ON THE DUALITY THEOREM ON AN ANALYTIC VARIETY

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ABSTRACT. The duality theorem for Coleff-Herrera products on a complex manifold says that if $f = (f_1, \ldots, f_p)$ defines a complete intersection, then the annihilator of the Coleff-Herrera product μ^f equals (locally) the ideal generated by f. This does not hold unrestrictedly on an analytic variety Z. We give necessary, and in many cases sufficient conditions for when the duality theorem holds. These conditions are related to how the zero set of f intersects certain singularity subvarieties of the sheaf $\mathcal{O}_Z$.

1. INTRODUCTION

Let $f = (f_1, \ldots, f_p)$ be a tuple of holomorphic functions on an analytic variety Z, where we throughout the article will assume that Z has pure dimension. The *Coleff-Herrera product* of f, as introduced in [CH], can be defined by

$$(1.1) \quad \mu^f = \bar{\partial}\frac{1}{f_p} \wedge \cdots \wedge \bar{\partial}\frac{1}{f_1}.\varphi := \int_Z \frac{\bar{\partial}|f_p|^{2\lambda} \wedge \cdots \wedge \bar{\partial}|f_1|^{2\lambda}}{f_p \cdots f_1} \wedge \varphi \bigg|_{\lambda=0}.$$

Here, φ is a test form, and the integral on the right-hand side is analytic in λ for $\operatorname{Re}\lambda \gg 0$, and has an analytic continuation to $\lambda = 0$, and $|_{\lambda=0}$ denotes this value. We denote the Coleff-Herrera product of f either by μ^f, or by $\bar{\partial}(1/f_1) \wedge \cdots \wedge \bar{\partial}(1/f_p)$. The definition (1.1) is different from the original one, but in the case we focus on here, that f defines a *complete intersection*, i.e., that $\operatorname{codim} Z_f = p$, various different definitions including this definition and the original definition by Coleff and Herrera coincide, also on a singular variety, see [LS].

If f defines a complete intersection, the *duality theorem*, proven by Dickenstein and Sessa, [DS], and Passare, [P], gives a close relation between the Coleff-Herrera product of f and the ideal $\mathcal{J}(f_1, \ldots, f_p)$ generated by f. This is done by means of the annihilator, $\operatorname{ann}\mu^f$, of μ^f, i.e., the holomorphic functions g such that $g\mu^f = 0$.

Theorem 1.1. *Let $f = (f_1, \ldots, f_p)$ be a holomorphic mapping on a complex manifold defining a complete intersection. Then locally,*

$$\mathcal{J}(f_1, \ldots, f_p) = \operatorname{ann}\mu^f.$$

The Coleff-Herrera product of a holomorphic mapping is a current on Z. Currents on singular varieties can be defined in a similar way as on manifolds, i.e., as linear functionals on test forms, see for example [L]. However, currents on Z also has a characterization in terms of currents in the ambient space: If $i : Z \to \Omega$ is the inclusion, $\operatorname{codim} Z = k$, and μ is a (p, q)-current on Z, then $i_*\mu$ is a $(k + p, k + q)$-current on Ω that vanishes on all forms that vanish on Z. Conversely, if T is a $(k + p, k + q)$-current on Ω, that vanishes on all forms that vanish on Z, then T defines a unique (p, q)-current T' on Z such that $i_*T' = T$. When we consider the Coleff-Herrera product in the ambient space, i.e., $i_*\mu^f$, we will denote it by

$$\bar{\partial}\frac{1}{f_p} \wedge \cdots \wedge \bar{\partial}\frac{1}{f_1} \wedge [Z],$$

and in fact, by analytic continuation, it can be defined by

$$(1.2) \qquad \bar{\partial}\frac{1}{f_p} \wedge \cdots \wedge \bar{\partial}\frac{1}{f_1} \wedge [Z] = \frac{\bar{\partial}|f_p|^{2\lambda} \wedge \cdots \wedge \bar{\partial}|f_1|^{2\lambda}}{f_p \cdots f_1} \wedge [Z]\bigg|_{\lambda=0}.$$

On an analytic variety, one can find rather simple examples of functions annihilating the Coleff-Herrera product of a complete intersection without lying in the ideal. However, we have an inclusion in one of the directions, see [CH], Theorem 1.7.7.

Theorem 1.2. *If $f = (f_1, \ldots, f_p)$ are holomorphic on Z, defining a complete intersection, then $\mathcal{J}(f_1, \ldots, f_p) \subseteq \operatorname{ann} \mu^f$.*

In this article, we discuss this inclusion, and give conditions for when the inclusion is an equality, and when the inclusion is strict.

Throughout this article, we will only discuss the duality theorem for strongly holomorphic functions on Z, i.e., functions f on Z, which are locally the restriction of holomorphic functions in the ambient space, denoted $f \in \mathcal{O}(Z)$. When we say holomorphic functions, we refer to strongly holomorphic functions. However, we will sometimes refer to them as strongly holomorphic functions, to make a distinction to weakly holomorphic, which we use in the introduction to provide examples. Recall that a function $f : Z_{\mathrm{reg}} \to \mathbb{C}$ is *weakly holomorphic* on Z, denoted $f \in \tilde{\mathcal{O}}(Z)$, if f is holomorphic on Z_{reg}, and f is locally bounded at Z_{sing}. Recall also that a germ of a variety, (Z, z), is said to be *normal* if $\mathcal{O}_{Z,z} = \tilde{\mathcal{O}}_{Z,z}$, and that the *normalization* of a variety Z is the unique (up to analytic isomorphism) normal variety Z' together with a finite proper surjective holomorphic map $\pi : Z' \to Z$ such that $\pi|_{Z' \setminus \pi^{-1}(Z_{\mathrm{sing}})} : Z' \setminus \pi^{-1}(Z_{\mathrm{sing}}) \to Z_{\mathrm{reg}}$ is a biholomorphism, see for example [D], Section II.7.

One of the reasons we do not have equality in Theorem 1.2 is because of weakly holomorphic functions, namely if $f = (f_1, \ldots, f_p)$ is strongly holomorphic and defining a complete intersection, and $g = \sum a_i f_i$ is

strongly holomorphic while the functions a_i are only weakly holomorphic, then by Theorem 4.3 in [L] (the analogue of Theorem 1.2 for weakly holomorphic functions), $g\mu^f = 0$, but it might very well happen that the a_i cannot be chosen to be strongly holomorphic. For example, let $Z = \{z^3 = w^2\} \subseteq \mathbb{C}^2$, which has normalization $\pi(t) = (t^2, t^3)$, and let $f \in \tilde{\mathcal{O}}(Z)$ be such that $\pi^* f = t$. Then $f^2 = z$ and $f^3 = w$ on Z, so that $f^2, f^3 \in \mathcal{O}(Z)$ and $f^3 \bar{\partial}(1/f^2) = 0$ (note that since f^2 is strongly holomorphic on Z, we see this as a current on Z, as explained above), while $f^3 \neq gf^2$ for any $g \in \bar{\mathcal{O}}(Z)$, since $f \notin \mathcal{O}(Z)$. That $f^3 \bar{\partial}(1/f^2) = 0$ can be seen either by going back to the normalization, where we get $t^3 \bar{\partial}(1/t^2)$, which is 0 by the (smooth) duality theorem, or by seeing it as a current in the ambient space, and using the Poincaré-Lelong formula as in Example 1 below.

Let us now consider a germ of a normal variety (Z, z), and the Coleff-Herrera product of one holomorphic function. Assume that $g \in \operatorname{ann} \bar{\partial}(1/f)$. Since $\bar{\partial}(1/f)$ is just $\bar{\partial}$ of $1/f$ in the current sense and g is holomorphic, we get that

$$\bar{\partial}\left(g\frac{1}{f}\right) = 0.$$

In the smooth case, by regularity of the $\bar{\partial}$-operator on 0-currents, $g(1/f)$ would be a holomorphic function. This will not hold in general on a singular space (as the example above shows). However, we get that $g/f \in \mathcal{O}(Z_{\mathrm{reg}})$. If (Z, z) is normal, then $\operatorname{codim}(Z_{\mathrm{sing}}, z) \geq 2$ in Z, and any function holomorphic on an analytic variety outside some subvariety of codimension ≥ 2 is locally bounded, see [D], Proposition II.6.1. Thus, g/f is weakly holomorphic, and since (Z, z) is normal, $g/f \in \mathcal{O}_{Z,z}$, i.e., $g \in \mathcal{J}(f)$. Combined with Theorem 1.2, we get that the duality theorem holds for the Coleff-Herrera product of one holomorphic function on (Z, z) if it is normal.

Assume now that (Z, z) is not normal. Then, there exists $\phi \in \tilde{\mathcal{O}}_{Z,z} \setminus \mathcal{O}_{Z,z}$. Since weakly holomorphic functions are meromorphic, we can write $\phi = g/h$ for some strongly holomorphic functions g and h. Then $g\bar{\partial}(1/h) = 0$, by Theorem 4.3 in [L] (the analogue of Theorem 1.2 for weakly holomorphic functions). However, since $g/h = \phi \in \tilde{\mathcal{O}}_{Z,z} \setminus \mathcal{O}_{Z,z}$, $g \notin \mathcal{J}(h)$ (in $\mathcal{O}_{Z,z}$).

Hence, in the case of the Coleff-Herrera product of one single holomorphic function on a germ of an analytic variety (Z, z), we get that the duality theorem holds for all f if and only if (Z, z) is normal. The next example shows that this characterization does not extend to tuples of holomorphic functions.

Example 1. Let $Z = \{z_1^2 + \cdots + z_k^2 = 0\} \subseteq \mathbb{C}^k$, where $k \geq 3$. Then Z is normal since Z is a reduced complete intersection with $Z_{\mathrm{sing}} = \{0\}$, and a reduced complete intersection is normal if and only if $\operatorname{codim} Z_{\mathrm{sing}} \geq 2$ (see the discussion after Definition 1). Let $\mu = \bar{\partial}(1/z_{k-1}) \wedge \cdots \wedge \bar{\partial}(1/z_1)$

(seen as a current on Z). We claim that $z_k\mu = 0$. To see this, we consider this as a current in the ambient space, i.e., $i_*(z_k\mu)$, and use the Poincaré-Lelong formula,

$$i_*(z_k\mu) = z_k\bar{\partial}\frac{1}{z_{k-1}} \wedge \cdots \wedge \bar{\partial}\frac{1}{z_1} \wedge \frac{1}{2\pi i}\bar{\partial}\frac{1}{z_1^2 + \cdots + z_k^2} \wedge d(z_1^2 + \cdots + z_k^2).$$

Then, $z_k dz_i^2 = 2z_i z_k dz_i$ and $z_i z_k \in \mathcal{J}(z_1, \cdots, z_{k-1}, z_1^2 + \cdots + z_k^2)$ for $i = 1, \ldots, k$, so each such term annihilates the current by Theorem 1.2. However, $z_k \notin \mathcal{J}(z_1, \cdots, z_{k-1})$ in $\mathcal{O}(Z)$.

We will show that depending on certain singularity subvarieties of the analytic sheaf $\mathcal{O}_Z$, compared to the zero set of f, we can give sufficient (and in many cases necessary) conditions for when the duality theorem holds on an analytic variety. This condition can be seen as a generalization of normality, coinciding with the usual notion of normality in the case $p = 1$.

Given a coherent ideal sheaf $\mathcal{J}$, there exists locally a finite free resolution

$$(1.3) \qquad 0 \to \mathcal{O}(E_N) \xrightarrow{\varphi_N} \mathcal{O}(E_{N-1}) \to \ldots \xrightarrow{\varphi_1} \mathcal{O}(E_0)$$

where $\mathcal{O}(E_k)$ is the sheaf associated to the vector bundle E_k. We define Z_k as the set of points where φ_k does not have optimal rank. If $Z = Z(\mathcal{J})$ and $p = \mathrm{codim}\, Z$, then $Z_1 = \cdots = Z_p = Z$ and $Z_{k+1} \subseteq Z_k$, see [E], Corollary 20.12. If $\mathcal{J} = \mathcal{J}_Z$, the ideal of holomorphic functions vanishing on Z, then we define

$$(1.4) \qquad Z^0 := Z_{\mathrm{sing}} \quad \text{and} \quad Z^k := Z_{p+k} \quad \text{for } k \geq 1,$$

where $p = \mathrm{codim}\, Z$. These sets are in fact independent of the choice of resolution by the uniqueness of minimal free resolutions in a local Noetherian ring, and from Lemma 3.1 and the remark following it in [AW3], Z^k are independent of the local embedding of Z into $\mathbb{C}^n$. Hence they are intrinsic subvarieties of Z. We will use the convention that $\mathrm{codim}\, Z^k$ refers to the codimension in Z, while by $\mathrm{codim}\, Z_k$, we refer to the codimension in the ambient space.

Theorem 1.3. *Let $f = (f_1, \ldots, f_p)$ be a holomorphic mapping on a germ of an analytic variety (Z, z) defining a complete intersection. If $\mathrm{codim}\,(Z^k \cap Z_f) \geq k + p + 1$ for $k \geq 0$, then $\mathrm{ann}\, \mu^f = \mathcal{J}(f_1, \ldots, f_p)$.*

The proof of Theorem 1.3 is in Section 4.

One might conjecture that this equality of the annihilator and the ideal holds if and only if the conditions in the theorem are satisfied. We have not been able to prove this in this generality, but have focused on a slightly weaker formulation of it. To do this, we introduce the notion of p-duality for an analytic variety.

Definition 1. *If (Z, z) is a germ of an analytic variety, we say that (Z, z) has p-duality if for all $f = (f_1, \ldots, f_p) \in \mathcal{O}_{Z,z}^{\oplus p}$ defining a complete intersection, we have $\mathrm{ann}\, \mu^f = \mathcal{J}(f_1, \ldots, f_p)$.*

Theorem 1.3 implies the following statement:

$(*)$ (Z, z) has p-duality if codim $Z^k \geq p + k + 1$, for $k \geq 0$.

We believe that the converse of $(*)$ holds, and we will discuss this throughout the rest of this introduction. We show that indeed, in many cases, the converse of $(*)$ holds, and if the condition in $(*)$ is not a precise condition for p-duality, it is at least very close to being so.

We saw above that 1-duality is equivalent to that Z is normal. The condition codim $Z^k \geq k + 2$ in $(*)$ is exactly the condition that Z is normal. This is proved in [M], but can also be seen using the conditions R1 and S2 in Serre's criterion for normality. Indeed, one can verify that the conditions R1 and S2 are equivalent to the condition codim $Z^k \geq k + 2$. Thus, the converse of $(*)$ holds when $p = 1$.

Recall that a germ (Z, z) is said to be *Cohen-Macaulay* if the ring $\mathcal{O}/\mathcal{J}_{Z,z}$ is Cohen-Macaulay. More concretely, this means that $\mathcal{O}/\mathcal{J}_{Z,z}$ has a free resolution of length $p = \mathrm{codim}\,(Z, z)$. Equivalently, $Z^k = \emptyset$ for $k \geq 1$. Hence, if (Z, z) is Cohen-Macaulay, the condition codim $Z^k \geq p + k$ for $k \geq 0$ becomes just codim $Z_{\mathrm{sing}} \geq p$. In case (Z, z) is Cohen-Macaulay, the converse of $(*)$ holds.

Proposition 1.4. *Assume that (Z, z) is Cohen-Macaulay and that* codim $Z_{\mathrm{sing}} = k$. *If $q \geq k$, then there exists $f = (f_1, \ldots, f_q) \in \mathcal{O}_{Z,w}^{\oplus q}$, for some w arbitrarily close to z, defining a complete intersection, and $g \in \mathcal{O}_{Z,w}$ such that $g \in \mathrm{ann}\,\mu^f$, but $g \notin \mathcal{J}(f_1, \ldots, f_q)$.*

Remark 1. In general, we need to move to a nearby germ in order to find the counterexample, however, if Z_{sing} is a complete intersection in Z, we can take $w = z$.

In particular, if (Z, z) is a reduced complete intersection, then (Z, z) is Cohen-Macaulay since the Koszul complex is a free resolution of length codim (Z, z), see [GH, p. 688].

In Example 1, $(Z, 0)$ is Cohen-Macaulay (since it is a reduced complete intersection) and $Z_{\mathrm{sing}} = \{0\}$, which has codimension $k - 1$ in $(Z, 0)$. Proposition 1.4 then says that there exists a complete intersection $f = (f_1, \ldots, f_{k-1})$ and $g \notin \mathcal{J}(f_1, \ldots, f_{k-1})$ such that $g \in \mathrm{ann}\,\mu^f$. Then $f = (z_1, \ldots, z_{k-1})$ and $g = z_k$ is exactly such an example, while for any complete intersection of codimension $< k - 1$, the duality theorem holds by Theorem 1.3.

If (Z, z) is not Cohen-Macaulay, we get the converse of $(*)$ only for the least p such that the condition in $(*)$ is not satisfied.

Proposition 1.5. *Assume that (Z, z) satisfies* codim $Z^k \geq k + p$ *for all $k \geq 0$, with equality for some $k \geq 1$. Then there exists $f = (f_1, \ldots, f_p) \in \mathcal{O}_{Z,z}^{\oplus p}$ defining a complete intersection, and $g \in \mathcal{O}_{Z,z}$, such that $g \in \mathrm{ann}\,\mu^f$, but $g \notin \mathcal{J}(f_1, \ldots, f_p)$.*

If $p = 1$, then the weakly holomorphic functions give rise to counterexamples as described above.

The proofs of Proposition 1.4 and Proposition 1.5 are in Section 7 and Section 8 respectively. To prove Proposition 1.4, we use Theorem 6.3, which says that there exists a tuple ξ of holomorphic $(p, 0)$-forms such that

$$(1.5) \qquad\qquad [Z] = \sum \xi_i \wedge R_i^Z,$$

where $[Z]$ is the integration current on Z, and $R^Z = (R_1^Z, \ldots, R_N^Z)$ is a tuple of currents such that $\mathcal{J}_Z = \cap_{i=1}^N \operatorname{ann} R_i^Z$, and the current R^Z is defined by means of a free resolution of $\mathcal{O}/\mathcal{J}_Z$, see Section 3. The existence of such ξ_i is proved in [A3], but the tuple ξ is not explicitly given. What we prove in Theorem 6.3 is that if R^Z is the current associated with a *minimal* free resolution, then all ξ_i vanish at Z_{sing}. This result can be seen as a generalization of the Poincaré-Lelong formula from the reduced complete intersection case to the Cohen-Macaulay case. In the reduced complete intersection case, the representation (1.5) is given by the Poincaré-Lelong formula, and since in that case, ξ is explicitly given, the fact that ξ vanish at Z_{sing} follows from the implicit function theorem, see Section 2.

Summarizing Theorem 1.3 and Propositions 1.4 and 1.5, we get the following.

Corollary 1.6. *Assume that* $\operatorname{codim} Z^k \geq k + p$ *for all* $k \geq 0$, *with equality for some* k. *Then* (Z, w) *has q-duality for $q < p$ and all w in some neighborhood of z, and (Z, w) does not have q-duality for $q = p$ for some w arbitrarily close to z. In addition, if* $\operatorname{codim} Z_{\mathrm{sing}} = p$, *that is, we have equality for $k = 0$, then (Z, w) does not have q-duality for $q > p$ for some w arbitrarily close to z.*

Proof. The only part that does not follow immediately from Theorem 1.3, Proposition 1.4 and Proposition 1.5 is if $q > p$, (Z, z) is not Cohen-Macaulay but there is equality in $\operatorname{codim} Z^k \geq k + p$ for $k = 0$. However, in that case, $\operatorname{codim} Z^0 = p$ and $\operatorname{codim} Z^1 \geq p + 1$, so since $Z^0 \supseteq Z^1$, there is some $w \in Z^0$ arbitrarily close to z such that (Z, w) is Cohen-Macaulay (i.e., $w \in Z^0 \setminus Z^1$), and we can apply Proposition 1.4. $\qquad\square$

2. The case of a reduced complete intersection

We begin by showing how to prove Corollary 1.6 in the case when Z is a reduced complete intersection, i.e., that $Z = \{h_1 = \cdots = h_r = 0\}$, where $r = \operatorname{codim} Z$, and $dh_1 \wedge \cdots \wedge dh_r \neq 0$ generically on Z.

Proposition 2.1. *Let (Z, z) be a reduced complete intersection and assume that* $\operatorname{codim} Z_{\mathrm{sing}} = p$. *Then, for all w in some neighborhood of z, (Z, w) has q-duality for $q < p$, and there exists w arbitrarily close to z such that (Z, w) does not have q-duality for $q \geq p$.*

In this case, the main ideas of the proof in the general case appear, but it only involves the Coleff-Herrera product, and hence we avoid many of the technicalities of the proof in the general case.

By the Poincaré-Lelong formula, see Section 3.6 in [CH],

$$(2.1) \qquad \frac{1}{(2\pi i)^r} \bar{\partial} \frac{1}{h_r} \wedge \cdots \wedge \bar{\partial} \frac{1}{h_1} \wedge dh_1 \wedge \cdots \wedge dh_r = [Z].$$

Now, let $f = (f_1, \ldots, f_q)$ be a complete intersection on Z, and consider μ^f as a current in the ambient space, as given by (1.2). By considering the regularization of μ^f in (1.2), using the Poincaré-Lelong formula (2.1) on $[Z]$, and also regularizing μ^h in (2.1), we get

$$(2.2)$$
$$i_* \mu^f = \left. \frac{\bar{\partial} |f_q|^{2\lambda_2} \wedge \cdots \wedge \bar{\partial} |f_1|^{2\lambda_2} \wedge \bar{\partial} |h_r|^{2\lambda_1} \wedge \cdots \wedge \bar{\partial} |h_1|^{2\lambda_1}}{f_q \ldots f_1 h_r \ldots h_1} \wedge \eta \right|_{\lambda_1 = 0, \lambda_2 = 0},$$

where $\eta = (2\pi i)^{-r} dh_1 \wedge \cdots \wedge dh_r$. Note that f being a complete intersection on Z means that (f, h) is a complete intersection on $\mathbb{C}^n$. In this case, by results of Samuelsson, [S], the right-hand side of (2.2) is continuous in (λ_1, λ_2) near $(0, 0)$. In particular, we can instead take the analytic continuation where $\lambda_1 = \lambda_2 = \lambda$ to $\lambda = 0$, which equals the Coleff-Herrera product of (f, h), i.e.,

$$(2.3) \qquad i_* \mu^f = \bar{\partial} \frac{1}{f_q} \wedge \cdots \wedge \bar{\partial} \frac{1}{f_1} \wedge \bar{\partial} \frac{1}{h_r} \wedge \cdots \wedge \bar{\partial} \frac{1}{h_1} \wedge \eta.$$

The representation (2.3) of the Coleff-Herrera product will be the basis of proving Proposition 2.1. First, we consider the case when $q < p$. Since Z is a reduced complete intersection, $\eta = (2\pi i)^{-r} dh_1 \wedge \cdots \wedge dh_r$ is non-vanishing on Z_{reg}. Thus, if $g \in \operatorname{ann} \mu^f$, i.e., by considering g in the ambient space, $g i_* \mu^f = 0$, we get from (2.3) that g annihilates the Coleff-Herrera product $\mu^{(f,h)}$ on Z_{reg}. The Coleff-Herrera product belongs to a class of currents called pseudomeromorphic currents, see Section 3. This class of currents is closed under multiplication with smooth functions, and have the property that if T is a pseudomeromorphic $(*, k)$-current with support on a variety of codimension $> k$, then $T = 0$, see Proposition 3.3. Thus, the current $g \mu^{(f,h)}$ is in fact 0, since it is a $(0, q+r)$-current with support on Z_{sing} which has codimension $p + r$ (in $\mathbb{C}^n$). By the duality theorem (on $\mathbb{C}^n$), $g \in \mathcal{J}(f, h)$, i.e., $g \in \mathcal{J}(f)$ in $\mathcal{O}_Z = \mathcal{O}/\mathcal{J}(h)$. Hence, Z has q-duality if $q < p$.

We now consider the case when $q \geq p$. We can find w arbitrarily close to z, and a complete intersection $f = (f_1, \ldots, f_q)$ on (Z, w) such that $Z(f) \subseteq Z_{\text{sing}}$, see Section 5, and in particular Lemma 5.3. Let $\mathcal{I} = \mathcal{J}(f_1, \ldots, f_q)$, and $V = Z(I)$. It follows from the Nullstellensatz that there exists a holomorphic function g such that $g \notin \mathcal{I}$, but $g \in \mathcal{J}_V$ and $g \mathcal{J}_V \subseteq \mathcal{I}$, see the proof of Proposition 1.4 in Section 7.

We claim the g annihilates μ^f, and since $g \notin \mathcal{J}(f)$, this proves the second part of Proposition 2.1. To prove this claim, note first that by

the implicit function theorem, $\eta = (2\pi i)^{-r} dh_1 \wedge \cdots \wedge dh_r$ vanishes on Z_{sing}, i.e., if $\eta = \sum_{|I|=r} h_I dz_I$, then each $h_I \in \mathcal{J}_{Z_{\text{sing}}}$. Since $V \subseteq Z_{\text{sing}}$, we get that $\mathcal{J}_{Z_{\text{sing}}} \subseteq \mathcal{J}_V$. Hence, $gh_I \in g\mathcal{J}_{Z_{\text{sing}}} \subseteq g\mathcal{J}_V \subseteq \mathcal{I}$, where the last inclusion follows by the choice of g. Thus, we get from multiplying (2.3) by g that g annihilates μ^f, since each term gh_I from $g\eta$ annihilates the Coleff-Herrera product $\mu^{(f,h)}$.

3. Residue currents and free resolutions

When Z is a reduced complete intersection defined by h, the Coleff-Herrera product μ^h is a natural current associated to Z, and in Section 2, the factorization of the integration current $[Z]$ in terms of μ^h was the starting point of the argument. We want to find a corresponding current R^Z and a factorization of the integration current $[Z]$ also when Z is not a complete intersection, see Theorem 6.3 below. To do this, we use a construction by Andersson and Wulcan of currents associated to free resolutions of ideals, [AW1].

Let $\mathcal{J}$ be a coherent ideal sheaf, and let (E, φ) be a locally free resolution of the sheaf $\mathcal{O}/\mathcal{J}$ as in (1.3). Mostly, we will use the case when $\mathcal{J} = \mathcal{J}_Z$, the sheaf of holomorphic functions vanishing on the analytic variety Z.

In particular, if Z is a reduced complete intersection, and $\mathcal{J}_Z = \mathcal{J}(h_1, \ldots, h_p)$, then the Koszul complex of h is a free resolution of $\mathcal{O}/\mathcal{J}_Z$. In this case, the current associated to the Koszul complex of h equals the Coleff-Herrera product μ^h, Theorem 3.2.

To construct the current associated to E, one first defines, outside of $Z = Z(\mathcal{J})$, right inverses $\sigma_k : E_{k-1} \to E_k$ to φ_k which are minimal with respect to some metric on E, i.e., $\varphi_k \sigma_k|_{\text{Im}\,\varphi_k} = \text{Id}_{\text{Im}\,\varphi_k}$, $\sigma_k = 0$ on $(\text{Im}\,\varphi_k)^\perp$, and $\text{Im}\,\sigma_k \perp \ker \varphi_k$. One lets

$$u = \sigma_1 + \sigma_2 \bar{\partial}\sigma_1 + \cdots + \sigma_N \bar{\partial}\sigma_{N-1} \ldots \bar{\partial}\sigma_1.$$

Then, if $F \not\equiv 0$ is a holomorphic function vanishing at Z, R^E is defined by

$$(3.1) \qquad\qquad R^E = \bar{\partial}|F|^{2\lambda} \wedge u|_{\lambda=0},$$

where for $\text{Re}\,\lambda \gg 0$, this is a (current-valued) analytic function in λ, and $|_{\lambda=0}$ denotes the analytic continuation to $\lambda = 0$. See [AW1] for more details.

Let R_k^E denote the part of R^E with values in E_k, i.e., R_k^E is a E_k-valued $(0, k)$-current. If $Z = Z(\mathcal{J})$, and $\text{codim}\,Z = p$, then we will in fact have that

$$(3.2) \qquad\qquad R^E = R_p^E + \cdots + R_N^E,$$

where N is the length of the free resolution (E, φ).

The current R^E has the following crucial property, [AW1], Theorem 1.1.

Theorem 3.1. *Let R^E be the current associated to a free resolution (E, φ) of an ideal $\mathcal{J}$. Then $\operatorname{ann} R^E = \mathcal{J}$.*

If Z is an analytic subvariety, we will denote by R^Z the current associated with a free resolution of $\mathcal{J}_Z$ of minimal length. Note that this current is not in general uniquely defined, as it might depend on the choice of metrics.

In this article, we are only concerned with local (or semi-local) statements, so the reader may very well assume the vector bundles are in fact free modules. However, we still keep the notation of vector bundles, partly to keep a consistent notation, but also since it is advantageous to be able to refer to the specific vector bundle E_k and not just the free module $\mathcal{O}^{\oplus r_k}$.

If $f = (f_1, \ldots, f_p)$ defines a complete intersection, the Coleff-Herrera product coincides with the so called *Bochner-Martinelli current* of f, as introduced by Passare, Tsikh and Yger in [PTY] in the smooth case. It was also developed in the case of an analytic variety in [BVY]. If f defines a complete intersection, the Bochner-Martinelli current of f, denoted R^f, can be defined as the current associated with the Koszul complex of f. In fact, in [AW1], currents associated with any generically exact complex of vector bundles are defined, and not only free resolutions as described above, and then the Bochner-Martinelli current for an arbitrary f can be defined as the current associated with the Koszul complex of f, see [A1]. This equality of the Coleff-Herrera product and the Bochner-Martinelli current makes the Coleff-Herrera product fit in the framework of residue currents associated with a free resolution, and this substitution will be used throughout the arguments. The theorem below is Theorem 4.1 in [PTY] in the smooth case, and Theorem 6.3 in [L] in the singular case.

Theorem 3.2. *If $f = (f_1, \ldots, f_p)$ defines a complete intersection on Z, then the Bochner-Martinelli current R^f of f equals the Coleff-Herrera product μ^f of f.*

Pseudomeromorphic currents were introduced in [AW2]. A current of the form

$$\frac{1}{z_{i_1}^{k_1}} \cdots \frac{1}{z_{i_m}^{k_m}} \bar{\partial} \frac{1}{z_{i_{m+1}}^{k_{m+1}}} \wedge \cdots \wedge \bar{\partial} \frac{1}{z_{i_p}^{k_p}} \wedge \alpha,$$

where α is a smooth form with compact support, is called an elementary current. A current T is said to be a *pseudomeromorphic current*, denoted $T \in \mathcal{PM}$, if it is a locally finite sum of push-forwards of elementary currents under compositions of smooth modifications and open inclusions. As can be seen from their construction, the Coleff-Herrera product μ^f and the current R^E associated with a free resolution are pseudomeromorphic. We will need the following two properties of pseudomeromorphic currents, see Proposition 2.3 and Corollary 2.4 in [AW2].

Proposition 3.3. *If $T \in \mathcal{PM}$ is of bidegree $(0, p)$ and T has support on a variety of codimension $\geq p + 1$, then $T = 0$.*

Proposition 3.4. *If $T \in \mathcal{PM}$ has support on Z, and if f is a holomorphic function vanishing on Z, then $\bar{f}T = 0$.*

We will use results from [A2], that one can define products of the currents R^f and R^Z, and that under certain conditions, the annihilator of the product $R^f \wedge R^Z$ equals the sum of the ideals $\mathcal{J}(f) + \mathcal{J}_Z$. This type of product can be defined more generally for currents R^E and R^F associated with two free resolutions E and F. If R^E is defined by

$$R^E := \bar{\partial}|G|^{2\lambda} \wedge u|_{\lambda=0},$$

then $R^E \wedge R^F$ can be defined by

$$R^E \wedge R^F := \bar{\partial}|G|^{2\lambda} \wedge u \wedge R^F|_{\lambda=0}.$$

Remark 2. If we consider $R^f \wedge R^Z$, where $f = (f_1, \ldots, f_p)$ is a strongly holomorphic mapping on Z, then this depends a priori on the choice of representatives of f in the ambient space. We will only need that under certain conditions, ann $R^f \wedge R^Z = \mathcal{J}(f) + \mathcal{J}_Z$, which is independent of the choice of representatives. However, one can in fact show that $R^f \wedge R^Z$ does not depend on the choice of representatives, essentially due to that R^Z is annihilated by both holomorphic and anti-holomorphic functions vanishing on Z.

If

$$0 \to E_n \xrightarrow{\varphi_n} E_{n-1} \to \ldots \xrightarrow{\varphi_1} E_0 \to 0$$

and

$$0 \to F_m \xrightarrow{\psi_m} F_{m-1} \to \ldots \xrightarrow{\psi_1} F_0 \to 0$$

are two complexes, then one can form the tensor product of the complexes, denoted $(E \otimes F, \varphi \otimes \psi)$, by letting $(E \otimes F)_k = \oplus_{i+j=k} E_i \otimes F_j$ and $(\varphi \otimes \psi)(\xi \otimes \eta) = \varphi_i \xi \otimes \eta + (-1)^i \xi \otimes \psi_j \eta$ if $\xi \otimes \eta \in E_i \otimes F_j$.

The following theorem, Theorem 4.1 and Remark 8 in [A2], and its corollary gives conditions for when the annihilator of $R^E \wedge R^F$ coincides with the sum of the annihilators, and when the tensor product of two (minimal) free resolutions is a (minimal) free resolution.

Theorem 3.5. *Let (E, φ) and (F, ψ) be free resolutions of ideal sheaves $\mathcal{I}$ and $\mathcal{J}$, and let $Z_k^{\mathcal{I}}$ and $Z_l^{\mathcal{J}}$ be the associated sets where φ_k and ψ_l does not have optimal rank. If $\mathrm{codim}\,(Z_k^{\mathcal{I}} \cap Z_l^{\mathcal{J}}) \geq k + l$ for all $k, l \geq 1$, then $\mathrm{ann}\,R^E \wedge R^F = \mathcal{I} + \mathcal{J}$ and $(E \otimes F, \varphi \otimes \psi)$ is a free resolution of $\mathcal{I} + \mathcal{J}$. In addition, if both E and F are minimal free resolutions at some point z, then the tensor product is a minimal free resolution.*

To be precise, the last statement is not included in [A2]. However, if the tensor product is a free resolution, it follows immediately from the definition of minimality at some z, that $\mathrm{Im}\,\varphi_k \subseteq \mathfrak{m}_z \mathcal{O}(E_{k-1})$ (where $\mathfrak{m}_z$ denotes the maximal ideal of $\mathcal{O}_{\mathbb{C}^n, z}$), that it is minimal.

Corollary 3.6. *If $f = (f_1, \ldots, f_p)$ is a reduced complete intersection on Z, and $\operatorname{codim} Z_f \cap Z^l \geq p + l$ for $l \geq 1$, then $\operatorname{ann} R^f \wedge R^Z = \mathcal{J}(f) + \mathcal{J}_Z$, and the tensor product of the Koszul complex of f and a free resolution of $\mathcal{J}_Z$ is a free resolution of $\mathcal{J}(f) + \mathcal{J}_Z$. In addition, if the free resolution of $\mathcal{J}_Z$ is minimal at some point z, then the tensor product is a minimal free resolution.*

Proof. If f is a complete intersection, then the Koszul complex of f is a minimal free resolution, and its associated singularity subvarieties Z_k^f are equal to Z_f for $k \leq p$, and empty for $k > p$. Since $Z_l = Z$ for $l \leq \operatorname{codim} Z$, the condition $\operatorname{codim} Z_f \cap Z_l \geq p + l$ is automatic for $l \leq \operatorname{codim} Z$ since f is a complete intersection on Z. Thus, the condition $\operatorname{codim} Z_k^f \cap Z_l \geq k + l$ becomes just $\operatorname{codim} Z_f \cap Z^l \geq p + l$. $\quad\square$

4. Proof of Theorem 1.3

The inclusion $\mathcal{J}(f_1, \ldots, f_p) \subseteq \operatorname{ann} \mu^f$ follows from Theorem 1.2 (also without the conditions on $Z^k \cap Z_f$), so we only need to prove the reverse inclusion. Assume that $Z \subseteq \Omega \subseteq \mathbb{C}^n$ and that $\operatorname{codim} Z = q$. Then $i_* \mu^f = \mu^f \wedge [Z]$, where $i : Z \to \Omega$ is the inclusion, and by Theorem 3.2, $\mu^f \wedge [Z] = R^f \wedge [Z]$. We will show that $g \in \operatorname{ann}(R^f \wedge [Z])$ implies that $g \in \operatorname{ann}(R^f \wedge R^Z)$ (which does not hold in general, but does under the conditions of the theorem). By (3.21) in [AS], outside of Z_{sing} there exists a smooth $(q, 0)$-vector field γ such that $\gamma \neg [Z] = R_q^Z$. Then, outside of Z_{sing},

$$gR^f \wedge R_q^Z = gR^f \wedge (\gamma \neg [Z]) = \gamma \neg (gR^f \wedge [Z]) = 0.$$

Hence $gR^f \wedge R_q^Z$ is a $(0, p + q)$-current with support on $Z_f \cap Z_{\mathrm{sing}}$, so by Proposition 3.3, it is 0 since $Z_f \cap Z_{\mathrm{sing}}$ has codimension $\geq p + q + 1$.

Outside of Z^{k+1}, there exists a smooth $\operatorname{Hom}(E_{q+k}, E_{q+k+1})$-valued smooth $(0, 1)$-form α_{q+k+1} such that $R_{q+k+1}^Z = \alpha_{q+k+1} R_{q+k}^Z$, see [AW1]. We will prove by induction that

$$(4.1) \qquad\qquad gR^f \wedge R_{q+k}^Z = 0.$$

Above we proved this for $k = 0$, so let us assume that it is proved for k. Then, outside of $Z^f \cap Z^{k+1}$,

$$gR^f \wedge R_{q+k+1}^Z = \alpha_{q+k+1}(gR^f \wedge R_{q+k}^Z) = 0.$$

Thus $gR^f \wedge R_{q+k+1}^Z$ has support on $Z^f \cap Z^{k+1}$ which has codimension $\geq p + q + k + 2$, and since it is a pseudomeromorphic current of bidegree $(0, p + q + k + 1)$, it is 0 by Proposition 3.3. Thus we have proven that $g \in \operatorname{ann}(R^f \wedge R^Z)$. By Corollary 3.6, $\operatorname{ann}(R^f \wedge R^Z) = \mathcal{J}(f) + \mathcal{J}_Z$, and hence we get that $g \in \mathcal{J}(f) + \mathcal{J}_Z$.

5. Complete intersections and choice of coordinates

This section contains several lemmas about choices of coordinates and existence of complete intersections containing a certain variety. They will be used throughout the rest of the sections. This first lemma, which is based on the first lemma in Section 5.2.2 in [GR], is the basis for the rest of them.

Lemma 5.1. *Assume that $(V, z) \subseteq (Z, z)$, where (Z, z) has pure dimension, V has codimension ≥ 1 in Z and that there exists $f = (f_1, \ldots, f_m)$ such that $(V, z) = (Z, z) \cap \{f_1 = \cdots = f_m = 0\}$. Then there exists a finite union, E, of proper linear subspaces of $\mathbb{C}^m$, such that $(Z, z) \cap \{a \cdot f = 0\}$ has codimension 1 in (Z, z) if $a \in \mathbb{C}^m \setminus E$.*

Proof. The set E of $a \in \mathbb{C}^m$ such that $(Z, z) \cap \{a \cdot f = 0\} = (Z, z)$ is a linear subspace of $\mathbb{C}^m$, and since $(Z, z) \cap \{f_1 = \cdots = f_m = 0\}$ has positive codimension, it must be a proper subspace. If (Z, z) is irreducible, there thus exists a proper subspace $E \subseteq \mathbb{C}^m$ such that $(Z, z) \cap \{a \cdot f = 0\}$ has codimension 1 in (Z, z) if $a \in \mathbb{C}^m \setminus E$. If (Z, z) is reducible, then there exists such subspaces E_i for each irreducible component (Z_i, z) of (Z, z), and thus we can take $E = \cup E_i$. $\square$

The following two lemmas are about existence of certain complete intersections containing a given variety, and their existence are the basis for the counterexamples to the duality theorem.

Lemma 5.2. *Assume that $(V, z) \subseteq (Z, z)$, where (Z, z) has pure dimension, $\operatorname{codim} V = p$ in Z, and let $f = (f_1, \ldots, f_m)$ be such that $(V, z) = (Z, z) \cap \{f_1 = \cdots = f_m = 0\}$. Then there exists $f' = (f'_1, \ldots, f'_p)$, a complete intersection on Z, such that $(V, z) \subseteq (V', z) := (Z, z) \cap \{f'_1 = \cdots = f'_p = 0\}$, where $f'_i = \sum a_{i,j} f_j$.*

Proof. By Lemma 5.1, there exists $E \subseteq \mathbb{C}^m$ such that $(Z, z) \cap \{a \cdot f = 0\}$ has codimension 1 in (Z, z) for $a \in \mathbb{C}^m \setminus E$. We choose $f'_1 = a \cdot f$, for some $a \in \mathbb{C}^m \setminus E$. Proceeding in the same way with $(Z, z) \cap \{f'_1 = 0\}$ instead of (Z, z), we get f'_2 such that $(Z, z) \cap \{f'_1 = f'_2 = 0\}$ has codimension 2 in Z. Repeating this, $f' = (f'_1, \ldots, f'_p)$ will be the desired complete intersection. $\square$

Lemma 5.3. *Assume that $(V, z) \subseteq (Z, z)$, where (V, z) has codimension p in (Z, z) and $\dim(Z, z) = d$. Then, for some w arbitrarily close to z, there exists a complete intersection $f = (f_1, \ldots, f_d) \in \mathcal{O}_{Z,w}^{\oplus d}$ such that $(V, w) = (Z, w) \cap \{f_1 = \cdots = f_p = 0\}$.*

Proof. By Lemma 5.2, there exists $f = (f_1, \ldots, f_p)$ a complete intersection on (Z, z) such that $(V, z) \subseteq (V', z)$, where $V' = \{f_1 = \cdots = f_p = 0\}$. Since the set where V' is reducible has codimension $> p$, there exists some w arbitrarily close to z such that $(V, w) = (V', w)$. Then we apply Lemma 5.2 again to $(\{w\}, w) \subseteq (V, w)$ to find $(f_{p+1}, \ldots, f_d)$,

a complete intersection on (V, w), so that $f = (f_1, \ldots, f_d)$ is the desired complete intersection. $\qquad\square$

This last lemma is about the existence of a certain choice of coordinates, which is used in the proof of Theorem 6.3.

Lemma 5.4. *Let $(Z, 0) \subseteq (\mathbb{C}^n, 0)$ and assume that Z has pure dimension d. Then we can choose coordinates w on $\mathbb{C}^n$ such that $(Z, 0) \cap \{w_I = 0\} = \{0\}$ for all $I \subseteq \{1, \ldots, n\}$ with $|I| = d$.*

Proof. We will choose the coordinates w on $\mathbb{C}^n$ inductively. By Lemma 5.1, there exists E such that $(Z, 0) \cap \{a \cdot z = 0\}$ has codimension 1 in Z if $a \notin E$, and we choose $w_1 = a \cdot z$ for some $a \notin E$. Now, we assume by induction that we have chosen coordinates $(w_1, \ldots, w_k)$ such that $(Z, 0) \cap \{w_I = 0\}$ has codimension $|I|$ for each $I \subseteq \{1, \ldots, k\}$ with $|I| \leq d$. For each $I \subseteq \{1, \ldots, k\}$ with $|I| \leq d - 1$, we can then find E_I by Lemma 5.1 such that $(Z, 0) \cap \{w_I = 0\} \cap \{a \cdot z = 0\}$ has codimension 1 in $(Z, 0) \cap \{w_I = 0\}$ if $a \notin E_I$. Since each E_I is a finite union of proper subspaces of $\mathbb{C}^n$, we can find $a \in \mathbb{C}^n \setminus \cup E_I$, and we then let $w_{k+1} = a \cdot z$. Proceeding in this way, $w = (w_1, \ldots, w_n)$ will be the desired choice of coordinates. $\qquad\square$

6. Representations of the integration current in the Cohen-Macaulay case

To prove Proposition 1.4, we will use the following representation of the integration current $[Z]$ on Z in terms of the current R^Z. Assume that Z is Cohen-Macaulay, and that $\mathrm{codim}\, Z = p$, so that $R^Z = R_p^Z$ by (3.2). By Example 1, [A3], there exist holomorphic $(p, 0)$-forms ξ_i such that

$$(6.1) \qquad [Z] = \sum \xi_i \wedge R_{p,i}^Z,$$

where $R_{p,i}^Z$ are the various components of R^Z, i.e., given a local frame $(e_1, \ldots, e_N)$ of $\mathcal{O}(E_p)$, $R_p^Z = \sum R_{p,i}^Z e_i$.

If Z is a reduced complete intersection defined by $f = (f_1, \ldots, f_p)$, then $R^Z = \mu^f$ by Theorem 3.2, and by the Poincaré-Lelong formula, see [CH], we have

$$[Z] = \frac{1}{(2\pi i)^p} \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} \wedge df_1 \wedge \cdots \wedge df_p.$$

Thus, we can take $\xi = df_1 \wedge \cdots \wedge df_p$, and then it is clear by the implicit function theorem that ξ vanishes at Z_{sing}. We will show that this is the case also when Z is Cohen-Macaulay. This is Theorem 6.3, and the proof will use the following lemmas. Recall that the *socle* of module M over a local ring $(R, \mathfrak{m}, k)$ is defined as $\mathrm{Hom}\,_R(k, M)$, see [BH]. We will use the following characterization of the socle, which is immediate from the definition:

$$(6.2) \qquad \mathrm{Hom}\,_R(k, M) \cong \{\alpha \in M \mid \mathfrak{m}\alpha = 0\}.$$

Lemma 6.1. *Let $\mathfrak{q}$ be a germ of an ideal at 0 such that $\sqrt{\mathfrak{q}} = \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal at 0, and let*

$$(6.3) \qquad 0 \to \mathcal{O}(E_n) \xrightarrow{\varphi_n} \cdots \xrightarrow{\varphi_1} \mathcal{O}(E_0) \to \mathcal{O}/\mathfrak{q} \to 0$$

be a minimal free resolution of $\mathcal{O}/\mathfrak{q}$, where $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n,0}$. Then

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q}) = \mathrm{rank}\, E_n.$$

Proof. We have

$$\mathrm{rank}\, E_n = \dim \mathrm{Tor}_n(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q})$$

since $\mathrm{Tor}_n(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q})$ is just the n:th homology of the complex (6.3) tensored with $\mathcal{O}/\mathfrak{m}$. This is $\mathbb{C}^{\mathrm{rank}\, E_n}$ since the free resolution is minimal so that if

$$\tilde{\varphi}_n : \mathcal{O}(E_n) \otimes \mathcal{O}/\mathfrak{m} \to \mathcal{O}(E_{n-1}) \otimes \mathcal{O}/\mathfrak{m},$$

then $\tilde{\varphi}_n = 0$ since $\mathrm{Im}\, \varphi_n \subseteq \mathfrak{m} E_{n-1}$ by definition of minimality of a free resolution. However, $\mathrm{Tor}_n(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q})$ can also be computed by taking a free resolution of $\mathcal{O}/\mathfrak{m}$, tensoring it with $\mathcal{O}/\mathfrak{q}$ and taking homology. Since the Koszul complex of $(z_1, \ldots, z_n)$ is a free resolution of $\mathcal{O}/\mathfrak{m}$, we get

$$\mathrm{Tor}_n(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q}) \cong \mathrm{Ker} \left(\overset{n}{\bigwedge} \mathcal{O}/\mathfrak{q} \xrightarrow{\delta_z} \overset{n-1}{\bigwedge} \mathcal{O}/\mathfrak{q} \right)$$

$$\cong \{\alpha \in \mathcal{O}/\mathfrak{q} \mid \mathfrak{m}\alpha = 0\} \cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q}),$$

where the last equality is (6.2). $\qquad\qquad\square$

Lemma 6.2. *Assume that there exist pseudomeromorphic currents $\mu_1, \ldots, \mu_N$ such that $\mathfrak{q} = \cap \, \mathrm{ann}\, \mu_i$, where $\mathfrak{q}$ is an ideal such that $\sqrt{\mathfrak{q}} = \mathfrak{m}$. Then*

$$N \geq \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q}).$$

Proof. We claim that there exists a $\mathbb{C}$-linear injective mapping

$$\tilde{\mu} : \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q}) \to \mathbb{C}^N,$$

which proves the statement. We consider $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q})$ as (6.2). Since $\mathfrak{q} \subseteq \mathrm{ann}\, \mu_i$, the mapping $\alpha \mapsto \alpha\mu_i, \alpha \in \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q})$ is well-defined. Since $\mathfrak{m}\alpha = 0$, and $\overline{\mathfrak{m}}\mu_i = 0$ by Proposition 3.4, $\alpha\mu_i$ is a current of order 0 with support on $\{0\}$. Thus

$$(6.4) \qquad\qquad \alpha\mu_i = a_i R_0,$$

for some $a_i \in \mathbb{C}$, where R_0 is the current $\delta_{z=0} d\bar{z}$, that is, $R_0.\alpha dz = \alpha(0)$. We thus get a mapping

$$\tilde{\mu}(\alpha) = (a_1, \ldots, a_N),$$

where a_i are defined by (6.4). It only remains to see that $\tilde{\mu}$ is injective. However, if $\tilde{\mu}(\alpha) = 0$, then $\alpha \in \cap \, \mathrm{ann}\, \mu_i = \mathfrak{q}$, so $\alpha = 0$ in $\mathcal{O}/\mathfrak{q}$. $\qquad\square$

Combining Lemma 6.1 and Lemma 6.2, if f is a complete intersection on Z, where Z is Cohen-Macaulay, then none of the components in the decomposition $R^f \wedge R^Z = \sum R^f \wedge R^Z_{p,i}$ are redundant. This will be a crucial step in the proof of the following theorem.

Theorem 6.3. *Let $Z \subseteq \Omega \subseteq \mathbb{C}^n$ be a subvariety of Ω of codimension p, and assume that Z is Cohen-Macaulay. Then there exists holomorphic $(p,0)$-forms ξ_i such that*

$$[Z] = \sum \xi_i \wedge R^Z_{p,i},$$

and if R^Z is defined with respect to a minimal free resolution of $\mathcal{O}_Z$, then all ξ_i vanish at Z_{sing}.

Proof. As mentioned in the introduction of the section, the existence of ξ_i is Example 1 in [A3], so we only need to prove that ξ_i vanish at Z_{sing} if R^Z is defined with respect to a minimal free resolution. Assume that $0 \in Z_{\mathrm{sing}}$. We begin by choosing coordinates in $\mathbb{C}^n$ such that $\{w_J = 0\} \cap Z = \{0\}$ for all $J \subseteq \{1, \dots, n\}$ with $|J| = n - p$, which is possible by Lemma 5.4. We have

(6.5) $$[Z] = \sum_{i, |I| = p} \xi_{I,i} dw_I \wedge R^Z_{p,i},$$

where $\xi_{I,i}$ are holomorphic functions, and we are done if we can prove that $\xi_{I,i}(0) = 0$ for all $\xi_{I,i}$.

Fix some $I \subseteq \{1, \dots, n\}$ with $|I| = p$. Let $w' = (w_{J_1}, \dots, w_{J_{n-p}})$, where $J = I^c$. By the Poincaré-Lelong formula applied to w' on Z, see [CH], Section 1.9, we have that

$$\frac{1}{(2\pi i)^p} R^{w'} \wedge dw' \wedge [Z] = k[0]$$

for some $k \geq 1$. Combined with the Poincaré-Lelong formula applied to w in $\mathbb{C}^n$, we get

$$R^w \wedge dw = (2\pi i)^n [0] = ((2\pi i)^{n-p}/k) R^{w'} \wedge dw' \wedge [Z].$$

Since by (6.5)

$$dw' \wedge [Z] = \pm \sum_i \xi_{I,i} dw \wedge R^Z_{p,i}$$

we get that

(6.6) $$R^w = C \sum_i \xi_{I,i} R^{w'} \wedge R^Z_{p,i}$$

for some constant $C \neq 0$.

We first consider the case when R^Z consists of one single component R^Z_p. By Corollary 3.6, $\mathrm{ann}(R^{w'} \wedge R^Z_p) = \mathcal{J}(w') + \mathcal{J}_Z$. We claim that the inclusion $\mathcal{J}(w)_0 \supseteq (\mathcal{J}(w') + \mathcal{J}_Z)_0$ is strict. If the inclusion is not strict, then w' generates the maximal ideal $\mathfrak{m}_{Z,0}$ in $\mathcal{O}_{Z,0}$, which is a contradiction by Proposition 4.32 in [D], since the number of functions needed to

generate the maximal ideal at a singular point must be strictly larger than the dimension. Thus there exists a g in

$$\mathcal{J}(w)_0 \setminus (\mathcal{J}(w') + \mathcal{J}_Z)_0 = (\operatorname{ann} R^w)_0 \setminus (\operatorname{ann}(R^{w'} \wedge R_p^Z))_0.$$

Multiplying (6.6) by g, we get that $g\xi_I \in \operatorname{ann}(R^{w'} \wedge R_p^Z)$, and hence we must have $\xi_I(0) = 0$.

Now we consider the case when R_p^Z consists of more than one component. By Corollary 3.6, the tensor product of the Koszul complex of w' and the minimal free resolution of $\mathcal{J}_Z$ is a minimal free resolution of $\mathfrak{q} := \mathcal{J}(w') + \mathcal{J}_Z$, and the rank N of its left-most non-zero module is equal to the rank of the left-most non-zero module in the free resolution of $\mathcal{J}_Z$ since the left-most non-zero module of the Koszul complex has rank 1. By Corollary 3.6, we have

$$(6.7) \qquad \mathfrak{q} = \cap_{i=1}^N \operatorname{ann}(R^{w'} \wedge R_{p,i}^Z).$$

By Lemma 6.1, $N = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q})$ and by Lemma 6.2, if $\mathfrak{q} = \cap_{i=1}^m \operatorname{ann} \mu_i$, then $m \geq N$. Thus, if we remove one term $\operatorname{ann}(R^{w'} \wedge R_{p,i}^Z)$ from the intersection in (6.7), we get something strictly larger, i.e., for any i,

$$(6.8) \qquad (\cap_{j \neq i} \operatorname{ann}(R^{w'} \wedge R_{p,j}^Z)) \setminus (\operatorname{ann} R^{w'} \wedge R_{p,i}^Z) \neq \emptyset.$$

We fix some $i = 1, \ldots, n$, and take g_i in (6.8) and multiply (6.6) by g_i. Since $g_i \in \cap_{j \neq i} \operatorname{ann}(R^{w'} \wedge R_{p,j}^Z)$, we must have $g_i \in \mathfrak{m}$, so $g_i R^w = 0$. Thus we get

$$g_i \xi_{I,i} R^{w'} \wedge R_{p,i}^Z = 0.$$

Since $g_i \notin \operatorname{ann}(R^{w'} \wedge R_{p,i}^Z)$ but $g_i \xi_{I,i} \in \operatorname{ann}(R^{w'} \wedge R_{p,i}^Z)$, we must have $\xi_{I,i} \in \mathfrak{m}$, and we are done. $\qquad \square$

7. Proof of Proposition 1.4

By moving to a nearby germ (Z, w), we can assume that Z_{sing} has pure codimension k, and that there exists a complete intersection $f = (f_1, \ldots, f_q)$ on (Z, w) such that $(Z_{\text{sing}}, w) = \{f_1 = \cdots = f_k = 0\} \cap (Z, w)$, see Lemma 5.3. We let $\mathcal{I} = \mathcal{J}(f_1, \ldots, f_q)_w$ and $V = Z(\mathcal{I})$, and since $q \geq k$, $V \subseteq Z_{\text{sing}}$. Since $\mathcal{J}_{V,w}$ is finitely generated over $\mathcal{O}_{Z,w}$, we get from the Nullstellensatz that $\mathcal{J}_{V,w}^m \subseteq \mathcal{I}$ for m sufficiently large. Now, we choose m to be minimal such that this inclusion holds. Thus, there exists a function $g \in \mathcal{J}_{V,w}^{m-1} \setminus \mathcal{I}$, such that $g\mathcal{J}_{V,w} \subseteq \mathcal{I}$. Since $g \notin \mathcal{I}$, we are done if we can show that $g\mu^f \wedge [Z] = 0$.

By Theorem 3.2, we can replace μ^f by R^f, and instead show that $gR^f \wedge [Z] = 0$. By Theorem 6.3

$$gR^f \wedge [Z] = g \sum \xi_i \wedge R^f \wedge R_i^p,$$

where ξ_i are holomorphic $(p, 0)$-forms vanishing on Z_{sing}. Thus $\xi_i = \sum \xi_{I,i} dw_I$, where $\xi_{I,i}$ are holomorphic functions vanishing at Z_{sing}. Since $g\mathcal{J}_{V,w} \subseteq \mathcal{I}$ and $\mathcal{J}_{Z_{\text{sing}},w} \subseteq \mathcal{J}_{V,w}$, we get that $g\xi_{I,i} \in \mathcal{I}$ in $\mathcal{O}_{Z,w}$. By

Corollary 3.6, ann $R^f \wedge R^Z = \mathcal{I} + \mathcal{J}_{Z,w}$. Since if $g\xi_{I,i} \in \mathcal{I}$ in $\mathcal{O}_{Z,w}$, then $g\xi_{I,i} \in \mathcal{I} + \mathcal{J}_{Z,w}$ in $\mathcal{O}_{\mathbb{C}^n,w}$, we get that $gR^f \wedge [Z] = 0$.

8. Singularity subvarieties and counterexamples in the non Cohen-Macaulay case

We will recall the notion of singularity subvarieties of analytic sheaves from [ST]. Let R be a local Noetherian ring and $M \neq 0$ a finitely generated R-module. A *regular M-sequence* in an ideal $I \subseteq R$ is a sequence $(f_1, \ldots, f_p)$ in I such that f_i is not a zero-divisor in $M/(f_1, \ldots, f_{i-1})M$ for $i = 1, \ldots, p$. The *depth* of an ideal I on a module M, denoted $\mathrm{depth}_I M$ is the maximal length of a regular M-sequence in I. By $\mathrm{depth}_R M$, we will denote the depth of the maximal ideal $\mathfrak{m}$ of R on M. This is also called the homological codimension of M. The *homological dimension* of M, denoted $\mathrm{dh}_R M$, is defined as the minimal length of any free resolution of M.

A *regular local ring* is a local ring R such that the maximal ideal $\mathfrak{m}$ of R is generated by $n = \dim R$ elements, where $\dim R$ is the Krull-dimension of R, that is, the maximal length of a strict chain of prime ideals in R. In particular, if Z is an analytic variety, then $\mathcal{O}_{Z,z}$ is a regular local ring if and only if $z \in Z_{\mathrm{reg}}$, see Proposition 4.32 in [D]. The following is Theorem 19.9 in [E], the Auslander-Buchsbaum formula.

Proposition 8.1. *If R is a regular local ring, and M is a finitely generated R-module, then $\mathrm{dh}_R M + \mathrm{depth}_R M = \dim R$.*

Let $\mathcal{F}$ be a coherent analytic sheaf on $\Omega \subseteq \mathbb{C}^n$, and let $\mathcal{O}_z$ denote the ring of germs of holomorphic functions at z in Ω. The *singularity subvarieties*, S_m, of $\mathcal{F}$ are defined by

$$S_m(\mathcal{F}) = \{z \in \Omega; \mathrm{depth}_{\mathcal{O}_z} \mathcal{F}_z \leq m\},$$

where we use the convention that $\mathrm{depth}_R M = \infty$ if $M = 0$, so that $S_m \subseteq \mathrm{supp}\,\mathcal{F}$. We will use the following alternative definition of the sets Z_k associated with an analytic sheaf above:

(8.1) $$Z_k(\mathcal{F}) = \{z \in \Omega; \mathrm{dh}_{\mathcal{O}_z} \mathcal{F}_z \geq k\}$$

(in the introduction, we defined the sets Z_k if $\mathcal{F}$ was of the form $\mathcal{O}/\mathcal{J}$, where $\mathcal{J}$ was an coherent ideal sheaf, but the same definition works for any coherent analytic sheaf). To see this, note first that if $\mathrm{rank}\,\varphi_k(z)$ is constant in a neighborhood of some $z_0 \in \Omega$ (i.e., $z_0 \notin Z_k$), then $\mathcal{O}(E_{k-1})/\mathrm{Im}\,\varphi_k$ is free in a neighborhood of z_0, so $\mathcal{O}/\mathcal{J}$ has a free resolution of length $k - 1$. Conversely, by the uniqueness of minimal free resolutions, $\mathrm{rank}\,\varphi_k(z)$ must be constant in a neighborhood of z if $k > \mathrm{dh}_{\mathcal{O}_z} \mathcal{F}_z$.

Proposition 8.2. *If $\mathcal{F}$ is coherent analytic sheaf on some open set in $\mathbb{C}^n$, we have $S_k(\mathcal{F}) = Z_{n-k}(\mathcal{F})$.*

Proof. This follows from Proposition 8.1 and (8.1).   □

Let $\Omega \subseteq \mathbb{C}^n$ be an open set, A a subvariety of Ω with ideal sheaf $\mathcal{J}_A$, and $\mathcal{F}$ a coherent analytic sheaf in Ω. For $z \in \Omega$, we define

$$\mathrm{depth}_{A,z} \, \mathcal{F} = \left\{ \begin{array}{ll} \infty & \text{if } \mathcal{F}_z = 0 \\ \mathrm{depth}_{\mathcal{J}_{A,z}} \, \mathcal{F} & \text{otherwise} \end{array} \right. .$$

and

$$\mathrm{depth}_A \, \mathcal{F} = \inf_{z \in A} \mathrm{depth}_{A,z} \, \mathcal{F}$$

The following is (part of) Theorem 1.14 in [ST].

Theorem 8.3. *Let $\Omega \subseteq \mathbb{C}^n$ be some open set, A a subvariety of Ω, and $\mathcal{F}$ a coherent analytic sheaf in Ω. Then for $q \geq 1$, we have $\mathrm{depth}_A \, \mathcal{F} \geq q$ if and only if $\dim A \cap S_{k+q}(\mathcal{F}) \leq k$ for all k.*

In particular, if we let Z be an analytic subvariety of Ω, $\mathcal{F} = \mathcal{O}_Z$, and $A = Z^1$, where the sets Z^k associated with Z are defined as in (1.4), we get the following.

Corollary 8.4. *For $q \geq 1$, we have $\mathrm{depth}_{Z^1} \mathcal{O}_Z \geq q$ if and only if $\mathrm{codim} \, Z^k \geq q + k$ in Z for all $k \geq 1$*

Proof. If we apply Theorem 8.3 to $A = Z^1$ and $\mathcal{F} = \mathcal{O}_Z$, then we only need to prove that $\mathrm{codim} \, Z^k \geq q + k$ for $k \geq 1$ is equivalent to $\dim Z^1 \cap S_{k+q}(\mathcal{O}_Z) \leq k$. We can write the last condition as $\dim(Z^1 \cap Z_{n-k-q}) \leq k$ by Proposition 8.2. If we replace $\dim V$ by $n - \mathrm{codim} \, V$ and set $k' = n-k-q$, we get $\mathrm{codim} \, (Z^1 \cap Z_{k'}) \geq q+k'$. Since $Z_k = Z$ for $k \leq p$, where $p = \mathrm{codim} \, Z$, and $Z^1 = Z_{p+1}$, this condition for $k \leq p$ is equivalent to $\mathrm{codim} \, Z_{p+1} \geq p+q+1$ (in Ω), and since $Z_k \subseteq Z_{p+1} = Z^1$ for $k > p+1$, this is equivalent to $\mathrm{codim} \, Z_{p+k} \geq p+q+k$ for $k \geq 2$. $\square$

In $\mathbb{C}^n$, it is a standard result that a tuple $f = (f_1, \ldots, f_p)$ of holomorphic functions is a complete intersection if and only if it is a regular sequence (see for example [dJP], Corollary 4.1.20). However, Corollary 8.4 says that this is not always the case on a singular variety. We will illustrate this with an example.

Example 2. Let $\pi(t_1, t_2) = (t_1, t_1 t_2, t_2^2, t_2^3)$, and let $Z = \pi(\mathbb{C}^2)$. Then $Z_{\mathrm{sing}} = \{0\}$, because outside of $\{t_1 = t_2 = 0\}$, one can construct a holomorphic inverse to π, and we will see that Z is not normal at 0, so $0 \in Z_{\mathrm{sing}}$. The function f such that $\pi^* f = t_2$ is weakly holomorphic on Z, since when $t_1 \neq 0$, $f = z_2/z_1$, and when $t_2 \neq 0$, $f = z_4/z_3$, so that $f \in \mathcal{O}(Z_{\mathrm{reg}})$, and it is clear that f is locally bounded near $Z_{\mathrm{sing}} = \{0\}$. However, f is not strongly holomorphic at 0, because if $f = h$ on Z in a neighborhood of 0, where h is holomorphic in a neighborhood of 0 in $\mathbb{C}^4$, then by taking pull-back by π to $\mathbb{C}^2$, we get

$$t_2 = h(t_1, t_1 t_2, t_2^2, t_2^3),$$

which can be seen to be impossible by a Taylor expansion of h at 0.

Since Z has pure dimension, codim $Z^k \geq k + 1$ for $k \geq 1$ by [E], Corollary 20.14b. Hence, $Z^k = \emptyset$ for $k \geq 2$. Since Z is not normal, it does not satisfy the condition

$$\text{(8.2)} \qquad \qquad \text{codim } Z^k \geq k + 2, \quad k \geq 0$$

for normality (see the introduction). However, since $Z^0 = Z_{\text{sing}} = \{0\}$, the condition (8.2) is satisfied for all $k \neq 1$. Thus, since $Z^1 \subseteq Z_{\text{sing}}$, and codim $Z^1 \not\geq 3$, we must have $Z^1 = \{0\}$. By Corollary 8.4, there does not exist a regular $\mathcal{O}_Z$-sequence $f = (f_1, f_2)$ in $\mathcal{J}_{Z^1}$, since any such sequence has length ≤ 1. In particular, if we take $f = (z_1, z_3)$, then f is a complete intersection since $Z \cap \{z_1 = z_3 = 0\} = \{0\}$, but f is not a regular sequence. We claim that one can also see this more directly. To begin with, it is clear that $z_3 \notin (z_1)$ in $\mathcal{O}_Z$ since $Z \cap \{z_1 = 0\} \not\subseteq Z \cap \{z_3 = 0\}$. We also have that $z_2 \notin (z_1)$ in $\mathcal{O}_Z$, since if $z_2 \in (z_1)$, then by taking pull-back to $\mathbb{C}^2$ as above, we get

$$t_1 t_2 = t_1 h(t_1, t_1 t_2, t_2^2, t_2^3),$$

which is easily seen to be impossible. However, since $z_2 z_3 = z_1 z_4$ in $\mathcal{O}_Z$, we get that $z_2 z_3 \in (z_1)$ in $\mathcal{O}_Z$. Thus, z_3 is a zero-divisor in $\mathcal{O}_Z/(z_1)$, i.e., (z_1, z_3) is not a regular $\mathcal{O}_Z$-sequence in $\mathcal{J}_{Z^1}$.

Lemma 8.5. *Let $f = (f_1, \dots, f_k)$ be a complete intersection on (Z, z). If*

$$\text{ann}\left(\bar{\partial} \frac{1}{f_r} \wedge \dots \wedge \bar{\partial} \frac{1}{f_1} \right) = \mathcal{J}(f_1, \dots, f_r) \text{ for all } r < k,$$

then $(f_1, \dots, f_k)$ is a regular $\mathcal{O}_{Z,z}$-sequence.

Proof. If $k = 1$, this is clear since $\mathcal{O}_{Z,z}$ is reduced and f is assumed to be a complete intersection. By induction over k, we can assume that $(f_1, \dots, f_{k-1})$ is a regular $\mathcal{O}_{Z,z}$-sequence. Assume that $(f_1, \dots, f_k)$ is not a regular sequence in $\mathcal{O}_{Z,z}$. Then, since $f_k \notin \mathcal{J}(f_1, \dots, f_{k-1})$, there exist $g \notin \mathcal{J}(f_1, \dots, f_{k-1})$ such that $f_k g \in \mathcal{J}(f_1, \dots, f_{k-1})$. But since $g \in \mathcal{J}(f_1, \dots, f_{k-1})$ outside of $\{f_k = 0\}$, we get that

$$\text{supp}\left(g \bar{\partial} \frac{1}{f_{k-1}} \wedge \dots \wedge \bar{\partial} \frac{1}{f_1} \right) \subseteq \{f_1 = \dots = f_k = 0\}$$

by Theorem 1.2. But then by Proposition 3.3, we get that

$$g \in \text{ann } \bar{\partial} \frac{1}{f_{k-1}} \wedge \dots \wedge \bar{\partial} \frac{1}{f_1} = \mathcal{J}(f_1, \dots, f_{k-1}),$$

which is a contradiction. $\qquad \square$

Proof of Proposition 1.5. By Lemma 5.2, there exists a complete intersection $(f_1, \dots, f_{p+1})$ such that $Z^1 \subseteq \{f_1 = \dots = f_{p+1} = 0\}$. By Corollary 8.4, $(f_1, \dots, f_{p+1})$ is not a regular $\mathcal{O}_{Z,z}$-sequence in $\mathcal{J}(f_1, \dots, f_{p+1})_z$. Thus by Lemma 8.5, we must have that

$$\text{(8.3)} \qquad \text{ann}\left(\bar{\partial} \frac{1}{f_k} \wedge \dots \wedge \bar{\partial} \frac{1}{f_1} \right) \supsetneq \mathcal{J}(f_1, \dots, f_k)$$

for some $k \leq p$. However, by Theorem 1.3, we have equality for $k \leq p-1$. Thus we must have strict inclusion in (8.3) for $k = p$. $\qquad\square$

Acknowledgments

I would like to thank my advisor Mats Andersson for valuable discussions in the preparation of this article.

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