

RESEARCH

Open Access

A note on a paper of Harris concerning the asymptotic approximation to the eigenvalues of $-y'' + qy = \lambda y$, with boundary conditions of general form

Mahdi Hormozi

Correspondence:
hormozi@chalmers.se
Department of Mathematical
Sciences, Division of Mathematics,
Chalmers University of Technology
and University of Gothenburg,
Gothenburg 41296, Sweden

Abstract

In this article, we derive an asymptotic approximation to the eigenvalues of the linear differential equation

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (a, b)$$

with boundary conditions of general form, when q is a measurable function which has a singularity in (a, b) and which is integrable on subsets of (a, b) which exclude the singularity.

Mathematics Subject Classification 2000: Primary, 41A05; 34B05; Secondary, 94A20.

Keywords: Sturm-Liouville equation, boundary condition, Prüfer transformation.

1. Introduction

Consider the linear differential equation

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (a, b), \quad (1.1)$$

where λ is a real parameter and q is real-valued function which has a singularity in (a, b) . According to [1], an eigenvalue problem may be associated with (1.1) by imposing the boundary conditions

$$y(a) \cos \alpha - y'(a) \sin \alpha = 0, \quad \alpha \in [0, \pi), \quad (1.2)$$

$$y(b) \cos \beta - y'(b) \sin \beta = 0, \quad \beta \in [0, \pi). \quad (1.3)$$

In [2], Atkinson obtained an asymptotic approximation of eigenvalues where y satisfies Dirichlet and Neumann boundary conditions in (1.1). Here, we find asymptotic approximation of eigenvalues for all boundary condition of the forms (1.2) and (1.3). To achieve this, we transform (1.1) to a differential equation all of whose coefficients belong to $L_1[a, b]$. Then we employ a Prüfer transformation to obtain an approximation of the eigenvalues. In this way, many basic properties of singular problems can be inferred from the corresponding regular ones. In [3], Harris derived an asymptotic approximation to the eigenvalues of the differential Equation (1.1), defined on the

interval $[a, b]$, with boundary conditions of general form. But, he demands the condition, $q \in L^1[a, b]$. Atkinson and Harris found asymptotic formulae for the eigenvalues of spectral problems associated with linear differential equations of the form (1.1), where $q(x)$ has a singularity of the form αx^{-k} with $1 \leq k < \frac{4}{3}$ and $1 \leq k < \frac{3}{2}$ in [2,4] respectively. Harris and Race [5] generalized those results for the case $1 \leq k < 2$. In [6], Harris and Marzano derived asymptotic estimates for the eigenvalues of (1.1) on $[0, a]$ with periodic and semi-periodic boundary conditions. The reader can find the related results in [7-10]. We consider $q(x) = Cx^{-K}$ where $1 \leq K < 2$ and an asymptotic approximation to the eigenvalues of (1.1) with boundary conditions of general form. Our technique in this article follows closely the technique used in [2-5]. Let $U = [a, 0) \cup (0, b]$ and $q \in L_{1,Loc}(U)$. As Harris did in [[5], p. 90], suppose that there exists some real function f on $[a, 0) \cup (0, b]$ in $AC_{Loc}([a, 0) \cup (0, b])$ which regularizes (1.1) in the following sense. For f which can be chosen in Section 2, define quasi-derivatives, $y^{[i]}$ as follows:

$$y^{[0]} := y, \quad y^{[1]} := y' + fy,$$

y is a solution of (1.1) with boundary conditions (1.2) and (1.3) if and only if

$$\begin{pmatrix} y^{[0]} \\ y^{[1]} \end{pmatrix} = \begin{pmatrix} -f & 1 \\ f' + q - f^2 - \lambda f \end{pmatrix} \begin{pmatrix} y^{[0]} \\ y^{[1]} \end{pmatrix} \tag{1.4}$$

The object of the regularization process is to choose f in such way that

$$f \in L^1(a, b) \quad \text{and} \quad -F := q - f^2 + f' \in L^1(a, b). \tag{1.5}$$

Having rewritten (1.1) as the system (1.4), we observe that, for any solution y of (1.1) with $\lambda > 0$, according to [2,4], we can define a function $\theta \in AC(a, b)$ by

$$\tan \theta = \frac{\lambda^{\frac{1}{2}} y}{y^{[1]}}.$$

When $y^{[1]} = 0$, θ is defined by continuity [[5], p. 91]. It makes sense to mention that one can find full discussions and nice examples about the choice of f in [2,4,5]. Atkinson in [2] noticed that the function θ satisfies the differential equation

$$\theta' = \lambda^{\frac{1}{2}} - f \sin(2\theta) + \lambda^{-\frac{1}{2}} F \sin^2(\theta). \tag{1.6}$$

Let $\lambda > 0$ and the n -th eigenvalue λ_n of (1.1-1.3), then according to [[1], Theorem 2], Dirichlet and non-Dirichlet boundary conditions can be described as bellow:

$$\left\{ \begin{array}{l} \text{in Case 1 } (\alpha = 0, \beta = 0) : \quad \theta(b, \lambda) - \theta(a, \lambda) = (n + 1)\pi; \\ \text{in Case 2 } (\alpha = 0, \beta \neq 0) : \quad \theta(b, \lambda) - \theta(a, \lambda) = (n + \frac{1}{2})\pi - \lambda^{-\frac{1}{2}} \cot \beta + O\left(\lambda^{-\frac{3}{2}}\right); \\ \text{in Case 3 } (\alpha \neq 0, \beta = 0) : \quad \theta(b, \lambda) - \theta(a, \lambda) = (n + \frac{1}{2})\pi + \lambda^{-\frac{1}{2}} \cot \alpha + O\left(\lambda^{-\frac{3}{2}}\right); \\ \text{in Case 4 } (\alpha \neq 0, \beta \neq 0) : \quad \theta(b, \lambda) - \theta(a, \lambda) = n\pi + \lambda^{-\frac{1}{2}} (\cot \alpha - \cot \beta) + O\left(\lambda^{-\frac{3}{2}}\right). \end{array} \right.$$

It follows from (1.5-1.6) that large positive eigenvalues of either the Dirichlet or non-Dirichlet problems over $[a, b]$ satisfy

$$\lambda^{\frac{1}{2}} = \frac{\theta(b) - \theta(a)}{(b - a)} + O(1). \tag{1.7}$$

Our aim here is to obtain a formula like (1.7) in which the $O(1)$ term is replaced by an integral term plus and error term of smaller order. We obtain an error term of $o\left(\lambda^{-\frac{N}{2}}\right)$ ($N \geq 1$). To achieve this we first use the differential Equation (1.6) to obtain estimates for $\theta(b) - \theta(a)$ for general λ as $\lambda \rightarrow \infty$.

2. Statement of result

We define a sequence $\xi_j(t)$ for $j = 1, \dots, N + 1, t \in [a, b]$ by

$$\begin{aligned} \xi_1(t) &:= \left| \int_0^t |f(s)| + |F(s)| ds \right| \\ \xi_j(t) &:= \left| \int_0^t (|f(s)| + |F(s)|) \xi_{j-1}(s) ds \right| \end{aligned} \tag{2.1}$$

and note that in view of $f, F \in L(a, b)$,

$$\xi_j(t) \leq c \xi_{j-1}(t) \quad \text{for } t \in [a, b], \quad 2 \leq j \leq N + 1 \tag{2.2}$$

Suppose that for some $N \geq 1$,

$$\begin{aligned} f' \xi_{N+1}, \quad f^2 \xi_N, \quad fF \xi_N &\in L[a, b]; \\ f(t) \xi_{N+1}(t) &\rightarrow 0 \quad \text{as } t \rightarrow 0. \end{aligned} \tag{2.3}$$

We define a sequence of approximating functions a

$$\theta_0(x) := \theta(a) + \lambda^{\frac{1}{2}}(x - a); \tag{2.4}$$

$$\theta_j(0) := \theta(0); \tag{2.5}$$

$$\theta_{j+1}(x) := \theta(a) + \lambda^{\frac{1}{2}}(x - a) - \int_a^x f \sin(2\theta_j(t)) dt + \lambda^{-\frac{1}{2}} \int_a^x F \sin^2(\theta_j(t)) dt. \tag{2.6}$$

for $j = 0, 1, 2, \dots$ and for $a \leq x \leq b$. We measure the closeness of the approximation in the next result. Thus

$$\theta'_{j+1} = \lambda^{\frac{1}{2}} - f \sin(2\theta_j) + \lambda^{-\frac{1}{2}} F \sin^2(\theta_j) \tag{2.7}$$

The following lemma appears in [2,5].

Lemma 2.1. *If $g \in L^1$ then for any j and $a \leq x \leq b$*

$$\int_a^x g(t) \sin(2\theta_j(t)) dt = o(1)$$

as $\lambda \rightarrow \infty$.

By using Lemmas 5.1 and 5.2 of [5] we conclude the following lemma

Lemma 2.2. *There exists a suitable constant C such that*

$$|a_{j+1} - \theta_j| \leq C \sup_{a \leq x \leq b} |\theta - \theta_j| \xi_{j+1}(x)$$

Now, we prove an elementary lemma.

Lemma 2.3. *If $g \in L^1$ and $\theta(x) - \theta_j(x) = \lambda^{-\frac{1}{2}} \int_a^x g\{\sin^2(\theta(t)) - \sin^2(\theta_j(t))\}dt$ then*

$$|\theta(x) - \theta_{j+1}(x)| \leq \lambda^{-\frac{1}{2}} \sup_{a \leq x \leq b} |\theta(x) - \theta_j(x)| \int_a^x g dt$$

Proof.

$$\begin{aligned} \theta(x) - \theta_{j+1}(x) &= \lambda^{-\frac{1}{2}} \int_a^x g\{\sin^2(\theta(t)) - \sin^2(\theta_j(t))\}dt \\ &= \frac{1}{2} \lambda^{-\frac{1}{2}} \int_a^x g\{\cos(2\theta_j(t)) - \cos(2\theta(t))\}dt \\ &= -\lambda^{-\frac{1}{2}} \int_a^x g \sin(\theta_j(t) - \theta(t)) \sin(\theta_j(t) + \theta(t)) dt \\ &\leq \lambda^{-\frac{1}{2}} \sup_{a \leq x \leq b} |\theta(x) - \theta_j(x)| \int_a^x g dt \end{aligned}$$

Remark 2.4. *Lemma 2.2 shows that if $|\theta(x) - \theta_j(x)| = o\left(\lambda^{-\frac{j}{2}}\right)$ then*

$$|\theta(x) - \theta_{j+1}(x)| = o\left(\lambda^{-\frac{(j+1)}{2}}\right)$$

Lemma 2.5. *There exists a suitable constant C such that*

$$\int_a^x f(\sin(2\theta_j(t)) - \sin(2\theta(t))) dt \leq C \lambda^{-\frac{1}{2}} \sup_{a \leq x \leq b} |\theta(x) - \theta_j(x)|, \quad x \in (a, b),$$

Proof.

$$\begin{aligned} \int_a^x f(\sin(2\theta(t))(1) - \sin(2\theta_j(t))(1)) dt &= \lambda^{-\frac{1}{2}} \int_a^x f\{\sin(2\theta)\theta' - \sin(2\theta_j)\theta'_j\} dt \\ &\quad + \lambda^{-\frac{1}{2}} \int_a^x f^2\{\sin^2(2\theta) - \sin(2\theta_j)\sin(2\theta_{j-1})\} dt \\ &\quad - \lambda^{-1} \int_a^x fF\{\sin(2\theta)\sin^2(\theta) - \sin(2\theta_j)\sin^2(\theta_{j-1})\} dt \\ &=: I_1 + I_2 - I_3. \end{aligned}$$

But

$$I_1 = \lambda^{-\frac{1}{2}} [f(t)(\sin^2(\theta(t)) - \sin^2(\theta_j(t)))]_a^x - \lambda^{-\frac{1}{2}} \int_a^x f'(t)\{\sin^2(\theta) - \sin^2(\theta_j)\} dt$$

By using Lemma 2.1 we have

$$I_1 \leq C_1 \lambda^{-\frac{1}{2}} \sup_{a \leq x \leq b} |\theta(x) - \theta_j(x)|.$$

Applying Lemmas 2.1 and 2.2 we have

$$\begin{aligned} I_2 &:= \lambda^{-\frac{1}{2}} \int_a^x f^2(t) \{\sin(2\theta) - \sin(2\theta_j)\} \sin(2\theta) dt \\ &\quad + \lambda^{-\frac{1}{2}} \int_a^x f^2(t) \{\sin(2\theta) - \sin(2\theta_j)\} \sin(2\theta_j) dt \\ &\quad + \lambda^{-\frac{1}{2}} \int_a^x f^2(t) \{\sin(2\theta_j) - \sin(2\theta_{j-1})\} \sin(2\theta_j) dt \\ &\leq C_2 \lambda^{-\frac{1}{2}} \sup_{a \leq x \leq b} |\theta(x) - \theta_j(x)| \end{aligned}$$

Finally, using Lemma 2.1, we conclude

$$\begin{aligned} I_3 &:= \lambda^{-1} \int_a^x fF \{\sin(2\theta) - \sin(2\theta_j)\} \sin^2(\theta) dt \\ &\quad + \lambda^{-1} \int_a^x fF (\sin(\theta) - \sin(\theta_{j-1})) (\sin(\theta) + \sin(\theta_{j-1})) \sin(2\theta_j) dt \\ &\leq C_3 \lambda^{-\frac{1}{2}} \sup_{a \leq x \leq b} |\theta(x) - \theta_j(x)| \end{aligned}$$

This ends the proof of Lemma 2.5.

Theorem 2.6. *Suppose that (2.3) hold for some positive integer N , then*

$$\theta(b) - \theta(a) - (b-a)\lambda^{\frac{1}{2}} = - \int_a^b f \sin(2\theta_N(x)) dx + \left(\lambda^{-\frac{1}{2}}\right) \int_a^b F \sin^2(\theta_N) dx + o\left(\lambda^{-\frac{N}{2}}\right)$$

as $\lambda \rightarrow \infty$.

Proof. We integrate (1.5) over $[a, x]$ and obtain

$$\theta(x) - \theta(a) = \lambda^{\frac{1}{2}}(x-a) - \int_a^x f \sin(2\theta(t)) dt + \lambda^{-\frac{1}{2}} \int_a^x F \sin^2(\theta(t)) dt$$

In particular

$$\theta(b) - \theta(a) = \lambda^{\frac{1}{2}}(b-a) - \int_a^b f \sin(2\theta(t)) dt + \lambda^{-\frac{1}{2}} \int_a^b F \sin^2(\theta(t)) dt$$

and so,

$$\begin{aligned} \theta(b) - \theta(a) - (b - a)\lambda^{\frac{1}{2}} &= - \int_a^b f \sin(2\theta_N(x)) dx + \left(\lambda^{\frac{-1}{2}}\right) \int_a^b F \sin^2(\theta_N) dx \\ &\quad + \int_a^b f \{\sin(2\theta_N(x)) - \sin(2\theta(x))\} dx \\ &\quad + \left(\lambda^{\frac{-1}{2}}\right) \int_a^b F \{\sin^2(\theta) - \sin^2(\theta_N)\} dx. \end{aligned}$$

We need to prove that two last terms are $o\left(\lambda^{\frac{-N}{2}}\right)$ as $\lambda \rightarrow \infty$. Applying Lemmas 2.2 and 2.4 we have

$$\begin{aligned} I &:= \int_a^b f(x) \{\sin(2\theta_N(x)) - \sin(2\theta(x))\} dx + \left(\lambda^{\frac{-1}{2}}\right) \int_a^b F(x) \{\sin^2(\theta) - \sin^2(\theta_N)\} dx \\ &\leq C\lambda^{-\frac{1}{2}} \sup_{a \leq x \leq b} |\theta(x) - \theta_N(x)| + C \left(\lambda^{\frac{-1}{2}}\right) \int_a^b F \sup_{a \leq x \leq b} |\theta(x) - \theta_N(x)| dx \end{aligned}$$

When $N = 1$, applying Lemma 2.5, $|\theta(x) - \theta_1(x)| = o\left(\lambda^{\frac{-j}{2}}\right)$. Now By using Lemma 2.3 and induction we achieve that $I = o\left(\lambda^{\frac{-N}{2}}\right)$ as $\lambda \rightarrow \infty$.

Remark 2.7. By using the discussions of choice of f in [5], the condition (2.3) let us to consider q as the form $q(x) \sim x^{-K}$ where $1 \leq K < 2$.

Acknowledgements

The author would like to thank Professor Grigori Rozenblum for useful comments.

Competing interests

The author declares that they have no competing interests.

Received: 19 October 2011 Accepted: 12 April 2012 Published: 12 April 2012

References

- Atkinson, FV, Fulton, CT: Asymptotics of Sturm-Liouville eigenvalues for problems on a finite interval with one limit-circle singularity I. *Proc R Soc Edinburgh Sect A*. **99**(1-2):51-70 (1984). doi:10.1017/S0308210500025968
- Atkinson, FV: Asymptotics of an eigenvalue problem involving an interior singularity. *Research program proceedings ANL-87-26*, 2. pp. 1-18. Argonne National Lab. Illinois (1988)
- Harris, BJ: Asymptotics of eigenvalues for regular Sturm-Liouville problems. *J Math Anal Appl*. **183**, 25-36 (1994). doi:10.1006/jmaa.1994.1128
- Harris, BJ: A note on a paper of Atkinson concerning the asymptotics of an eigenvalue problem with interior singularity. *Proc Roy Soc Edinburgh Sect A*. **110**(1-2):63-71 (1988). doi:10.1017/S0308210500024859
- Harris, BJ, Race, D: Asymptotics of eigenvalues for Sturm-Liouville problems with an interior singularity. *J Diff Equ*. **116**(1):88-118 (1995). doi:10.1006/jdeq.1995.1030
- Harris, BJ, Marzano, F: Eigenvalue approximations for linear periodic differential equations with a singularity. *Electron J Qual Theory Diff Equ* 1-18 (1999). **No. 7**
- Coskun, H, Bayram, N: Asymptotics of eigenvalues for regular Sturm-Liouville problems with eigenvalue parameter in the boundary condition. *J Math Anal Appl*. **306**(2):548-566 (2005). doi:10.1016/j.jmaa.2004.10.030
- Fix, G: Asymptotic eigenvalues of Sturm-Liouville systems. *J Math Anal Appl*. **19**, 519-525 (1967). doi:10.1016/0022-247X(67)90009-1
- Fulton, CT, Pruess, SA: Eigenvalue and eigenfunction asymptotics for regular Sturm-Liouville problems. *J Math Anal Appl*. **188**(1):297-340 (1994). doi:10.1006/jmaa.1994.1429

10. Fulton, CT: Two point boundary value problems with eigenvalue parameter contained in the boundary conditions. *Proc Roy Soc Edinburgh Sect A*. **77**, 293–308 (1977)

doi:10.1186/1687-2770-2012-40

Cite this article as: Hormozi: A note on a paper of Harris concerning the asymptotic approximation to the eigenvalues of $-y'' + qy = \lambda y$, with boundary conditions of general form. *Boundary Value Problems* 2012 **2012**:40.

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- ▶ Convenient online submission
- ▶ Rigorous peer review
- ▶ Immediate publication on acceptance
- ▶ Open access: articles freely available online
- ▶ High visibility within the field
- ▶ Retaining the copyright to your article

Submit your next manuscript at ▶ springeropen.com
