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An Analytical Method to Calculate Borehole Fluid Temperatures for Time-scales from Minutes to Decades

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ABSTRACT

Knowledge of borehole exit fluid temperature is required to optimize the design and performance of ground source heat pump systems. The borehole exit fluid temperature depends upon the prescribed heat injection and extraction rates. This paper presents a method to determine the fluid temperature of a single or a multiple borehole heat exchanger for any prescribed heat injection or extraction rate. The fluid temperature, from minutes to decades, is determined using step response functions. An analytical radial solution is used for shorter times. A finite line-source solution is used for longer times. The line-source response function has been reduced to one integral only. The derivative, the weighting function, is given by an explicit formula both for single boreholes and any configuration of vertical boreholes.

INTRODUCTION

Optimizing the design and performance of ground source heat pump (GSHP) system requires accurate knowledge of the fluid temperatures exiting the borehole heat exchanger. The fluid temperature exiting a borehole heat exchanger depends upon the short-term and the long-term thermal response of the borehole and the ground surrounding the borehole, respectively. For a multiple borehole heat exchanger, the exiting fluid temperature also depends upon the thermal interactions between the boreholes. The development of the thermal response of the ground surrounding the borehole field is a slow process and depends upon the injections and extractions of ground heat, over time. Because both the thermal mass and the thermal capacity of the ground surrounding a borehole field are very large, the changes in ground temperatures are very slow. A time resolution of months or years is typically used to study the temperature development of the ground. On the other hand, the borehole heat exchanger itself has limited thermal mass and capacity and, consequently, the heat transfer inside the borehole is more sensitive to any changes in the required injection or extraction rates. As a result, the thermal response of the borehole is quite rapid and, therefore, is studied using a time resolution ranging from minutes to hours. Development of thermal interactions between different boreholes is again a slow and long-term process and, thus, requires monthly or yearly time resolution. Determining the accurate borehole fluid temperatures is an intricate procedure as it involves thermal processes that vary from short- to long-term intervals, with time resolutions ranging from minutes to years. At present, no single model exists that can effectively calculate both the short-term thermal response of the borehole and the long-term development of surrounding ground temperatures.

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EXISTING SOLUTIONS

Traditionally, the focus of borehole heat transfer related research has been to determine the long-term response of the borehole heat exchanger. A number of analytical and numerical methods, including the classical line and cylindrical source solutions (Ingersoll et al., 1954), have been developed to model the development of the ground temperature surrounding the borehole. The classical line and cylindrical source methods provide solutions to the radial transient heat transfer problem in the ground, assuming the borehole to be a line or a cylindrical heat source of infinite length. Various discrepancies occur when applying these two solutions to model the borehole heat transfer. These solutions not only ignore the end effects of their heat sources, they also ignore the thermal properties of the borehole elements. Moreover, these solutions are inaccurate when determining the short-term response of the borehole because of their underlying assumptions regarding geometry and the length of their heat sources. Some of these issues were addressed by Eskilson (1987), who used the finite line-source approach to develop the non-dimensional thermal response solutions, also known as *g-functions*. The *g-functions* were developed using a numerical approach that considered the transient radial-axial heat transfer in the borehole heat exchanger. The *g-functions* are valid for times longer than 200 hours (Yavuzturk, 1999). Eskilson also determined the thermal interactions between boreholes using intricate superposition of numerical solutions for each borehole. The use of *g-functions* to determine the borehole fluid temperature is somewhat restricted by the fact that these functions need to be computed numerically, which is a time-consuming and computationally-intensive task. Hence, these functions are pre-computed for different borehole heat exchanger geometries and configurations and are stored as databases in ground loop design software.

Lately, several researchers have also attempted to develop analytical and semi-analytical *g-functions* to address the flexibility issues of numerically-developed *g-functions*. Zeng et al. (2002) developed an analytical *g-function* expression using a constant value of borehole wall temperature, taken at the middle of the finite line-source. Lamarche and Beauchamp (2007) developed another expression for analytical *g-function* using the integral mean temperature along the finite line-source. The authors compared their analytical *g-function* to numerically obtained *g-functions* for different cases. They concluded that using the integral mean temperature along the borehole length, instead of the temperature at the middle of the borehole, gives more accurate results. Bandos et al. (2009) have developed simple approximate solutions for the cases considered by Zeng and Lamarche and Beauchamp.

In the last decade or so, the calculation of short-term response to optimize the design and performance of a borehole heat exchanger has also attracted the interest of many researchers. Yavuzturk (1999) extended the work of Eskilson and developed *g-functions* for times between 2.5 min and 200 hours using a numerical approach. Xu and Spitler (2006) developed a numerical model with variable convective resistance and the thermal mass of the fluid to determine short-term borehole response. Beier and Smith (2003) and Bandyopadhyay et al. (2008) developed semi-analytical solutions based on Laplace transforms. With regard to long-term response, the numerical and semi-analytical solutions used to determine the short-term response of a borehole are also computationally intensive. Recently, Javed and Claesson (2011) developed an analytical approach to determine the short-term response of borehole heat exchangers.

PROBLEM STATEMENT AND SOLUTION METHODOLOGY

The performance optimization of a GSHP system requires knowledge of fluid temperature for any prescribed heat injection or extraction rate. The fluid temperatures can be simulated using a short-term response solution. However, at present, the use of short-term borehole response solutions to determine fluid temperature is largely limited to a few software programs used for ground loop design. These programs use short-term solutions to determine the minimum and maximum fluid temperatures under peak load conditions when calculating the required length of the borehole heat exchanger. This approach, though adequate to design a ground heat exchanger, is not well-suited to determining the resulting fluid temperatures for a prescribed heat injection rate. This paper presents a simple, but accurate, method to calculate the borehole fluid temperature for any *prescribed* heat injection rate $q(t)$ (W/m). Both single and multiple borehole heat exchangers are considered. The required fluid temperature, at any time t, depends upon the value of the injection rate, at time t, and on the preceding sequence of heat injection.

In this analysis, the so-called step response solution becomes an important tool. This step response solution helps determine the required fluid temperature for a *constant* injection rate q_0. Next, the fluid temperature for any $q(t)$ is given by an integral of $q(t\text{-}\tau)$, multiplied by the time derivative of the step-response solution taken at time τ. The integration in τ is taken from zero to sufficiently large values. This means that the time derivative of the step response shows how the preceding extraction rates influence the current fluid temperature; it is a *weighting function* for the preceding injection rates.

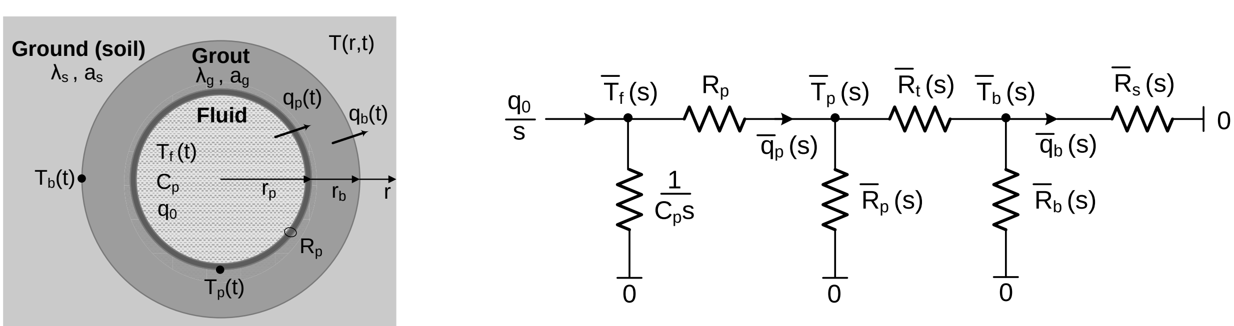
This paper provides a methodology to calculate the response function from very short times (minutes) to very long times (years, or longer). For short times, up to 100 hours, an analytical radial solution is used. After this point, a solution based on the finite line-source is used. It is important to note that the line-source response function has been reduced to one integral only. The derivative, the weighting function, is given by an explicit formula both for single boreholes and any configuration of vertical boreholes.

SHORT-TERM RESPONSE

Javed and Claesson (2011) developed a new analytical solution, which they used to calculate the short-term response of the borehole. The solution models the two legs of the U-tube as a single equivalent-diameter pipe and uses a single average value to represent the fluid temperatures entering and exiting the U-tube. The resulting radial heat transfer problem is shown in Figure 1. The heat flux q_0 is injected into the circulating fluid with temperature $T_f(t)$. The fluid has a thermal capacity of C_p. The pipe thermal resistance is R_p, and the pipe's outer boundary temperature is $T_p(t)$. The heat flux $q_p(t)$ flows through the pipe wall to the grout. The thermal conductivity and the thermal diffusivity of the grout are λ_g and a_g, respectively. The heat flux $q_b(t)$ flows across the borehole boundary to the surrounding ground (soil). The borehole boundary temperature is $T_b(t)$. The thermal conductivity and the thermal diffusivity of the ground (soil) are λ_s and a_s, respectively. The heat transfer problem, shown in Figure 1a, can be represented by means of the thermal network shown in Figure 1b. The network involves a sequence of composite resistances. The Laplace transform for the fluid temperature, $\bar{T}_f(s)$, is readily obtained from the thermal network. Finally, the fluid temperatures in time domain are obtained from $\bar{T}_f(s)$ using an inversion formula. The short-term response solution has been fully validated using both simulated and experimental data. Further details of the solution can be found elsewhere (Javed and Claesson, 2011).

LONG-TERM RESPONSE

The long-term step response is obtained from a continuous line heat source with the strength q_0 (W/m) along the borehole $x = 0$, $y = 0$, and $D < z < D+H$. The initial ground temperature is zero and the heat emission starts at $t = 0$. The solution is obtained by an integration of a point heat source along the borehole and integration in time from zero to t. The solution is:



(a) (b)

Figure 1 *(a) Geometry, temperatures, heat fluxes and thermal properties of the borehole. (b) The thermal network for the radial heat flow process for a borehole in the Laplace domain.*

$$T(r,z,t) = \int_0^t dt' \int_D^{D+H} dz' \frac{q_0}{\rho c[4\pi a(t-t')]^{1.5}} \cdot e^{-\frac{r^2+(z-z')^2}{4a(t-t')}} \qquad r = \sqrt{x^2+y^2} \tag{1}$$

The temperature is zero at the ground surface z = 0. This is achieved by introducing a mirror sink above the ground surface or subtracting $T(r,-z,t)$ from the solution obtained above. With the substitution $s = 1/\sqrt{4a(t-t')}$, the line-source solution may be written in the following way:

$$T_{ls}(r,z,t) = \frac{q_0}{4\pi\lambda} \int_{1/\sqrt{4at}}^{\infty} ds \cdot e^{-r^2 s^2} \cdot \frac{2}{\sqrt{\pi}} \int_D^{D+H} dz' \left[e^{-s^2 (z-z')^2} - e^{-s^2 (z+z')^2}\right] \tag{2}$$

The second exponential in the second integral represents the mirror sink. The *mean* temperature over the heat source length $D < z < D+H$ at any radial distance r is of particular interest.

$$\bar{T}_{ls}(r,t) = \frac{1}{H} \cdot \int_D^{D+H} T_{ls}(r,z,t)\, dz \tag{3}$$

Substituting $T_{ls}(r,z,t)$ from Equation 2 into Equation 3 gives:

$$\bar{T}_{ls}(r,t) = \frac{q_0}{4\pi\lambda} \int_{1/\sqrt{4at}}^{\infty} ds \cdot e^{-r^2 s^2} \cdot \underbrace{\frac{2}{H\sqrt{\pi}} \cdot \int_D^{D+H} dz \int_D^{D+H} dz' \left[e^{-s^2 (z-z')^2} - e^{-s^2 (z+z')^2}\right]}_{I} \tag{4}$$

Next, the double integral I in the expression for $\bar{T}_{ls}(r,t)$ must be evaluated. Applying the substitutions $sz = sD+u$ and $sz' = sD+v$, results in:

$$I = \frac{1}{Hs^2} \cdot \frac{2}{\sqrt{\pi}} \cdot \int_0^{Hs} du \int_0^{Hs} dv \left[e^{-(u-v)^2} - e^{-(2\cdot Ds+u+v)^2}\right] \tag{5}$$

Equation 5 can be rewritten as:

$$I = \frac{1}{Hs^2} \cdot I_{ls}(Hs, Ds) \tag{6}$$

When evaluating the double integral $I_{ls}(h,d)$, $h = Hs$, and $d = Ds$, the integration in v gives error functions with u in the argument. The second integration in u gives integrals of the error function, as follows:

$$\mathrm{erf}(X) = \frac{2}{\sqrt{\pi}} \int_0^X e^{-v^2} dv \qquad \mathrm{ierf}(X) = \int_0^X \mathrm{erf}(u)\, du = X \cdot \mathrm{erf}(x) - \frac{1}{\sqrt{\pi}} (1 - e^{-X^2}) \tag{7}$$

The final expression for the double integral becomes:

$$I_{ls}(h,d) = 2 \cdot \mathrm{ierf}(h) + 2 \cdot \mathrm{ierf}(h+2d) - \mathrm{ierf}(2h+2d) - \mathrm{ierf}(2d) \tag{8}$$

The mean temperature (4) over the borehole length can now be represented as a single integral:

$$\bar{T}_{ls}(r,t) = \frac{q_0}{4\pi\lambda} \cdot \int_{1/\sqrt{4at}}^{\infty} ds \cdot e^{-r^2 s^2} \cdot \frac{I_{ls}(Hs, Ds)}{Hs^2} \tag{9}$$

LONG-TERM STEP RESPONSE FOR SINGLE AND MULTIPLE BOREHOLES

The mean temperature at the borehole radius r_b gives the long-term response for a single borehole:

$$T_1(t) = \bar{T}_{ls}(r_b, t) \tag{10}$$

The time derivative of the response temperature $T_1(t)$ is readily obtained since time only occurs in the lower limit of the integral:

$$\frac{dT_1}{dt} = \frac{q_0}{4\pi\lambda} \cdot e^{-\frac{r^2}{4at}} \cdot I_{ls}\left(\frac{H}{\sqrt{4at}}, \frac{D}{\sqrt{4at}}\right) \cdot \frac{1}{H} \cdot \sqrt{\frac{a}{t}}, \tag{11}$$

The last factor involves the derivative of $1/\sqrt{4at}$. It is gratifying that the time derivative, which gives the weighting functions, is obtained as an explicit formula.

Now, consider N vertical boreholes at the positions (x_j, y_j, z), $D < z < D+H$, $j = 1, 2, \ldots, N$. The total temperature field becomes:

$$T(x, y, z, t) = \sum_{j=1}^{N} T_{ls}\left(\sqrt{\left(x - x_j\right)^2 + \left(y - y_j\right)^2}, z, t\right) \tag{12}$$

The mean temperature is needed along the borehole wall (bw) for any borehole i.

$$\bar{T}_{\mathrm{bw},i}(t) = \sum_{j=1}^{N} \bar{T}_{ls}\left(r_{i,j}, t\right) \tag{13}$$

Here $r_{i,j}$ denotes the radial distance between borehole i and j ($i \neq j$). The contribution from the own heat source of the borehole i is obtained for the radial distance r_b.

$$r_{i,i} = r_b\,, \qquad r_{i,j} = \sqrt{\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2} \qquad i \neq j \tag{14}$$

The mean borehole wall temperature for the entire set of N boreholes is:

$$\frac{1}{N}\sum_{i=1}^{N} \bar{T}_{\mathrm{bw},i}(t) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} \bar{T}_{ls}\left(r_{i,j}, t\right) \tag{15}$$

This mean temperature is used as the response function. Using Equation 9, the response function for N boreholes may now be written in the following way:

$$T_N(t) = \frac{q_0}{4\pi\lambda} \cdot \int_{1/\sqrt{4at}}^{\infty} ds \cdot I_e(s) \cdot \frac{I_{ls}(Hs, Ds)}{Hs^2}, \tag{16}$$

Here, the function $I_e(s)$ involves a double sum in the exponentials:

$$I_e(s) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} e^{-r_{i,j}^2 s^2} \tag{17}$$

The time derivative of the response functions for N boreholes is now obtained in the same way as it was for a single borehole. Here, I_e and I_{ls} are given by (17) and (7-8), respectively:

$$\frac{dT_N}{dt} = \frac{q_0}{4\pi\lambda} \cdot I_e\left(\frac{1}{\sqrt{4at}}\right) \cdot I_{ls}\left(\frac{H}{\sqrt{4at}}, \frac{D}{\sqrt{4at}}\right) \cdot \frac{1}{H} \cdot \sqrt{\frac{a}{t}} \quad (18)$$

Here, I_e and I_{ls} are given by (17) and (7-8), respectively:

The following examples show how the exponent $I_e(s)$ can be obtained for different configurations of multiple borehole heat exchangers. The first example considers 3 boreholes in a straight line, separated by the spacing *B*. The double sum in Equation 17 involves nine terms. The exponent involves the distances $r_{i,j}$. Three terms involve r_b, four terms involve *B*, and two terms involve *2B*. Therefore:

$$I_e(s) = \frac{1}{3}\left[3 \cdot e^{-r_b^2 s^2} + 4 \cdot e^{-B^2 s^2} + 2 \cdot e^{-(2B)^2 s^2}\right] \quad (19)$$

The second example considers 9 boreholes in a square with spacing *B*. The double sum (17) now involves 9 x 9 = 81 terms. The exponent involves the distances r_b, *B*, $\sqrt{2}B$, 2*B*, $\sqrt{5}B$, and $\sqrt{8}B$. For example, in Figure 2, the diagonal distance $\sqrt{8}B$ occurs four times between boreholes: 1 to 9, 3 to 7, 7 to 3, and 9 to 1. Counting the number of occurrences for each distance gives:

$$I_e(s) = \frac{1}{9}\left[9 \cdot e^{-r_b^2 s^2} + 24 \cdot e^{-B^2 s^2} + 16 \cdot e^{-(\sqrt{2}B)^2 s^2} + 12 \cdot e^{-(2B)^2 s^2} + 16 \cdot e^{-(\sqrt{5}B)^2 s^2} + 4 \cdot e^{-(\sqrt{8}B)^2 s^2}\right] \quad (20)$$

The sum of the coefficients before the exponentials is 81 in Equation 20.

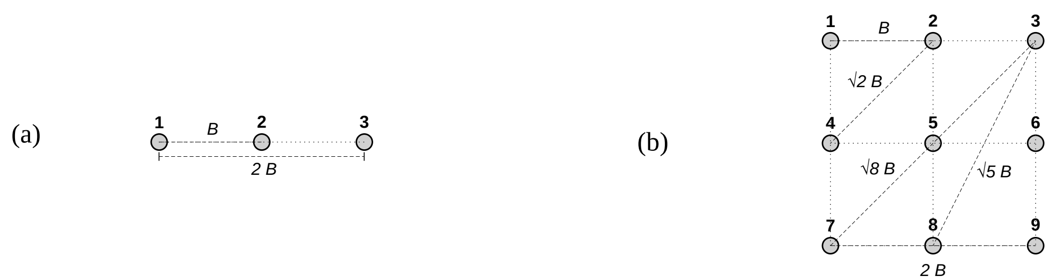


Figure 2 *Radial distances between boreholes: (a) Three boreholes in a straight line; (b) Nine boreholes in a square.*

COMBINED STEP RESPONSE

The final step response that accounts for both short and long term is obtained in the following way. Up to a certain time, the radial short-term response is used. After that time period, the long-term response from the line-source solution is used. One complication that arises is that the line-source solution does not account for the local thermal processes in the borehole. Figure 3 illustrates this problem. The top curve shows the radial solution for a single borehole and the lower curve shows the corresponding line-source solution. As shown, the slope of the two curves is very similar between 10 and 1000 hours.

The borehole has thermal resistances over the pipe and the grout. These resistances cause an increase in the fluid temperature. This means that the line-source solution should be shifted upwards to account for this temperature increase. The temperature difference at a suitable breaking time (t_{bt}) is added to the line-source solution so that the radial and the line-source solutions coincide at the breaking time. In the final step response, the radial solution is used, up to the breaking time. After that, the line-source solution, including the upward shift, is used. The choice of the breaking time is not critical since the two curves are parallel over a large time span. A reasonable choice is t_{bt} = 100 hours.

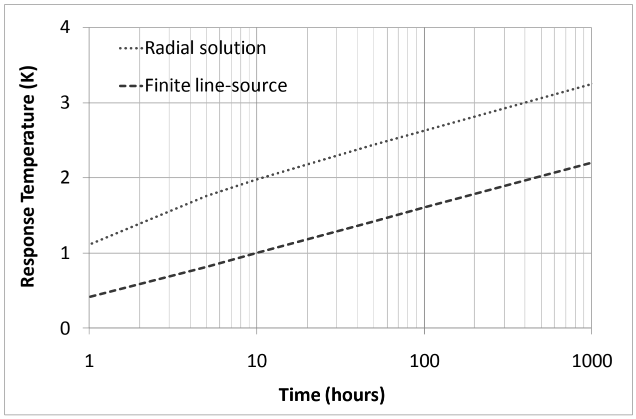


Figure 3 *Short- and medium-term fluid temperatures using the radial solution and the finite line-source solution.*

EXAMPLES

In this study, three examples are considered: 1 borehole, 3 boreholes in a line, and 9 boreholes in a square (Figure 2). Table 1 presents the parameters used for the examples. Figure 4 shows the response functions for the three cases with the logarithm of time on the horizontal axis. The time span ranges from 10^{-2} to 10^{6} hours. The three curves are identical below the breaking time. The curves start to deviate from each other after 500 hours.

Figure 4 also presents a comparison of the long-term and the short-term fluid temperatures predicted by the new method with those predicted by Eskilson's *g-functions* (1987) and a numerical model (Javed & Claesson, 2011), respectively. For the first two cases of a single borehole and for the 3 boreholes in a straight line, the long-term fluid temperatures, predicted by the new method and Eskilson's *g-functions*, are in very good agreement up to 25 years. For the 9 boreholes in a square, the agreement is very good up to 10 years and reasonably good afterwards. The difference between the fluid temperatures that are predicted by the new method and Eskilson's *g-functions* increases with time and with the number of boreholes. However, the difference is relatively small for up to 25 years. For all three cases, the short-term fluid temperature predicted by the new model is identical to the short-term fluid temperature predicted by the numerical solution.

Table 1. Parameters Considered for Examples

Property	Value
Heat injection rate (q_0)	10 W/m (10.4 Btu/h·ft)
Borehole radius (R_b)	55 cm (22 in)
Pipe radius (R_b)	28 cm (11 in)
Ground (soil)	
thermal conductivity (λ_s)	3.0 W/m·K (1.73 Btu/h·ft·°F)
density (ρ_s)	2500 kg/m^3 (156 lb/ft^3)
heat capacity (c_s)	750 J/kg·K (0.18 Btu/lb·°F)
Grout	
thermal conductivity (λ_g)	1.5 W/m·K (0.87 Btu/h·ft·°F)
density (ρ_g)	1550 kg/m^3 (97 lb/ft^3)
heat capacity (c_g)	2000 J/kg·K (0.48 Btu/lb·°F)

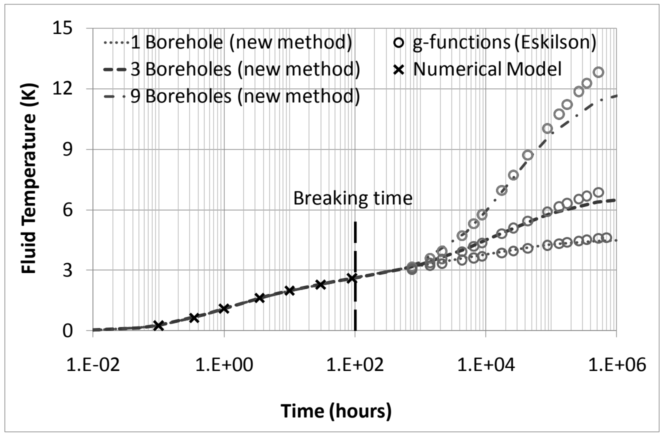
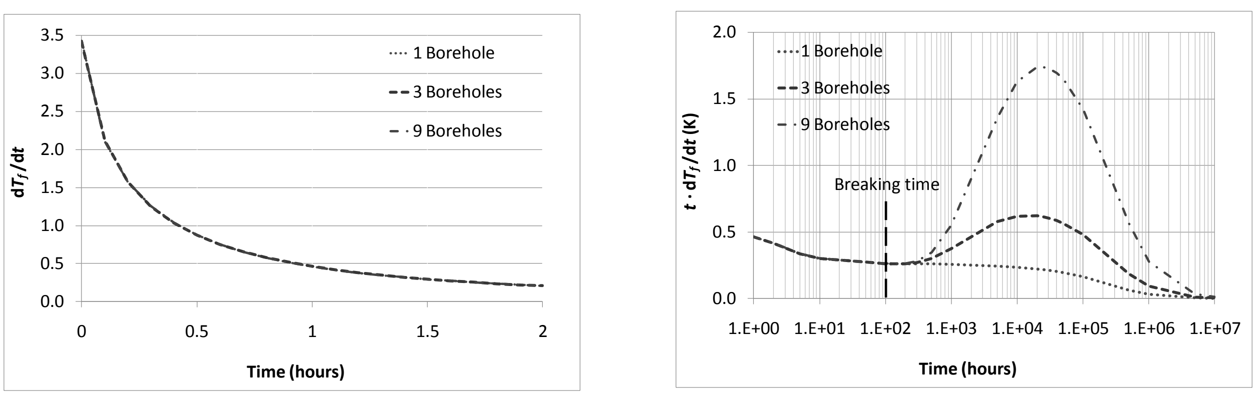


Figure 4 *Response functions for 1, 3, and 9 boreholes using a combination of radial and finite line-source solutions.*

Figure 5 shows the time derivative of the response function, which gives the weighting functions. As discussed in the problem statement, these weighting functions are the key element in determining the fluid temperature for the prescribed heat injection or heat extraction rate. Figure 5a shows the weighting function during the first two hours (from the radial solution). It can be seen that, during these two hours, the function falls by the factor 10. While it will continue to fall strongly with time, the function is still needed when applied to very long times. Therefore, Figure 5b shows the function multiplied by time *t*.



(a) (b)

Figure 5 *Time derivatives of response functions (weighting functions) for 1, 3 and 9 boreholes.*

CONCLUSION

Knowledge of the borehole exit fluid temperature is critical to the design and the performance optimization of GSHP systems. The exit fluid temperature depends upon both the short-term response of the borehole and the long-term response of the surrounding ground. This paper presents a simple analytical method to calculate fluid temperatures for times ranging from minutes to decades. The short-term borehole response is calculated using a recently developed and well-validated analytical solution. The long-term response is calculated using a finite line-source solution. For a single borehole, a closed form formula (Equations 10 and 7-9) has been developed to determine the long-term step response. For multiple boreholes, a simple and systematic approach (Equations 16, 17 and 7-9) is introduced to calculate the long-term response. The long-term response predicted by the new method is in good agreement with the response obtained from Eskilson's *g-functions*. The total response from minutes to decades is obtained by joining the long-term response to the short-term response at a suitable breaking time. The choice of breaking time is not critical and any time between 10 and 1000 hours may be selected. Finally, the time derivative of the step response is given as an explicit expression to be used for modelling. This expression shows the effect of the preceding extraction rates on the current fluid temperature.

NOMENCLATURE

a	=	thermal diffusivity (m^2/s or ft^2/h)
B	=	spacing between boreholes (m or ft)
C	=	thermal capacity per unit length (J/m·K or Btu/ft·°F)
c	=	specific heat capacity (J/kg·K or Btu/lb·°F)
D	=	starting point of active borehole depth (m or ft)
H	=	active borehole height (m or ft)
λ	=	thermal conductivity (W/m·K or Btu/h·ft·°F)
q	=	rate of heat transfer per unit length (W/m or Btu/h·ft)
R	=	thermal resistance (m·K/W or h·ft·°F/Btu)
$\bar{R}(s)$	=	thermal resistance in the Laplace domain (m·K/W or h·ft·°F/Btu)
r	=	radius (m or ft)
ρ	=	density (kg/m^3 or lb/ft^3)
s	=	Laplace transform variable (in the short-term response) and
	=	$1/\sqrt{4a(t-t')}$ (in the long-term response)
T	=	temperature (K or °F)
$\bar{T}$	=	mean temperature (K or °F)
$\bar{T}(s)$	=	Laplace transform of T (K·s or °F·h)
t	=	time (s or h)
z	=	vertical coordinate

Subscripts

b	=	borehole
bw	=	borehole wall
f	=	fluid
g	=	grout
ls	=	line-source
p	=	pipe
s	=	ground (soil)

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