THESIS FOR THE DEGREE OF LICENTIATE OF PHILOSOPHY

Topics on Game Theory

Emilio Bergroth

Department of Mathematical Sciences
Division of Mathematical Statistics
Chalmers University of Technology and University of Gothenburg
Göteborg, Sweden 2011

Topics on Game Theory
Emilio Bergroth
NO 2011:8
ISSN 1652-9715

©Emilio Bergroth, 2011

Department of Mathematical Sciences
Division of Mathematical Statistics
Chalmers University of Technology and University of Gothenburg
SE-412 96 Göteborg
Sweden
Telephone +46 (0)31 772 1000

Typeset with LaTeX.
Printed in Göteborg, Sweden 2011

Abstract

In this thesis we study two different non-cooperative two-person games. First, we study a zero-sum game played by players I and II on a $n \times n$ random matrix, where the entries are iid standard normally distributed random variables. Given the realization of the matrix, player I chooses a row i and player II a column j. The entry at position (i, j) represents the winnings and losings of the players. Let $\mathbf{p} = [p_1, ..., p_n]^T$ denote the optimal strategy of player I. We show that $P\left(\max_{i \in [n]} p_i > \frac{c}{\sqrt{n}}\right) \to 0$ as $n \to \infty$ for any $c > 10\sqrt{\pi}\left(1 + \sqrt{2\log 4}\right)\sqrt{\log 4}$.

The other game studied here is a spatial game in which each player, represented by a vertex in a given graph, plays the repeated prisoner's dilemma game against its neighbors. At time one, each player chooses a strategy at random independently of each other. At time $t = 2, 3, ...$, each player, looking at its neighborhood (including the player itself), uses the strategy of the player that scored highest in the previous round. We study the game played on some deterministic graphs. For certain graphs and choices of the parameters of the game, we find the probability that a given player cooperates as time tends to infinity. We also analyze the iterated prisoner's dilemma played on the binomial random graph. In particular, we study the asymptotic distribution of cooperation when the number of players tends to infinity. More precisely, if n stands for the numbers of players and r is the probability that two players play against each other, we show that asymptotically almost surely (*a.a.s.*) cooperation dies out if $r = \frac{1}{n^c}$, $c < 1$, whereas it survives *a.a.s.* in the largest component if $r = \lambda\frac{\log n}{n}$ and $\lambda = \lambda(n) \leq \frac{1}{6}$ such that $\lambda \log n \to \infty$ as $n \to \infty$.

Keywords: zero-sum game, optimal strategy, random matrix, iterated prisoner's dilemma, spatial game, cooperation, deterministic graph, binomial random graph.

Acknowledgments

First I would like to thank my supervisor Johan Jonasson for introducing me to Game Theory and the problems studied in this thesis. Thank you as well for taking the time to talk to me and to help me, and for giving me useful comments and suggestions concerning this thesis.

Then I would like to thank Erik Broman, Daniel Ahlberg and Oscar Hammar for giving me feedback on the manuscript. Many thanks also to all my colleagues at Mathematical Sciences for good times, in particular to Oscar H., Daniel A., Janeli S., Erik J., Ottmar C., Malin Ö., Alexandra J., Marcus I., Erik B., Sofia T., Wengang M., Frank E., Magnus R., Anastassia B., Vera L., Anna R., Dawan M., Peter H. and Erik K.

Finally, and most of all, I would like to thank my family and best friends: Hilda, Alma and Máxima Bergroth. Thank you for always encouraging me and for bringing so much love and happiness to my life!

Contents

Chapter 1

Introduction

Game Theory is a branch of applied mathematics originally related to economical and political problems. In its origins, John von Neumann and Oskar Morgenstern intended to study human behavior when making strategic decisions. The assumption was that these decisions were based on rationality. Nowadays Game Theory is related to other areas such as ecology and biology, in particular related to evolution. In these areas, the individuals behavior does not rely on rationality but on other aspects such as fitness and natural selection.
In Game Theory, the decisions makers are called players [10]. Typical objects of study are games called Two-person games. The players, player I and player II, have a choice to make, and each player's score depends on its own choice and the choice of the other player [9]. If the players' actions are independent, then the game is called non-cooperative. A very simple description of a non-cooperative two-person game is given by $(S_I, S_{II}, A_I, A_{II})$, where

1. S_I is a nonempty set, the set of possible moves of player I;
2. S_{II} is a nonempty set, the set of possible moves of player II;
3. A_I and A_{II} are the score functions (real valued functions) defined on $S_I \times S_{II}$.

This way to describe the game is known as the "Strategic Form" [10]. The interpretation is as following: at the same time and without having information about the other player's choice, player I chooses $s_I \in S_I$ and player II chooses $s_{II} \in S_{II}$, resulting in that player I wins $A_I(s_I, s_{II})$ and player II wins $A_{II}(s_I, s_{II})$. When the total score to both players adds to zero, i.e. $A_I(s_I, s_{II}) = -A_{II}(s_I, s_{II})$ for all $s_I \in S_I$, $s_{II} \in S_{II}$, the game is called a zero-sum game. Two-person zero-sum games are also known as matrix games, because the scores can be represented by a matrix. If $S_I = \{s_{I,1}, ..., s_{I,n}\}$ and $S_{II} = \{s_{II,1}, ..., s_{II,m}\}$, then the matrix

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nm} \end{pmatrix}$$

is the Payoff matrix of the game, with

$$x_{ij} = A_I(s_{I,i}, s_{II,j}) = -A_{II}(s_{I,i}, s_{II,j}).$$

The matrix is known to both players. Player I chooses a row, i $(s_{I,i})$, and player II chooses a column, j $(s_{I,i})$; then player II pays x_{ij} to player I if $x_{ij} > 0$ or player I pays $|x_{ij}|$ to player II if $x_{ij} < 0$.
We distinguish between pure strategies and mixed strategies. Pure strategies for player I (II) are just deterministic choices of a move or element of S_I (S_{II}). In a mixed strategy $\mathbf{p} = [p_1, ..., p_n]^T$ $(\mathbf{q} = [q_1, ..., q_n]^T)$ for player I (II), each element p_i (q_j) is a probability, so that when I (II) makes its choice of move, it does so according to these probabilities: move $s_{I,i}$ $(s_{II,j})$ is chosen with probability p_i (q_j). Mixed strategies are also known as randomized strategies. According to the well-known Minimax Theorem of von Neumann and Morgenstern there exist a number V, called the value of the game, and mixed strategies $\mathbf{p} = [p_1, ..., p_n]^T$ and $\mathbf{q} = [q_1, ..., q_n]^T$, called optimal strategies or minimax strategies, for both players respectively. These strategies have the following properties: when player I plays $\mathbf{p}$ then its expected winnings are at least V independently of what player II plays, and if player II plays $\mathbf{q}$ then its expected loses are at most V [1]. A central issue in Game Theory is the study of the optimal strategies. In this sense, Jonasson in [1] analyzed the optimal strategies $\mathbf{p} = [p_1, ..., p_n]^T$ and $\mathbf{q} = [q_1, ..., q_n]^T$ of a two-person zero-sum game played on a random $n \times n$-matrix $\mathcal{X} = [X_{ij}]_{1 \leq i,j \leq n}$, where the X_{ij}'s are iid normally distributed random variables. More precisely,

the author shows that if Z is the number of rows in the support of the optimal strategy for player I given the realization of the matrix, then there exists $a < \frac{1}{2}$ such that

$$P\left(\left(\frac{1}{2} - a\right) n < Z < \left(\frac{1}{2} + a\right) n\right) \quad \to \quad 1,$$

as $n \to \infty$. It is also shown that $\mathbb{E}Z = \left(\frac{1}{2} + o(1)\right) n$. In **Chapter 2**, we study the same game, establishing a result about the maximal probability assigned to a row/column in the optimal strategy. More precisely, we find that for any $c > 10\sqrt{\pi}\left(1 + \sqrt{2\log 4}\right)\sqrt{\log 4}$,

$$P\left(\max_{i\in[n]} p_i > \frac{c}{\sqrt{n}}\right) \to 0,$$

as $n \to \infty$.

The second part of this thesis concerns a model introduced by Lindgren and Nordahl [3], in which the well-known "Prisoner's dilemma" is studied in a spatial setting. Merrill Flood and Melvin Dresher stated the Prisoner's dilemma for first time in 1950. However, the "prisoner's dilemma" name was given by Albert W. Tucker who formalized the game using a payoff matrix to describe it [13]. The following is the prisoner's dilemma: two suspects are arrested by the police. The police offers the same deal to each prisoner. If one testifies against the other (defects) and the other remains silent (cooperates), the defector goes free and the cooperator gets a 5-year sentence. If both prisoners cooperate, each of them receive a 1-year sentence. If each prisoner testifies against the other, each receives a 3-years sentence. This means that there are only two actions for each prisoner, to defect (D) or to cooperate (C). What should the prisoners do? Assuming that each prisoner wants only to minimize his own time in prison, then the best action to take is to defect, whatever the other prisoner does. On the other hand, it is clear that if the two prisoners were to act for their common good, they should cooperate.

This need of choice between defection and cooperation is present in many social and biological contexts. In fact, the prisoner's dilemma is present at all levels, as explained by Nowak in [4]: "Replicating molecules had to cooperate to form the first cells. Single cells had to cooperate to form the first multicellular organisms. The soma cells of the body cooperate and help the cells of the germ line to reproduce. Animals cooperates to form social structures... Humans cooperate on large scale, giving raise to cities, states and countries. Cooperation allows specialization. Nobody needs to know everything. But cooperation is always vulnerable to exploitation by defectors."

Lindgren and Nordahl model the evolution of cooperation in a spatial setting: each player, associated with a vertex in a given graph, plays the prisoner's dilemma game against its neighbors. In their work, the authors made simulations in order to understand the behavior of cooperation as the game is played repeatedly. In this thesis, we establish some rigorous results concerning the iterated prisoner's dilemma. Here the game is played with the following rules: (i) at time 0 each player chooses independently strategy C with probability p, and strategy D with probability $1-p=q$; (ii) at time $t=1,2,...$ each player plays the game against its neighbors; (iii) at time $t=2,3,...$ each player, looking at its own neighborhood (the player itself and its neighbors), uses the same strategy as the player with highest score at time $t-1$. For each player, the payoff matrix of a single game is

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix} \end{array}$$

with $0<a<1, 1<b<2, a+b\leq 2$. We are interested in the probability that cooperation survives as the game is played repeatedly, in particular that a given player i survives as a cooperator as $t\to\infty$. More formally, we study the limit $\pi_p(C)=\lim_{t\to\infty}P(s_t^i=C)$, where s_t^i stands for the strategy used by player i at time t. The structure of the population is determined by different graphs, characterized by vertices and edges, which represents players and interactions respectively. In **Chapter 3** we study some deterministic graphs, in which there are infinitely many players and the number of neighbors is equal for all players. Examples of such graphs are trees, d-dimensional lattices, etc. In most cases $\pi_p(C)$ depends on the parameters a and b. We study for each graph, as it is possible, the evolution of cooperation for different values of a and b. However, there are cases that $\pi_p(C)$ is independent of a and b. One such case is the one-dimensional lattice, where each player plays against its two neighbors. We show in this case that

$$\pi_p(C) \quad = \quad \left(q+p^2\right)^2 p^3 \left(3-2p\right).$$

But this is just the special case of the $n-1$-nary tree, in which each player plays the game against its n neighbors, for which given the conditions $a+(n-1)b \leq n$ and $(n-1)a+b > n-1$, we show that

$$\begin{aligned} \pi_p(\mathrm{C}) \quad = \quad & p^{n+1}x^{n(n-1)}\left(1-p^{n-1}x^{2-3n+n^2}\right)^n \\ & -px^n\left(\left(1-p^n x^{(n-2)n}\right)^n-1\right), \end{aligned}$$

where $x = p + q(1-p^{n-1})$.

In **Chapter 4** we study the iterated prisoner's dilemma played on a random graph known as the *binomial random graph.* In this graph, denoted by $\mathbb{G}(n,r)$, there are n players which play with each other. This interaction is given by the result of $\binom{n}{2}$ independent coins flipping, each of them with probability of success equal to $r \in (0,1)$ [6]. We are interested in the behavior of cooperation as n tends to infinity, with r as a function of n. Our results are also independent of a and b. We show, for example, that the probability that cooperation survives in some part of the graph tends to zero if $r \geq \frac{1}{n^c}, 0 \leq c < 1$.

Chapter 2

The optimal strategy of a two-person zero-sum game

Consider a two-person game, played by player I and player II, which is described by $(S_I, S_{II}, A_I, A_{II})$, where

1. $S_I = \{r_1, ..., r_n\}$ is the set of moves of player I;
2. $S_{II} = \{c_1, ..., c_n\}$ is the set of moves of player II;
3. A_I and A_{II} are the score functions defined on $S_I \times S_{II}$;
4. $X_{ij} = A_I(r_i, c_j) = -A_{II}(r_i, c_j)$, $i, j = 1, 2..., n$. The X_{ij}'s are iid random variables with a standard normal distribution.

We interpret this as following [1]: the game is played on a $n \times n$-matrix $\mathcal{X} = [X_{ij}]_{1 \leq i,j \leq n}$ where the X_{ij}'s are iid random variables with a standard normal distribution. The matrix is known to both players; player I chooses a row, i (r_i), and player II chooses a column, j (c_j); then player II pays X_{ij} to player I if $X_{ij} > 0$. If $X_{ij} < 0$, player I pays $|X_{ij}|$ to player II. We allow both players to use mixed strategies.

Definition 1. *A mixed strategy for player I is a vector probability* $\boldsymbol{p}^* = [p_1^*, ..., p_n^*]^T$, *with*

1. $p_i^* \geq 0$, $i = 1, ..., n$;
2. $\sum_{i=1}^{n} p_i^* = 1$,

where p_i^* is the probability that player I chooses the move r_i. Moreover, there are mixed strategies, called *optimal* strategies, with the following properties [10]:

Theorem 2.0.1. *For every finite two-person zero-sum game played on a matrix $X = [x_{ij}]_{1 \leq i,j \leq n}$,*

1. *there is a number V called the value of the game;*
2. *there is a mixed strategy $\boldsymbol{p} = [p_1, ..., p_n]^T$ for player I, called an optimal strategy, such that*
$$\sum_{i=1}^{n} p_i x_{ij} \geq V \qquad \textit{for all } j = 1, 2, ..., n;$$
3. *there is a mixed strategy $\boldsymbol{q}$ for player II, called an optimal strategy, such that*
$$\sum_{j=1}^{n} q_j x_{ij} \leq V \qquad \textit{for all } i = 1, 2, ..., n.$$

This theorem, due to von Neumann, is the well-known Minimax Theorem. In fact, the so-called Equilibrium theorem [10] says that if player I plays $\mathbf{p}$ and player II plays $\mathbf{q}$, then

$$\sum_{i=1}^{n} p_i x_{ij} = V \qquad \text{for all } j = \text{such that } q_j > 0,$$

and

$$\sum_{j=1}^{n} q_j x_{ij} = V \qquad \text{for all } i \text{ such that } p_i > 0.$$

In this chapter, we study the optimal strategy of a game described as above. More precisely, we establish the following result concerning the maximal probability assigned to a row in the optimal strategy $\mathbf{p} = [p_1, ..., p_n]^T$ for player I given the realization of the payoff matrix.

Theorem 2.0.2. *For any* $c > 10\sqrt{\pi}\left(1 + \sqrt{2\log 4}\right)\sqrt{\log 4}$,

$$P\left(\max_{i\in[n]} p_i > \frac{c}{\sqrt{n}}\right) \quad \to \quad 0 \qquad \textit{as } n \to \infty.$$

Proof. Consider any $k \times k$ sub-matrix A_k of $\mathcal{X}$. Since the X_{ij}'s are standard normally distributed, the distribution of ${A_k}^{-1}$ is invariant under multiplication by any orthogonal matrix [1]. In particular, the distribution of ${A_k}^{-1}$ is invariant under multiplication by a rotation matrix.

Let $\xi_1, \xi_2, ..., \xi_k$ be *i.i.d.* standard normally distributed random variables. Let also

$$x_i = \frac{\xi_i}{\sqrt{\sum_{j=1}^{k} \xi_j^2}}.$$

It is well known that x $= [x_1, ..., x_k]^T$ is uniformly distributed on the surface of the (unit) k-sphere.

Let now $E_k^+ = \{\mathbf{y}_k = {A_k}^{-1}1 \in \mathbb{R}_+^k\}$, $E_k^- = \{\mathbf{y}_k = {A_k}^{-1}1 \in \mathbb{R}_-^k\}$ and $E_k = E_k^+ \cup E_k^-$, where $\mathbf{y}_k = [y_1, ..., y_k]^T$.
Let also $N^+ = \{x_1 > 0, ..., x_k > 0\}$ and $N^- = \{x_1 < 0, ..., x_k < 0\}$. Since the x_i's are the coordinates of a uniformly distributed point on the surface of the sphere, we have that for $c, d > 0$,

$$\begin{aligned}
P\left(\frac{|y_1|}{|y_1| + ... + |y_k|} \geq \frac{cd}{\sqrt{n}} | E_k\right) &= P\left(\frac{|x_1|}{|x_1| + ... + |x_k|} \geq \frac{cd}{\sqrt{n}} | N^+\right) \\
&= P\left(\frac{|\xi_1|}{|\xi_1| + ... + |\xi_k|} \geq \frac{cd}{\sqrt{n}} | N^+\right) \\
&\leq P\left(|\xi_1| \geq c\sqrt{n}\right) \\
&\quad + P\left(d(|\xi_2| + ... + |\xi_k|) \leq n\right).
\end{aligned}$$

Since the ξ_i's are $N(0,1)$ r.v., $\mathbb{E}(|\xi_1|) = \sqrt{\frac{2}{\pi}}$, $\mathbb{E}({\xi_1}^2) = 1$ and for $x > 0$ $P(|\xi_1| \geq x) \leq \frac{2}{x} e^{-\frac{x^2}{2}}$ [5]. In order to proceed, we need the following Chernoff bound [7]:

Theorem 2.0.3. *If $X_1, X_2, ..., X_n$ are nonnegative independent random variables, we have the following bound for the sum $X = \sum_{i=1}^n X_i$:*

$$P(X \leq \mathbb{E}(X) - \lambda) \quad \leq \quad e^{-\frac{\lambda^2}{2\sum_{i=1}^n \mathbb{E}({X_i}^2)}}.$$

Letting $S_k = \sum_{j=2}^k |\xi_j|$, we then have

$$\begin{aligned} P\left(\frac{|y_1|}{|y_1| + ... + |y_k|} \geq \frac{cd}{\sqrt{n}} |E_k\right) &\leq P\left(|\xi_1| \geq c\sqrt{n}\right) \\ &\quad + P\left(dS_k \leq \mathbb{E}(dS_k) - \left(d\sqrt{\frac{2}{\pi}} \frac{k-1}{n} - 1\right) n\right) \\ &\leq \frac{2}{c\sqrt{n}} e^{-\frac{c^2}{2} n} + e^{-\frac{\left(d\sqrt{\frac{2}{\pi}}\frac{k-1}{n} - 1\right)^2}{2(k-1)} n^2}. \end{aligned}$$

Let now $Z = |\{i \in [n] : p_i > 0\}|$. Finally, we have that

$$\begin{aligned} P\left(\max_{i \in [n]} p_i \geq \frac{cd}{\sqrt{n}}\right) &= P\left(\bigcup_{i=1}^n \left\{p_i \geq \frac{cd}{\sqrt{n}}\right\}\right) \\ &= \sum_{k=1}^n P\left(\bigcup_{i=1}^n \left\{p_i \geq \frac{cd}{\sqrt{n}}\right\}, Z = k\right) \\ &\leq P(Z \leq \lfloor 0.1n \rfloor) \\ &\quad + \sum_{k=\lfloor 0.1n \rfloor + 1}^n \binom{n}{k}^2 P\left(\bigcup_{i=1}^k \left\{\frac{y_i}{y_1 + ... + y_k} \geq \frac{cd}{\sqrt{n}}\right\} |E_k^+\right) 2^{1-k} \end{aligned}$$

$$
\begin{aligned}
&\leq P(Z \leq \lfloor 0.1n \rfloor) \\
&\quad + \sum_{k=\lfloor 0.1n \rfloor+1}^{n} k \binom{n}{k}^2 2^{1-k} \left(\frac{2}{c\sqrt{n}} e^{-\frac{c^2}{2}n} + e^{-\frac{\left(d\sqrt{\frac{2}{\pi}}\frac{k-1}{n}-1\right)^2}{2(k-1)}n^2} \right) \\
&\leq P(Z \leq \lfloor 0.1n \rfloor) \\
&\quad + n^2 \left(\frac{2}{c\sqrt{n}} e^{-\frac{c^2}{2}n} + e^{-\frac{\left(d\sqrt{\frac{2}{\pi}}\left(0.1-\frac{1}{n}\right)-1\right)^2}{2}n} \right) \left(\max_{k\in[\lfloor 0.1n \rfloor+1;n]} \binom{n}{k}^2 2^{1-k} \right).
\end{aligned}
$$

It is shown in [1] that $P(Z \leq \lfloor 0.1n \rfloor) \to 0$ as $n \to \infty$, and since

$$
\max_{k\in[\lfloor 0.1n \rfloor+1;n]} \binom{n}{k}^2 2^{1-k} \leq 4^n,
$$

we have that

$$
\begin{aligned}
P\left(\max_{i\in[n]} p_i \geq \frac{cd}{\sqrt{n}}\right) &\leq P(Z \leq \lfloor 0.1n \rfloor) \\
&\quad + n^2 4^n \left(\frac{2}{c\sqrt{n}} e^{-\frac{c^2}{2}n} + e^{-\frac{\left(d\sqrt{\frac{2}{\pi}}\left(0.1-\frac{1}{n}\right)-1\right)^2}{2}n} \right) \\
&\to 0,
\end{aligned}
$$

as $n \to \infty$ if $c > \sqrt{2\log 4}$ and $d > \frac{10}{\sqrt{\frac{2}{\pi}}}\left(1+\sqrt{2\log 4}\right)$. $\square$

Chapter 3

The iterated Prisoner's dilemma on deterministic graphs

In this chapter, we study the well-known prisoner's dilemma in a spatial setting. The prisoner's dilemma is a two-person game described by $(S_I, S_{II}, A_I, A_{II})$, where

1. $S_I = S = \{C, D\}$ is the set of possible moves of player I;
2. $S_{II} = S = \{C, D\}$ is the set of possible moves of player II;
3. A_I and A_{II} are the score functions defined on $S_I \times S_{II}$, and
4. $A_I(i,j) = A_{II}(j,i) = x_{ij}$, $i,j \in S$, where $x_{C,C} = 1$, $x_{C,D} = 0$, $x_{D,C} = b$ and $x_{D,D} = a$, with $0 < a < 1, 1 < b < 2, a + b \leq 2$.

Here C stands for "cooperate" and D for "defect". An alternative way to write this is in terms of the payoff bi-matrix

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 1,1 & 0,b \\ b,0 & a,a \end{pmatrix} \end{array}$$

with $0 < a < 1, 1 < b < 2, a + b \leq 2$.
Consider a graph $G(V, E)$, where V and E are the vertex set and the edge set respectively. Let vertices and edges represent players and interaction respectively. More precisely, two vertices connected by an edge represent two players playing against each other. We use $i \leftrightarrow j$ to indicate that player i and player j play against each other, $i, j \in V$ (i.e. that there is an edge between i and j). We define the neighbors of player $i \in V$ as those players whose vertices are connected by an edge to the vertex of player i.
Let $s_t^i \in \{C, D\}$ be the move chosen by player i at time $t = 0, 1, ...,$ $i \in V$. Let also $N_i = \{i, N_i^1, ..., N_i^{k_i}\}$ be the neighborhood of player i, where $N_i^1, N_i^2, ..., N_i^{k_i}$ are the k_i neighbors of player i. Consider now a game played with the following rules:

- at time zero, each player $i \in V$ chooses, independently, move C with probability p or move D with probability $q = 1 - p$;
- at time $t = 1, 2, ...,$ each player $i \in V$ plays the prisoner's dilemma against its neighbors;
- at time one, player i chooses s_0^i, and at time $t = 2, 3, ...,$ it chooses move according to the following updating rule:

$$s_i^{t+1} = \begin{cases} C & \text{if } \nexists k \in N_i : v_k^t = \max_{j \in N_i} v_j^t, s_k^t = D \\ D & \text{if } \nexists k \in N_i : v_k^t = \max_{j \in N_i} v_j^t, s_k^t = C \\ s_i^t & \text{otherwise,} \end{cases}$$

where v_i^t is the total score of player i at time $t = 1, 2, ...$ More formally, $v_i^t = v_i^{t,C} \mathbf{1}_{\{s_i^t = C\}} + v_i^{t,D} \mathbf{1}_{\{s_i^t = D\}}$, where

$$v_i^{t,C} = \sum_{j: i \leftrightarrow j} \mathbf{1}_{\{s_j^t = C\}}$$

and

$$v_i^{t,D} = b \sum_{j: i \leftrightarrow j} \mathbf{1}_{\{s_j^t = C\}} + a \sum_{j: i \leftrightarrow j} \mathbf{1}_{\{s_j^t = D\}}$$

are the total score of player i at time t given that it chooses move C and D respectively.

In short, the interpretation is as follows: at time $t = 2, 3, ...$, player $i \in V$ chooses the move of the player (within its neighborhood) with highest score in the previous round. If there are two or more players with highest score and different move, then player i chooses the same move as the one chosen in the previous round.

In this chapter, we study the repeated prisoner's dilemma played on some deterministic graphs which have the common property that they contain infinitely many players and where the number k_i of neighbors is equal for all $i \in V$. As we have seen above, cooperation is a weak strategy against defection. The advantage of spatial structure is that it can make it possible for cooperators to form clusters protecting themselves against defectors [12][4]. This happens because the winnings of mutual cooperation can overbalance losings against defection [12]. Here, we study only cases when a D cannot be defeated so that the system is stable after a finite number of rounds and the calculations become more simple.
We are mainly interested in the probability that cooperation survives as the game is played repeatedly, in particular that a given player $i \in V$ survives as a cooperator as $t \to \infty$. More formally, we study the limit $\pi_p(C) = \lim_{t\to\infty} P(s_t^i = C)$. To simplify, we refer to a player which chooses move C as a C. In the same way, we refer to a player which chooses move D as a D. We are also interested in the size of the cluster in which player $i \in V$ survives as C. We may regard such a cluster as a community of cooperators. In this sense, we define a cluster as a connected component of players such that all players choose same move. Let $\mathcal{C}_i$ be the number of C's in the cluster which contains player i, as $t \to \infty$. In some cases, we find the distribution of $\mathcal{C}_i$.

3.1 One dimensional lattice

Consider firstly the one dimensional lattice, where the players are the elements of $\mathbb{Z}$ and where the game is played between each pair of players whose Euclidean distance is one. This means that player i plays against player $i+1$ and $i-1$, $i \in \mathbb{Z}$. This is a nice case since it is independent of a and b, as we see in the following theorem:

Theorem 3.1.1. *For all a and b such that $0 < a < 1, 1 < b < 2, a+b \leq 2$,*

$$\pi_p(C) \quad = \quad \left(q+p^2\right)^2 p^3 \left(3-2p\right).$$

Proof. When the game is played on the one dimensional lattice, each player has only two neighbors. This means that the total score for player $i \in \mathbb{Z}$, at each time, is 2, 1 or 0 if it chooses move C; and it is $2b$, $b+a$ or $2a$ if it chooses move D. Since $b > 1$, a D cannot change to C. Since $a+b \leq 2$ and $2b > 2$, a player which plays C at time t plays C also at time $t+1$ if and only if at time t: (i) it plays against two C's ; or (ii) it plays against a C and a D which play against another C and another D respectively. In particular, this means that player i has to belong, at each time, to a cluster of at least three C's. Figure 3.1 shows initial configurations which give a final cluster of three C's. Consequently, $P(\mathcal{C}_i = 1) = P(\mathcal{C}_i = 2) = 0$, and for $s = 3, 4, \ldots$, we have that

$$P(\mathcal{C}_i = s) = sp^s q^4 + 2sp^{s+2}q^3 + sp^{s+4}q^2 = sp^s q^2 \left(q+p^2\right)^2.$$

Then,

$$\begin{aligned}
\pi_p(\mathrm{C}) &= \sum_{s=1}^{\infty} P(\mathcal{C}_i = s) \\
&= \sum_{s=3}^{\infty} sp^s q^2 \left(q+p^2\right)^2 \\
&= pq^2 \left(q+p^2\right)^2 \left(\sum_{s=1}^{\infty} \frac{d}{dp} p^s - 1 - 2p\right) \\
&= \left(q+p^2\right)^2 p^3 \left(3-2p\right).
\end{aligned}$$

□

The expectation of $\mathcal{C}_i$ is

$$\begin{aligned}
\mathbb{E}(\mathcal{C}_i) &= \sum_{s=3}^{\infty} s^2 \left(p^s q^4 + 2p^{s+2} q^3 + p^{s+4} q^2\right) \\
&= \sum_{s=3}^{\infty} s(s+1) \left(p^s q^4 + 2p^{s+2} q^3 + p^{s+4} q^2\right) \\
&\quad -(q+p^2)^2 p^3 (3-2p) \\
&= \sum_{s=1}^{\infty} s(s+1) \left(p^s q^4 + 2p^{s+2} q^3 + p^{s+4} q^2\right) \\
&\quad -2\left(pq^4 + 2p^3 q^3 + p^5 q^2\right) - 6\left(p^2 q^4 + 2p^4 q^3 + p^6 q^2\right) \\
&\quad -(q+p^2)^2 p^3 (3-2p) \\
&= \left(q^4 p + 2q^3 p^3 + q^2 p^5\right) \frac{d^2}{dp^2} \sum_{s=1}^{\infty} p^{s+1} \\
&\quad -2\left(pq^4 + 2p^3 q^3 + p^5 q^2\right) - 6\left(p^2 q^4 + 2p^4 q^3 + p^6 q^2\right) \\
&\quad -(q+p^2)^2 p^3 (3-2p) \\
&= \left(q^4 p + 2q^3 p^3 + q^2 p^5\right) \left(\frac{2}{q^3} - 2 - 6p\right) - (q+p^2)^2 p^3 (3-2p) \\
&= \frac{p^3 (1-qp)^2 (9 + p(4p-11))}{q}.
\end{aligned}$$

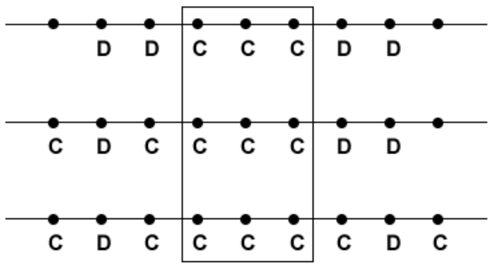

Figure 3.1: Some configurations which produce a final cluster of three C's.

3.2 $\mathbb{Z} \times \mathbb{Z}_2$

As a slightly more complex structure involving a second dimension in the simplest possible form, consider now the lattice on $\mathbb{Z} \times \mathbb{Z}_2$, as we see in Figure 3.2. Let each element $(i, k) \in \mathbb{Z} \times \mathbb{Z}_2$ be a player. For each pair of players, let the players play against each other if their Euclidean distance is one. We refer to the set of the players (i, k), $i \in \mathbb{Z}$, as the players in the k^{th} row, $k = 1, 2$. This case, as in most cases, is not independent of

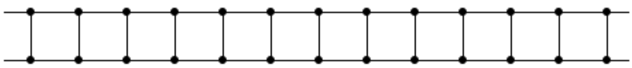

Figure 3.2: A part of the lattice on $\mathbb{Z} \times \mathbb{Z}_2$.

the parameters of the payoff matrix, as we see shall now see.

Theorem 3.2.1. *If* $3 \geq 2a + b > 2, a + 2b > 3$, *then*

$$\pi_p(C) = \left(q^2 + p^4\right)^2 q^4 p^2 \left(\frac{1}{\left(1 - p^2\right)^2} - 1 - 2p^2\right).$$

Proof. Consider a player which chooses move C. Since $3 \geq 2a + b$ and $a+2b > 3$, at each time this player survives as C if and only if: (i) it plays against three C's; or (ii) at least one of its neighbors is a C which plays against two other C's and each D it plays against, plays only against other D's. Since $2a + b > 2$, it is not possible for a D to change to C. Suppose now that player i belongs, at time zero, to a cluster of s C's, $s = 1, 2, \ldots$. Suppose also that there is at least one D which plays against two or more C's of this cluster. Because of the structure of this graph, such a cluster cannot overbalance the losses against defection. Figure 3.3 shows an initial cluster of eight C's which cannot survive. Consequently, player i survives as C if and only if it belongs to an initial cluster of $s \times 2$ C's, $s = 3, 4, \ldots$ Figure 3.4 shows some initial configurations which produce a final cluster of six C's.

Now it is clear that $P(\mathcal{C}_i = 1) = \ldots = P(\mathcal{C}_i = 5) = 0$, and for $s = 3, 4, \ldots$, $P(\mathcal{C}_i = 2s - 1) = 0$ and $P(\mathcal{C}_i = 2s) = sp^{2s}q^8 + 2sp^{2(s+2)}q^6 + sp^{2(s+4)}q^4$.

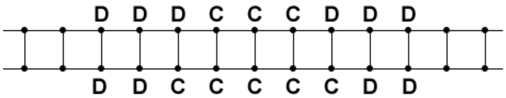

Figure 3.3: An initial cluster of eight C's which cannot resist the attack of the defectors.

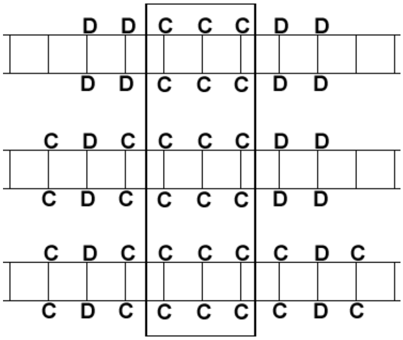

Figure 3.4: Some configurations which give a final cluster of six C's.

Thus,

$$
\begin{aligned}
\pi_p(\mathrm{C}) &= \sum_{s=1}^{\infty} P(\mathcal{C}_i = s) \\
&= \sum_{s=3}^{\infty} s p^{2s} q^8 + 2 \sum_{s=3}^{\infty} s p^{2(s+2)} q^6 + \sum_{s=3}^{\infty} s p^{2(s+4)} q^4 \\
&= \left(q^2 + p^4\right)^2 q^4 \sum_{s=3}^{\infty} s p^{2s} \\
&= \left(q^2 + p^4\right)^2 q^4 \left(\frac{p^2}{(1-p^2)^2} - p^2 - 2p^4 \right)
\end{aligned}
$$

□

In this case, the expectation of $\mathcal{C}_i$ is

$$\begin{aligned}\mathbb{E}(\mathcal{C}_i) &= 2\sum_{s=3}^{\infty} s\left(sp^{2s}q^8 + 2sp^{2(s+2)}q^6 + sp^{2(s+4)}q^4\right)\\ &= \frac{2qp^6(9-11p^2+4p^4)(1+p(p+p^3-2))^2}{(1+p)^3}.\end{aligned}$$

We now study a different case with the condition $a+2b \leq 3$ instead of $a+2b > 3$. The new condition means that a cluster of four C's in which one C plays against the three other C's remains intact if and only the C's do not play, at time one, against a D which plays against all C's.

Theorem 3.2.2. *If $2a+b>2$ and $a+2b \leq 3$, then*

$$\begin{aligned}\pi_p(C) = &\left(q+p^2(1-qp)\right)^2\left(p^4(4-4p^2)\right)\\ &+p^6\left(q^2+2pq^2+p^2\right)\left(2p\left(q+p^2(1-qp)\right)-\left(q^2+2pq^2+p^2\right)\right).\end{aligned}$$

Proof. Since $2a+b>2$, a necessary condition for player i to survive as C is that it belongs to a cluster of C's in which every time there is at least one C with total score equal to 3. Since $a+2b \leq 3$, a D invades that cluster if and only if it plays against all C's, and this can only happen at time one. We recall that since $2a+b>2$, a D cannot change to C. Let $A_j, j=1,...,4$, be the event that player i survives as C in a cluster of the form shown in figure 3.5.

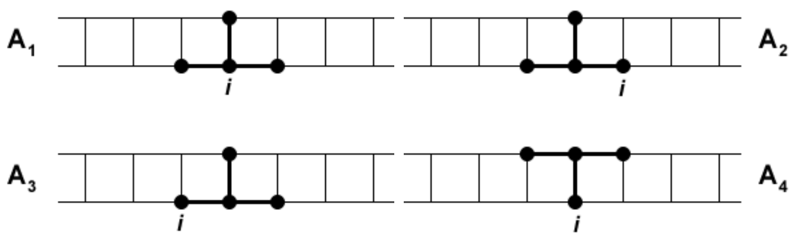

Figure 3.5: Player i belonging to different clusters of four C's in which one C has total score 3.

From the above considerations, $P(A_1) = ... = P(A_4) = p^4\big(q + p^2(1-qp)\big)^2$. We also have that $P(A_1 \cap A_2) = P(A_1 \cap A_3) = p^6\big(q + p^2(1-qp)\big)^2$, $P(A_1 \cap A_4) = p^6\big(q^2 + 2pq^2 + p^2\big)^2$, $P(A_2 \cap A_3) = p^7\big(q + p^2(1-qp)\big)^2$, $P(A_2 \cap A_4) = P(A_3 \cap A_4) = p^6\big(q + p^2(1-qp)\big)^2$, $P(A_1 \cap A_2 \cap A_3) = P(A_2 \cap A_3 \cap A_4) = p^7\big(q + p^2(1-qp)\big)^2$, $P(A_1 \cap A_2 \cap A_4) = P(A_1 \cap A_3 \cap A_4) = p^7\big(q + p^2(1-qp)\big)\big(q^2 + 2pq^2 + p^2\big)$ and
$P(A_1 \cap A_2 \cap A_3 \cap A_4) = p^7\big(q + p^2(1-qp)\big)^2$.
The inclusion-exclusion formula gives

$$\begin{aligned}
\pi_p(\mathrm{C}) = P\left(\bigcup_{i=1}^{4} A_i\right) & \\
=\ & \sum_{i=1}^{4} P(A_i) - \sum_{i<j} P(A_i \cap A_j) \\
& + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - P(A_1 \cap A_2 \cap A_3 \cap A_4) \\
=\ & 4p^4\big(q + p^2(1-qp)\big)^2 - 2p^6\big(q + p^2(1-qp)\big)^2 \\
& - p^6\big(q^2 + 2pq^2 + p^2\big)^2 - p^7\big(q + p^2(1-qp)\big)^2 \\
& - 2p^6\big(q + p^2(1-qp)\big)^2 + 2p^7\big(q + p^2(1-qp)\big)\big)^2 \\
& + 2p^7\big(q + p^2(1-qp)\big)\left(q^2 + 2pq^2 + p^2\right) - p^7\big(q + p^2(1-qp)\big)^2 \\
=\ & \left(q + p^2(1-qp)\right)^2 \left(p^4(4 - 4p^2)\right) \\
& + p^6\left(q^2 + 2pq^2 + p^2\right)\left(2p\left(q + p^2(1-qp)\right) - \left(q^2 + 2pq^2 + p^2\right)\right).
\end{aligned}$$

□

Remark. In this case $(2a + b > 2, a + 2b \leq 3)$, we do not compute the expectation of $\mathcal{C}_i$ because it becomes much more complicated to obtain $P(\mathcal{C}_i = s)$ when s is large.
In the previous cases, we had the condition $2a+b > 2$, which means that player i survives as C if and only if it belongs to a cluster in which every time there is at least one C with total score 3. We now study a different case, still on $\mathbb{Z} \times \mathbb{Z}_2$, with the new condition $2a + b = 2$. This allows player i to belong to a cluster in which there is at least one C with total score 2 or 3.

Theorem 3.2.3. *If $2a+b=2$ and $a+2b>3$, then*

$$\begin{aligned}\pi_p(C) &= \left(q^2+p^4\right)^2 q^4 \left(\frac{p^2}{\left(1-p^2\right)^2}-p^2\right) \\ &\quad + \frac{q^3p^3(3-2p+2p^2)}{\left(q+p^2\right)^2}\left(q^3+pq\right)^2.\end{aligned}$$

Proof. Given these conditions, a C survives as C if and only if at each time there is at least one C in its neighborhood with total score 2 or 3 and every time some player of this cluster plays against a D, that D plays only against other D's. Thus, there are two kind of clusters in which player i can survive as C. One is when the C's are positioned in the same row. Given this graph, a C survives in a cluster of s C's in a row, $s=3,4,...$, if and only if there are only D's in the other row, and in the ends of the rows there is no D which plays against two or more C's. Figure 3.6 shows initial configurations for which player i survives as C in a cluster of 3 C's in a row.

Then, the probability that player i survives as C in a cluster of s C's in a row, $s=3,4,...$, is

$$\begin{aligned}&sp^sq^{s+6}+2sp^{s+1}q^{s+4}(pq+pq^2+p^2q(1-q^2)) \\ &+sp^{s+2}q^{s+2}(pq+pq^2+p^2q(1-q^2))^2.\end{aligned}$$

Another kind of cluster in which player i can survive as C consists of $2s$ C's, $s=2,3,...$, with s of them in each row. Such a cluster remains intact if and only if, at each time, the cluster plays against two C's or two D's (which play against two other D's) in each end. Thus, the probability that player i survives as C in such a cluster of $2s$ C's, $s=2,3,...$, is

$$sp^{2s}q^8+2sp^{2(s+2)}q^6+sp^{2(s+4)}q^4.$$

Then,

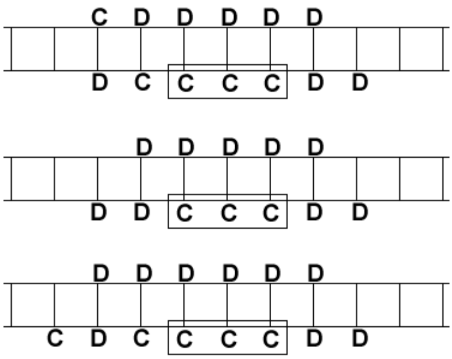

Figure 3.6: Some initial configurations which give a final cluster of three C's.

$$
\begin{aligned}
\pi_p(\mathrm{C}) &= \sum_{s=2}^{\infty} sp^{2s}q^8 + 2\sum_{s=2}^{\infty} sp^{2(s+2)}q^6 + \sum_{s=2}^{\infty} sp^{2(s+4)}q^4 \\
&+ \sum_{s=3}^{\infty} sp^s q^{s+6} + 2\sum_{s=3}^{\infty} sp^{s+1}q^{s+4}(pq + pq^2 + p^2q(1-q^2)) \\
&+ \sum_{s=3}^{\infty} sp^{s+2}q^{s+2}(pq + pq^2 + p^2q(1-q^2))^2 \\
&= \sum_{s=2}^{\infty} sp^{2s}q^4\left(q^4 + 2p^4q^2 + p^8\right) + \sum_{s=3}^{\infty} sp^s q^s\left(q^6 + 2pq^4 + p^2q^2\right) \\
&= \left(q^2 + p^4\right)^2 q^4 \left(\frac{p^2}{(1-p^2)^2} - p^2\right) \\
&+ \frac{q^3p^3(3 - 2p + 2p^2)}{(q+p^2)^2}\left(q^3 + pq\right)^2.
\end{aligned}
$$

□

We study the same case $(2a + b = 2, a + 2b > 3)$ further in order to find the distribution of $\mathcal{C}_i$. From the considerations above, it is clear that $P(\mathcal{C}_i = 1) = P(\mathcal{C}_i = 2) = 0$, and for $s = 2, 3, ...,$

$$
\begin{aligned}
P(\mathcal{C}_i = 2s-1) &= (2s-1)p^{2s-1}q^{2s+5} \\
&\quad +2(2s-1)p^{2s}q^{2s+3}(pq+pq^2+p^2q(1-q^2)) \\
&\quad +(2s-1)p^{2s+1}q^{2s+1}(pq+pq^2+p^2q(1-q^2))^2,
\end{aligned}
$$

and

$$
\begin{aligned}
P(\mathcal{C}_i = 2s) &= sp^{2s}q^8 + 2sp^{2(s+2)}q^6 + sp^{2(s+4)}q^4 \\
&\quad +2sp^{2s}q^{2s+6} + 4sp^{2s+1}q^{2s+4}(pq+pq^2+p^2q(1-q^2)) \\
&\quad +2sp^{2s+2}q^{2s+2}(pq+pq^2+p^2q(1-q^2))^2.
\end{aligned}
$$

The expectation of $\mathcal{C}_i$ is

$$
\begin{aligned}
\mathbb{E}(\mathcal{C}_i) &= \sum_{s=2}^{\infty} 2sP(\mathcal{C}_i = 2s) \\
&\quad + \sum_{s=2}^{\infty}(2s-1)P(\mathcal{C}_i = 2s-1) \\
&= \frac{2qp^4\left(4-3p^2+p^4\right)\left(1+p\left(-2+p+p^3\right)\right)^2}{(1+p)^3} \\
&\quad + \frac{1}{(1-qp)^3}q^7p^3(9-qp(11-4qp)) \times \\
&\quad \times \left(-1+p\left(1+(-2+p)p\left(1+p^2\right)\right)\right)^2.
\end{aligned}
$$

3.3 The binary tree

In this section, we study the iterated prisoner's dilemma played on the binary tree. Let each vertex represent a player, and let those players whose vertices are connected by an edge play against each other. This means that each player plays against three other players. Our first result about the tree is the following theorem:

Theorem 3.3.1. *If* $2a+b>2, a+2b\leq 3$ *then*

$$\pi_p(C) \;=\; p^4x^6\left(4-3p^2x^2(1+px-p^2x^2)\right)$$

where $x=q(1-p^2)+p$.

Proof. Since $2a+b>2$ and $a+2b\leq 3$, player i survives as C if and only if it belongs to a cluster of C's in which at each time there is one C with score equal to 3 and every time some player of this cluster plays against a D, that D does not play against two other C's. This means such a cluster can be invaded only at time one.
Consider the four possible positions of player i in a cluster of four C's, in which one C plays against the other three C's, as shown in figure 3.7. Let A_j, $j=1,2,3,4$, be the event that player i survives as C in such a cluster of four C's with i at position j. Set $x=q(1-p^2)+p$. Then, by independence in the initial configuration, $P(A_j)=p^4x^6$, $P(A_1\cap A_2)=\ldots=P(A_1\cap A_4)=p^6x^8$, $P(A_2\cap A_3)=\ldots=P(A_3\cap A_4)=p^7x^9$, $P(A_1\cap A_2\cap A_3)=\ldots=P(A_1\cap A_3\cap A_4)=p^8x^{10}$ and $P(A_2\cap A_3\cap A_4)=P(A_1\cap A_2\cap A_3\cap A_4)=p^{10}x^{12}$. The inclusion-exclusion formula gives:

$$\begin{aligned}
\pi_p(\mathrm{C}) &= P\left(\bigcup_{i=1}^{4} A_i\right)\\
&= 4P(A_1)-3P(A_1\cap A_2)-3P(A_2\cap A_3)\\
&\quad +3P(A_1\cap A_2\cap A_3)+P(A_2\cap A_3\cap A_4)\\
&\quad -P(A_1\cap A_2\cap A_3\cap A_4)\\
&= 4p^4x^6-3p^6x^8-3p^7x^9+3p^8x^{10}\\
&= p^4x^6\left(4-3p^2x^2(1+px-p^2x^2)\right).
\end{aligned}$$

□

The second part of this section is concerned with the case $2a+b>2, a+2b>3$. Proposition 3.3.2 gives not an exact result, but it presents an estimation of $\pi_p(C)$. Before we proceed, consider player i and its neighbors and second neighbors $i_1,i_2,i_3,i_{11},i_{12},i_{21},i_{22},i_{31}$ and i_{32}, as shown in Figure 3.8. Let B_i be the event that player i and its three neighbors are C's at time zero. Obviously, $P(B_i)=p^4$. Consider now

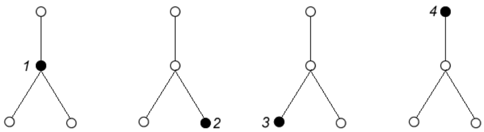

Figure 3.7: Different positions in a cluster of four C's in which there is a C with total score equal to 3.

player i_{jk}, $j = 1, 2, 3$, $k = 1, 2$. Let also F_{jk} be the event that $v^t_{i_{jk}} \leq 3$ for all $t = 1, 2, ...$, supposing that player i_j only plays against i_{jk}. Set now $x = P(F_{jk}|B_i)$.

Proposition 3.3.2. *If* $3 > 2a + b > 2, a + 2b > 3$, *then*

$$\pi_p(C) \quad = \quad p^4 x^6 \left(4 - 3p^2 x^2 \left(1 + px - p^2 x^2\right)\right).$$

In addition, $q^2 \leq x \leq q^3 + p$.

Proof. Consider a player which chooses move C. At each time, in order to survive as C, this player has to belong to a cluster of C's in which at least one C has total score three. Also, if a player of this cluster plays against a D, that D plays only against other D's. Since $2a + b > 2$, a D cannot change to C. Consider also the four positions in such a cluster of four C's in which one C plays against the three other C's, as shown in Figure 3.7. Let again A_j, $j = 1, 2, 3, 4$, be the event that player i survives as C with position j. Consider firstly the event A_1. Then we have that

$$P(A_1) \quad = \quad P(B_i) P \left(\bigcap_{j=1}^{3} \bigcap_{k=1}^{2} F_{jk} | B_i \right).$$

Consider now each player i_{jk}, $j = 1, 2, 3$, $k = 1, 2$. If player i_{jk} chooses at time zero move D, then the cluster is not invaded if and only if i_{jk}'s two other neighbors also choose at time zero move D. If player i_{jk} instead chooses at time zero move C, then the cluster is not invaded if (but not only if) i_{jk}'s two other neighbors choose at time zero move D. Thus,

$q^2 \leq x \leq q^3 + p$. Since the i_{jk}'s do not have neighbors in common (excluding the i_j's), it is then clear that

$$P\left(\bigcap_{j=1}^{3}\bigcap_{k=1}^{2} F_{jk}|B_i\right) = x^6.$$

By independence in the initial configuration and using the same arguments, it is easily seen that $P(A_1) = ... = P(A_4) = p^4x^6$, $P(A_1 \cap A_2) = ... = P(A_1 \cap A_4) = p^6x^8$, $P(A_2 \cap A_3) = ... = P(A_3 \cap A_4) = p^7x^9$, $P(A_1 \cap A_2 \cap A_3) = ... = P(A_1 \cap A_3 \cap A_4) = p^8x^{10}$ and $P(A_2 \cap A_3 \cap A_4) = P(A_1 \cap A_2 \cap A_3 \cap A_4) = p^{10}x^{12}$. The inclusion exclusion formula gives,

$$\begin{aligned}
\pi_p(\mathrm{C}) &= P\left(\bigcup_{j=1}^{4} A_j\right) \\
&= 4p^4x^6 - 3p^6x^8 - 3p^7x^9 + 3p^8x^{10} \\
&= p^4x^6\left(4 - 3p^2x^2\left(1 + px^1 - p^2x^2\right)\right).
\end{aligned}$$

□

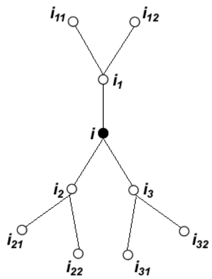

Figure 3.8: The neighbors and second neighbors of player i.

3.4 The $(n-1)$-nary tree

As a generalization of much of the above, we now study the evolution of cooperation when the iterated prisoner's dilemma is played on the $(n-1)$-nary tree, $n = 2, 3, ...$ Note that the one-dimensional lattice, studied in the beginning of this chapter, is the special case of the $(n-1)$-nary tree with $n = 2$. The following theorem generalizes **Theorem 3.1.1** and **Theorem 3.3.1**.

Theorem 3.4.1. *If* $a + (n-1)b \leq n, (n-1)a + b > n - 1$ *then*

$$\begin{aligned}\pi_p(C) &= p^{n+1}x^{n(n-1)}\left(1 - p^{n-1}x^{2-3n+n^2}\right)^n \\ &\quad -px^n\left(\left(1 - p^n x^{(n-2)n}\right)^n - 1\right),\end{aligned}$$

where $x = p + q(1 - p^{n-1})$.

Proof. In the $(n-1)$-nary tree, each player plays against its n neighbors. Since $(n-1)a + b > n - 1$, player i, in order to survive as a C, has to belong to a cluster in which, at each time, there is at least one C having total score equal to n. Consider now the $n+1$ possible positions of player i, in a cluster of $(n+1)$ C's, in which one C plays against the other n C's (compare Figure 3.7). Since $a + (n-1)b \leq n$, a D invades that cluster if and only if it plays against all C's. Thus, such a cluster can only be invaded at time one. Since $(n-1)a + b > n - 1$, a D cannot change to C. Let now A_j, $j = 1, ..., n+1$, be the event that player i survives as C in such a cluster of $(n+1)$ C's with i at position j. Here position 1 refers to the position of the C playing against the other n C's. By independence at time zero, it is clear that $P(A_1) = ... = P(A_{n+1}) = p^{n+1}(p + q(1 - p^{n-1}))^{(n-1)n}$. Set $x = p + q(1 - p^{n-1})$. Then, we have that $P(A_1 \cap A_2) = ... = P(A_1 \cap A_{n+1}) = p^{2n}x^{2(n-1)^2}$ and in general, $P(A_1 \cap A_2 \cap ... \cap A_m) = p^{n+1+(m-1)(n-1)}x^{(m-1)(n-1)^2+(n-1)(n-(m-1))}$, $m = 2, ..., n+1$. We also have that $P(A_2 \cap A_3) = p^{2n+1}x^{2(n-1)^2+n-2}$ and in general, $P(A_2 \cap ... \cap A_m) = p^{(m-1)n+1}x^{(m-1)(n-1)^2+(n-(m-1))}$,

$m = 3, ..., n+1$. The inclusion-exclusion formula gives in this case:

$$
\begin{aligned}
\pi_p(\mathrm{C}) &= P\left(\bigcup_{i=1}^{n+1} A_i\right) \\
&= (n+1)P(A_1) - nP(A_1 \cap A_2) - \binom{n}{2} P(A_2 \cap A_3) \\
&\quad +... + (-1)^{n-1} P(\cap_{j=1}^{n} A_j) \\
&= (n+1)P(A_1) + \sum_{i=1}^{n} \binom{n}{i} (-1)^i P(A_1 \cap ... \cap A_{i+1}) \\
&\quad + \sum_{i=2}^{n} \binom{n}{i} (-1)^{i-1} P(A_2 \cap ... \cap A_{i+1}) \\
&= (n+1)p^{n+1} x^{(n-1)n} \\
&\quad + \sum_{i=1}^{n} \binom{n}{i} (-1)^i p^{n+1+i(n-1)} x^{i(n-1)^2+(n-1)(n-i)} \\
&\quad + \sum_{i=2}^{n} \binom{n}{i} (-1)^{i-1} p^{in+1} x^{i(n-1)^2+n-i} \\
&= (n+1)p^{n+1} x^{(n-1)n} \\
&\quad + p^{n+1} x^{n(n-1)} \sum_{i=1}^{n} \binom{n}{i} (-1)^i p^{i(n-1)} x^{i(2-3n+n^2)} \\
&\quad - p x^n \sum_{i=2}^{n} \binom{n}{i} (-1)^i p^{in} x^{i(n-2)n}.
\end{aligned}
$$

Applying the Binomial theorem to the last expression, we finally get

$$
\begin{aligned}
\pi_p(\mathrm{C}) &= (n+1)p^{n+1} x^{(n-1)n} \\
&\quad + p^{n+1} x^{n(n-1)} \left(\left(1 - p^{n-1} x^{2-3n+n^2}\right)^n - 1 \right) \\
&\quad - p x^n \left(\left(1 - p^n x^{(n-2)n}\right)^n - 1 - n p^n x^{(n-2)n} \right) \\
&= p^{n+1} x^{n(n-1)} \left(1 - p^{n-1} x^{2-3n+n^2}\right)^n \\
&\quad - p x^n \left(\left(1 - p^n x^{(n-2)n}\right)^n - 1 \right).
\end{aligned}
$$

□

3.5 The d-dimensional lattice

So far we have studied the evolution of cooperation in relatively simple networks. One natural step towards more complex graphs are the d-dimensional lattices, $d = 2, 3...$, where the players are the elements of $\mathbb{Z}^d$ and those players which are adjacent play against each other. As we have seen, spatial structure can make cooperation survive. This success occurs when cooperators form clusters protecting themselves against defectors. When the game is played on a lattice, it becomes of immediate interest to know if there is positive probability that a given player survives in an infinite cluster of cooperators. We interpret the existence of such a cluster as if cooperation dominates the community.
Here, we analyze the case when $(2d-1)a+b > 2d-1$ and $a+(2d-1)b \leq 2d$. We study this case because of its simplicity: in this case, a C located inside a large enough cluster of C's cannot change to D. Firstly, we establish that there is $p \in (0, 1)$ such that the probability that player i survives as C in an infinite cluster of C's is greater than zero. This is closely related to Percolation theory. In fact, we use very similar arguments of the proof of **Theorem 1.10** in [8] to show **Theorem 3.5.1**. Let $\theta(p) = P(\mathcal{C}_i = \infty)$ and $p_c = \sup\{p : \theta(p) = 0\}$.

Theorem 3.5.1. *If* $(2d-1)a + b > 2d-1$ *and* $a + (2d-1)b \leq 2d$, *then*

$$0 < p_c(d) < 1,$$

for $d = 2, 3, ...$

Proof. Firstly, we show that $p_c(d) > 0$. Since $(2d-1)a + b > 2d-1$, a D cannot change strategy. This means that, in order to have an infinite cluster of C's as $t \to \infty$, a necessary condition is that player i belongs to an infinite cluster of C's at time zero. Let $\sigma(n)$ be the number of paths which have length n and which start at the origin. It is clear that $\sigma(n)$ is at most $2d(2d-1)^{n-1}$ [8]. Let also $N(n)$ be the number of those paths which contain only C's. Then

$$\begin{aligned}
\theta(p) &\leq P_p(N(n) \geq 1) \\
&\leq E_p(N(n)) \\
&= p^n \sigma(n) \\
&\leq p^n 2d(2d-1)^{n-1},
\end{aligned}$$

for all n. This means that $\theta(p) = 0$ if $p < \frac{1}{2d-1}$. Thus, $p_c(d) > 0$.
Now, we show that $p_c(d) < 1$. Consider firstly a symmetric $5 \times 5 \times ... \times 5$ d-dimensional box which contains in the center the origin. Let all players contained in the box be cooperators. Consider now the player located in the center of this box, i.e. player $(0, ..., 0)$. Note that the minimal number of edges needed to connect this player to a player located outside the box is three. Since $a + (2d-1)b \leq 2d$, player $(0, ..., 0)$ cannot change strategy. This means that if at time zero there is an infinite connected component of such boxes, then cooperation survives in an infinite cluster independently of what happens to the other players. Let $\hat{\theta}(p)$ be the probability that there exists an infinite connected component of disjoint symmetric $5 \times 5 \times ... \times 5$ d-dimensional boxes such that all players are C's, and the origin is contained in the center of one of these boxes. Clearly, $\theta(p) \geq \hat{\theta}(p)$. Such a cluster is finite if and only if all paths to infinity are blocked by a connected path of disjoint $5 \times 5 \times ... \times 5$ d-dimensional boxes which (each of them) contain at least one D. Note that diagonals are also used to connect boxes in such a path (since the boxes block the paths from the origin to infinity). Call the box which contains the origin, box 0. In the same way, call the box which contains player $(5k, 0, ..., 0)$ in the center, box k, $k \in \mathbb{Z}$. If the there is such a component enclosing the origin, then it must contain some box k, $k = 1, 2, ...$ Note now that if box k belongs to such a component, there is at least one self-avoiding path starting from that box, passing through some box m, $m = -1, -2, ...$ and returning to box k. Then, such a path contains n boxes only if $k < n$. Since each box is connected to one of the $3^d - 1$ closest located boxes and since the path is a self-avoiding walk, there are no more than $\left(3^d - 1\right)\left(3^d - 2\right)^{n-1}$ such paths. Therefore,

$$\begin{aligned}
1 - \theta(p) &\leq 1 - \hat{\theta}(p) \\
&\leq \sum_{n=1}^{\infty} \left(1 - p^{5^d}\right)^n n \left(3^d - 1\right)\left(3^d - 2\right)^{n-1} \\
&= \left(3^d - 1\right)\left(1 - p^{5^d}\right) \sum_{n=1}^{\infty} n \left(\left(1 - p^{5^d}\right)\left(3^d - 2\right)\right)^{n-1} \\
&= \frac{\left(3^d - 1\right)\left(1 - p^{5^d}\right)}{\left(1 - \left(3^d - 2\right)\left(1 - p^{5^d}\right)\right)^2} \qquad \text{if } \left(1 - p^{5^d}\right) < \frac{1}{\left(3^d - 2\right)}.
\end{aligned}$$

This means that $1-\theta(p) \to 0$ as $1-p^{5^d} \to 0$. Thus, we can find $0 < p_0 < 1$ such that $1-\theta(p) \leq \frac{1}{2}$ if $p > p_0$ and then $p_c \leq p_0$. □

As explained above, if $(2d-1)a+b > 2d-1$ and $a+(2d-1)b \leq 2d$, then a player located in the center of a $5 \times 5 \times ... \times 5$ d-dimensional box of cooperators survives as C independently of what happens to the other players, so obviously, $\pi_p(C) \geq p^{5^d}$. In order to give a better bound, we study the same case further. We only do this for $d = 2$ because the calculations become much more complicated for larger d.

Proposition 3.5.2. *If $3a+b > 3$ and $a+3b \leq 4$, then*

$$y < \pi_p(C) < 5y,$$

where

$$\begin{aligned} y \;=\; & p(2+p(-4+p(-4+p(10+p(7+p(-14+p\times \\ & \times(-10+p(20+p(-46+p(172+p(-203+p(-154+p\times \\ & \times(617+p(-746+p(777+p(-926+p(863+p(-506+p\times \\ & \times(147+p(70+p(-135+p(86-23p+p^3)))))))))))))))))))))). \end{aligned}$$

Proof. Since $3a+b > 3$, player $i \in \mathbb{Z}^2$ survives as C only if, at each time, it belongs to a cluster of five C's in which there is at least one C with total score equal to 4. Since $a+3b \leq 4$, this cluster remains intact if and only if it never plays against a D which plays against all C's. This means that a D can invade such a cluster of five C's only at time one. Now, consider the five positions of such cluster and let A_j be the event that player i survives as C, with position $j = 1, 2, ..., 5$, where position one refers to the C which plays against the four other C's. Note that since $3a+b > 3$, a D cannot be defeated by a C. Obviously, $P(A_1) = ... = P(A_5)$. We show in the **Appendix** that $P(A_i) = y$, y as given above. Since $P(A_i) < \pi_p(C) < 5P(A_i)$, the proof is complete. □

3.6 $\mathbb{Z} \times C_n$

In this section, we study the iterated prisoner's when the presence of defectors at time zero is rare. More precisely, we study the asymptotic distribution of cooperation as p tends to one. Firstly, consider the lattice on $\mathbb{Z} \times \mathbb{Z}_n$, $n = 2, 3, ...$ Let each element $(i, j) \in \mathbb{Z} \times \mathbb{Z}_n$ be a player. For each pair of players, let the players play against each other if their Euclidean distance is one. We refer to the set of the players (i, j), $i \in \mathbb{Z}$, as the players in the j^{th} row, $j = 1, 2, ..., n$. In the same way, we refer to the set of players (i, j), $j = 1, 2, ..., n$, as the players in the i^{th} column. In order to have symmetric conditions for all players, let each player located in the n^{th} row play against the player located in the same column and 1^{st} row. This means that player (i, n) plays against player $(i, 1)$, $i \in \mathbb{Z}$. We call a graph constructed in this way C_n. Note that the lattice on $\mathbb{Z} \times \mathbb{Z}_2$ is the particular case of this graph with $n = 2$.

We choose to study this graph instead of the square lattice because this one facilitates the calculations when p tends to one. In particular, we analyze three different cases with the common property that a D cannot change to C. With this property, it is easier to capture the evolution of defection when one single defector appears inside a cluster of cooperators. In other cases, when a D can change to C, we cannot establish rigorous results concerning $\lim_{p\to 1} \pi_p(C)$.
Let A be the event that player i starts as C in the center of a cluster of $5 \times n$ C's. Obviously, $P(A) \to 1$ as $p \to 1$. Since $P(s_i^t = C) = P(s_i^t = C|A)P(A) + P(s_i^t = C|A^c)P(A^c)$, we have that $\lim_{p\to 1} \pi_p(C) = \lim_{p\to 1} P(s_i^t = C|A)$. Now, we can show the following results:

Proposition 3.6.1. *If $a + 3b \leq 4$, then*

$$\lim_{p\to 1} \pi_p(C) \quad = \quad 1.$$

Proof. Consider a cluster of $5 \times n$ C's. Since $a + 3b \leq 4$, a D invades this cluster if and only if it plays against all C's. Thus, it is clear that $P(s_i^t = C|A) = 1$ for $t = 1, 2, ...$ $\square$

Theorem 3.6.2. *If $a + 3b > 4$ and $3a + b > 3$, then*

$$\lim_{p\to 1} \pi_p(C) = \begin{cases} 0 & \textit{if } n = 3, 4, 5 \\ 15/36 & \textit{if } n = 6 \\ 20/49 & \textit{if } n = 7 \\ 1 - \frac{6}{n} + \frac{9}{n^2} & \textit{if } n = 8, 9, ... \end{cases}$$

Proof. We show the case when $n \geq 8$. The other cases are shown in a similar way. Let the column of player i be column 0, and enumerate the columns to the right as 1,2,..., and to the left, as -1,-2,... Let the row of player i be row 0 and enumerate the rows upwards as $1, 2, ..., \lfloor \frac{n-1}{2} \rfloor$, and downwards as $-1, -2, ..., -\lceil \frac{n-1}{2} \rceil$. Let ξ_1 and ξ_2 be the first column to the right and to the left respectively which contains, at time zero, one or more D's. Let also X_k be the the number of D's in column k at time zero, $k \in \mathbb{Z}$. Obviously, $P(\xi_1 < \infty) = P(\xi_2 < \infty) = 1$ for all $p \in (0, 1)$. We also have that $P(X_k = 1|\xi_1 = k) = P(X_m = 1|\xi_2 = m) = \frac{n(1-p)p^{n-1}}{1-p^n} \to 1$ as $p \to 1$, $k = 1, 2, ...$, $m = -1, -2, ...$ This means that the probability that there are more than one D in the same column tends to zero as p tends to one. In fact, for any $n < \infty$, $P(X_{k+1} = 0, X_{k+2} = 0, ..., X_{k+n} = 0|\xi_1 = k) \to 1$ as $p \to 1$.
Since $a+3b > 4$ and $3a+b > 3$, if there is a single D inside a cluster of C's, then that D starts a cluster of D's which invade the C's. That cluster forms a cross whose arms have width equal to three rows/columns, as we can see in figure 3.9. Since other D's starting far away will not have effect on player i, we can assume that there is, at time zero and as p tends to one, only one D on each side of player i.

Such a cluster of D's grows if and only if at least one of the D's located in one of the ends plays against three C's. This means that player i survives as C if:

- at time zero, there is no D in row $-2, -1, 0, 1, 2$. This happens, as $p \to 1$, with probability $\left(\frac{n-5}{n}\right)^2$;
- there is a D which starts a cluster of D's in row 2 (-2) on one side, and there is no D starting in row -3,-2,-1,0,1 (3,2,1,0,-1) on the other side. This event has probability, as $p \to 1$, equal to $4\frac{n-6}{n^2} + \frac{2}{n^2}$;
- there is a D starting a cluster of D's in row 1 (-1) on one side, and this cluster is stopped by another cluster of D's started in row 2 or 3 (-2 or -3) on the other side. The probability for this is $8\frac{1}{n^2} \lim_{p\to 1} P(\xi_1 \geq \xi_2)$. The cluster of D's is stopped by another cluster of D's started in the same row, without defection of player i, only if $\xi_1 = \xi_2$. The probability for the last event, as $p \to 1$, is

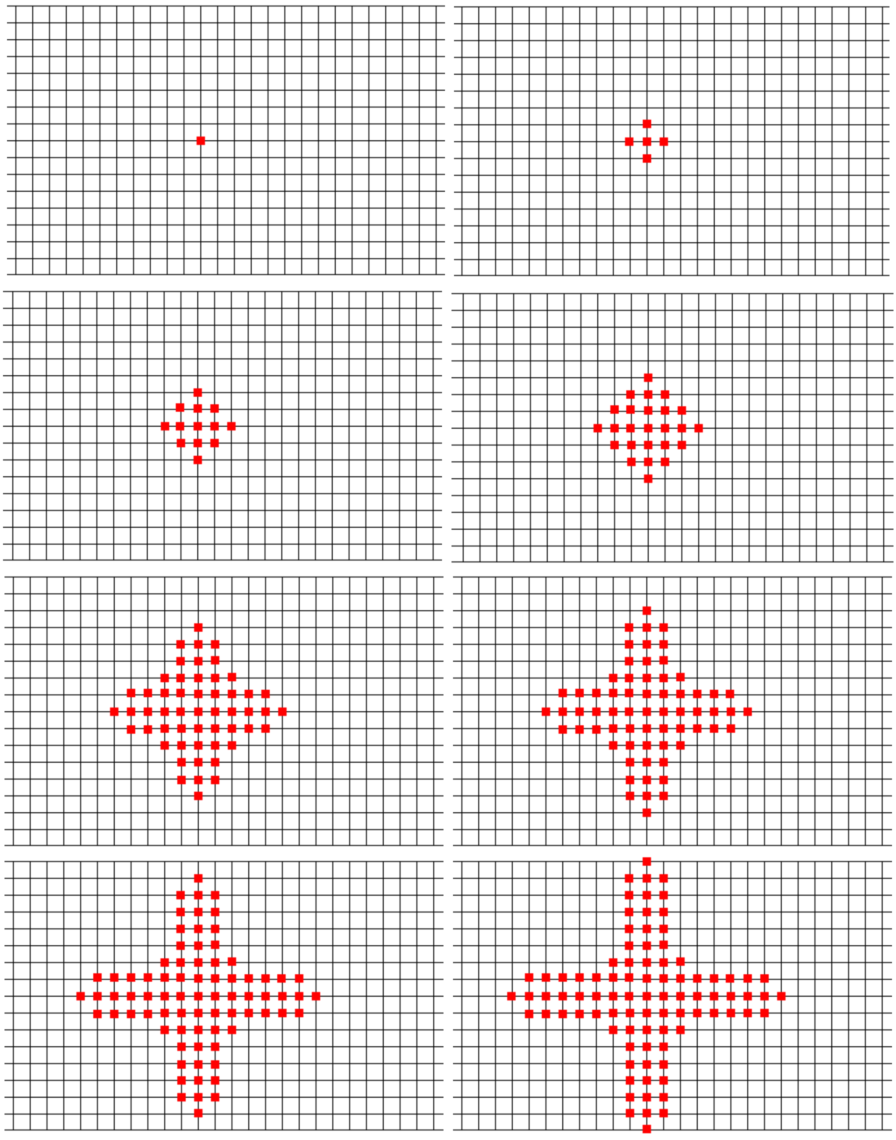

Figure 3.9: Evolution of a cluster of D's started by a single D, with $n = 17$ and the conditions $a + 3b > 4$ and $3a + b > 3$.

$2\frac{1}{n^2}\lim_{p\to 1} P(\xi_1 = \xi_2)$;

- there is a D starting a cluster of D's in row 0 on one side, and this cluster is stopped by another cluster of D's started by a D in row 2 or -2 on the other side. This has probability $4\frac{1}{n^2}\lim_{p\to 1} P(\xi_1 \geq \xi_2 + 2)$.

The random variables ξ_1 and ξ_2 are independent and geometrically distributed with parameter $1-p^n$. This means that $P(\xi_1 = \xi_2) = \frac{(1-p^n)}{1+p^n} \to 0$ as $p \to 1$. We also have that $P(\xi_1 \geq \xi_2 + k) = \sum_{t=1}^{\infty}\sum_{s=t+k}^{\infty} p^{n(s-1)}(1-p^n)p^{n(t-1)}(1-p^n) = \frac{p^{nk}}{1+p^n} \to \frac{1}{2}$ as $p \to 1$, $k = 1, 2, ...$ Thus,

$$\begin{aligned} \lim_{p\to 1} \pi_p(\mathrm{C}) &= \left(\frac{n-5}{n}\right)^2 + 4\frac{n-6}{n^2} + \frac{2}{n^2} \\ &\quad +8\frac{1}{n^2}\frac{1}{2} + 2\frac{1}{n^2}\frac{1}{2} + 4\frac{1}{n^2}\frac{1}{2} \\ &= 1 - \frac{6}{n} + \frac{9}{n^2}. \end{aligned}$$

□

Theorem 3.6.3. *If $a + 3b > 4$ and $3a + b = 3$, then*

$$\lim_{p\to 1} \pi_p(C) = \begin{cases} 0 & \text{if } n = 3, 4 \\ 10/25 & \text{if } n = 5 \\ 15/36 & \text{if } n = 6 \\ 1 - \frac{6}{n} + \frac{13}{n^2} & \text{if } n = 7, 8, ... \end{cases}$$

Proof. We show the case when $n \geq 7$. Given these conditions, a single D starting inside a cluster of C's will invade the C's forming a cross. As in the previous case, the arms of the cross will also have width equal to three rows/columns. Since $3a + b = 3$, a C can survive as C in a cluster of C's which has width two rows/columns (in the previous case the cluster of C's needs to have, at least, width three rows/columns), as shown in Figure 3.10.

The arms of such a cluster of D's grow only if the D's located in the ends play against three C's. Thus, player i survives as C, as $t \to \infty$, if at time zero:

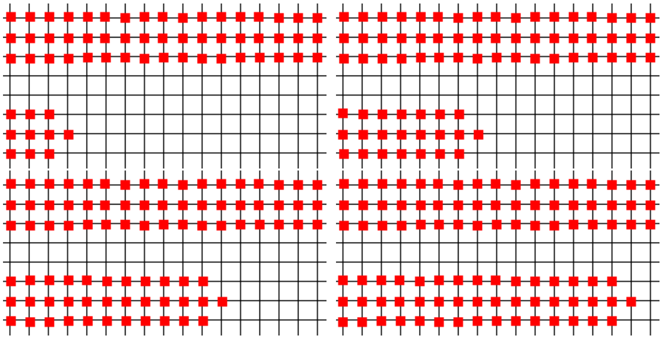

Figure 3.10: A cluster of C's with width two (rows) surrounded by D's.

- there is no D starting a cluster of D's in row $-2, -1, 0, 1, 2$;
- there is a D which starts a cluster of D's in row 2 (-2) on one side, and there is no D starting in row -2,-1,0,1 (2,1,0,-1) on the other side. This has probability, as $p \to 1$, equal to $4\frac{n-5}{n^2} + \frac{2}{n^2}$;
- there is a D starting in row 1 (-1) on one side. The cluster of D's is stopped by another cluster of D's started in row 2 or 3 (-2 or -3) on the other side. This happens, as $p \to 1$, with probability $\frac{4}{n^2}$;
- there is a D which starts a cluster of D's in row 0 on one side, and this cluster is stopped by another cluster of D's started in row 2 or -2 on the other side. The probability of this event is, as $p \to 1$, equal to $\frac{2}{n^2}$.

We can conclude that

$$\lim_{p \to 1} \pi_p(\mathrm{C}) = \left(\frac{n-5}{n}\right)^2 + 4\frac{n-5}{n^2} + \frac{8}{n^2}.$$

□

Chapter 4

The iterated Prisoner's dilemma on random graphs

In this chapter, once again we study the iterated Prisoner's dilemma in a spatial setting. As described previously, each player associated with a vertex in a given graph plays the game against its neighbors. Now, the game is played in a more typical network than those regular graphs analyzed in the previous chapter. More precisely, it is played on a graph constructed, at time zero, by random mechanisms.

4.1 Binomial random graph

Let Ω be the set of all graphs on vertex set $V = \{1, 2, ..., n\}$, and for each graph $\omega \in \Omega$, set

$$P(\omega) \;\; = \;\; r^{e_\omega}(1-r)^{\binom{n}{2}-e_\omega},$$

where e_ω is the number of edges of ω and r is the probability that two vertex are connected. The graph chosen according to the $P(\omega)$'s, introduced by Erdös and Rényi in 1961, is called the *binomial random graph* and denoted $\mathbb{G}(n, r)$ [6]. Let nodes and edges represent players and interaction respectively in the usual way.

Let $Y_C^i \in \mathrm{Bi}(n-1, pr)$ ($Y_D^i \in \mathrm{Bi}(n-1, qr)$) be the number of C's (D's) which play against player $i \in V$ at time 1. Let also $Y^i = Y_C^i + Y_D^i$ and, as in **Chapter 3**, let $v_i^t = v_i^{t,C}\mathbf{1}_{\{s_i^t=C\}} + v_i^{t,D}\mathbf{1}_{\{s_i^t=D\}}$ be the total score for player i at time t, where $v_i^{t,s}$ is the total score of player i at time t given it uses strategy $s \in S = \{C, D\}$. That is,

$$v_i^{t,C} = \sum_{j:i\leftrightarrow j} \mathbf{1}_{\{s_j^t=C\}},$$

and

$$v_i^{t,D} = b\sum_{j:i\leftrightarrow j} \mathbf{1}_{\{s_j^t=C\}} + a\sum_{j:i\leftrightarrow j} \mathbf{1}_{\{s_j^t=D\}},$$

where the notation $i \leftrightarrow j$ indicates that there is an edge between i and j. Clearly, $\mathbb{E}(v_i^{1,C}) = (n-1)pr$ and $\mathbb{E}(v_i^{1,D}) = (bp+aq)(n-1)r$.
Here, the rules of the game are as described in **Chapter 3**, with the only difference that the players compare their *average* score. More formally, for $t = 1, 2, ...,$

$$s_i^{t+1} = \begin{cases} C & \text{if } Y^i > 0, \nexists k \in N_i : \frac{v_k^t}{Y^k} = \max_{j\in N_i} \frac{v_j^t}{Y^j}, s_k^t = D \\ D & \text{if } Y^i > 0, \nexists k \in N_i : \frac{v_k^t}{Y^k} = \max_{j\in N_i} \frac{v_j^t}{Y^j}, s_k^t = C \\ s_i^t & \text{otherwise,} \end{cases}$$

where N_i is the neighborhood of player i (including i itself).
In this chapter, we study the asymptotic distribution of cooperation as $n \to \infty$ and show some rigorous results concerning the survival of cooperation with $r = r(n)$ as a function of n. These results are independent of a and b.
Firstly, we note that since $b > 1$, then $\pi_p(C) = p^n \to 0$ as $n \to \infty$ if $r = 1$. This may suggest that too much interaction rules out cooperation. Indeed, this turns out to be (at least partially) true, as we see in the following theorem.

Theorem 4.1.1. *If $r \geq \frac{1}{n^c}$, $0 \leq c < 1$, then for $t = 2, 3, ...,$*

$$\lim_{n\to\infty} P(\cup_{i=1}^n \{s_t^i = C\}) = 0.$$

Proof. It is clear that for $t = 2, 3, ...,$

$$
\begin{aligned}
&P(\cup_{i=1}^n \{s_t^i = C\}) \leq o(1) \\
&\qquad +nP\left(Y_C^1 > 0, Y_D^1 > 0, \left\{\exists j : 1 \leftrightarrow j, \frac{v_1^{1,C}}{Y^1} \geq \frac{v_j^{1,D}}{Y^j}\right\}\right) \\
&\leq \; o(1) \\
&\qquad +n^2P\left(v_1^{1,C}Y^2 \geq v_2^{1,D}Y^1, Y^2 \geq b^{1/2}(n-1)r, Y^1 \leq \frac{1}{b^{1/2}}(n-1)r\right) \\
&\qquad +n^2P\left(v_1^{1,C}Y^2 \geq v_2^{1,D}Y^1, Y^2 \leq b^{1/2}(n-1)r, Y^1 \leq \frac{1}{b^{1/2}}(n-1)r\right) \\
&\qquad +n^2P\left(v_1^{1,C}Y^2 \geq v_2^{1,D}Y^1, Y^2 \geq b^{1/2}(n-1)r, Y^1 \geq \frac{1}{b^{1/2}}(n-1)r\right) \\
&\qquad +n^2P\left(v_1^{1,C}Y^2 \geq v_2^{1,D}Y^1, Y^2 \leq b^{1/2}(n-1)r, Y^1 \geq \frac{1}{b^{1/2}}(n-1)r\right) \\
&\leq \; o(1) + 2n^2P\left(Y^2 \geq b^{1/2}(n-1)r\right) \\
&\qquad +n^2P\left(Y^1 \leq \frac{1}{b^{1/2}}(n-1)r\right) + n^2P\left(v_1^{1,C}b^{1/2} \geq v_2^{1,D}\frac{1}{b^{1/2}}\right) \\
&= \; o(1) + 2n^2P\left(Y^2 \geq \mathbb{E}\left(Y^2\right) + (b^{1/2}-1)(n-1)r\right) \\
&\qquad +n^2P\left(Y^1 \leq \mathbb{E}\left(Y^1\right) - \left(1 - \frac{1}{b^{1/2}}\right)(n-1)r\right) \\
&\qquad +n^2P\left(v_1^{1,C} \geq v_2^{1,D}\frac{1}{b}, v_2^{1,D} \leq b(n-1)pr + \frac{a}{2}(n-1)qr\right) \\
&\qquad +n^2P\left(v_1^{1,C} \geq v_2^{1,D}\frac{1}{b}, v_2^{1,D} > b(n-1)pr + \frac{a}{2}(n-1)qr\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \; o(1) + 2n^2P\left(Y^2 \geq \mathbb{E}\left(Y^2\right) + (b^{1/2}-1)(n-1)r\right) \\
&\qquad +n^2P\left(Y^1 \leq \mathbb{E}\left(Y^1\right) - \left(1 - \frac{1}{b^{1/2}}\right)(n-1)r\right) \\
&\qquad +n^2P\left(v_2^{1,D} \leq \mathbb{E}\left(v_2^{1,D}\right) - \frac{a}{2}(n-1)qr\right) \\
&\qquad +n^2P\left(v_1^{1,C} \geq \mathbb{E}\left(v_2^{1,C}\right) + \frac{a}{2b}(n-1)qr\right).
\end{aligned}
$$

The proof is now completed by the following Chernoff bounds [7]:

Theorem 4.1.2. *Let $X_1, X_2, ..., X_n$ be independent random variables with $P(X_i = 1) = p_i$, $P(X_i = 0) = 1 - p_i$. We consider the sum $X = \sum_{i=1}^n X_i$, with expectation $E(X) = \sum_{i=1}^n p_i$. Then we have*

$$P(X \geq \mathbb{E}(X) + u) \quad \leq \quad e^{-\frac{u^2}{2(\mathbb{E}(X)+\frac{u}{3})}}.$$

Theorem 4.1.3. *If $X_1, X_2, ..., X_n$ are non-negative independent random variables, we have the following bounds for the sum $X = \sum_{i=1}^n X_i$:*

$$P(X \leq \mathbb{E}(X) - u) \quad \leq \quad e^{-\frac{u^2}{2\sum_{i=1}^n \mathbb{E}(X_i^2)}}.$$

Thus,

$$\begin{aligned} P(\cup_{i=1}^n s_t^i = C) &= o(1) + 2n^2 e^{-\frac{\left((b^{1/2}-1)(n-1)r\right)^2}{2\left((n-1)r+\frac{b^{1/2}-1}{3}(n-1)r\right)}} \\ &+ n^2 e^{-\frac{\left(\left(1-\frac{1}{b^{1/2}}\right)(n-1)r\right)^2}{2((n-1)r)}} + n^2 e^{-\frac{\left(\frac{a}{2}(n-1)qr\right)^2}{2\left(b^2(n-1)pr+a^2(n-1)qr\right)}} \\ &+ n^2 e^{-\frac{\left(\frac{a}{2b}(n-1)qr\right)^2}{2\left((n-1)pr+\frac{a}{6b}(n-1)qr\right)}} \\ &\rightarrow \; 0, \end{aligned}$$

as $n \to \infty$.

□

As we see in the previous result, too much interaction kills cooperation. But is it possible at all for cooperation to survive when $r > 0$? An immediate answer comes by noting that $\lim_{n\to\infty} (1-r)^{n-1} = 1$ if $0 < r < \frac{1}{n^c}$, $c > 1$, so $\lim_{n\to\infty} \pi_p(C) = p$. So, yes a given player can be an isolated cooperator when r is small enough. Indeed, it is easily seen that if $r = c/n$, $c > 0$, then for $t = 1, 2, ...,$

$$\lim_{n\to\infty} P(s_t^i = C, Y^i = 0) = pe^{-c}.$$

Moreover, it is well known that there are asymptotically almost surely ($a.a.s.$) isolated players when $r = \frac{\log n}{cn}$, $c > 1$, a special case of the following theorem (Erdös and Rényi [6]):

Theorem 4.1.4. *Let T_v be the number of $v-$vertex isolated trees in $\mathbb{G}(n,r)$, $v = 1, 2, ...,$ and $c_n = vnr - \log n - (v-1)\log\log n$. Then,*

$$P(T_v > 0) \to \begin{cases} 0 & \text{if } n^v r^{v-1} \to 0 \text{ or } c_n \to \infty, \\ 1 & \text{if } n^v r^{v-1} \to \infty \text{ and } c_n \to -\infty. \end{cases}$$

Moreover, if $n^v r^{v-1} \to c \in (0,\infty)$ or $c_n \to c > 0$, then $T_v \xrightarrow{d} Po(\lambda)$, where $\lambda = \lim_{n\to\infty} \mathbb{E}(T_v) \in (0,\infty)$.

According to **Theorem 4.1.4** there is positive probability that, as n tends to infinity, there are isolated cooperators if $r = \frac{\log n}{cn}$, $c > 1$. Of course, this is not of so much interest since we are mainly interested in the probability that cooperation survives as a result of interaction between the players. But it may still suggest that limited interaction leaves room for cooperation to survive. In order to ensure interaction, we investigate this further by studying the game played on the *largest* component with r big enough so that this component is *a.a.s.* unique. Such a component may be regarded as the biggest community of players. Firstly, we study the case when $r = \frac{c}{n}$, $c > 1$, as we see in the theorem bellow. Note that in this case the limit is the Poisson distribution. Let B_i be the event that player i survives as C, as $t \to \infty$, in a component which has at least one more player than any other component.

Theorem 4.1.5. *If $r = \frac{c}{n}$ and $c > 1$, then*

$$\lim_{n\to\infty} P(B_i) \quad > \quad 0.$$

Proof. Firstly, we recall that when the game is played on the one dimensional lattice there are some local configurations which protect cooperation as t tends to infinity. Consider now player $i \in V$, and suppose that this player plays only against one player, say i_1. Let player i_1 play only against one more player, i_2. Let also player i_2 play only against one more player, i_3, and so on until we reach a connected component of seven players. Let now the seventh player, which we call i_6, be connected to the largest component through other players. Then, the interaction for these players is only described by $i \leftrightarrow i1, i1 \leftrightarrow i2, i2 \leftrightarrow i3, i3 \leftrightarrow i4, i4 \leftrightarrow i5, i5 \leftrightarrow i6, i6 \leftrightarrow \cdot$, where $i6 \leftrightarrow \cdot$ denotes that player $i6$ is connected to the largest component. We find such a structure in Figure 4.1.

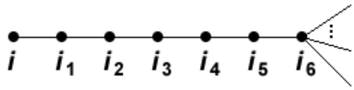

Figure 4.1: A subgraph of the largest component which can ensure cooperation for player i as $t \to \infty$.

If player $i, i1$ and $i2$ choose at time 0 strategy C, then they survive as C's if and only if at time 0: i) player $i3$ and $i4$ choose strategy D; ii) player $i3$ and $i4$ choose strategy C; or iii) player $i3$ and $i4$ choose strategy C and D respectively. Given one of these configurations, players i, i_1 and i_2 survive as C's independently of what happens to the other players.

Consequently, we only need to show that such a local tree of seven players, connected to the largest component, occurs with positive probability. The following theorem gives the size of the largest component [6]:

Theorem 4.1.6. *If* $r = \frac{c}{n}$ *and* $c > 1$*, then* $\mathbb{G}(n, r)$ *contains a giant component of* $(1 + o_r(1))\beta n$*, where* β *is the unique solution of*

$$\beta + e^{-\beta c} = 1.$$

Furthermore, asymptotically almost surely (a.a.s.) the size of the second largest component of $\mathbb{G}(n, r)$ *is at most* $\frac{16c}{(c-1)^2} \log n$*.*

Now let L_j be the event that player $j \in V$ belongs to a component which has at least one more player than any other component. It is then clear that given the local tree of seven players shown in Figure 4.1,

$$\lim_{n \to \infty} P(L_{i_6}) = \beta.$$

Since for any $k < \infty$, $\text{Bin}(n - k, \frac{c}{n}) \to \text{Poi}(c)$ as $n \to \infty$, then the probability that a given player belongs to such a structure as player i in Figure 4.1 (for any $i_1, ..., i_6 \in V$), tends to $c^6 e^{-6c}\beta$ as $n \to \infty$. In the same way, the probability that a given player belongs to such a subgraph as player i_j , $j = 1, 2$, tends to $\frac{c^6 e^{-6c}\beta}{2}$ as $n \to \infty$. By independence in the initial configuration, it is then clear that

$$\lim_{n\to\infty} P(B_i) \geq \beta 2p^3(1-pq)c^6 e^{-6c}.$$

$\square$

As described above, when the limit is the Poisson distribution a given vertex has positive probability to belong to such a subgraph as the local tree shown in Figure 4.1. If we have instead $r = r(n)$ such that $\lim_{n\to\infty} \frac{1}{nr} = 0$, then that probability tends to zero as n tends to infinity. With this in mind, it is now of interest to know if cooperation can still survive in the largest component when r is of bigger order than $\frac{c}{n}$. As we saw in **Theorem 4.1.1**, cooperation dies out *a.a.s.* if $r = \frac{1}{n^c}$, $c < 1$. Now we study the intermediate case when $r = \lambda \frac{\log n}{n}$, with $\lambda = \lambda(n)$ as a function of n. The theorem below, weaker than **Theorem 4.1.5**, says that cooperation survives *a.a.s.* in some part of the biggest community of players if we choose $\lambda = \lambda(n)$ in a proper way. More precisely,

Theorem 4.1.7. *If $r = \lambda \frac{\log(n)}{n}$ and $\lambda \leq \frac{1}{6}$ such that $\lambda \log n \to \infty$ as $n \to \infty$, then*

$$\lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} B_i\right) = 1.$$

Proof. In this case, the probability that a given player $j \in V$ is connected to a component which has at least one more player than any other component tends to one as n tends to infinity. So if we consider again the subgraph shown in Figure 4.1, then $P(L_{i_6}) = 1 - o(1)$. Consider again the local tree described in the proof of **Theorem 4.1.5** and let R_j be the event that player $j \in V$ belongs to such a structure as player i in Figure 4.1 (for any $i_1, ..., i_6 \in V$). Then, we have that

$$\begin{aligned} P(R_j) &= \left(\prod_{k=1}^{6}(n-k)\right) r^6 (1-r)^{\sum_{k=2}^{7}(n-k)}(1-o(1)) \\ &\sim n^6\lambda^6 \frac{(\log n)^6}{n^6}(1-r)^{6n} \\ &\sim \lambda^6 (\log n)^6 e^{-6\lambda \log n} \\ &= \lambda^6 (\log n)^6 n^{-6\lambda}, \end{aligned}$$

where $a_n \sim b_n$ denotes $0 < \liminf \frac{a_n}{b_n} \leq \limsup \frac{a_n}{b_n} < \infty$. Let now $X = \sum_{j=1}^{n} \mathbf{1}_{R_j}$. If $\lambda \leq \frac{1}{6}$ such that $\lambda \log n \to \infty$ as $n \to \infty$, then

$$\begin{aligned}\mathbb{E}(X) &= nP(R_1) \\ &\to \infty\end{aligned}$$

as $n \to \infty$. We also have that

$$\begin{aligned}\mathrm{Var}(X) &= \sum_{j=1}^{n} \mathrm{Var}(\mathbf{1}_{R_i}) + \sum_{i=j}^{n} \sum_{i \neq j} \mathrm{Cov}(\mathbf{1}_{R_j}, \mathbf{1}_{R_i}) \\ &= nP(R_1)(1 - P(R_1)) \\ &\quad + n(n-1)(\mathbb{E}(\mathbf{1}_{R_1 \cap R_2}) - \mathbb{E}(\mathbf{1}_{R_1})^2).\end{aligned}$$

Now, for large enough n

$$\begin{aligned}\mathbb{E}\left(\mathbf{1}_{R_1 \cap R_2}\right) &\leq \left(\prod_{k=2}^{13} (n-k)\right) r^{12} (1-r)^{\sum_{k=3}^{14} (n-k)} \\ &\leq \left(\prod_{k=1}^{6} (n-k)^2\right) r^{12} \frac{(1-r)^{2 \sum_{k=2}^{7} (n-k)}}{(1-r)^{48}} \\ &= \frac{\mathbb{E}(\mathbf{1}_{R_1})^2}{(1-o(1))^2 (1-r)^{48}} \\ &= \mathbb{E}(\mathbf{1}_{R_1})^2 (1 + o(1)).\end{aligned}$$

Consequently,

$$\mathrm{Var}(X) \leq \mathbb{E}(X)((1 - P(R_1)) + \mathbb{E}(X) o(1)).$$

Before we complete the proof, we present the *second moment method* which says that for every random variable Y with $E(Y) > 0$, then

$$P(Y = 0) \leq \frac{\mathrm{Var}(Y)}{\mathbb{E}(Y)^2}.$$

Therefore, if $\lambda \leq \frac{1}{6}$ such that $\lambda \log n \to \infty$ as $n \to \infty$, then

$$\begin{aligned} P(X = 0) &\leq \frac{\operatorname{Var}(X)}{\mathbb{E}(X)^2} \\ &\to 0, \end{aligned}$$

as $n \to \infty$. □

Remark. The ideas in the proof of Theorem 4.1.7 were taken from the proof of the well known connectivity theorem of Erdös and Rényi [11].

Appendix

Proof of Proposition 3.5.2

Proof. In order to complete the proof of **Theorem 3.5.2**, we need to calculate $P(A_1)$. Consider firstly the eight nearest neighbors of a cluster of five C's where one C plays against the other four C's. Call these players $1, 2, ..., 8$, as shown in figure 4.2. Note the difference between players $1, 3, 5, 7$ and $2, 4, 6, 8$. Here, a player denoted by an even number plays against two C's of this cluster, while a player given an odd number plays only against one C of the cluster. Since defection of such a cluster

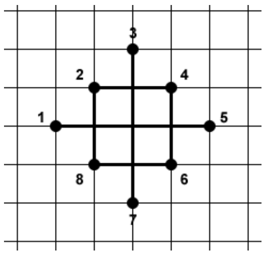

Figure 4.2: The eight neighbors of a group of 5 C's.

can only happen at time one, then $P(A_1)$ is the probability that, at time zero, player i and its four neighbors are C's and no one of these eight neighbors is a D surrounded by four C's. Let now $S = (s_0^1, s_0^2, ..., s_0^8)$ be the initial configuration of the neighbors of that cluster. Therefore, we have that

$$
\begin{aligned}
P(A_i) &= P(A_i \cap \{S = (C, ..., C)\}) \\
&+ \sum_{k=1}^{4} \binom{4}{k} P\left(A_i \bigcap_{i=1}^{k} \{s_0^{2i-1} = D\} \bigcap_{i=k+1}^{4} \{s_0^{2i-1} = C\} \bigcap_{j=1}^{4} \{s_0^{2i} = C\} \right) \\
&+ \sum_{k=1}^{4} \binom{4}{k} P\left(A_i \bigcap_{i=1}^{k} \{s_0^{2i} = D\} \bigcap_{i=k+1}^{4} \{s_0^{2i} = C\} \bigcap_{j=1}^{4} \{s_0^{2i-1} = C\} \right) \\
&+ 8P(A_i \cap \{S = (D, D, C, C, C, C, C, C)\}) \\
&+ 8P(A_i \cap \{S = (D, C, C, D, C, C, C, C)\}) \\
&+ 4P(A_i \cap \{S = (D, D, C, C, C, C, C, D)\}) \\
&+ 16P(A_i \cap \{S = (D, D, C, D, C, C, C, C)\}) \\
&+ 4P(A_i \cap \{S = (D, C, C, D, C, D, C, C)\}) \\
&+ 4P(A_i \cap \{S = (D, D, D, C, C, C, C, C)\}) \\
&+ 16P(A_i \cap \{S = (D, D, C, C, C, C, D, C)\}) \\
&+ 4P(A_i \cap \{S = (D, C, D, C, C, D, C, C)\}) \\
&+ 8P(A_i \cap \{S = (D, D, C, C, C, C, D, D)\}) \\
&+ 8P(A_i \cap \{S = (D, D, C, C, C, D, D, C)\}) \\
&+ 4P(A_i \cap \{S = (D, D, C, C, D, C, C, D)\}) \\
&+ 4P(A_i \cap \{S = (D, D, D, C, C, D, C, C)\}) \\
&+ 8P(A_i \cap \{S = (D, C, D, D, C, D, C, C)\}) \\
&+ 8P(A_i \cap \{S = (D, D, C, D, C, C, C, D)\}) \\
&+ 8P(A_i \cap \{S = (D, C, D, D, C, C, D, C)\}) \\
&+ 4P(A_i \cap \{S = (D, D, C, D, C, D, C, D)\}) \\
&+ 8P(A_i \cap \{S = (D, D, D, C, C, D, C, D)\}) \\
&+ 4P(A_i \cap \{S = (D, D, D, D, C, C, C, D)\}) \\
&+ 8P(A_i \cap \{S = (D, D, C, D, D, C, C, D)\}) \\
&+ 4P(A_i \cap \{S = (D, C, D, D, C, D, C, D)\}) \\
&+ 4P(A_i \cap \{S = (D, D, D, C, C, C, D, D)\}) \\
&+ 8P(A_i \cap \{S = (D, C, D, C, C, D, D, D)\}) \\
&+ 4P(A_i \cap \{S = (D, C, D, D, C, D, D, C)\}) \\
&+ 8P(A_i \cap \{S = (D, D, D, C, C, D, D, C)\}) \\
&+ 4P(A_i \cap \{S = (D, D, D, C, D, C, D, C)\})
\end{aligned}
$$

$$
\begin{aligned}
&+4P(A_i \cap \{S = (D,D,D,D,C,D,C,D)\})\\
&+2P(A_i \cap \{S = (D,D,C,D,D,D,C,D)\})\\
&+8P(A_i \cap \{S = (C,D,D,D,D,D,D,C)\})\\
&+8P(A_i \cap \{S = (D,D,D,D,C,D,D,C)\})\\
&+4P(A_i \cap \{S = (D,D,D,D,D,C,D,C)\})\\
&+2P(A_i \cap \{S = (D,D,D,C,D,D,D,C)\})\\
&+4P(A_i \cap \{S = (D,D,D,D,D,D,C,D)\})\\
&+4P(A_i \cap \{S = (D,D,D,D,D,D,D,C)\})\\
&+P(A_i \cap \{S = (D,D,D,D,D,D,D,D)\})
\end{aligned}
$$

Independence in the initial configuration gives:

$$
\begin{aligned}
P(A_i) = {} & p^5\left(p^8 + \sum_{k=1}^{4}\binom{4}{k}p^{8-k}q^k(1-p^3)^k\right)\\
&+p^5\sum_{k=1}^{4}\binom{4}{k}p^{8-k}q^k(1-p^2)^k + 8p^{11}q^2\left(q+pq(1-p^2)\right)\\
&+8p^{11}q^2\left((1-p^3)(1-p^2)\right) + 4p^{10}q^3\left(q^2+2pq^2+p^2q^3\right)\\
&+16p^{10}q^3\left(q+pq(1-p^2)\right)(1-p^2) + 4p^{10}q^3(1-p^3)(1-p^2)^2\\
&+4p^{10}q^3\left(q^2+2pq(1-p^2)\right) + 16p^{10}q^3\left(q+pq(1-p^2)\right)(1-p^3)\\
&+4p^{10}q^3(1-p^2)(1-p^3)^2 + 8p^9q^4\left(q^2+2pq^2(1-p^2)+p^2q^2(1-p^2)\right)\\
&+8p^9q^4\left(q+pq(1-p^2)\right)^2 + 4p^9q^4\left(q^2+2pq^2+p^2q^3\right)(1-p^3)\\
&+4p^9q^4\left(q^2+2pq(1-p^2)\right)(1-p^2)\\
&+8p^9q^4(1-p^3)\left(q+pq(1-p^2)\right)(1-p^2)\\
&+8p^9q^4\left(q^2+2pq^2+p^2q^3\right)(1-p^2) + 8p^9q^4\left(q+pq(1-p^2)\right)(1-p^3)^2\\
&+4p^8q^5(1-p^2)^2\left(q^2+2pq^2+p^2q^3\right)\\
&+8p^8q^5\left(q^2+2pq^2(1-p^2)+p^2q^2(1-p^2)\right)(1-p^2)\\
&+4p^8q^5\left(q^2(1-p^2)^2+2pq(1-p^2)(q+pq^2)\right)(1-p^2)\\
&+8p^8q^5\left(q+pq(1-p^2)\right)\left(q^2+2pq^2+p^2q^3\right)\\
&+4p^8q^5\left(q+pq(1-p^2)\right)^2(1-p^2)
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & 4p^8q^5\left(q^2(1-p^3)^2+2pq^2(1-p^3)+p^2q^3\right)\\
+ \quad & 8p^8q^5(1-p^3)\left(q^2(1-p^2)+2pq^2(1-p^3)+p^2q^3\right)\\
& +4p^8q^5(1-p^3)\big(q+pq(1-p^2)\big)^2\\
+ \quad & 8p^8q^5\left(q+pq(1-p^2)\right)\left(q^2+2pq(1-p^2)\right)\\
+ \quad & 4p^8q^5\left(q^2+2pq(1-p^2)\right)(1-p^3)^2\\
+ \quad & 4p^7q^6\left(q^2(1-p^2)^3+2pq(1-p^2)^2(q+pq^2)\right)\\
+ \quad & 2p^7q^6\big(q^2+2pq^2+p^2q^3\big)^2\\
+ \quad & 8p^7q^6\Big(q^2(1-p^3)(q+(2-p)p(1-p^2))+pq^2+(pq)^2(1-p^2)(1+q)\Big)\\
+ \quad & 8p^7q^6\left(q+pq(1-p^2)\right)\left(q^2(1-p^2)+2pq^2+p^2q^3\right)\\
+ \quad & 4p^7q^6\left(q+pq(1-p^2)\right)\left(q^2(1-p^2)+pq(1-p^2)^2+pq(q+pq^2)\right)\\
+ \quad & 2p^7q^6\big(q^2+2pq(1-p^2)\big)^2\\
+ \quad & 4p^6q^7\big(q^2(q+pq(1-p^2))^2+2pq^2(q+pq(1-p^2))(1-p^2)\\
& +p^2q^3(1-p^2)^2\big)\\
+ \quad & 4p^6q^7\big(q^2(q+pq(1-p^2))^2\\
& +2pq^2(q+pq(1-p^2))(q^2+pq^2+pq(1-p^2))\big)\\
+ \quad & p^5q^8\big(q^4(1-p^3)^2+4pq^4(1-p^3)\\
& +4p^2q^4+2p^2q^4(1-p^3)+4p^3q^4+p^4q^4\big)\\
= \quad & p(2+p(-4+p(-4+p(10+p(7+p(-14+p\times\\
& \times(-10+p(20+p(-46+p(172+p(-203+p(-154+p\times\\
& \times(617+p(-746+p(777+p(-926+p(863+p(-506+p\times\\
& \times(147+p(70+p(-135+p(86-23p+p^3))))))))))))))))))))).
\end{aligned}
$$

□

Remark. To deduce the last equality, we used Wolfram Mathematica 7.

Bibliography

[1] Jonasson, J. *On the optimal strategy in a random game.* Electronic Communications in Probability 9, 132-139 (2004).

[2] Lopez-Ruiz, R. and Calbet, X. *Why the Maxwellian Distribution is the Attractive Fixed Point of the Boltzmann Equation.* Preprint in Arxiv: nlin.CD/0611044 v1 (2006).

[3] Lindgren, K. and Nordahl, M. G. *Evolutionary dynamics of spatial games.* Physica D 75, (1994) 292-309.

[4] Novak, M. A. *Evolutionary dynamics: exploring the equations of life.* Harvard University Press, (2006).

[5] Durrett, R. *Probability: Theory and Examples.* Duxbury Advanced Series, third edition, (2005).

[6] Janson, S., Luczak, T. and Rucinski, A. *Random Graphs.* John Wiley & Sons, Inc., (2000).

[7] Fan Chung and Linyuan Lu *Concentration Inequalities and Martingale Inequalities: A Survey.* Internet Mathematics Vol. 3, No. 1: 79-127.

[8] Grimmett *Percolation.* Springer, Second edition, (1999)

[9] Myerson, R. B. *Game theory: analysis of conflict.* Harvard University Press, (1991)

[10] Ferguson, T. S. *Game Theory.* Lecture notes, University of California at Los Angeles.

[11] Acemoglu, D. and Ozdaglar, A. *Lecture 3: Erdös-Rényi graphs and Branching processes.* MIT, 6.207/14.15: Networks, Lecture notes, 2009.

[12] Brauchli K., Killingback T. and Doebeli M. *Evolution of Cooperation in Spatially Structured Populations.* J. theor. Biol. 200, 405-417, 1999.

[13] Poundstone, W. *Prisoner's Dilemma.* Doubleday, NY NY, 1992.