions. Simulation results show a promising performance for the proposed methods, which also have reasonable complexities compared to existing approaches.**

Index Terms– Wireless sensor network, localization, received signal strength, path-loss exponent, transmit power, general trust region subproblem.

I. Introduction

Localizing an unknown sensor node, henceforth called the target node, using received signal strength (RSS) is a popular technique in the literature in the context of location aware services [1], [2]. In this approach a number of fixed sensors at known positions, called reference nodes, measure the power of the signal transmitted by a target node and estimate the location of the target node. In the literature the received power is commonly modeled by the log-normal shadowing [3]. During the past few years a huge number of algorithms have been proposed to solve the localization problem based on the RSS measurements. The maximum likelihood estimator (MLE) derived for this problem is highly nonlinear and nonconvex [4]. To avoid difficulty in solving the MLE, a number of suboptimal estimators have been suggested in the literature, e.g., algorithms based on the semidefinite relaxation (SDR) [5]–[7], the linear least squares (LLS) [8], [9], the constrained linear least squares [10], and projection onto convex sets [11], [12], just to cite a few.

The power of the received signal mainly depends on the transmit power of the target node and the path-loss exponent. The transmit power of a target node depends on, e.g., its battery and radiation pattern and the path-loss exponent depends on the environment. A number of researchers tackled the localization problem when channel parameters are unknown, e.g., [13], [14], and derived suboptimal estimators. When only the transmit power is unknown, there are a number of approaches to deal with the localization problem, e.g., techniques based on eliminating the common term [15], [16]. A joint estimation technique based on the SDR and the LLS was proposed in [17], [18], which shows good performance compared to recently suggested approaches. Although the proposed approaches show good performance in some scenarios, it is still required to improve the performance of the estimators when channel parameters are unknown.

In this study, we consider the RSS-based localization problem when the channel parameters, i.e., the transmit power or path-loss exponent, are unknown. Different from [19], we model the unknown transmit power and path-loss exponent as fixed nuisance unknown parameters. Similar to our previous work [17] using suitable approximations, we obtain a nonlinear least squares objective function, which is smoother than the original MLE objective function but still is nonconvex. We, then, formulate the localization problem as a general trust region subproblem. In fact in this step, instead of relaxing the problem to a convex problem, which has been done, e.g., in [17], [18] when the transmit power is unknown, we transform the problem to a quadratic program and employ a technique developed in the numerical optimization literature for solving such a problem [20], [21]. Under mild conditions, the proposed approach will attain the optimal solution of the considered problem. In this paper, we first propose techniques to solve the localization problem when either the transmit power or the path-loss exponent is unknown. We, then, extend the proposed techniques to a general case when both channel parameters are unknown. Simulation results show that the proposed approach outperforms previous techniques

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and demonstrate that the suggested techniques are very close to Cramér-Rao lower bound (CRLB) in some scenarios. Complexity analyses show a reasonable cost for implementing the proposed techniques compared to the exisiting approaches.

The reminder of this paper is organized as follows. The signal model is briefly studied in Section II. Section III introduces estimators to estimate an unknown transmit power or path-loss exponent along with the location of the target. The complexity analyses of different approaches are presented in Section IV. Simulation results are presented in Section V and finally some concluding remarks and future work are discussed in Section VI.

II. System model

Let us consider a 2D network[1] consisting of a target node at unknown position, $\mathbf{x} \in \mathbb{R}^2$, and N reference nodes at known locations, $\mathbf{a}_i \in \mathbb{R}^2,\ i = 1, \ldots, N$. We assume that the target node transmits a signal and the reference nodes are able to measure the power of the received signal from the target node. The received power (in dBm) of the signal transmitted by the target node at the ith reference node, P_i, under the log-normal shadowing model, is given by [4], [6], [22], [23]

$$P_i = P_0 - 10\beta \log_{10} \frac{d_i}{d_0} + n_i, \quad i = 1, \ldots, N \tag{1}$$

where P_0 (in dBm) is the reference power at distance d_0 from the target node, β is the path-loss exponent, $d_i \triangleq \|\mathbf{x} - \mathbf{a}_i\|$ is the Euclidean distance between the target node and the ith reference node, and n_i are the log-normal shadowing terms modeled as independent and identically distributed zero-mean Gaussian random variables with standard deviation σ_{dB}, i.e., $n_i \sim \mathcal{N}(0, \sigma^2_{\text{dB}})$. Without loss of generality, we assume that $d_0 = 1$ m.

In this study, we assume that the transmit power or path-loss exponent is unknown and investigate approaches to estimate the location of the target node. We also assume that P_0 and β are fixed during the localization process. Since the distribution of the RSS measurement is Gaussian, assuming the transmit power, P_0, or path-loss exponent, β, as an unknown parameter, the MLE to estimate the location of the target based on the model in (1) is obtained by the following nonconvex optimization problem [24]:

$$\hat{\boldsymbol{\theta}}_{\text{MLE}} = \underset{\boldsymbol{\theta} \in \mathcal{D}}{\arg\min} \sum_{i=1}^{N} \left(P_i - P_0 + 10\beta \log_{10} d_i\right)^2, \tag{2}$$

where $\boldsymbol{\theta} = [\mathbf{x}^T\ P_0]^T$, $\boldsymbol{\theta} = [\mathbf{x}^T\ \beta]^T$, or $\boldsymbol{\theta} = [\mathbf{x}^T\ P_0\ \beta]^T$ and the set $\mathcal{D}$ defines a set in which the unknown parameters belong, e.g., $\boldsymbol{\theta} = [\mathbf{x}^T\ P_0]^T$, then $\mathcal{D} = \{[z_1\ z_2\ z_3]^T \in \mathbb{R}^3 : z_3 > 0\}$. As it is observed, the MLE is highly nonconvex and difficult to solve, especially when β is unknown. In the next section, we formulate the localization problem as the least squares problem, which is nonconvex but smoother than the MLE in (2).

III. Suboptimal Algorithms

We first study the localization problem when either P_0 or β is unknown. Then, we extend the results to a general case when both P_0 and β are unknown. We formulate different cases as general trust region subproblems. For details of solving the trust region subproblem, we refer the reader to, e.g., [20], [25], [26].

Note that in the localization literature a fixed transmission power and path-loss exponent are usually assumed for different links, e.g., see [6], [22], [23], [27], [28] and references therein.

A. Unknown transmit power

This section describes the procedure of approximating the MLE of (2) for the case when β is known (i.e., when $\boldsymbol{\theta} = [\mathbf{x}^T\ P_0]^T$) into a nonlinear least squares (NLS) problem[2], which can be solved exactly under mild conditions. We divide both sides of (1) by 5β and reformulate Eq. (1) as

$$\log_{10} h_i \lambda_i = \frac{P_0}{5\beta} + \frac{n_i}{5\beta}, \tag{3}$$

where $h_i \triangleq d_i^2$, $\lambda_i \triangleq 10^{P_i/5\beta}$, and $\alpha \triangleq 10^{P_0/5\beta}$. Taking the power of 10 on both sides yields

$$h_i \lambda_i = \alpha 10^{n_i/5\beta}. \tag{4}$$

For sufficiently small noise, the right hand side of (4) can be approximated using the first-order Taylor series expansion as[3]

$$h_i \lambda_i \simeq \alpha\big(1 + \frac{\ln 10}{5\beta} n_i\big). \tag{5}$$

Eq. (5) can be, alternatively, written as

$$h_i \lambda_i = \alpha + n'_i, \tag{6}$$

where n'_i is a zero-mean Gaussian random variable with variance $(\ln 10)^2 \alpha^2 \sigma^2_{dB} / 25\beta^2$. In this step, we apply the nonlinear least squares criterion to the model in (6) to estimate the unknown parameters. The corresponding

[1]The generalization to 3D networks is straightforward, but is not explored in this paper.

[2]The least-absolute mean approach can also be employed for obtaining a robust estimator. For that approach, we can use techniques introduced in [17] for solving the problem.

[3]Note that $a^x = 1 + x \ln a + \ldots + \frac{(x \ln a)^n}{n!} + \ldots,\ -\infty < x < \infty$.

NLS estimator of the unknown parameters $[\mathbf{x}^T \ \alpha]$ in (6) is [24, Ch. 8]

$$[\hat{\mathbf{x}}^T \ \hat{\alpha}] = \underset{[\mathbf{x}^T \ \alpha]\in\mathbb{R}^3}{\arg\min} \sum_{i=1}^{N} (h_i\lambda_i - \alpha)^2. \tag{7}$$

The cost function (7) is still nonlinear and noncovex, but it is much smoother than the MLE objective function in (2). For a discussion on the behavior of the both objective functions, see [17]. Let us write the problem (7) as

$$\begin{aligned} &\underset{z,\mathbf{x},\alpha}{\text{minimize}} \sum_{i=1}^{N} (\lambda_i(z - 2\mathbf{a}_i^T\mathbf{x} + \|\mathbf{a}_i\|^2) - \alpha)^2 \\ &\text{subject to } z = \|\mathbf{x}\|^2. \end{aligned} \tag{8}$$

Now, we express the problem in (8) as a quadratic program as follows:

$$\begin{aligned} &\underset{\mathbf{y}_1}{\text{minimize}} \quad \|\mathbf{A}_1\mathbf{y}_1 - \mathbf{b}_1\|^2 \\ &\text{subject to } \mathbf{y}_1^T\mathbf{D}_1\mathbf{y}_1 + 2\mathbf{f}_1^T\mathbf{y}_1 = 0 \end{aligned} \tag{9}$$

where $\mathbf{y}_1 \triangleq [\|\mathbf{x}\|^2 \ \mathbf{x}^T \ \alpha]^T$ and matrices $\mathbf{A}_1$ and $\mathbf{D}_1$, and vectors $\mathbf{b}_1$ and $\mathbf{f}_1$ are defined as

$$\begin{aligned} \mathbf{A}_1 &\triangleq \begin{bmatrix} \lambda_1 & -2\lambda_1\mathbf{a}_1 & -1 \\ \vdots & \vdots & \vdots \\ \lambda_N & -2\lambda_N\mathbf{a}_N & -1 \end{bmatrix}, \\ \mathbf{D}_1 &\triangleq \mathrm{diag}(0,1,1,0), \\ \mathbf{b}_1 &\triangleq \left[-\lambda_1\|\mathbf{a}_1\|^2 \ \ldots \ -\lambda_N\|\mathbf{a}_N\|^2\right]^T, \\ \mathbf{f}_1 &\triangleq \left[-\frac{1}{2} \ 0 \ 0 \ 0\right]^T. \end{aligned}$$

The problem in (9) minimizes a quadratic function over a quadratic constraint. This type of problem is called a generalized trust region subproblem [20]. It is known that the general trust region subproblem has no duality gap and the optimal solution can be extracted from the dual solution [20], [26], [29]. A necessary and sufficient condition for $\mathbf{y}_1^*$ to be optimal in (9) is that [26]

$$\begin{aligned} &(\mathbf{A}_1^T\mathbf{A}_1 + \gamma\mathbf{D}_1)\mathbf{y}_1^* = (\mathbf{A}_1^T\mathbf{b}_1 - \gamma\mathbf{f}_1), \\ &(\mathbf{y}_1^*)^T\mathbf{D}_1\mathbf{y}_1^* + 2\mathbf{f}_1^T\mathbf{y}^* = 0, \\ &(\mathbf{A}_1^T\mathbf{A}_1 + \gamma\mathbf{D}_1) \succeq 0. \end{aligned} \tag{10}$$

The last expression in (10) means that $(\mathbf{A}_1^T\mathbf{A}_1 + \gamma\mathbf{D}_1)$ is a positive semidefinite matrix. Under conditions considered in (10), the solution to the problem of (9) is given by

$$\mathbf{y}_1(\gamma) = (\mathbf{A}_1^T\mathbf{A}_1 + \gamma\mathbf{D}_1)^{-1}(\mathbf{A}_1^T\mathbf{b}_1 - \gamma\mathbf{f}_1). \tag{11}$$

In such a situation to find γ, we simply replace (11) into constraint $\mathbf{y}_1^T\mathbf{D}_1\mathbf{y}_1 + 2\mathbf{f}_1^T\mathbf{y}_1 = 0$, i.e.,

$$\phi(\gamma) = \mathbf{y}_1^T(\gamma)\mathbf{D}_1\mathbf{y}_1(\gamma) + 2\mathbf{f}_1^T\mathbf{y}_1(\gamma) = 0, \quad \gamma \in \mathcal{I} \tag{12}$$

where the interval $\mathcal{I}$ consists of all γ such that $\mathbf{A}_1^T\mathbf{A}_1 + \gamma\mathbf{D}_1 \succeq 0$. The interval of $\mathcal{I}$ is given by [21]

$$\mathcal{I} = (-1/\gamma_1, \infty), \tag{13}$$

with γ_1 representing the largest eigenvalue of $(\mathbf{A}_1^T\mathbf{A}_1)^{-1/2}\mathbf{D}_1(\mathbf{A}_1^T\mathbf{A}_1)^{-1/2}$ [20]. In summary, the solution to (8) is obtained as follows:

- Use a bisection search to find a root of $\phi(\gamma) = 0$, say γ^*. Note that $\phi(\gamma)$ is a strictly decreasing function with respect to γ [20].
- Replace γ^* in (11) to obtain $\mathbf{y}_1^* = \mathbf{y}_1(\gamma^*)$.
- Estimate the unknown parameters as $\hat{\mathbf{x}} = [\mathbf{y}_1^*]_{2:3}$, $\hat{P}_0 = 5\beta\log_{10}[\mathbf{y}_1^*]_3$, with $[\mathbf{v}]_{i:j}$ denoting the ith to the jth elements of vector $\mathbf{v}$.

Note that when $\gamma = 1/\gamma_1$, which is called *hard case* [25], can be suitably handled using techniques introduced in the literature [21], [25]. However, this case occurs rarely in practical situations; we have never observed it any of our numerous simulations. That the *hard case* is rare has also been noted in other studies, e.g., in [21].

B. Unknown path-loss exponent

In this section, we assume that the transmit power P_0 is known, but the path-loss exponent β is unknown. We propose a two-step estimator to find estimates of the location and path-loss exponent. We first jointly estimate the path-loss exponent and the location of the target node. Then, we update the estimate of both parameters. We assume that β belongs to an interval $\beta \in [\beta_1, \beta_2]$. In practice the path-loss exponent varies normally from 2 (free space) to 6 (e.g., in an indoor scenario). We express (1) (assuming $d_0 = 1$ m) as $d_i^2 = 10^{(P_0 - P_i + n_i)/(5\beta)}$. Similar to (5), we can write

$$d_i^2 = 10^{(P_0 - P_i)/(5\beta)}\big(1 + \frac{\ln 10}{5\beta}n_i\big). \tag{14}$$

Now, we write the path-loss exponent as $\beta = \beta_0(1 + (\beta - \beta_0)/\beta_0)$, where β_0 is chosen such that $|(\beta - \beta_0)/\beta_0|$ is as small as possible. Note that β is unknown and β_0 is a tuning parameter chosen by designer. We will see in the simulation section that how different values of β_0 can affect the performance of the algorithm. Let $\delta \triangleq (\beta - \beta_0)/\beta_0$ for any $\beta_0 \neq 0$. Hence, $\beta = \beta_0(1 + \delta)$, and

$$d_i^2 = 10^{(P_0 - P_i)/(5\beta_0(1+\delta))}\left(1 + \frac{\ln 10}{5\beta}n_i\right).$$

A Taylor series expansion around $\delta = 0$ and assuming that $|\delta|$ is small leads to the approximations

$1/(1+\delta) \approx 1-\delta$ and

$$d_i^2 \approx 10^{(P_0-P_i)(1-\delta)/(5\beta_0)}\left(1+\frac{\ln 10}{5\beta}n_i\right) = q_i^{(1-\delta)}\left(1+\frac{\ln 10}{5\beta}n_i\right), \quad (15)$$

where $q_i \triangleq 10^{(P_0-P_i)/(5\beta_0)}$. The model in (15) is still nonlinear and difficult to solve. To obtain a linear model based on the unknown parameters, we make yet another simplifying assumption. Considering a Taylor series expansion of $q_i^{-\delta}$ around $\delta = 0$, and assuming that $|\delta \ln q_i|$ is small, yields the approximation $q_i^{-\delta} \approx 1 - \delta \ln q_i$, which in turn allows us to further approximate d_i^2 as

$$d_i^2 \approx q_i(1-\delta \ln q_i)\left(1+\frac{\ln 10}{5\beta}n_i\right). \quad (16)$$

The approximation in (16) is valid as long as $|\delta \ln q_i|$ is small. For example, if δ is extremely small, which means β_0 is very close to β, the expression in (16) is a valid approximation. Otherwise, we can investigate for which networks the approximation in (16) is valid. In the shadow-free case, we have

$$\begin{aligned}\delta \ln q_i &= \delta \frac{P_0-P_i}{5\beta}\ln 10 \\ &= \delta \frac{10\beta}{5\beta_0}\log_{10}(d_i)\ln(10) \\ &= 2\delta(1+\delta)\log_{10}(d_i)\ln(10).\end{aligned}$$

Hence, the condition $|\delta \ln q_i| \ll 1$ is equivalent to

$$10^{-1/|2\delta(1+\delta)\ln(10)|} \ll d_i \ll 10^{1/|2\delta(1+\delta)\ln(10)|}.$$

Thus, given a certain δ, i.e., quality of our guess of β, we will have both a lower and an upper bound on d_i.

To find an optimal value of β_0, we assume that the path-loss exponent β has some distribution over an interval and we choose a value for β_0 (numerically) such that the location estimation error is minimized. For example, in the simulations, we will assume that the path-loss exponent is uniformly distributed over the interval $[2, 6]$ and we will see that there is an optimal β_0 minimizing the root-mean-square error of the estimation.

The two-step estimator is implemented as follows.

1) first step: In this step, we apply the least squares criterion to the model in (16) to estimate both location and δ. Then,

$$\begin{aligned}&\underset{z,\mathbf{x},\delta}{\text{minimize}} \sum_{i=1}^{N}(z - 2\mathbf{a}_i^T\mathbf{x} + \|\mathbf{a}_i\|^2 - q_i + q_i\delta\ln q_i)^2 \\ &\text{subject to } z = \|\mathbf{x}\|^2. \end{aligned} \quad (17)$$

Similar to the previous section, we can express (17) as a general trust region subproblem

$$\begin{aligned}&\underset{\mathbf{y}_2}{\text{minimize}} \quad \|\mathbf{A}_2\mathbf{y}_2 - \mathbf{b}_2\|^2 \\ &\text{subject to } \mathbf{y}_2^T\mathbf{D}_2\mathbf{y}_2 + 2\mathbf{f}_2^T\mathbf{y}_2 = 0, \end{aligned} \quad (18)$$

where $\mathbf{y}_2 \triangleq [\|\mathbf{x}\|^2 \ \mathbf{x}^T \ \delta]^T$, matrices $\mathbf{D}_2$ and $\mathbf{A}_2$, and vectors $\mathbf{b}_2$ and $\mathbf{f}_2$ are defined as

$$\mathbf{A}_2 \triangleq \begin{bmatrix} 1 & -2\mathbf{a}_1 & q_1\ln q_1 \\ \vdots & \vdots & \vdots \\ 1 & -2\mathbf{a}_N & q_N \ln q_N \end{bmatrix},$$

$$\mathbf{b}_2 \triangleq \begin{bmatrix} q_1 - \|\mathbf{a}_1\|^2 \\ \vdots \\ q_N - \|\mathbf{a}_N\|^2 \end{bmatrix},$$

$$\mathbf{D}_2 \triangleq \text{diag}(0,1,1,0),$$

$$\mathbf{f}_2 \triangleq \left[-\frac{1}{2} \ 0 \ 0 \ 0\right]^T.$$

In the sequel, we employ a similar technique as used in the previous section (Eqn. (11)–(13)) to solve (18). After solving the problem in (18), we obtain an estimate of the target location and the path-loss exponent as

$$\tilde{\mathbf{x}} = [\mathbf{y}_2^*]_{2:3}, \quad (19)$$

where $\mathbf{y}_2^*$ is the optimal solution of (18).

2) second step: In this step, we refine the estimates derived in the first step. Note that it is possible to estimate the path-loss exponent from the first step as $\hat{\beta} = \beta_0(1 + [\mathbf{y}_2^*]_4)$, but as we have observed, through simulations, that in the first step, the location is more accurately estimated compared to the path-loss exponent. Therefore in this step, we first update the path-loss exponent using a simple estimator based on the estimate of the location of the target obtained in (19). From (1) using the method of moment [24], we can estimate the path-loss exponent as

$$\tilde{\beta} \simeq \frac{\sum_{i=1}^{N}(P_0 - P_i)}{10\log\prod_{i=1}^{N}\tilde{d}_i}. \quad (20)$$

Note that since the true distance $d_i = \|\mathbf{x} - \mathbf{a}_i\|$ is not available, we instead used the approximate distance $\tilde{d}_i = \|\tilde{\mathbf{x}} - \mathbf{a}_i\|$ in (20), where $\tilde{\mathbf{x}}$ is the estimate of the target location obtained in the first step, i.e., Eq. (19). With an estimate of the path-loss exponent in (20), we back to (14) and write

$$d_i^2 = 10^{(P_0-P_i)/(5\tilde{\beta})}\left(1 + \frac{\ln 10}{5\tilde{\beta}}n_i\right), \quad i = 1, \ldots, N. \quad (21)$$

Now, we apply a weighted least squares criterion to (21) and express the problem as a general trust region subproblem as follows:

$$\begin{aligned}&\underset{\mathbf{y}_3}{\text{minimize}} \quad \|\mathbf{W}(\mathbf{A}_3\mathbf{y}_3 - \mathbf{b}_3)\|^2 \\ &\text{subject to } \mathbf{y}_3^T\mathbf{D}_3\mathbf{y}_3 + 2\mathbf{f}_3^T\mathbf{y}_3 = 0, \end{aligned} \quad (22)$$

where $\mathbf{y}_3 \triangleq [\|\mathbf{x}\|^2 \ \mathbf{x}^T]^T$ and matrices $\mathbf{D}_3$, $\mathbf{A}_3$, and $\mathbf{W}$ and vectors $\mathbf{b}_3$ and $\mathbf{f}_3$ are defined as

$$\mathbf{A}_3 \triangleq \begin{bmatrix} 1 & -2\mathbf{a}_1 \\ \vdots & \vdots \\ 1 & -2\mathbf{a}_N \end{bmatrix},$$
$$\mathbf{W} \triangleq \mathrm{diag}\left(10^{P_1/(5\tilde{\beta})}, 10^{P_2/(5\tilde{\beta})}, \ldots, 10^{P_N/(5\tilde{\beta})}\right),$$
$$\mathbf{D}_3 \triangleq \mathrm{diag}(0, 1, 1),$$
$$\mathbf{b}_3 \triangleq \left[-\|\mathbf{a}_1\|^2 \ \ldots \ -\|\mathbf{a}_N\|^2\right]^T,$$
$$\mathbf{f}_3 \triangleq \left[-\frac{1}{2} \ 0 \ 0\right]^T,$$

where the operator diag denotes a diagonal matrix. Again, we employ a similar technique as used before (Eqn.(11)–(13)) to solve (22). The target location now is estimated as

$$\bar{\bar{\mathbf{x}}} = [\mathbf{y}_3^*]_{2:3}, \tag{23}$$

where $\mathbf{y}_3^*$ is the optimal solution of (22).

C. Unknown path-loss exponent and transmit power

In this section, we consider a general case when both channel parameters, P_0 and β, are unknown and we investigate a two-step estimator to solve the localization problem.

1) first step: We first assume that P_0 belongs to an interval $[P_{0_1}, P_{0_2}]$. Let us pick one point in this interval, say $\bar{P}_0$, and using a similar technique as used before, we can express (1) as

$$d_i^2 = 10^{(\bar{P}_0 - P_i)/(5\beta_0)(1-\delta)}\gamma(1 + \frac{\ln 10}{5\beta} n_i), \tag{24}$$

where $\gamma \triangleq 10^{(P_0 - \bar{P}_0)/(5\beta)}$. Suppose that $(\bar{P}_0 - P_i)/(5\beta_0)\delta \ln 10$ is small. Similar to (16), we can express (24) as

$$d_i^2 = g_i\gamma(1 - \delta \ln g_i)(1 + \frac{\ln 10}{5\beta} n_i), \tag{25}$$

where $g_i \triangleq 10^{(\bar{P}_0 - P_i)/(5\beta_0)}$. Therefore, we can obtain a linear model as

$$[1 - 2\mathbf{a}_i^T \ -g_i \ g_i \ln g_i]\mathbf{y}_4 = -\|\mathbf{a}_i\|^2 + \epsilon_i, \tag{26}$$

with $\mathbf{y}_4 \triangleq [\|\mathbf{x}\|^2 \ \mathbf{x}^T \ \gamma \ \gamma\delta]^T$ and $\epsilon_i \triangleq g_i\gamma(1 - \delta \ln g_i)\ln 10 n_i/(5\beta)$. Similar to the previous section, we apply a nonlinear least squares criterion and then transform the corresponding NLS to a general trust region subproblem as

$$\begin{aligned} &\underset{\mathbf{y}_4}{\text{minimize}} \quad \|\mathbf{A}_4\mathbf{y}_4 - \mathbf{b}_4\|^2 \\ &\text{subject to } \mathbf{y}_4^T\mathbf{D}_4\mathbf{y}_4 + 2\mathbf{f}_4^T\mathbf{y}_4 = 0 \end{aligned} \tag{27}$$

where matrices $\mathbf{D}_4$ and $\mathbf{A}_4$, and vectors $\mathbf{b}_4$ and $\mathbf{f}_4$ are defined as

$$\mathbf{A}_4 \triangleq \begin{bmatrix} 1 & -2\mathbf{a}_1 & -g_1 & g_1 \ln g_1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & -2\mathbf{a}_N & -g_N & g_N \ln g_N \end{bmatrix},$$
$$\mathbf{D}_4 \triangleq \mathrm{diag}(0, 1, 1, 0, 0),$$
$$\mathbf{b}_4 \triangleq \left[-\|\mathbf{a}_1\|^2 \ \ldots \ -\|\mathbf{a}_N\|^2\right]^T,$$
$$\mathbf{f}_4 \triangleq \left[-\frac{1}{2} \ 0 \ 0 \ 0 \ 0\right]^T.$$

Here, we apply a similar procedure as employed for Eqn. (11)–(13) to solve the problem in (27). We obtain an estimate of the target location as

$$\breve{\mathbf{x}} = [\mathbf{y}_4^*]_{2:3}, \tag{28}$$

where $\mathbf{y}_4^*$ is the optimal solution of (27).

2) second step: In this step, we first obtain new estimates of the transmit power and path-loss exponent as follows. From the model in (1), we write

$$P_i \simeq P_0 - 10\beta \log_{10} \breve{d}_i + n_i, \quad i = 1, \ldots, N, \tag{29}$$

where $\breve{d}_i = \|\breve{\mathbf{x}} - \mathbf{a}_i\|$ with $\breve{\mathbf{x}}$ given in (28). Now, we apply a linear least squares technique[4] to find an estimate of the transmit power and path-loss exponent for the linear model of (29). Therefore,

$$[\breve{P}_0 \ \breve{\beta}]^T = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\mathbf{h}, \tag{30}$$

where

$$\mathbf{G} \triangleq \begin{bmatrix} 1 & -10 \log \breve{d}_1 \\ \vdots & \vdots \\ 1 & -10 \log \breve{d}_N \end{bmatrix},$$
$$\mathbf{h} \triangleq [P_1 \ \ldots \ P_N]^T. \tag{31}$$

Based on the estimate in (30) and from the model in (1), we can write

$$d_i^2 \simeq 10^{(\breve{P}_0 - P_i)/(5\breve{\beta})}(1 + \frac{\ln 10}{5\breve{\beta}} n_i). \tag{32}$$

Therefore, we obtain a similar model as (21) except P_0 and $\tilde{\beta}$ are respectively replaced with $\breve{P}_0$ and $\breve{\beta}$ (given in Eq. (30)). Again, we employ a weighted least squares technique and then transform the problem to a general trust region subproblem similar to (22). The only difference is that the weighting matrix $\mathbf{W}$ is replaced with the following matrix:

$$\overline{\mathbf{W}} = \mathrm{diag}\big(10^{P_1/(5\breve{\beta})}, 10^{P_2/(5\breve{\beta})}, \ldots, 10^{P_N/(5\breve{\beta})}\big). \tag{33}$$

[4]If there are a large number of RSS measurements, we can apply a total least squares technique [30] to find a more accurate estimates of P_0 and β.

Thus, an estimate of the target location now is obtained by solving the trust region subproblem (22) in which the weighting matrix $\mathbf{W}$ is replaced with $\overline{\mathbf{W}}$. Therefore,

$$\bar{\bar{\mathbf{x}}} = [\bar{\mathbf{y}}_3^*]_{2:3}, \tag{34}$$

where $\bar{\mathbf{y}}_3^*$ is the optimal solution of (22) by replacing $\mathbf{W}$ with $\overline{\mathbf{W}}$.

IV. Complexity Analysis

In this section, we study the complexity of the proposed technique and compare the cost of different approaches in terms of floating point operations (*flops*) and running time. We compare the complexity of the MLE, the LLS, the SDR, the proposed method in Section III-A. The complexity of the algorithms proposed in Sections III-B and III-C can be computed similarly. Here, we compute the worst-case complexity. To compute the complexity of the MLE, we assume that a good initial point is available, and an iterative algorithm such as Gauss-Newton (GN) method is used to find the global minimum after a number of iterations. Of course, finding a good initial point for the MLE is a challenging task and this study aims to tackle it. The most complex part of the GN approach is to compute the Newton step [31]. After K_{GN} iterations (usually less than 50 iterations), the solution of the MLE (assuming a good initial point) is obtained. It can be verified that the complexity of the MLE is the order of N^3 for every Newton step. Then the total cost can be computed as $O(K_{GN}N^3)$. The worst-case complexity of the SDP can be computed as $O(K_{SDP}N^4 \log(1/\epsilon))$, where the number of iterations K_{SDP} is commonly approximated by $O(N^{1/2})$ [32], [33] and ϵ is an accuracy tolerance. The complexity of the LLS can be computed as $O(34N)$ for this problem.

For the proposed approach, we need to use a bisection search to solve (11), which is the most complex part of the algorithm. We first decompose $\mathbf{A}_1^T\mathbf{A}_1$ using the singular value decomposition technique, i.e., $\mathbf{A}_1^T\mathbf{A}_1 = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$, where $\mathbf{U}$ is an orthogonal matrix and $\mathbf{\Lambda}$ is a diagonal matrices. Therefore, $(\mathbf{A}_1^T\mathbf{A}_1 + \gamma\mathbf{D})^{-1}$ can be computed as $\mathbf{U}(\mathbf{\Lambda} + \gamma\mathbf{D})^{-1}\mathbf{U}^T$. Hence, in every bisection step, we need to compute the inverse of the diagonal matrix $(\mathbf{\Lambda} + \gamma\mathbf{D})$. Suppose that the bisection search takes K_{GTR} steps, then the total cost of the proposed approach can be approximated as $36K_2 + 34N$. In the simulation, we have observed that the bisection search algorithm usually takes 20 to 30 iterations to find the optimal value of γ. Table I summarizes the complexity of the different approaches.

In a similar way, the complexity of the proposed algorithm for unknown path-loss or both unknown path-loss and transmission power can be computed. The total complexity is the sum of the complexity for each step.

We have also measured the average running time of different algorithms for a network consisting of 5 reference nodes as considered in Section V. The algorithms have been implemented in Matlab 2012 on a MacBook Pro (Processor 2.3 GHz Intel Core i7, Memory 8 GB 1600 MHz DDR3). To implement the MLE, we use the Matlab function *lsqnonlin* [34] initialized with the estimate of the proposed estimator. To implement the SDP, we use the *CVX* toolbox [35]. We have run the algorithms for 500 realizations of the network and computed the average running time in ms as shown in Table II. It is observed that the proposed approach has a reasonable complexity compared to other approaches.

V. Simulation results

A 20 m by 20 m area was considered for the simulation. Five reference nodes were placed at fixed positions $(0, 0), (20, 0), (0, 20), (20, 20)$, and $(10, 10)$, all in meters. A target node is randomly placed inside the area. In the simulations for every realization, the transmit power, P_0, and the path-loss exponent, β, are randomly drawn from $[-20, -15]$ dBm and from $[2, 6]$, respectively. To compare different approaches, we consider the root-mean-square-error (RMSE). In the simulations, we examine different scenarios.

A. Unknown transmit power

In this section, we compare the proposed method with the corresponding CRLB computed in Appendix A, the SDR, and the LLS (the least squares followed by a correction technique [36], [37]). For details of the SDR and LLS, please see [17].

Fig. 1(a) shows the RMSE of the location estimate for different approaches versus the variances of the shadowing. As the figure shows, the proposed method outperforms other approaches and is very close to the CRLB. Fig. 1(b) illustrates the RMSE of the transmit power estimation for different approaches. As can be observed, both proposed approach and the LLS outperform the SDR and are close to the CRLB.

In the next simulation, we study the robustness of the algorithm against the perturbation in transmission power. We model the transmit power as a Gaussian random variable with mean $\bar{P}_0$ and standard deviation ξ, i.e., $P_0 \sim \mathcal{N}(\bar{P}_0, \xi^2)$. Then, the algorithm tries to jointly estimate the mean power $\bar{P}_0$ and the location.

Fig. 2 illustrates the RMSE of the location and transmission power $\bar{P}_0$ estimates for different values of standard deviation of perturbation. It is observed that the perturbation in power transmission can be absorbed in the showing terms, especially for low standard deviation of perturbation, and the behavior of estimates remains the same. It is observed when the variance of the

TABLE I
COMPLEXITY OF DIFFERENT APPROACHES; K_{GN} AND K_{GTR} ARE RESPECTIVELY THE NUMBER OF ITERATIONS FOR THE GN AND THE BISECTION APPROACHES TO CONVERGE. ϵ IS AN ACCURACY PARAMETER.

Method	Cost
MLE	$O(K_{GN}N^3)$
SDP	$O(K_{SDP}N^4 \log(1/\epsilon))$, $K_{SDP} = O(N^{1/2})$
LS	$O(34N)$
Proposed technique	$O(36K_{GTR} + 34N)$

TABLE II
AVERAGE RUNNING TIME FOR DIFFERENT ALGORITHMS.

Method	Time (ms)
MLE	14
SDP	64
LS	0.12
Proposed technique	1.4

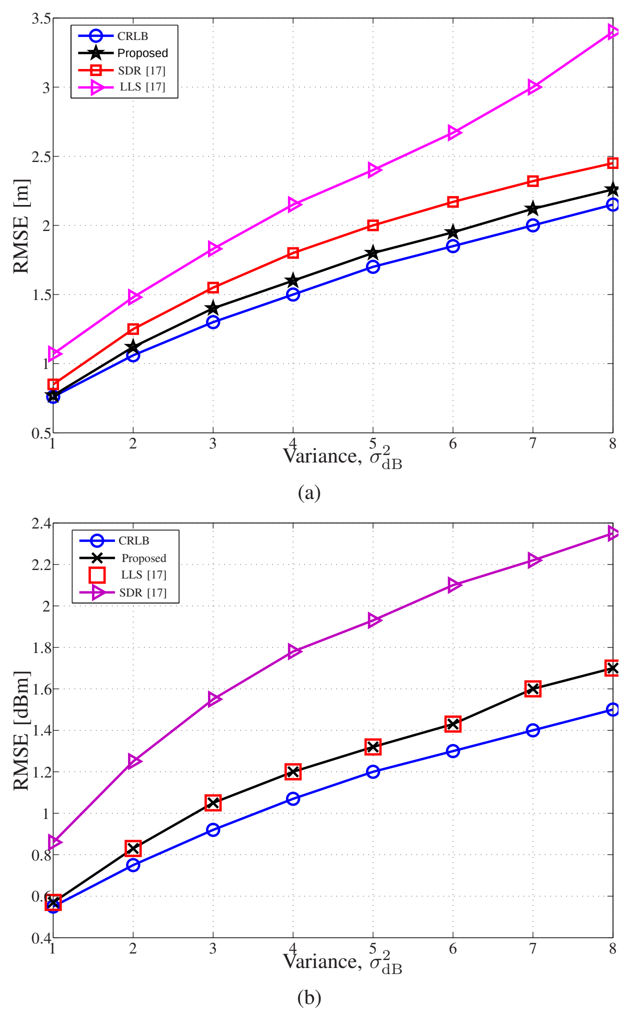

Fig. 1. The RMSE of different approaches for (a) the location estimate and (b) the power estimate.

shadowing is small, the performance is mainly affected by the perturbation noise.

B. Unknown path-loss exponent

In the next simulations, we assume that the transmit power P_0 is known and we estimate both the path-loss exponent and the location of the target node. We compare the proposed method with the corresponding CRLB (derived in Appendix A).

Fig. 3(a) shows the CRLB and the RMSE of the location estimation for the proposed technique. In this simulation, we set $\beta_0 = 5$, that is, $\delta = (\beta - 5)/5$. As can be seen, the proposed approach is close to the CRLB. The gap between the CRLB and the proposed method is mainly because of the approximations used in different steps. Fig. 3(b) shows the RMSE of the path-loss exponent estimation for the proposed method and the corresponding CRLB. Although there is a gap between the CRLB and the proposed method, the performance of the proposed method seems to be acceptable. To further improve the estimate, we have implemented the MLE using *lsqnonlin* [34] initialized with the estimate given by the proposed algorithm. We have also implemented the MLE initialized with true values of the target location and path-loss exponent for comparison. As it is observed from Fig. 3(b), the estimate can be considerably improved. It is seen that there is a gap between the MLE and the CRLB. The reason is that the MLE asymptotically attains the CRLB. That is, for low variances of noise or large number of measurements, the MLE is optimal.

To study the effect of parameter β_0, we first evaluate the validity of the approximation used in (16). In Fig. 4, we plot the cumulative distribution function (CDF) of $(P_0 - P_i)(1+\delta)\delta \ln 10/(5\beta_0)$ for different values of β_0. As can be seen, the value of β_0 considerably affects the validity of the approximation. For instance, $\beta_0 = 5$

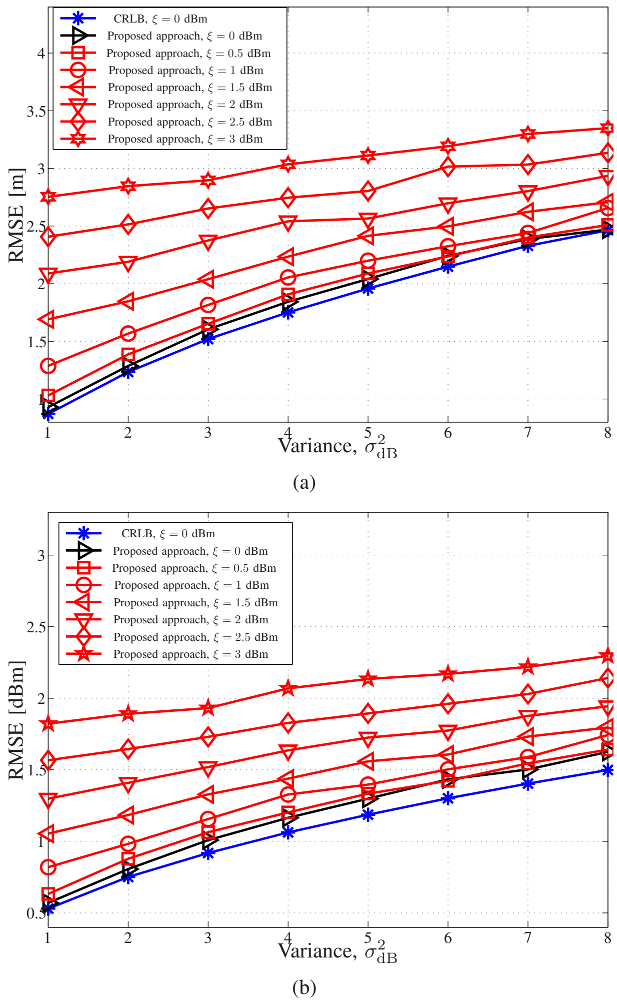

Fig. 2. The RMSE of the proposed approach for transmission power modeled as a Gaussian random variable with mean $\bar{P}_0$ and standard deviation ξ in dBm; (a) the location estimation and (b) the mean of the power, $\bar{P}_0$, estimation.

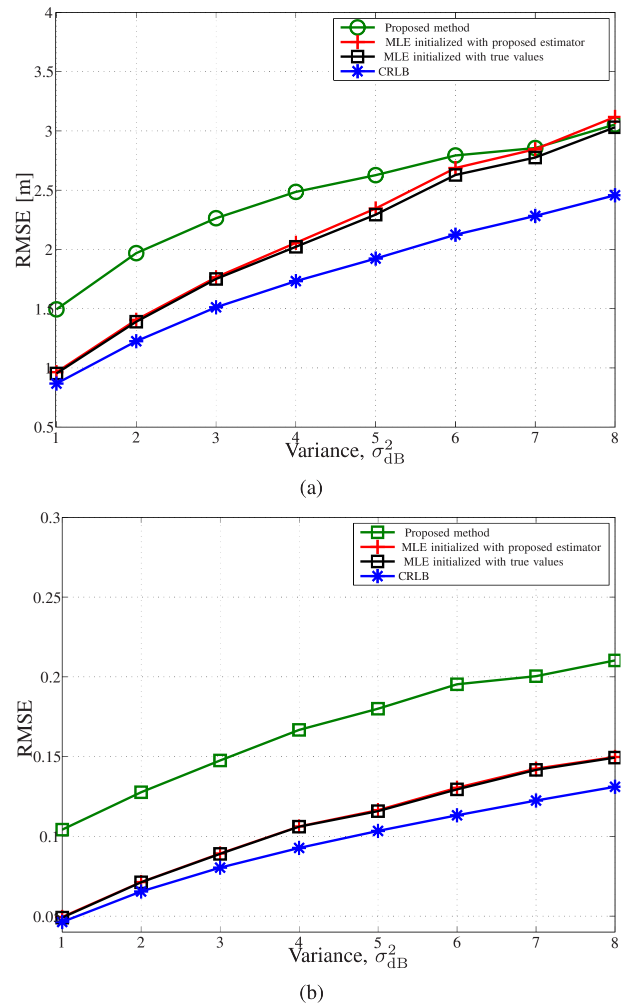

Fig. 3. The RMSE of the proposed approach and the CRLB (unknown path-loss exponent) for (a) the location estimation and (b) the path-loss exponent estimation.

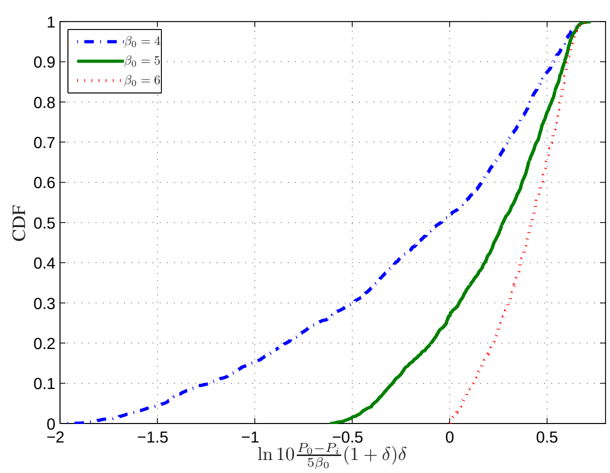

Fig. 4. The CDF of $(P_0 - P_i)(1+\delta)\delta \ln 10/(5\beta_0)$ for $\beta \sim \mathcal{U}(2, 6)$ ($\sigma_{\mathrm{dB}} = 1$).

seems a good choice in this scenario. In Fig. 5(a) and Fig. 5(b), we plot the RMSE of the location and path-loss exponent estimation versus β_0 for different variances σ^2_{dB}. As it is seen, there is a critical value for β_0 such that the estimation errors for the location and the path-loss exponent are minimized. This phenomena is clearly seen in Fig. 5(b). Considering the definition of $\delta = (\beta - \beta_0)/\beta_0$, we see that both small and large values of β_0 make δ be large. Therefore, the approximation in (16) may not be valid.

In the next simulation, we compare the performance of the proposed approach in this study with the one proposed in [38]. Note that in [38], the authors assume different path-loss exponents for every link and propose an iterative approach to solve the problem. That is, they first obtain an estimate of the location and then update the path-loss exponent. In the simulation, we assume that

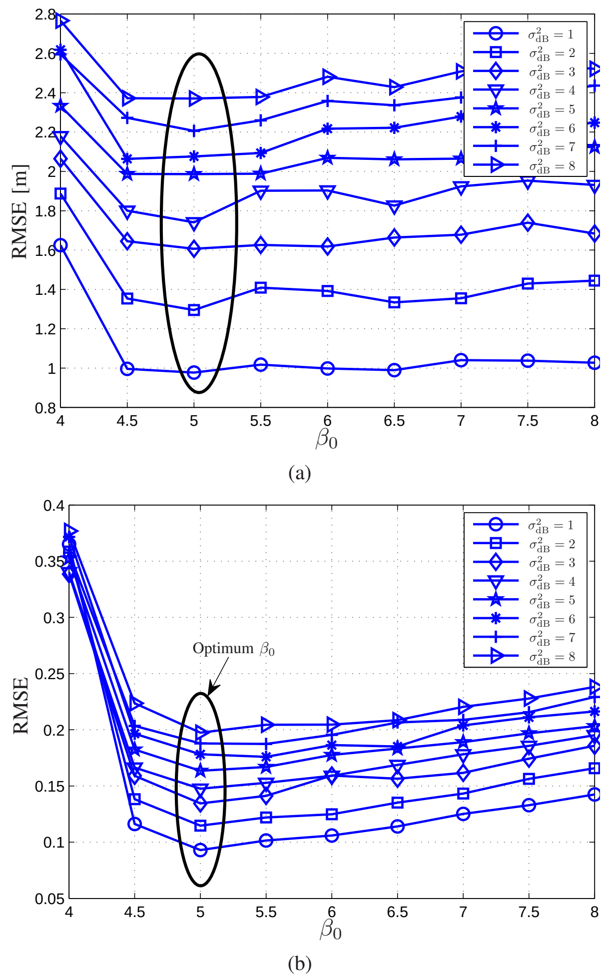

Fig. 5. The RMSE of the proposed approach for the (a) location estimation, (b) path-loss exponent.

the path-loss is fixed for different links, resulting a single unknown parameter in optimization problem in [38]. We iterate the procedure suggested in [38] three times. Note that it is needed to have a reasonable interval for the path-loss and an initial estimate of the path-loss at the beginning. We set both the initial value and β_0 equal to 5. It is noted here that we have not chosen an optimal value for β_0 in the simulation.

Fig. 6 shows the RMSE of the location and path-loss exponent estimates for different approaches when the path-loss exponent is uniformly distributed over an interval, noted in the title of figures. It is observed that the proposed approach outperforms the method in [38], especially for the location estimate. Note that as the ambiguity about path-loss increases, i.e., a larger interval, the performance of the proposed technique in [38] considerably degrades, while the proposed technique in this study is quite robust.

C. *Unknown transmit power and path-loss exponent*

In this section, we consider the previous network except we add one more reference node at location $(10\text{m}, 20\text{m})$. In this simulation, we set $\bar{P}_0 = -17.5$ dBm and $\beta_0 = 5.2$.

Fig. 7 shows the RMSE of the location estimate of the first and the second steps and the corresponding CRLB (derived in Appendix A). It is seen that the second step improves the accuracy of the estimation compared to the first step for medium to high variaces of shadowing. In fact, for a low σ^2_{dB}, the joint estimation works well and the second step may deteriorate the accuracy of the estimation. Then, for low σ^2_{dB}s the first step is preferred and for high σ^2_{dB}s the two-step estimator is more efficient than the first-step estimator. Similar to the previous section, we have implemented the MLE using *lsqnonlin* with the initial estimate from the second step of the proposed estimator. As the figure shows the performance is considerably improved, especially for when the noise variances are low.

VI. Conclusions

In this paper, we have studied the localization problem based on RSS measurements when the transmit power or path-loss exponent is unknown. The maximum likelihood estimator (MLE) is highly nonconvex and difficult to solve. Using approximations, we have changed the MLE objective function to an approximate MLE. We have, then, formulated the problem as a general trust region subproblem, which can be solved exactly under mild conditions. To find the solution, we first need to run a one-dimensional bisection search to find the optimal Lagrange dual parameter, which in turn is used to compute the location estimate using a closed-form expression. Simulation results show that the proposed methods outperform recently proposed techniques with reasonable complexities. One open problem for future studies is to mathematically obtain the optimal value of the tuning parameter β_0. Generalizing the RSS model in which the path-loss or transmission power is different for every link is also worth to investigate in future studies.

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Appendix A
Cramér-Rao Lower bound

In this section, we compute the CRLB for the location estimate and unknown nuisance parameters (P_0 or β).

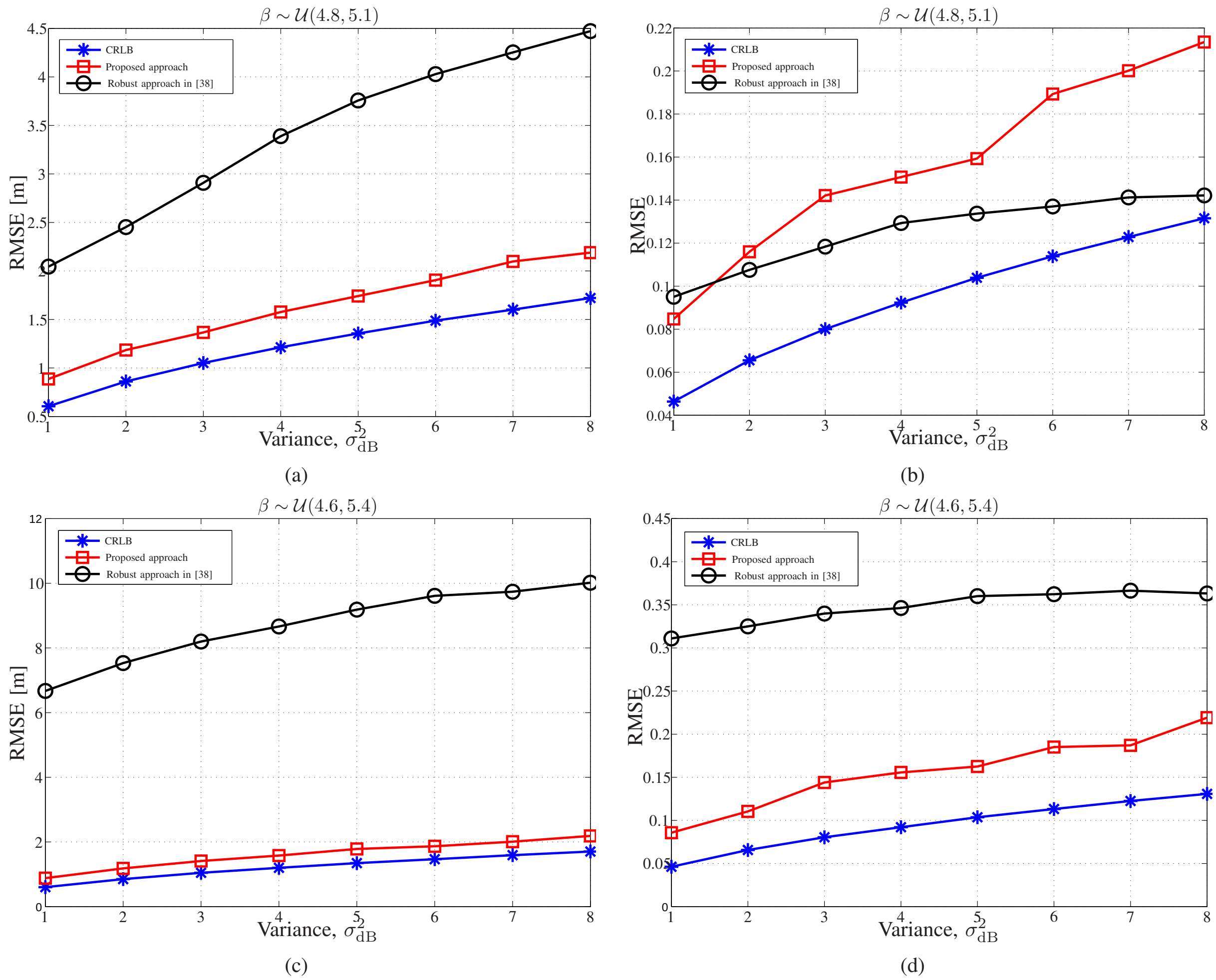

Fig. 6. Comparison between the performance of the proposed technique and the approach proposed in [38] for (a) location estimate for $\beta \sim \mathcal{U}(4.8, 5.1)$, (b) path-loss exponent for $\beta \sim \mathcal{U}(4.8, 5.1)$, (c) location estimate for $\beta \sim \mathcal{U}(4.6, 5.4)$, and (d) path-loss exponent for $\beta \sim \mathcal{U}(4.6, 5.4)$.

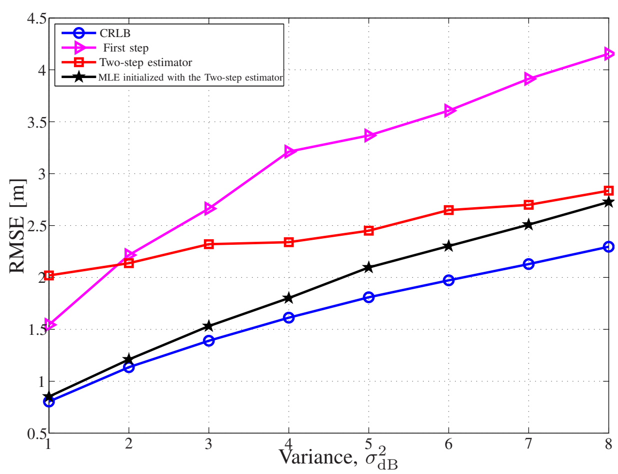

Fig. 7. The RMSE of the proposed approach and the CRLB for location estimation versus the variance of shadowing.

For the Gaussian distribution, the Fisher information matrix can be computed as [24, Ch. 3]

$$J_{nm} = [\mathbf{J}]_{nm} = \left[\frac{\partial \boldsymbol{\mu}}{\partial \theta_n}\right]^T \mathbf{C}^{-1} \left[\frac{\partial \boldsymbol{\mu}}{\partial \theta_m}\right], \quad n, m = 1, \ldots, L, \tag{35}$$

where $\mathbf{C} = \sigma_{dB}^2 \mathbf{I}_N$ with $\mathbf{I}_N$ as the N by N identity matrix, $\boldsymbol{\mu} = [\mu_1 \ \ldots \ \mu_N]^T$ with $\mu_i = P_0 - 10\beta \log d_i$, $\boldsymbol{\theta} = \left[\mathbf{x}^T \ P_0\right]^T$, $\boldsymbol{\theta} = \left[\mathbf{x}^T \ \beta\right]^T$, or $\boldsymbol{\theta} = \left[\mathbf{x}^T \ P_0 \ \beta\right]^T$, and the derivative $\partial \mu_i / \partial \theta_n$ is given as

$$\begin{aligned} \frac{\partial \mu_i}{\partial x_1} &= -10\beta \frac{x_1 - a_{i,1}}{\ln 10 \ d_i^2}, \\ \frac{\partial \mu_i}{\partial x_2} &= -10\beta \frac{x_2 - a_{i,2}}{\ln 10 \ d_i^2}, \\ \frac{\partial \mu_i}{\partial \beta} &= -10 \log_{10} d_i, \\ \frac{\partial \mu_i}{\partial P_0} &= 1, \end{aligned} \tag{36}$$

where $\mathbf{x} = [x_1 \ x_2]^T$, $\mathbf{a}_i = [a_{i,1} \ a_{i,2}]^T$. The CRLB, then, can be computed as

$$\mathrm{Var}(\hat{\theta}_i) \geq \left[\mathbf{J}^{-1}\right]_{i,i}. \quad (37)$$

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