An axiomatization of the leveling tax-transfer policy

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Abstract: We show that the leveling tax-transfer policy is the only consistent policy that exploits the maximum allowed transfers, as long as these are no more than the amount sufficient to equalize after-tax incomes. The leveling policy stands in contrast to equal sacrifice policies, which are also consistent but do not conduct any transfers.

1. Introduction

Most countries employ direct taxation in order to fund government activities and to redistribute income. A natural question is what is a just taxation policy.

The theory of public economics in general, and of optimal taxation in particular (e.g., Mirrlees (1971)), often assumes that government maximizes a social welfare function. The social welfare function takes into account only the after-tax situation. The distribution of initial endowments may thus influence the social optimum only due to restrictions it places on the set of feasible outcomes. The maximization of social welfare in this manner goes back to the Utilitarian doctrine of Bentham (1948, originally from 1789), and was supported by Edgeworth (1897) and Pigou (1928).

Consider an economy characterized by a list of pre-tax incomes, and an amout to be taxed. Assume that the government should collect the given amout, and may redisribute the remaining incomes as it pleases. Maximizing an equality minded symmetric social welfare function implies equal after-tax incomes. If in addition there is a maximum limit on the total amount transferred among agents, the maximization implies the leveling of incomes above and below certain amounts, determined by the feasibility constraints. A version of this result was observed by Edgeworth (1897), and a generalization is due to Fei (1981).

This is by no means the only possible approach to the tax-transfer problem. Mill (1848) promoted an alternative — the Equal Sacrifice Principle. This principle suggests that all individuals should sacrifice the same amount of utility with respect to a common utility function from money. The Equal Sacrifice Principle leads to the other extreme: no transfer payments are ever made.

A well established method for choosing the appropriate taxation policy is the axiomatic method.

It was applied successfully to the taxation problem by Young (1988). He presented a set of axioms that characterize the class of Equal Sacrifice policies. The implication of Young's result is clear: if one accepts the suggested axioms, then one must accept the Equal Sacrifice Principle.

We characterize the leveling tax-transfer policy in a similar framework. Young (1988) required in his model that no agent shall receive money. We relax this assumption, and assume that transfers may be allowed but their total is subject to an exogeneous constraint.

A key axiom both in Young (1988) and in our characterization is consistency. Assume that the policy recommends a certain tax program to a given economy. Further assume that a set of agents wishes to reallocate taxes among its members, given that all the other agents' taxes are fixed, and that government's budget constraints are met. Consistency requires that if the same policy will be applied to this reduced economy, its recommendation would not change.

In addition to consistency, we require that the policy always exploits the maximum allowed transfers, as long as these are no more than the amount sufficient to equalize after-tax incomes.

2. The model and the theorem

An *economy* is a triple ($\mathbf{x}$, T, B), where $\mathbf{x} \in \mathbb{R}^I_+$, I is a nonempty finite set of natural numbers, $T \le \sum_{i\in I} x_i$, $0 \le B \le [(\sum_{i\in I} |x_i - \bar{y}|) - |T|] \,/\, 2$, where $\bar{y} = [(\sum_{i\in I} x_i) - T] \,/\, \#I$, $|\alpha|$ is the absolute value of α, and $\#S$ is the number of members in S.

I is the set of agents in the economy, and $\mathbf{x}$ is the list of taxable incomes. T is the total amount to be taxed (government spending), and B is the maximum feasible amount of net transfers among agents. B is assumed to be not larger than the amount sufficient to equalize after-tax incomes of all agents. When applying the consistency axiom to a set of transfer receivers, the total tax of this economy would be negative. Therefore, T is allowed to be negative.

A *program* in an economy ($\mathbf{x}$, T, B) is a list $\mathbf{t} \in \mathbb{R}^I$, that satisfies $\mathbf{t} \le \mathbf{x}$, $\sum_{i\in I} t_i = T$, and $\sum_{i\in I} |t_i| \le |T| + 2B$.

A program $\mathbf{t}$ of ($\mathbf{x}$, T, B) is *transfer budget exhaustive* (*tb-exhaustive*) if it satisfies $\sum_{i\in I} |t_i| = |T| + 2B$.

The *Leveling Program* of an economy ($\mathbf{x}$, T, B), denoted by L($\mathbf{x}$, T, B), is the unique tb-exhaustive program $\mathbf{t}$ satisfying:

$$t_i = \begin{cases} x_i - \lambda_H & , \quad \lambda_H < x_i \\ 0 & , \quad \lambda_L \le x_i \le \lambda_H \\ x_i - \lambda_L & , \quad x_i < \lambda_L \end{cases}$$

for some well chosen $\lambda_L \le \bar{y} \le \lambda_H$. It is shown in the proofs section, that the Leveling Program is indeed unique and well defined.

All agents in the economy whose taxable incomes are higher than λ_H pay taxes such that their net income would be λ_H. All agents whose income is lower than λ_L receive transfer payments such that their net income would be λ_L.

Now we discuss consistency. First, we define reduced economies. Let $(\mathbf{x}, T, B)$ be an economy with the set of agents I, let S be a non-empty subset of I, and let $\mathbf{t}$ be a program of $(\mathbf{x}, T, B)$. The *reduced economy* of $(\mathbf{x}, T, B)$ with respect to S and $\mathbf{t}$ is:

$$(\mathbf{x}, T, B)^{S, \mathbf{t}} = (\mathbf{x}|S, T^*, B^*)$$

where

$$T^* = \sum_{i \in S} t_i$$

and

$$B^* = \frac{1}{2} \min \left\{ |T| - |T^*| + 2B - \sum_{i \in I \setminus S} |t_i|, \sum_{i \in S} \left| x_i - \overline{y^*} \right| - |T^*| \right\}$$

and where $\overline{y^*}$ is the after-tax per capita income of members of S.

The reduced economy $(\mathbf{x}, T, B)^{S, \mathbf{t}}$ is the economy whose members are the members of S; the set of available programs is the one induced by the ongoing agreement $\mathbf{t}|I \setminus S$ with the rest of the agents and by the government's constraints T and B.

The definition of T^* needs no explanation. In order to understand the definition of B^*, note that if $\tilde{\mathbf{t}}$ is a program of the reduced economy, we would like $\left(\tilde{\mathbf{t}}, \mathbf{t}|I \setminus S\right)$ to be a program of $(\mathbf{x}, T, B)$, in other words $\tilde{\mathbf{t}}$ must satisfy:

$$\sum_{i \in S} \left|\tilde{t}_i\right| + \sum_{i \in I \setminus S} |t_i| \leq |T| + 2B \qquad (2.1)$$

or

$$\sum_{i \in S} \left|\tilde{t}_i\right| \leq |T| + 2B - \sum_{i \in I \setminus S} |t_i| \qquad (2.2)$$

Let B^{**} be the transfers budget we would like to have for the reduced economy. Then $\tilde{\mathbf{t}}$ must satisfy:

$$\sum_{i \in S} \left|\tilde{t}_i\right| \leq |T^*| + 2B^{**} \qquad (2.3)$$

Equating the right hand sides of (2.2) and (2.3) gives:

$$B^{**} = \frac{1}{2}\left(|T| - |T^*| + 2B - \sum_{i \in I \setminus S} |t_i|\right) \qquad (2.4)$$

Note that B^{**} may be larger than the amount sufficient to equalize after-tax incomes in the reduced economy. This can happen when $\mathbf{t}$ is not tb-exhaustive, and the reduced economy has an excess amount of feasible transfers. This is the reason why B^* is defined as the minimum between B^{**} and the amount that allows to equalize after-tax incomes in the reduced economy.

Let Ω be a set of economies. A *policy* on Ω is a function that assigns a unique program to every economy in Ω.

A policy σ on Ω is *consistent* if for all economies $(\mathbf{x}, T, B)$ in Ω, then $\mathbf{t} = \sigma(\mathbf{x}, T, B)$ and $(\mathbf{x}, T, B)^{S, \mathbf{t}} \in \Omega$ imply

$$\mathbf{t}|S = \sigma\left[(\mathbf{x}, T, B)^{S, \mathbf{t}}\right] \qquad (2.5)$$

Consistency requires that if a policy recommends a program to an economy, then it would recommend the reduced program to the reduced economy. A weaker condition is *bilateral consistency* that requires that the condition (2.5) will hold only for subsets S that contain exactly two members.

Let N be a non-empty set of natural numbers. Denote by Ω^N the family of all economies whose members form a finite non-empty subset of N. The set N is the set of potential agents. Note that Ω^N is "closed under reduction," i.e., Ω^N contains all the reduced economies of those economies that are contained in it.

Now we state the main result.

Theorem: Let N be a set of natural numbers that contains at least three members. The Leveling Policy is the unique bilaterally consistent policy on Ω^N that always assigns tb-exhaustive programs.

The theorem employs two axioms: consistency and tb-exhaustiveness. Both axioms are closely related to optimization. This is discussed in the next section.

3. Discussion and related work

Optimization, equality-mindedness, and tb-exhaustiveness

Fei (1981, Theorem 8) showed that when maximizing an anonymous social welfare function that satisfies Dalton's (1920) principle of transfer, we get the Leveling Program as the unique solution.

Dalton's principle of transfer is a property of "equality mindedness," which states that "more equal" income distributions yield a higher level of social welfare. Before going into the formal definition, we would like to note that even without having a definition of "equality mindedness", we can agree that for two person economies an income distribution (x_i, x_j) is more equal than (y_i, y_j) if $x_i + x_j = y_i + y_j$ and $|x_i - x_j| < |y_i - y_j|$. Thus, the Leveling Program of a two person economy "minimizes inequality." This leads to the following definition of "equality mindedness."

Let I be a finite set of natural numbers which represent the members of an economy. A *social welfare function* is a function that assigns a real number to every income distribution. A social welfare function f is *equality-minded* if for any two income distributions **x** and **y** that satisfy:

1. $i, j \in I$ with $x_i + x_j = y_i + y_j$ and $|x_i - x_j| < |y_i - y_j|$,
2. $(k \neq i, k \neq j) \Rightarrow x_k = y_k$,

it is also satisfied that $f(\mathbf{x}) > f(\mathbf{y})$.

Dalton's principle of transfer is weaker, since it also requires that the ordering of agents by income would be the same in both income distributions. However, Dalton's principle and anonymity imply equality-mindedness, as it is defined here.

It follows from equality-mindedness that an optimal program is tb-exhaustive. The argument is that if f is equality-minded, then for any income distribution **y** that is not the equal distribution, there exists in any neighborhood of **y** another distribution **z** with $\sum_{i \in I} z_i = \sum_{i \in I} y_i$, and $f(\mathbf{z}) > f(\mathbf{y})$.

Consistency and optimization

Analogous properties to the consistency axiom were applied to many other models, and were powerful in the characterization of important solution concepts in coalitional game theory, such as the core and prekernel by Peleg (1985,1986) and the Nash bargaining solution by Lensberg (1988). Moreover, consistency was studied in connection with various fair allocation problems in economic situations, for example Thomson (1988,1994), and also in connection with bankruptcy problems by Aumann and Maschler (1985), Young (1987), and Dagan and Volij (1997). For a survey about consistency properties and their applications see Thomson (2004).

Lensberg (1987) studied the class of collectively rational bargaining solutions. A collectively rational solution is derived by maximizing a continuous separable-additive, strictly quasi-concave social welfare function of "utility distributions." Lensberg showed that the class of these solutions is equivalent to the class of continuous, Pareto optimal, and consistent solutions.

This result gives an important support for welfare maximization, however, his results hold only when one considers the large domain of all bargaining problems. This calls for considering other classes of problems, and in particular bargaining problems that are induced by certain economic situations.

A model which is closely related to ours is the one of bankruptcy problems. See Thomson (2003) for a survey on the diverse literature on this topic. A bankruptcy problem constitutes a claims list $\mathbf{c} \geq \mathbf{0}$ and an estate $E \geq 0$, such that $\sum_{i \in I} c_i \leq E$ An allocation in such problem is a list $\mathbf{y}$ that satisfies $\sum_{i \in I} y_i = E$, and $\mathbf{0} \leq \mathbf{y} \leq \mathbf{c}$.

Taxation policies as defined in this paper may be applied to bankruptcy problems as well. To do this, associate to each bankruptcy problem $(\mathbf{c}, E)$ the economy $\left(\mathbf{c}, \sum_{i \in I} c_i - E, 0\right)$. Obviously, the axioms that characterize the Leveling Policy do not characterize it for the restricted class of bankruptcy problems. That is because tb-exhaustiveness is trivially satisfied in all allocations in all bankruptcy problems.

Young (1987, Theorem 2) showed that a bankruptcy rule is symmetric, continuous, and consistent if and only if it is derived by maximizing a symmetric, continuous, separable-additive, strictly quasi-concave function of $\mathbf{c}$ and $\mathbf{y}$.

Note that Young's function depends also on $\mathbf{c}$, and not only on the "after tax" income distribution $\mathbf{y}$. Young's (1987, Theorem 2) may be applied to Equal Sacrifice taxation-policies, thus these policies can be derived by maximizing social welfare functions which are also contingent on the initial income distribution $\mathbf{x}$.

We believe that the above mentioned results demonstrate that consistency is closely related to optimization. Our theorem gives additional support for this relation.

The Leveling Policy in bankruptcy

The Leveling Policy applied to bankruptcy gives the Constrained Equal Awards (CEA) rule, an ancient rule that appears in Jewish legal literature. This rule was derived as the Nash bargaining solution of an associeted bargaining problem by Dagan and Volij (1993), and was further axiomatized by Dagan (1996), Lahiri (2001) and Yeh (2004, 2006). These axiomatizations employ a variety of axioms most of which are relevant to bankruptcy more than to taxation.

Recently, Thomson (2007) compared different rules by the extent of inequality they recommend, by using Lorenz dominance. He identifies several classes of rules, and provides a Lorenz dominance comparison within each class. In particular, the CEA rule belongs to all these classes, and gives the most equal outcome compared to all the other rules. Thomson's (2007) results are also relevant to a taxation scenario, when transfers are not allowed.

4. Proofs

Definition of the Leveling Program

We first show that the Leveling Program is well defined for every economy.

Consider three cases. In the first $\max_{i \in I}\{x_i\} < \bar{y}$. In this case the average after-tax income is higher than the maximum pre-tax income. When this happens we have $T = \sum_{i \in I} (x_i - \bar{y})$ and $B = 0$. In this case set $\lambda_L = \bar{y} = \lambda_H$, and $t_i = x_i - \bar{y}$. In the second case $\min_{i \in I}\{x_i\} > \bar{y}$. Namely, the minimum pre-tax income is higher than the average after-tax income. Similarly, when this happens we also have $T = \sum_{i \in I} (x_i - \bar{y})$ and $B = 0$. In this case too set $\lambda_L = \bar{y} = \lambda_H$, and $t_i = x_i - \bar{y}$.

In the third and last case $\min_{i \in I}\{x_i\} \leq \bar{y} \leq \max_{i \in I}\{x_i\}$. Set λ_H and λ_L to solve the equations:

$$\sum_{i \in I} \max\{0, x_i - \lambda_H\} = \max\{0, T\} + B \qquad (4.1)$$

$$\sum_{i \in I} \min\{0, x_i - \lambda_L\} = \min\{0, T\} - B \qquad (4.2)$$

Equation (4.1) has a unique solution if and only if $\max\{0, T\} + B > 0$. If $\max\{0, T\} + B = 0$ then any solution λ_H is greater or equal to the maximum pre-tax income. Similarly, equation (4.2) has a unique solution if and only if $\min\{0, T\} - B < 0$. If $\min\{0, T\} - B = 0$, then any solution λ_L is smaller or equal to the minimum pre-tax income. The upper bound on B in the definition of an economy, guarantees that $\lambda_L \leq \bar{y} \leq \lambda_H$. In all cases the tax-transfer program is uniquely defined. Q.E.D.

Several Lemmas

Lemma 1: Let Ω be a non-empty set of economies. The Leveling Policy on Ω is consistent.

Proof: Let $\mathbf{t} = L(\mathbf{x}, T, B)$. Consider a reduced economy, $(\mathbf{x}, T, B)^{S, \mathbf{t}} = (\mathbf{x}|S, T^*, B^*)$. By definition $T^* = \sum_{i \in S} t_i$, and as L is tb-exhaustive, $\sum_{i \in S} |t_i| + \sum_{i \in I \setminus S} |t_i| = |T| + 2B$. Hence, in the reduced economy we have $B^* = B^{**}$, and $\sum_{i \in S} |t_i| = |T^*| + 2B^*$. Let $\tilde{\mathbf{t}} = L\left[(\mathbf{x}, T, B)^{S, \mathbf{t}}\right]$. $\tilde{\mathbf{t}}$ is characterized by (λ_L, λ_H) that lead to $\sum_{i \in S} \tilde{t}_i = |T^*|$ and $\sum_{i \in S} |\tilde{t}_i| = |T^*| + 2B^*$. So we have $\sum_{i \in S} \tilde{t}_i = \sum_{i \in S} t_i$ and $\sum_{i \in S} |\tilde{t}_i| = \sum_{i \in S} |t_i|$. Thus, for the reduced economy we can choose identical (λ_L, λ_H) as in the original economy $(\mathbf{x}, T, B)$, and $\tilde{\mathbf{t}} = \mathbf{t}|S$. Q.E.D.

Another property of the Leveling Policy is converse-consistency. It requires that if a program solves for all two-agent reduced economies, this program solves for the whole economy. Formally: A policy σ on Ω is *conversely-consistent* if $\mathbf{t}$ is a program of $(\mathbf{x}, T, B) \in \Omega$, and for all two-agent coalitions S, $(\mathbf{x}, T, B)^{S, \mathbf{t}} \in \Omega$ and $\mathbf{t}|S = \sigma\left[(\mathbf{x}, T, B)^{S, \mathbf{t}}\right]$, then $\mathbf{t} = \sigma(\mathbf{x}, T, B)$.

Lemma 2: Let Ω be a non-empty set of economies. The Leveling Policy on Ω is conversely-consistent.

Proof: Let $\mathbf{t}$ be a program of an economy $(\mathbf{x}, T, B)$, such that $\mathbf{t}|S = L\left[(\mathbf{x}, T, B)^{S, \mathbf{t}}\right]$ for all two person coalitions S. As the Leveling Program is tb-exhaustive for the two person reduced economies, $\mathbf{t}$ is tb-exhaustive for the whole economy. Now, the members of the economy can be divided into three disjoint sets: those who pay a tax ($t_i > 0$), those who receive a transfer ($t_i < 0$) and those with $t_i = 0$. If the only non-empty set of the three is the tax payers or the transfer receipents, as **t** is the Leveling program in all two person economies, it follows that all members have an equal after tax income. As the Leveling Program equates all after tax incomes, if feasible, **t** must be the Leveling Program of $(\mathbf{x}, T, B)$. Now consider the complementary case. If the set of tax payers is non-empty, note that all tax payers receive an equal after-tax income; denote this income by λ_H, otherwise let $\lambda_H = \max_{i \in I}\{x_i\}$. Similarly, if there are some transfer receipents they all obtain the same after-tax income, and denote this income by λ_L. If there are no transfer receipients let $\lambda_L = \min_{i \in I}\{x_i\}$. As **t** solves all the two person economies, it must be that $\lambda_L \leq \lambda_H$ and that all memebers with $t_i = 0$ (if any) have $\lambda_L \leq x_i \leq \lambda_H$. To sum up, **t** is a tb-exhaustive program characterized by (λ_L, λ_H) as in the definition of the Leveling Program, hence it must be te Leveling Program of $(\mathbf{x}, T, B)$. Q.E.D.

Lemma 3: Let N be a non-empty set of natural numbers. If a policy on Ω^N is bilaterally consistent and coincides with the Leveling Policy for all two person economies, then it is the Leveling Policy.

Proof: Let σ be a bilaterally consistent policy on Ω^N that coincides with the Leveling Policy for all two person economies. Let $\mathbf{t} = \sigma(\mathbf{x}, T, B)$. By bilateral consistency, $\mathbf{t}|S = \sigma\left[(\mathbf{x}, T, B)^{S, \mathbf{t}}\right]$ for all S with two members. Since σ coincides with the Leveling Policy for all two person economies, $\mathbf{t}|S = L\left[(\mathbf{x}, T, B)^{S, \mathbf{t}}\right]$ for all S with two members. As Ω^N is closed under reduction, all these two person economies belong to Ω^N. Finally, as the Leveling Policy is conversely consistent, we have $\mathbf{t} = L(\mathbf{x}, T, B)$. Q.E.D.

A program is order preserving if the ordering of agents by their pre-tax incomes and by after-tax incomes is the same. Formally: A program **t** of $(\mathbf{x}, T, B)$ is order preserving if for all $i, j \in I$ we have $x_i \leq x_j \Rightarrow x_i - t_i \leq x_j - t_j$. A less intuitive, but weaker property is the following: A program **t** of $(\mathbf{x}, T, B)$ is *weakly order preserving* if for all $i, j \in I$ we have $\left(x_i \leq x_j,\ t_i < 0 < t_j\right) \Rightarrow$ $x_i - t_i \leq x_j - t_j$.

Lemma 4: Let N be a non-empty set of natural numbers. Let σ be a policy on Ω^N. If σ is bilaterally consistent and assigns tb-exhaustive programs, then σ assigns weakly order preserving programs.

Proof: Clearly, the lemma is true for economies with one or two members, since all programs for these economies are weakly order preserving. Now, assume by contradiction, that there is an economy in which weak order preserving is violated. By applying bilateral consistency to a pair of people for which the violation occurs, we get that the amount transferred among these two is larger than the one feasible in the reduced economy. A contradiction. Q.E.D.

Lemma 5: Let N be a set of natural numbers that contains at least three members. If a policy σ on Ω^N is bilaterally consistent and assigns tb-exhaustive programs, then it coincides with the Leveling Policy for all two person economies.

Proof: Let N be a set of natural numbers that contains at least three members. Let σ be a bilaterally consistent policy on Ω^N that assigns tb-exhaustive programs. By Lemma 4, it also assigns weakly order preserving programs.

Step 1: For all two person economies $[(x_i, x_j), T, B]$, if $x_j > 2x_i$, $T = x_j - x_i$ and $B = 0$, then $\sigma[(x_i, x_j), T, B] = L[(x_i, x_j), T, B]$.

Proof: Consider the three person economy $[(x_i, x_j, x_k), T', B']$, where $x_k = 0$, $T' = x_j - 2x_i$, and $B' = x_i$. Let $\mathbf{t} = \sigma[(x_i, x_j, x_k), T', B']$. Note that this economy has a unique tb-exhaustive weakly order preserving program: By tb-exhaustiveness there are members of the economy who pay (positive) taxes that amount to $T' + B' = x_j - x_i$. As $x_k = 0$, and $x_i < x_j - x_i$, it must be that j is a tax-payer, i.e., $t_j > 0$. Weakly order preserving implies that $t_i \geq 0$, thus by tb-exhaustiveness $t_k = -x_i$. Now, weak order preserving implies that $t_i = 0$, and $t_j = T' + B'$. Thus we have $\mathbf{t} = L[(x_i, x_j, x_k), T', B']$. By applying bilateral consistency to $\{i, j\}$ we get the required result.

Step 2: For all two person economies $[(x_i, x_j), T, B]$, if $x_i = 0$, $-x_j \leq T \leq 0$, and $B = 0$, then $\sigma[(x_i, x_j), T, B] = L[(x_i, x_j), T, B]$.

Consider the three person economy $[(x_i, x_j, x_k), T', B']$, where $x_k > 2x_j$, $T' = x_k - x_j + T$, and $B' = -T$. Let $\mathbf{t} = \sigma[(x_i, x_j, x_k), T', B']$. By a similar argument to the one given in Step 1, k must be a tax-payer and $t_j \geq 0$. Therefore, tb-exhaustiveness implies $t_i = -T$. By applying bilateral consistency to $\{j, k\}$ and by Step 1, $\mathbf{t} = L[(x_i, x_j, x_k), T', B']$. By applying bilateral consistency to $\{i, j\}$ we get the required result.

Step 3: For all two person economies $[(x_i, x_j), T, B]$, if $T \geq 0$, and $B = 0$, then $\sigma[(x_i, x_j), T, B] = L[(x_i, x_j), T, B]$.

Assume, without loss of generality, that $x_i \leq x_j$. Let t^* be the tax imposed on i by the Leveling Program. Consider the three person economy $[(x_i, x_j, x_k), T', B']$, where $x_k = 0$, $T' = T - (x_i - t^*)$, and $B' = x_i - t^*$. Let $\mathbf{t} = \sigma[(x_i, x_j, x_k), T', B']$. Assume that $t_i < 0$. Then by applying bilateral consistency to $\{i, k\}$, we get that $\mathbf{t}|\{i, k\} \neq L\left\{[(x_i, x_j, x_k), T', B']^{\{i,k\},\mathbf{t}}\right\}$ which is in contradiction to Step 2. Thus, $t_i \geq 0$, and by a similar argument $t_j \geq 0$. tb-exhaustiveness implies $t_k = t^* - x_i$, and weak order preserving implies $\mathbf{t} = L[(x_i, x_j, x_k), T', B']$. By applying bilateral consistency to $\{i, j\}$ we get the required result.

Step 4: For all two person economies $[(x_i, x_j), T, B]$, if $T \leq 0$, and $B = 0$, then $\sigma[(x_i, x_j), T, B] = L[(x_i, x_j), T, B]$.

Assume, without loss of generality, that $x_i \leq x_j$. Let $t^* \leq 0$ be the tax imposed on j by the Leveling Program. Consider the three person economy $[(x_i, x_j, x_k), T', B']$, where $x_k > 3(x_j - t^*)$, $T' = x_k - (x_j - t^*) + T$, and $B' = -T$. Let $\mathbf{t} = \sigma[(x_i, x_j, x_k), T', B']$. Assume that $t_i > 0$. Then by applying bilateral consistency to $\{i, k\}$, we get that $\mathbf{t}|\{i, k\} \neq L\left\{[(x_i, x_j, x_k), T', B']^{\{i,k\},\mathbf{t}}\right\}$ which is in contradiction to Step 3. Thus, $t_i \leq 0$, and by a similar argument $t_j \leq 0$. tb-exhaustiveness implies $t_k = T' + B' = x_k - (x_j - t^*)$, and weak order preserving implies $\mathbf{t} = L\left[(x_i, x_j, x_k), T', B'\right]$. By applying bilateral consistency to $\{i, j\}$ we get the required result.

Step 5: For all two person economies $\left[(x_i, x_j), T, B\right]$, if $B > 0$, then $\sigma\left[(x_i, x_j), T, B\right] = L\left[(x_i, x_j), T, B\right]$.

Assume, without loss of generality, that $x_i \leq x_j$. Let $t^* < 0$ be the tax imposed on i by the Leveling Program. Consider the three person economy $\left[(x_i, x_j, x_k), T', B'\right]$, where $x_k = 0$, $T' = T - (x_i - t^*)$, and $B' = B + (x_i - t^*)$. Let $\mathbf{t} = \sigma\left[(x_i, x_j, x_k), T', B'\right]$. The inequality $\max\{T', 0\} + B' > x_i$, and tb-exhaustiveness imply $t_j > 0$. Assume that $t_i \geq 0$. tb-exhaustiveness implies $t_k < -x_i$, which is in contradiction to weak order preserving. Thus, $t_i < 0$, and therefore $t_j = \max\{T', 0\} + B'$. The latter equation and order preserving imply $\mathbf{t} = L\left[(x_i, x_j, x_k), T', B'\right]$. By applying bilateral consistency to $\{i, j\}$ we get the required result.

Steps 3,4, and 5 complete the proof of Lemma 5. Q.E.D.

Proof of the theorem

First we show that the Leveling Policy satisfies the axioms. By Lemma 1 the Leveling Policy is consistent, and hence bilaterally consistent. By definition it assigns tb-exhaustive programs. Now we show the other direction. By Lemma 5, a policy satisfying the axioms coincides with the Leveling Policy on all two person problems, this implies, by Lemma 3, that it is the Leveling Policy. Q.E.D.

Independence of axioms

The axioms used in the theorem are independent. Equal Sacrifice Policies, as defined in Young (1988), may be easily extended to out model. These policies are consistent but do conduct any transfers. As for a policy that assigns tb-exhaustive programs, but is not consistent consider the one that assigns the unique tb-exhaustive program $\mathbf{t}$ satisfying:

$$t_i = \begin{cases} \lambda_H(x_i - \bar{y}) & , \quad x_i > \bar{y} \\ 0 & , \quad x_i = \bar{y} \\ \lambda_L(x_i - \bar{y}) & , \quad x_i < \bar{y} \end{cases}$$

For some well chosen $\lambda_L \geq 0$ and $\lambda_H \geq 0$.

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