Thesis for the Degree of Doctor of Philosophy

Topics in algorithmic, enumerative and geometric combinatorics

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ABSTRACT

This thesis presents five papers, studying enumerative and extremal problems on combinatorial structures.

The first paper studies Forman's discrete Morse theory in the case where a group acts on the underlying complex. We generalize the notion of a Morse matching, and obtain a theory that can be used to simplify the description of the G-homotopy type of a simplicial complex. As an application, we determine the $\mathfrak{S}_2 \times \mathfrak{S}_{n-2}$-homotopy type of the complex of non-connected graphs on n nodes. In the introduction, connections are drawn between the first paper and the evasiveness conjecture for monotone graph properties.

In the second paper, we investigate Hansen polytopes of split graphs. By applying a partitioning technique, the number of nonempty faces is counted, and in particular we confirm Kalai's 3^d-conjecture for such polytopes. Furthermore, a characterization of exactly which Hansen polytopes are also Hanner polytopes is given. We end by constructing an interesting class of Hansen polytopes having very few faces and yet not being Hanner.

The third paper studies the problem of packing a pattern as densely as possible into compositions. We are able to find the packing density for some classes of generalized patterns, including all the three letter patterns.

In the fourth paper, we present combinatorial proofs of the enumeration of derangements with descents in prescribed positions. To this end, we consider fixed point λ-coloured permutations, which are easily enumerated. Several formulae regarding these numbers are given, as well as a generalisation of Euler's difference tables. We also prove that except in a trivial special case, the event that π has descents in a set S of positions is positively correlated with the event that π is a derangement, if π is chosen uniformly in $\mathfrak{S}_n$.

The fifth paper solves a partially ordered generalization of the famous secretary problem. The elements of a finite nonempty partially ordered set are exposed in uniform random order to a selector who, at any given time, can see the relative order of the exposed elements. The selector's task is to choose online a maximal element. We describe a strategy for the general problem that achieves success probability at least 1/e for an arbitrary partial order, thus proving that the linearly ordered set is at least as difficult as any other instance of the problem. To this end, we define a probability measure on the maximal elements of an arbitrary partially ordered set, that may be interesting in its own right.

Keywords: Discrete Morse theory, simplicial G-complex, centrally symmetric polytope, split graph, derangement, pattern packing, composition, finite poset, optimal stopping, secretary problem.

Preface

This thesis consists of the following papers.

- ▷ **Ragnar Freij**,
  "Equivariant discrete Morse theory",
  in *Discrete Mathematics* **309** (2009), 3821–3829.
- ▷ **Ragnar Freij**, Matthias Henze, Moritz W. Schmitt, Günter M. Ziegler,
  "Face numbers of centrally symmetric polytopes produced from split graphs",
  submitted to *Electronic Journal of Combinatorics*, (2012).
- ▷ **Ragnar Freij**, Toufik Mansour,
  "Packing a binary pattern in compositions",
  in *Journal of Combinatorics* **2** (2011), 111–137.
- ▷ Niklas Eriksen, **Ragnar Freij**, Johan Wästlund,
  "Enumeration of derangements with descents in prescribed positions",
  in *Electronic Journal of Combinatorics* **16** (2009), #R32.
- ▷ **Ragnar Freij**, Johan Wästlund,
  "Partially ordered secretaries",
  in *Electronic Communications in Probability* **15** (2010), 504–507.

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Ragnar Freij
Göteborg, March 2012

Contents

Part I

INTRODUCTION

1 INTRODUCTION

The present thesis consists of five different papers. Roughly spoken, they treat three different research areas: pattern containment in words, combinatorial geometric structures occuring in graph theory, and optimal stopping. A common theme in the first four papers is the enumeration of combinatorial structures, in one way or another. Some of the papers can also be said to have a common flavour of extremal combinatorics, where we prove that certain "extremal structures" have some natural "standard form". These "extremal structures" are as diverse as

- Hansen polytopes of split graphs, that have few faces,
- compositions that densely pack a given pattern, and
- partial orders in which maximal elements are "difficult to identify".

In the following, we will discuss these and other aspects of the thesis more closely. Throughout the thesis, all graphs considered will be assumed to be finite. To avoid overloading words, a *vertex* will always mean a vertex of some geometric object, such as a simplicial complex or a polytope. The sites of graphs will be called *nodes*. Similarly, a *simplex* will always mean the convex hull of $d+1$ points in general position in Euclidean d-space, while the elements of an abstract simplicial complex will be called *cells*.

1.1 Discrete Morse theory and graph complexes

Discrete Morse theory was first introduced by Forman in [17]. In essence, it provides a method to drastically reduce the number of cells in a (usually combinatorially defined) simplicial complex, in order to simplify computation of its homotopy type. Let us start by a basic definition, to fix notation.

Definition 1 (Simplicial complexes). *An abstract simplicial complex is a collection Σ of finite sets (which we will call cells), such that if $\sigma \in \Sigma$ and $\tau \subseteq \sigma$, then $\tau \in \Sigma$. (The simplicial complexes occuring in this thesis will all be finite.)*

An abstract simplicial complex has a geometric realization $|\Sigma|$, obtained by embedding the points of $\cup_{\sigma\in\Sigma}(\sigma)$ in general position in Euclidean space, denoting by $|\sigma|$ the simplex spanned by the points in σ, and defining $|\Sigma| = \cup_{\sigma\in\Sigma}|\sigma|$.

Any two geometric realizations of Σ are clearly homeomorphic. This allows us to talk about the topology of an abstract simplicial complex, by which we simply mean the topology of any of its geometric realizations.

Just like a simplicial complex, a *CW complex* is constructed by gluing (homeomorphic images of) discs of different dimensions together. The main difference is that in a CW complex the boundary of cells may be decomposed into lower-dimensional cells in many ways, and in particular the same cell may appear many times along the boundary of a higher-dimensional cell. Hence, the notion of CW complexes is much more flexible than that of simplicial complexes, but also much less combinatorial in nature. For a proper definition and fundamental properties of CW complexes, consult any book on algebraic topology, eg. [31].

Discrete Morse theory describes how certain matchings on a (typically large) simplicial complex Σ induce simplicial collapses of Σ onto a (typically much smaller) CW complex, whose homology can hopefully be calculated more easily. More precisely, let $P(\Sigma)$ be the partially ordered set (poset) of the cells in Σ, ordered by inclusion. A *matching* on a poset P is a set M of pairs $(\sigma,\tau) \in P \times P$ such that:

- Every element in P is in exactly one pair in M.
- $\sigma \leq \tau$ for every $(\sigma,\tau) \in M$.
- There is no ρ such that $\sigma < \rho < \tau$ for any $(\sigma,\tau) \in M$.

In other words, M is a collection of intervals of height 1 or 2. If $(\sigma,\sigma) \in M$, then σ is said to be unmatched. For technical reasons, we will also say that σ is unmatched if $(\emptyset,\sigma) \in M$

If M is a matching on a poset P, let P/M be the digraph whose objects are the pairs in M, with an arrow $(\sigma,\tau) \to (\sigma',\tau')$ if $\tau > \sigma'$. It is possible that the digraph P/M has directed cycles, but if it does not, then we can think of it as a poset. With this notation, the following is the main theorem of discrete Morse theory.

THEOREM 1. *Let M be a matching on $P(\Sigma)$ such that $P(\Sigma)/M$ has no directed cycles. Then, there is a CW complex $\mathcal{M}$ that is homotopy equivalent to Σ, and whose cells are in one-to-one correspondence (with dimension preserved) with the unmatched cells in Σ.*

In particular, if there is a complete matching on $P(\Sigma)$, then Σ has trivial homotopy type. A matching satisfying the conditions in Theorem 1 is called a *Morse matching* on Σ.

1.1.1 Graph complexes

A graph property is a property defined on graphs on a fixed set of nodes, which is invariant under permutations of the nodes. Properties such as "the nodes labelled one and two being connected" are hence not graph properties. Examples of graph properties are being connected, being planar, being cycle-free, and so on.

Let $\mathcal{Q}$ be a graph property, which we identify with the set of graphs having this property. Assume that, whenever $G \in \mathcal{Q}$ and $H \subseteq G$ is obtained by deleting some edges from G (but keeping all the nodes), then $H \in \mathcal{Q}$. Then we say that the property $\mathcal{Q}$ is *monotone*. For example, the property of being non-connected is monotone.

A (non-trivial) monotone graph property $\mathcal{Q}$, defined on graphs with n nodes, can be viewed as an abstract simplicial complex $\Sigma_{\mathcal{Q}}$. Indeed, the vertices of $\Sigma_{\mathcal{Q}}$ are indexed by the edge set $E(K_n)$ of the complete graph K_n. A set $\sigma \subseteq E(K_n)$ is in $\Sigma_{\mathcal{Q}}$ if the graph with the corresponding edge set has the property $\mathcal{Q}$. A simplicial complex $\Sigma_{\mathcal{Q}}$ that is constructed in this way will be called a *graph complex*, and clearly contains all information about $\mathcal{Q}$. An excellent general reference on graph complexes is [33].

A very interesting open problem about graph properties is the evasiveness conjecture, normally attributed to Aanderaa, Karp and Rosenberg [50]. To state the conjecture, we must first introduce some terminology on decision trees for properties (or equivalently, for set systems or Boolean functions).

Let S be a finite set, and suppose we are interested in whether an unknown set $A \subseteq S$ has the property $\mathcal{Q} \subseteq 2^S$. The (deterministic) complexity of $\mathcal{Q}$ is the maximum number of questions of the form "is $x \in A$?" that you can be forced to ask, in order to determine whether $A \in \mathcal{Q}$. In other words, it is the smallest depth of a decision tree for the property $\mathcal{Q}$. It is clear that any property on subsets of S has complexity at most $|S|$, and a property is called *evasive* if its complexity equals $|S|$. We are now ready to state the evasiveness conjecture for graph properties [50].

CONJECTURE 1. *Any non-trivial monotone graph property is evasive.*

The trivial graph properties $\mathcal{Q} \subseteq 2^S$ are just $\emptyset$ and 2^S. Partial results on Conjecture 1 for special classes of properties are known. For example, Bollobás [6] proved that the property of containing a k-clique is evasive for every k and n, and Chakrabarti, Khot and Shi [9] proved that any non-trivial property that is closed under taking graph minors is evasive for large enough n.

As we have seen, monotone graph properties are special cases of simplicial complexes on $\binom{n}{2}$ vertices, with an $\mathfrak{S}_n$-action inherited from the natural $\mathfrak{S}_n$-action on the edge set of the complete graph K_n. It is remarkable that while there is no proof of Conjecture 1 for general graph properties, there is also (to our present knowledge) no counterexample known to the following stronger conjecture.

CONJECTURE 2. *Any non-trivial monotone property $\mathcal{Q} \subseteq 2^S$, which is invariant under some transitive group action on S, is evasive.*

Observe that $\mathfrak{S}_n$ acts transitively on the edges of K_n, so Conjecture 2 generalizes Conjecture 1. Conjectures 1 and 2 are both proven in the special cases where the number of nodes of the graph, respectively the cardinality of S, are prime powers [35, 49]. Note that the special case of Conjecture 2 where $|S|$ is a prime power has no relevance for Conjecture 1, since the number of vertices of a graph complex is $\binom{n}{2}$, which is never a prime power except in the trivial cases $n \leq 3$.

Interestingly, although the paper [35], which proved Conjecture 1 in the prime power case, preceded the introduction of discrete Morse theory in [17] by more than a decade, its methods can be described very well in the language of discrete Morse theory. Indeed, assume Σ is a subcomplex of a d-dimensional simplex (or equivalently a monotone Boolean function of $d + 1$ variables). Let T be a decision tree to decide whether an unknown set s of vertices span a cell in Σ. Then T induces a matching on Σ, as is explained and proven in [18]. The unmatched elements in this Morse matching are the cells for which all $d + 1$ questions are needed by T. Any other cell σ is matched to a cell τ from which it cannot be distinguished when T terminates. So by construction, any non-evasive simplicial complex has a Morse matching with no unmatched cells, and thus collapses to a point, and in particular has the homotopy type of a point.

To prove Conjecture 1, it would thus suffice to prove that any non-trivial graph complex never collapses to a point. This is exactly what Kahn, Saks and Sturtevant proved in [35] in the case of properties of graphs with n nodes, where n is a prime power. They applied results from [45] on fixed points of acyclic complexes, together with the trivial observation that if a vertex transitive action on a simplicial complex Σ has a fixed point, then Σ is a simplex.

Similarly, to prove Conjecture 2 it would suffice to prove that any non-trivial simplicial complex whose symmetry group acts transitively on the vertices, never collapses to a point. However, this appears to be a much harder task, since there exist non-trivial complexes with transitive group actions that have the homotopy type of a point (though they may not collapse simplicially). The first such example was provided by Oliver, cf. [35], and a collection of examples is presented in Lutz's thesis [41].

1.1.2 Equivariant discrete Morse theory

In Paper I we develop a method suitable for studying simplicial complexes of particular graph properties, although most of the theory is developed in a more general context. The general setting is a simplicial complex, with a group G acting on it. This action induces an G-module structure on the homology groups $H_*(\Sigma)$. In the graph complex case, $\Sigma = \Sigma_{\mathcal{Q}}$ will have $\binom{n}{2}$ nodes, and there will be a group action by the symmetric group $\mathfrak{S}_n$, induced by the natural $\mathfrak{S}_n$-action on the complete graph K_n.

When we study this G-module structure, all the operations we perform on $\Sigma_{\mathcal{Q}}$ must respect the group action, and unfortunately discrete Morse theory is ill suited for this. The main theorem of Paper I generalizes and strengthens Theorem 1. The generalization is that we consider more general "matchings", and the strengthening is that if such a matching is invariant under G, then the homotopy equivalence $\mathcal{M} \simeq \Sigma$ between the corresponding Morse complex $\mathcal{M}$ and the underlying complex Σ is G-equivariant. The theory relies on the notion of GCW complexes, which is described more carefully in Paper I. For here, it suffices to know that a GCW complex is a CW complex with an intrinsic G-action on it.

An interval of a poset is

$$[\sigma, \tau] = \{\rho : \sigma \leq \rho \leq \tau\}.$$

Just as we constructed P/M when M was a matching, we can also construct P/I where I is a partition of P into intervals, with an arrow $[\sigma, \tau] \to [\sigma', \tau']$ if $\tau > \sigma'$. As before, we will say that σ is unmatched by I if $[\sigma, \sigma] \in I$ or if $|\sigma| = 1$ and $[\emptyset, \sigma] \in I$.

THEOREM 2. *Let Σ be a simplicial complex with an action by G. Let I be a partition of $P(\Sigma)$ into intervals such that $P(\Sigma)/I$ has no directed loops. Assume that I is G-equivariant, i.e. if $[\sigma, \tau]$ is in I, then so is $[g\sigma, g\tau]$ for $g \in G$.*

Then, there is a GCW complex $\mathcal{M} \simeq \Sigma$, whose cells are in one-to-one correspondence (with dimension preserved) with the unmatched cells in Σ.

Assume that $\mathcal{Q}$ is a graph property on graphs with n nodes. For practical purposes, one can often not consider the full $\mathfrak{S}_n$ action on $\Sigma_{\mathcal{Q}}$, but must restrict attention to some subgroup. Indeed, one often wants to collapse $\Sigma_{\mathcal{Q}}$ to a wedge of spheres, and to do this G-equivariantly $G \subseteq \mathfrak{S}_n$ must fix some vertex of Σ, i.e. some edge of K_n. So the largest subgroup $G \subseteq \mathfrak{S}_n$ that we can use is in such cases $\Gamma := \mathfrak{S}_2 \times \mathfrak{S}_{n-2} \subseteq \mathfrak{S}_n$.

Paper I is essentially a rewritten version of [19], and contains a calculation of $H_*(\Sigma_{\mathcal{Q}})$, where $\mathcal{Q}$ is the collection of non-connected graphs on n nodes. The homology groups are considered as Γ–modules, where $\Gamma = \mathfrak{S}_2 \times \mathfrak{S}_{n-2} \subseteq \mathfrak{S}_n$ acts by permuting the nodes $\{1, 2\}$ and the nodes

$\{3, \ldots, n\}$ independently. The homology groups, without the group action by Γ, are well known, and were calculated in [55].

1.2 Centrally symmetric polytopes with few faces

In Paper II, we consider face numbers of certain centrally symmetric polytopes, motivated by the still unresolved 3^d-conjecture (Conjecture 3) by Kalai [36]. The following few definitions are intended to fix notation. For general questions about polytopes, we refer to [56].

DEFINITION 2 (Polytopes). *A polytope is the convex hull of finitely many points in Euclidean space.*

Equivalently, a polytope is the bounded intersection of finitely many half-spaces in Euclidean space.

The dimension of a polytope is the dimension of the smallest affine space containing it.

DEFINITION 3 (Faces). *A face of a polytope P is*

$$\{\boldsymbol{x} \in P \mid \langle \boldsymbol{f}, \boldsymbol{x} \rangle = c\},$$

where $\langle \boldsymbol{f}, \boldsymbol{x} \rangle \leq c$ is an inequality that holds for every $\boldsymbol{x} \in P$.

In particular, $\emptyset$ and P are faces of P, as are all the vertices of P. It is easy to prove (see [56]) that every polytope has finitely many faces, that every face is a polytope itself, and that the faces, ordered by inclusion, form a lattice. If $\dim(P) = d$, a maximal face $F \subsetneq P$ has dimension $d-1$ and is called a *facet*. We denote the number of i-dimensional faces of P by $f_i(P)$, and call the vector $(f_{-1}(P), f_0(P), \ldots, f_d(P))$ the f-vector of P. The empty set is considered to be a (-1)-dimensional face.

Two polytopes are said to be *combinatorially equivalent* if their face lattices are isomorphic as posets. They are said to be *affinely* (respectively *projectively*) equivalent if there is an invertible affine (projective) map between them. Combinatorial equivalence is thus a weaker equivalence relation than projective equivalence, which is in turn weaker than affine equivalence.

DEFINITION 4 (Full flags). *A sequence of d faces $F_0, F_1, \dots, F_{d-1}$ of a d-dimensional polytope P is called a full flag of P if* $\dim(F_i) = i$ *and* $F_i \subseteq F_{i+1}$ *for every* i.

Finally, the following notion of duality will be crucial in Paper II.

DEFINITION 5 (Polar polytope). *Let $P \subseteq \mathbb{R}^d$ be a polytope with the origin in its interior (so in particular P has dimension d). Define the polar of P to be the polytope*

$$P^* = \{\boldsymbol{x} \in \mathbb{R}^d \mid \forall \boldsymbol{v} \in P : \langle \boldsymbol{v}, \boldsymbol{x} \rangle \leq 1\}.$$

We will use the words "polar" and "dual" interchangeably. The face lattice of P^* is isomorphic to the reversal of the face lattice of P, so in particular we have $f_i(P^*) = f_{d-i-1}(P)$ for $i = -1, \dots, d$. The polar of the cube is called a cross-polytope, and is exactly the unit ball in the L_1 metric.

1.2.1 Combinatorially extremal polytopes

Questions about which polytopes have extremal f-vectors are very natural, and still not completely understood. It is trivial to prove by induction that when d is fixed, $\sum_i f_i$ is minimized by the simplex. Much more profound are the upper bound theorem by McMullen [43] and the lower bound theorem by Barnette [2, 3], giving optimal bounds on $f_k(P)$, when the dimension d and number of vertices $n = f_0$ are fixed. Interestingly, the upper bound is obtained by the so-called cyclic polytope $C_d(n)$ for every k simultanously. The lower bound is reached by the so-called stacked polytopes—constructed by stacking $(n - d)$ simplices on top of each other—also simultanously for every k. It should be said, however, that while the statements of these two theorems sound quite similar, their results are very different in spirit. More examples of extremal results on f-vectors are included in Ziegler's standard reference book [56].

Another set of interesting questions on polytopes is what more can be said about the combinatorics of polytopes satisfying some particular geometric constraints, in particular polytopes with certain symmetries. In this spirit lies the classical subject of regular polytopes, stemming from the ancient study of the platonic solids. A regular polytope is a polytope with a symmetry group that acts transitively on its full flags, and a great survey on this topic is Coxeter's book [12].

More modern, and less restrictive, is the study of *centrally symmetric* polytopes, by which we mean polytopes P such that $P = -P$. Many naturally occuring polytopes have centrally symmetric embeddings, but what restrictions central symmetry induces on the combinatorics of P, and in particular on the f-vector, is quite poorly understood. For example, there still does not seem to be any upper bound theorem for centrally symmetric polytopes, although there is a nice centrally symmetric analogue of the cyclic polytope [4, 5], which conjecturally should provide an upper bound, at least asymptotically.

One way in which the study of centrally symmetric polytopes is much more involved than that of generic polytopes, seems to be that there is no good centrally symmetric analogue to the simplex. A natural, recursively constructed, class of candidates are the so-called *Hanner polytopes.*

Definition 6 (Hanner polytopes). *A line segment is a Hanner polytope. A d-dimensional polytope P with $d > 1$ is a Hanner polytope if it can be written as the Cartesian product of two Hanner polytopes or as the polar of a Hanner polytope.*

A line segment has three non-empty faces, namely its two endpoints and the segment itself. The face number is preserved when taking polars, and is multiplicative when taking products. Hence any d-dimensional Hanner polytope will have 3^d non-empty faces. This is conjectured by Kalai [36] to be minimal among all centrally symmetric polytopes. Surprisingly enough, this conjecture is still open, and provides the main inspiration for Paper II.

Conjecture 3 (Kalai, 1989). *Any centrally symmetric d-dimensional polytope P has at least 3^d non-empty faces. Equality holds if and only if P is combinatorially equivalent to a Hanner polytope.*

This conjecture can be placed in the tradition of trying to determine the "least round" centrally symmetric convex body, where the notorious Mahler conjecture [42] may be the most famous.

Conjecture 4 (Mahler, 1939). *For any centrally symmetric convex body $P \subseteq \mathbb{R}^d$, let P^* be its polar body. Then $\mathrm{vol}(P) \cdot \mathrm{vol}(P^*) \geq 4^d/d!$. Equality holds if and only if P is affinely equivalent to a Hanner polytope.*

Notice that the product $\mathrm{vol}(P) \cdot \mathrm{vol}(P^*)$ is an affine invariant for centrally symmetric bodies, because scaling P by a factor λ along one axis, scales P^* by a factor λ^{-1} along the same axis. One should observe that the Mahler conjecture cannot be resolved to the positive using combinatorial methods only, since it is a statement about convex bodies in general, not necessarily polytopes. Still, some light can be shed on the Mahler conjecture by looking at particular examples of polytopes whose Mahler volume can be calculated explicitly. In fact, [20] includes some computations carried out in `polymake` [34], suggesting strong relationships between Mahler volume, face numbers and the number of full flags of the Hansen polytopes studied.

1.2.2 Independence complexes and Hansen polytopes

While Paper I studies simplicial complexes originating from graph properties, Paper II considers geometric structures constructed from particular graphs. Recall that a set $I \subseteq G$ of nodes is called independent in G, if there is no edge between two elements of I. The dual notion is that of a clique $C \subseteq G$, where every pair of nodes in C have an edge between them.

A graph G on n nodes gives an abstract simplicial complex on n vertices, whose simplices are the independent sets in G. This complex is called the *independence complex* of G, and is denoted Σ_G. It is worth observing that the 1-skeleton (i.e. the union of the 1-dimensional cells) of the independence complex of G is just the complement graph $\overline{G}$. In this respect, it would be more natural to look at the clique complex of G, whose 1-skeleton is G itself, but we stick to the independence complex for historical reasons.

For a finite abstract simplicial complex Σ, with vertex set $[n]$, one can define the dual simplicial complex

$$\overline{\Sigma} = \{\tau \subseteq [n] \mid \forall \sigma \in \Sigma : |\sigma \cap \tau| \leq 1\}.$$

For example, the dual of the independence complex of a graph is the clique complex of the same graph. It follows from the definition that $\Sigma \subseteq \overline{\overline{\Sigma}}$. The inclusion can be strict, as is seen in the following example.

Example 1. *Let* $\Sigma = \{\emptyset,\{1\},\{2\},\{3\}, \{1,2\},\{2,3\},\{3,1\}\}$ *be the complex whose geometric realisation is the empty triangle. Then* $\overline{\Sigma}$ *is the three point set, and* $\overline{\overline{\Sigma}} = \Sigma \cup \{1,2,3\}$ *is the filled triangle.*

To any finite abstract simplicial complex Σ with n vertices, Hansen [30] associates a polytope in $n + 1$ dimensions. This is constructed as

$$\mathrm{H}\,(\Sigma) = \operatorname{conv}\left\{ \pm \left(\boldsymbol{e}_0 + \sum_{i \in \sigma} \boldsymbol{e}_i \right) \mid \sigma \in \Sigma \right\}, \tag{1.1}$$

where $\{\boldsymbol{e}_i\}_{i=0}^{n}$ is the standard basis for $\mathbb{R}^{n+1}$. We see that $\mathrm{H}\,(\Sigma)$ has two vertices for every cell in Σ, and is centrally symmetric by construction. If Σ_G is the independence complex of G, we will abuse notation slightly, and write $\mathrm{H}(G)$ rather than $\mathrm{H}(\Sigma_G)$. It is elementary to see that $\mathrm{H}(K_n)$ is affinely equivalent to a cross-polytope, and $\mathrm{H}(\overline{K_n})$ is affinely equivalent to a cube, where K_n is the complete graph on n nodes.

It follows from the definition of $\overline{\Sigma}$ that, for any $C \in \overline{\Sigma}$, and any

$x \in \mathrm{H}(\Sigma)$, we have $-1 \leq -x_0 + 2\sum_{i\in C} x_i \leq 1$. It is easily seen that there are no redundancies among these inequalities, and it is natural to ask whether these conditions are sufficient, i.e. whether we have

$$\mathrm{H}(\Sigma) = \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1} \mid \forall C \in \overline{\Sigma} : -1 \leq -x_0 + 2\sum_{i\in C} x_i \leq 1 \right\}. \qquad (1.2)$$

In [30] it is proven that this is the case if and only if Σ is the independence complex of a so-called *perfect graph*.

Definition 7 (Perfect graphs). *A graph G is perfect if, for every induced subgraph $H \subseteq G$, the chromatic number χ_H equals the size of the largest clique in H.*

Equivalently, by the strong perfect graph theorem [11], G is perfect if and only if it contains no odd cycle C_{2k+1} or complement of an odd cycle $\overline{C_{2k+1}}$ as an induced subgraph, for $k \geq 2$.

A reason to study Hansen polytopes of perfect graphs is their combinatorial simplicity, which makes their face lattice relatively easy to understand. Also, they often turn out to have "few faces". More precisely, in [51], certain Hansen polytopes show up as counterexamples to the so-called B- and C-conjectures of Kalai, posed in [36]. These were stronger versions of Conjecture 3, the latter still being open.

We are now looking for possible counterexamples to Conjectures 3 and 4, and in particular to the first one, with its strong combinatorial flavour. In a centrally symmetric embedding, facets come in parallel pairs, and geometric intuition suggests that a vertex that is situated between two parallel facets would typically increase the face number. Hence it should not be a big restriction to only look at the following class of polytopes.

Definition 8 (Weakly Hanner polytopes). *A polytope is weakly Hanner if it is centrally symmetric, and each facet contains exactly half of the vertices.*

It is not hard to show that—as the names suggest—every Hanner polytope is weakly Hanner. A weakly Hanner polytope is clearly the twisted prism over any of its faces Q, meaning that

$$P \cong \operatorname{conv}(\{-1\} \times -Q, \{1\} \times Q),$$

where the equivalences are affine. Again, the intuition that a minimal polytope should not live in too many different hyperplanes suggests that we should focus on subpolytopes of a cube. So we assume $Q \subseteq C^{d-1}$, where C^d is the d-dimensional cube.

Very heuristically, "pushing Q to one corner of the cube" should add structure to the polytope, and decrease the risk of getting unnecessary faces. This means we should let Q be spanned by the indicator vectors $\{\sum_{i\in\sigma} \mathbf{e}_i \mid \sigma \in \Sigma\}$ of a simplicial complex Σ, so our twisted prism P becomes a Hansen polytope. Admittedly, this restriction from centrally symmetric 0-1–polytopes to Hansen polytopes is quite dubious, since we even lose some Hanner polytopes here. Indeed, the Cartesian product of two octahedra can be shown to not be combinatorially equivalent to a Hansen polytope. However, when looking for counterexamples to Conjecture 3, this heuristic seems to suggest a good place to start searching.

However, in [30], it is proven that (1.2) is equivalent not only to Σ being the independence complex of a perfect graph, but also to $\mathrm{H}(\Sigma)$ being a weakly Hanner polytope. So there are vague, heuristic, reasons to believe that we do not lose any counterexamples to the 3^d–conjecture in the chain of restrictions

$$\begin{aligned} \text{Centrally symmetric polytopes } &\supseteq \\ \text{0-1–polytopes } &\supseteq \\ \text{Hansen polytopes } &\supseteq \\ \mathrm{H}(G) \text{ for perfect graphs } G. & \end{aligned}$$

In Paper II, we consider Hansen polytopes of *split graphs*, which are graphs whose node sets can be partitioned into one clique and one independent set. It is easy to see that all split graphs are perfect. Moreover, computer simulations using `polymake` [34] suggest that Hansen polytopes of split graphs have remarkably few faces. However, we show that if S is a split graph on $d-1$ nodes, then $\mathrm{H}(S)$ has at least 3^d faces. Equality holds only if $\mathrm{H}(S)$ is indeed combinatorially equivalent to a Hanner polytope, and otherwise the difference is at least 16. The approach we take is to use a partitioning technique, that gives a neat combinatorial description of the number of faces of Hansen polytopes of split graphs.

We also consider the following, very natural operation on split graphs: Add a new node, and connect it to every node in the clique of S. We then get a new split graph S' (where the new node can be considered to be an element of either the clique or the independent set). We prove that $s(\mathrm{H}(S)) - 3^d = s(\mathrm{H}(S')) - 3^{d+1}$, so the number of "additional faces" is invariant under this construction.

Finally, we look at the special case where S can be obtained by applying the $S \mapsto S'$ operation repeatedly to a four-path. This graph gives exactly 16 "additional faces". In this case, some experimental results related to the Mahler conjecture are presented in [20], not included in this thesis.

1.3 Pattern containments in words

Papers III and IV concern enumerative problems related to pattern containments. Let $\pi = \pi_1 \cdots \pi_m$ and $\tau = \tau_1 \cdots \tau_\ell$ be two words over the positive integers. An *occurrence* of τ in π is a subsequence

$$1 \leq i_1 < i_2 < \cdots < i_\ell \leq m$$

such that $\pi_{i_1}, \ldots, \pi_{i_\ell}$ is order-isomorphic to τ. In such a context, τ is called a *pattern*.

A word of positive integers is *reduced* if it contains exactly the letters $1, \ldots, k$ for some k. Patterns are usually studied in the special case where π and τ are permutations, meaning that they are reduced and have no repeated letters. Patterns also behave reasonably well with respect to "permutation structure", for example an occurence of τ in π gives an occurence of τ^{-1} in π^{-1}. However, the definitions and most questions regarding pattern containment are just as naturally stated for words over a linearly ordered alphabet in general.

For a word $\tau = \tau_1 \cdots \tau_\ell$ over the totally ordered alphabet $[n] = \{1, \ldots, n\}$, we define its *reversal* $\overline{\tau}$ to be the word $\tau_\ell \cdots \tau_1$. We also define the *complement* of τ with respect to the alphabet $[n]$ to be the word $\tau^c = (n+1-\tau_1) \cdots (n+1-\tau_\ell)$. It is easy to see that an occurence of τ in π gives an occurence of $\overline{\tau}$ in $\overline{\pi}$, and of τ^c in π^c.

In [1], Babson and Steingrímsson introduced a notion of *generalized patterns*. A generalized pattern is a word τ with dashes - between some letters. An occurrence of τ in π is an occurrence of τ in π as an ordinary pattern, where the letters corresponding to τ_i and τ_{i+1} must occur immediately after each other in π, unless there is a dash between τ_i and τ_{i+1}. For example, the subsequence 243 in 2413 is an occurrence of 13-2, but is not an occurrence of 1-32. Note that among generalized patterns on permutations, there are no inverses. However, the reversal and complement operations remain, and it is still true that an occurence of τ in π gives an occurence of $\overline{\tau}$ in $\overline{\pi}$, and of τ^c in π^c.

There is a developing theory of partially ordered patterns [38], where the pattern τ is over a finite partially ordered set while π is still a word of integers, and an occurence of τ is a subsequence $1 \leq i_1 < \cdots < i_\ell \leq m$ such that the map $\tau_j \mapsto \pi_{i_j}$ is an order homomorphism. This theory has so far mainly been a tool to study ordinary pattern containment in permutations.

1.3.1 Packing patterns in words

Much of the work on permutation patterns has been concerned with fixing a pattern τ, and studying the distribution of τ on permutations. By this we mean the numbers $N(\tau,m,k)$ of m letter permutations with k occurences of τ, where k and m are parameters. Usually the case of pattern-avoiding permutations, i.e. the case $k = 0$, is much better understood than the situation for $k > 0$. A nice account of some of the results on pattern-avoiding permutations are given in [7].

The work in Paper III is in another direction, following [8]: Fix a pattern τ, and study the maximal number $\mu(\tau,n)$ of times that τ can occur in a word π, when $|\pi|$ is given. The norm $|\pi|$ can be measured in different ways, the most straightforward norm being the length of the word π, which is the situation studied in [8]. It is then fairly easy to see that $\mu(\tau,n)$ scales like $\binom{n}{\ell}$, where ℓ is the number of maximal dash-free subwords in τ, or simply the length of τ when τ is a classical pattern. Indeed, there are at most $\binom{n}{\ell}$ substrings of an n letter word that can appear as an occurence on τ. On the other hand, if we let π^k be the concatenation of k copies of π, then τ occurs at least $\binom{k}{\ell}$ times as often in π_k as in π. Since the size $|\pi^k|$ grows linearly with k, this proves that the number of occurences grows at least linearly in $\binom{n}{\ell}$. With a little more work, one sees that $\mu(\tau,n)/\binom{n}{\ell}$ actually approaches a limit as $n \to \infty$. We call this limit the *packing density* of τ, and denote it by $\delta(\tau)$. In [8], $\delta(\tau)$ is calculated for certain classes of patterns.

This can be generalized to assigning different weights to different letters, and it may be interesting to study how the packing density changes when adjusting these weights. The same argument as before shows that $\lim_{n\to\infty} \mu(\tau,n)/\binom{n}{\ell}$ exists also in this case. An interesting special case is assigning weight i to the letter i, so the size of π is $n = \|\pi\| = \sum_i \pi_i$. Looking at words (over $\mathbb{Z}_+$) of fixed norm n is thus equivalent to looking at integer compositions of n. In Paper III, we determine the packing densities into compositions of all patterns of length 3, and prove some more general results for patterns of special kinds.

1.3.2 Fixed points and descending runs

Among the most elementary patterns are descents 21 and inversions 2-1. Their distributions on permutations are well known. Indeed, $N(21, m, k)$ are the so-called Eulerian numbers, while $N(2\text{-}1, m, k)$ are called the Mahonian numbers, and while they have no simple closed form, recursive formulas are easy to derive. In Paper IV, we take a closer look at the descent statistic and study its joint distribution with the fixed point statistic.

Specifically, decompose $[n]$ in blocks of length a_i, with $a_1+\cdots+a_k = n$, and consider the set $\mathfrak{S}_{\boldsymbol{a}} = \mathfrak{S}_{(a_1,\ldots,a_k)}$ of permutations that descend within each of these blocks. It is clear that $|\mathfrak{S}_{\boldsymbol{a}}| = \binom{n}{a_1,\ldots,a_k}$, since a permutation in $\mathfrak{S}_{\boldsymbol{a}}$ is determined by which letters go in which block. For example, the 6 permutations in $\mathfrak{S}_{(2,2)}$ are

$$21|43,\ 31|42,\ 41|32,\ 32|41,\ 42|31 \text{ and } 43|21.$$

With this notation, the symmetric group $\mathfrak{S}_n$ is just $\mathfrak{S}_{(1,\ldots,1)}$, and should not be confused with $\mathfrak{S}_{(n)}$ (which contains only the strictly decreasing permutation).

Paper IV enumerates the *derangements* in $\mathfrak{S}_{\boldsymbol{a}}$, i.e. the permutations with no fixed points. We denote the set of derangements in $\mathfrak{S}_{\boldsymbol{a}}$ by $D_{\boldsymbol{a}}$. For example, we have

$$D_{(2,2)} = \{21|43, 31|42, 43|21\}.$$

It is well known, and easy to prove by an inclusion-exclusion argument, that the number of derangements in $\mathfrak{S}_n$ is the integer closest to $n!/\mathrm{e}$, for every $n \geq 1$.

If having descents in specified subsets of the positions and being fixed-point free were almost independent events, we would hence have

$$|D_{\boldsymbol{a}}| \cdot \prod_i a_i! \approx \frac{n!}{\mathrm{e}},$$

with the squig $\approx$ interpreted properly. However, the events can be pretty far from independent. To see this, consider again the one block composition $\boldsymbol{a} = (n)$. There is only one permutation in $\mathfrak{S}_{(n)}$, and this one is a derangement if and only if n is even.

For fixed k, the generating function of $|D_{\boldsymbol{a}}|$ is

$$\sum_{\boldsymbol{a}} |D_{\boldsymbol{a}}| x_1^{a_1} \cdots x_k^{a_k} = \frac{1}{(1+x_1)\cdots(1+x_k)(1-x_1-\cdots-x_k)}, \quad (1.3)$$

where the sum is taken over all compositions $\boldsymbol{a}$ with at most k blocks, allowing some of the blocks to be empty. The generating function was first given by Han and Xin in [28], using symmetric function methods. We reprove their theorem with a simple recursive method.

From the generating function, we derive a closed formula for $D_{\boldsymbol{a}}$, namely

$$|D_{\boldsymbol{a}}| = \frac{1}{\prod_i a_i!} \sum_{\boldsymbol{0} \leq \boldsymbol{b} \leq \boldsymbol{a}} (-1)^{\sum b_j} \left(n - \sum b_j\right)! \prod_i \binom{a_i}{b_i} b_i!. \quad (1.4)$$

Before stating further results of Paper IV, we need to introduce the number of *fixed point λ-coloured permutations* to be

$$f_\lambda(m) = \sum_{\pi \in \mathfrak{S}_m} \lambda^{\mathrm{fix}(\pi)},$$

where $\mathrm{fix}(\pi)$ is the number of fixed points in π, and we use the convention

$0^0 = 1$. It follows directly that $f_1(m) = m!$, and that $f_0(m) = |D_{(1,\ldots,1)}|$ is just the number of derangements of $[m]$. In fact, $f_\lambda(m)$ is just the special case $\alpha = 1$, $u = \lambda - 1$ of the Charlier polynomials [25]

$$C_n(\alpha,u) = \sum_k \binom{n}{k} \alpha(\alpha+1)\cdots(\alpha+k-1)u^{n-k}.$$

A key property of the numbers $f_\lambda(n)$ is that

$$\frac{d}{d\lambda} f_\lambda(n) = n f_\lambda(n-1). \tag{1.5}$$

Since the publication of Paper IV, the numbers $f_\lambda(n)$ and the differentiation rule 1.5 have been successfully applied by Sun and Zhuang [54] to give a unifying proof of Riordans formula [48] on tree enumeration, and other related identities. Riordan's formula says that

$$\sum_k \binom{n}{k} (k+1)!(n+1)^{n-k} = (n+1)^{n+1}, \tag{1.6}$$

and was generalized in [54] to

$$\sum_k \binom{n}{k} f_\lambda(k+1)(n+1)^{n-k} = (n+\lambda)^{n+1}. \tag{1.7}$$

Similarly, (1.4) still holds if we replace every occurence of $m!$ by $f_\lambda(m)$ in the summation, for any λ. The independence of λ in (1.4) is proven in two ways in Paper IV, by differentiation and by using bijections. The most interesting application of this independence is writing

$$|D_{\boldsymbol{a}}| = \frac{1}{\prod_i a_i!} \sum_{\boldsymbol{0} \le \boldsymbol{b} \le \boldsymbol{a}} (-1)^{\sum b_j} f_0\left(n - \sum b_j\right) \prod_i \binom{a_i}{b_i} f_0(b_i). \tag{1.8}$$

Noting that $f_0(1) = 0$, we reduce the number of non-zero terms in the summation quite substantially.

Finally, we look back at the question of dependence between the events of being in $\mathfrak{S}_a$ and being a derangement. We prove that, except in the trivial case where $\boldsymbol{a} = (n)$ and n is odd, the two events are always positively correlated. In other words, we always have

$$|D_{\boldsymbol{a}}| \geq \frac{1}{\prod_i a_i!} f_0(n).$$

Noting that the right hand side corresponds to the term with $\boldsymbol{b} = \mathbf{0}$ in (1.8), we consider all other terms as "correlation terms", and conclude that their sum is negative.

This correlation result is actually proven via a stronger monotonicity result. Consider $h(\boldsymbol{a}) = |D_{\boldsymbol{a}}| \prod_i a_i!$, which is the number of permutations in $\mathfrak{S}_n$, that become derangements when sorted decreasingly within the blocks. Suppose that $\boldsymbol{a}'$ is not a single block of odd length, and is constructed from $\boldsymbol{a}$ by moving a position from a smaller block to a larger one. Then we prove that $h(\boldsymbol{a}') \geq h(\boldsymbol{a})$. In particular, this implies that h is monotone with respect to the natural "containment order" on compositions of n.

Since the publication of Paper IV, there has been some further progress in the same direction. In [53], Steinhardt studies the more general concept of $(\boldsymbol{a}, S)$-permutations. As before, $\boldsymbol{a} = (a_1, \ldots, a_k)$ is a composition of n, and we let $S \subseteq [k]$ be a subset of the blocks. An $(\boldsymbol{a}, S)$-permutation is a permutation that descends in the blocks indexed by S, and that ascends within each of the other blocks. With this notation, $\mathfrak{S}_{\boldsymbol{a}}$ is just the set of $(\boldsymbol{a}, [k])$-permutations.

In [53], a bijection that goes back to [24] is used, to study $(\boldsymbol{a}, S)$-permutations according to their cycle structure. In particular, it is shown combinatorially that for any conjugacy class $\mathcal{C}$, and any permutation $\sigma \in \mathfrak{S}_k$, the $(a_1, \ldots, a_k, S)$-permutations in $\mathcal{C}$ are in bijection with the $(a_{\sigma(1)}, \ldots, a_{\sigma(k)}, \sigma^{-1}(S))$-permutations in $\mathcal{C}$. Since the class of derangements is just the union of all conjugacy classes with no 1-cycle, this answers the open Problem 3 posed in Paper IV, which asked for a direct combinatorial explanation of why the numbers $D_{\boldsymbol{a}}$ are invariant under permutations of the blocks in $\boldsymbol{a}$.

The fact that the function

$$\frac{1}{\prod_i a_i!} \sum_{\mathbf{0} \leq \boldsymbol{b} \leq \boldsymbol{a}} (-1)^{\sum b_j} f_\lambda \left(n - \sum b_j\right) \prod_i \binom{a_i}{b_i} f_\lambda(b_i)$$

is constant in λ is also given a neat combinatorial proof in [53]. Moreover, the generating function from Paper IV, and the closed formula that follows from it, are generalized to the case of $(\boldsymbol{a}, S)$-derangements. Indeed, it is shown that the generating function for $(\boldsymbol{a}, S)$-derangements is

$$\frac{\prod_{i \notin S}(1 - x_i)}{(1 - x_1 - \cdots - x_k) \prod_{i \in S}(1 + x_i)}.$$

This result answers our open Problem 4 from paper IV, and is proven in two ways by Steinhardt, one of which is completely analogous with our proof of the case $S = [k]$.

The bijection from [24] is also recycled to prove the following enumeration of $(\boldsymbol{a}, S)$-derangements: Let $c_{(\boldsymbol{a},S)}(\pi) = 0$ if π has any odd length cycle contained in a block of $\boldsymbol{a}$, or if it has *any* cycle contained in an *ascending* block of $\boldsymbol{a}$. Otherwise, let $c_{(\boldsymbol{a},S)}(\pi) = 2^m$, where m is the number of (even length) cycles contained in a (descending) block of $\boldsymbol{a}$. Then the number of $(\boldsymbol{a}, S)$-derangements is

$$\frac{1}{\prod a_i!} \sum_{\pi \in \mathfrak{S}_n} c_{(\boldsymbol{a},S)}(\pi).$$

Several results of Steinhardt's paper were also proven independently by Kim and Seo [37] and by Elizalde [13].

1.4 Optimal stopping on finite posets

1.4.1 The secretary problem

The secretary problem is a famous, motivating example for much of optimal stopping theory. It is also known under several other names, such as the marriage problem, the googol game and the best choice problem. A nice survey of the problem and related results is [15]. The formulation of the problem that motivates its name is the following. There are n candidates for a job being interviewed one after the other, in uniform random order. After each interview, the selector must decide whether or not to accept the present candidate, based only on the relative rankings of the candidates she has seen so far. The selector's task is to maximize the probability of hiring the very best candidate. Selecting any other candidate than the best one, or altogether failing to hire one, are considered failures.

It is well known that there is a strategy that succeeds with probability approaching $1/\mathrm{e}$ from above as n grows, and that this strategy is optimal. Proving this relies on the observation that an optimal strategy must have the form that you discard all the first $k = k(n)$ candidates regardless of their relative rankings, and after this threshold time you accept the first candidate that is better than everyone you have seen before. This is a (very simple) special case of the monotone case theorem [10], and once this observation has been made, simple calculations yield that the success probability is maximized when $k(n) = \lfloor n/\mathrm{e} \rfloor$, and is approximately $1/\mathrm{e}$.

Actually, calculations are further simplified if we, rather than letting n be known to the selector, assume that the candidates arrive at independent uniform random times in [0,1]. This formulation, which we will use henceforth, is clearly at least as difficult for the selector as the original problem, because if n were known the selector could produce n random times in [0,1] herself, and then follow her strategy from the continuous time case.

Since the problem first appeared, it has seen many generalizations and alternative versions. Some quite natural versions appear when the applicants do not only possess relative rankings, but have some absolute real valued ranking, drawn from a distribution that may or may not be known to the selector. The problem where the selector has full information about the distribution was treated in [26], where it is proven that the success rate decreases to a limit $\alpha \approx .58$ as $n \to \infty$. An interesting result by Petruccelli [46] shows that if the selector knows that the distribution is normal with variance 1, then the asymptotic success rate is still α, so knowing the mean gives no advantage asymptotically. In such versions, we may also want to maximize not only the probability of selecting the best candidate, but rather the expected ranking of the selected candidate. Other versions include letting the objective be to select one among the r best candidates for some $r > 1$, or to punish selection of a suboptimal candidate less harshly than failure to hire. A selection of such variations of the secretary problem were treated in [14].

1.4.2 The partially ordered secretary problem

Several generalizations and versions of the secretary problem ask for stategies for the selector, when the applicants are not necessarily linearly ordered, but are rather elements of some partially ordered set (poset). The selector then succeeds if the element she selects is a maximal element in P. We will denote the set of maximal elements in P by $\max(P)$. The *rank* $\mathrm{rk}(x)$ of an element $x \in P$ is the maximum length of a chain $x_n < x_{n-1} < \cdots < x_1 = x$ in P, and the *height* of P is defined to be

$$h(P) = \max\{\mathrm{rk}(x) : x \in P\}.$$

Analogously, the *width* of a poset is the largest size of a set of pairwise unrelated elements in P.

The secretary problem has been solved for particular posets, such as binary trees [44], stacks of antichains [21] and disjoint unions of chains with restrictions on the order in which the candidates appear [27, 40].

The very interesting problem on whether there exists a strategy that wins with probability bounded away from zero for any possible poset remained open for many years. Finally, in [47] Preater constructed a strategy that solves the problem even when the poset is not known to the selector. The strategy is the following:

STRATEGY 1 (Preater). *Wait until time* $1/2$ *without accepting any of the elements seen. Denote by* Y *the induced poset formed by the elements that have arrived by time* $1/2$*. Tag all the elements of rank* $h(Y)$ *in* Y*, and with probability* $1/2$ *tag also the elements of rank* $h(Y) - 1$ *in* Y*. Finally, accept the first candidate that dominates a tagged element, when it appears.*

Preater [47] proved that this strategy succeeds with probability at least 1/8 for any poset. From this point, the most interesting question about the partially ordered secretary problem was whether one can universally win with the same probability 1/e as one can if the poset is known to be linear. It might be natural to think so, since the linear order extends all other orders, and has the property (which is unfortunate for the selector) that all its k-element subposets are isomorphic. However, extracting a strategy that works for all posets from the strategy in the linear order case, turned out to be quite difficult.

In [23], Georgiou et al improved the analysis of Preater's strategy to prove that it actually wins with probability at least 1/4 for any poset. There are two parameters—the waiting time and the probability of tagging elements of second-to-maximal rank—with which Preater's strategy can be modified. However, by considering the behaviour on simple classes of posets, it can be shown that such modifications can never give a universal success probability of 1/e [32].

Kozik [39] constructed a quite different strategy, based on a sequence of threshold times, and proved that his strategy succeeds for any poset with at least a probability that is strictly greater than 1/4. The proof is quite technical, by division into cases. There is to our knowledge no better analysis of his strategy, but Kozik himself says that "although the improvement might be considerable, we do not think it is possible to approach the value e^{-1} in this way" [39]. Kozik's strategy is as follows:

STRATEGY 2 (Kozik). *If an element arrives at time t, and is one of m maximal elements in the induced poset seen at the time, accept it if $t > \mathrm{e}^{-1/m}$.*

1.4.3 The solution for general partial orders

In Paper V, we present the first strategy that does not depend on the poset, and has a success rate at least 1/e for any poset. Here, we will describe the strategy slightly differently from Paper V, to hint at further open questions. The strategy crucially depends on the *greedy maximum* distribution $\mu_{\mathbf{Gr}(P)}$, defined as the distribution of the random variable $\mathbf{Gr}(P) \in P$, constructed recursively for all posets P as follows: If P has only one element, then this element is $\mathbf{Gr}(P)$. Otherwise, first select x uniformly at random in P. If x is maximal in P, then $\mathbf{Gr}(P) = x$. Otherwise, select $\mathbf{Gr}(P)$ according to the distribution $\mu_{\mathbf{Gr}((x,\infty))}$, where (x,∞) is the order ideal $\{y \in P : y > x\}$.

We are now ready to define our strategy from Paper V, which we prove to succeed with probability at least 1/e for any poset P.

STRATEGY 3 (Freij, Wästlund). *Wait until time* 1/e *without accepting any of the elements seen. If an element x arrives at a time after* 1/e, *denote by P_x the induced poset seen when x arrives. Accept x with probability $\mu_{\mathbf{Gr}(P_x)}(x)$.*

For any fixed $x \in \max(P)$, and any $t \in [0,1]$, let $P_{t,x}$ be the random subposet of P that contains x, and for all other elements $y \in P \smallsetminus \{x\}$ contains y with probability t, independently for different y. We can think of this as the induced poset as seen at time t, conditioned on x arriving at time t. It is interesting that the only property that we need from the class of distributions $\{\mu_{\mathbf{Gr}(P)}\}_P$, for Strategy 3 is the following inequality for every $x \in \max(P)$ and $t \in [0,1]$:

$$\mathbb{E}[\mu_{\mathbf{Gr}(P_{t,x})}(x)] \geq \mu_{\mathbf{Gr}(P)}(x). \tag{1.9}$$

The expectation on the left hand side is obviously taken over the random subposet $P_{t,x}$. This inequality follows quite directly from the construction of $\{\mu_{\mathbf{Gr}(P)}\}_P$, and is proven in Paper V. It is natural to ask for what other classes of distributions on maximal elemets of posets, the inequality (1.9) holds.

It is easily seen that (1.9) does not hold for the uniform distribution on maximal elements. Indeed, trying Strategy 3 with $\mu_{\mathbf{Gr}(P)}$ replaced by uniform distribution fails for the following reason. Let $P_{k,n}$ be the poset with maximal elements $x_1, \dots, x_n$, and non-maximal elements $\{y_{i,j}\}_{1 \leq i \leq n}^{1 \leq j \leq k}$, and relations $x_i > y_{i,j}$ for every i, j. Then, if n is fixed and $k \to \infty$, the probability of accepting any x_i other than the last one to arrive will be negligible. On the other hand, the probability of accepting the last x_i will be exactly $1/n$. So this modified strategy does not even give success rates bounded away from zero.

On the other hand, let $\mu_{\mathbf{Lin}(P)}(x)$ be the probability that x is the unique maximal element of a uniformly chosen linear extension of P. We conjecture that (1.9) still holds with $\mu_{\mathbf{Gr}(P)}$ replaced by $\mu_{\mathbf{Lin}(P)}$. This does not seem to follow from any known correlation inequalities for the uniform distribution on linear extensions, such as the ones in [16]. It may be helpful to understand this inequality from the viewpoint of correlations between coordinates in the 0-1–polytope associated to P, perhaps using geometric methods similar to those in [52]. The 0-1–polytope associated to a poset P with elements $p_1, \dots, p_n$ is the n-dimensional polytope

$$K_P := \{\boldsymbol{x} \in \mathbb{R}^n \mid 0 \leq x_i \leq 1\,,\, x_i \leq x_j \text{ whenever } p_i < p_j \text{ in } P\}\,.$$

Note that the polytope associated to a linear order is a simplex, and that $K_Q \subseteq K_P$ if Q is an extension of P.

Independently of (but chronologically after) our results in Paper V, Garrod and Morris contributed to the partially ordered secretary problem in [22]. They present, for every poset, a strategy that succeeds with probability at least 1/e. Their result is significantly weaker than ours in the sense that the strategy depends on the poset. However, for special classes of posets their strategy performs better than strategy 3.

STRATEGY 4 (Garrod, Morris). *Assume that we know that P has m maximal elements. If $m = 1$, let $\tau = 1/\mathrm{e}$, and if $m > 1$, let $\tau = m^{-(m-1)/2}$. Wait until time τ without accepting any of the elements seen. If an element x arrives at a time after τ, denote by P_x the induced poset seen when x arrives. Accept x if $x \in \max(P_x)$ and $|\max(P_x)| \leq m$.*

For any poset, Strategy 4 has success rate at least 1/e, and if P has $m > 1$ maximal elements and width m, the success rate is at least $m^{-(m-1)/2}$. It is conjectured that the result holds even if the width assumption is ignored. Strategy 3 is not analyzed in depth for special classes of posets, but it is safe to say that Garrod's and Morris' strategy performs better for a known number $m > 1$ of maximal elements. However, the problem it solves is less general than the one considered in Paper V.

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