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Nonlinear evolution of two fast-particle-driven modes near the linear stability threshold

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A system of two coupled integro-differential equations is derived and solved for the non-linear evolution of two waves excited by the resonant interaction with fast ions just above the linear instability threshold. The effects of a resonant particle source and classical relaxation processes represented by the Krook, diffusion, and dynamical friction collision operators are included in the model, which exhibits different nonlinear evolution regimes, mainly depending on the type of relaxation process that restores the unstable distribution function of fast ions. When the Krook collisions or diffusion dominate, the wave amplitude evolution is characterized by modulation and saturation. However, when the dynamical friction dominates, the wave amplitude is in the explosive regime. In addition, it is found that the finite separation in the phase velocities of the two modes weakens the interaction strength between the modes. © *2011 American Institute of Physics*.
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I. INTRODUCTION

In fusion plasmas, high-energy ions arising from plasma heating as well as being generated in fusion reactions have thermodynamically non-equilibrium distributions and thus may lead to the occurrence of wave micro-instabilities.[1] These instabilities can in turn cause anomalous losses of fast ions and may have direct impact on the operation scenarios and ignition conditions.[2] Furthermore, the excited waves may also provide information about the burning plasma conditions that cannot be accessed directly.[3] In the framework of the plasma theory, investigation of these instabilities is connected with the identification of the stability thresholds with respect to wave excitations by fast ions as well as with the study of the non-linear dynamics of the wave-fast ion systems above the stability thresholds. The theory describing the nonlinear evolution of a single plasma mode driven resonantly by fast ions just above the linear instability threshold was developed in a number of papers,[4–8] and was applied to study the dynamical properties of both the toroidal Alfvén eigenmodes and the fishbone instability excited by energetic ions in tokamak plasmas.[9–12] The basic assumption of this Berk-Breizman model is that linear dissipation from background plasma γ_d and the energetic particle drive for instability γ_L give a net growth rate $\gamma = \gamma_L - \gamma_d > 0$ that satisfies the conditions $\gamma_L/\gamma_d \sim 1$ and $\gamma_L - \gamma_d \ll \gamma_L$, i.e., it is assumed that the mode excitation takes place just above the instability threshold. It was shown that the mode dynamics is determined by an interplay between the wave electric field, that tends to flatten the distribution function of the fast ions $F(t, x, v)$, and the relaxation processes modeled by collision operator $\left(\frac{\partial F}{\partial t}\right)_{coll}$ restoring the distribution function, which are represented as

$$\left(\frac{\partial F}{\partial t}\right)_{coll} = -\nu F + \alpha^2 \frac{\partial F}{\partial v} + \beta^3 \frac{\partial^2 F}{\partial v^2}. \tag{1}$$

It was also found that in the case, when the dominant relaxation processes are modeled via an "annihilation" (Krook) and/or diffusion collisional operators (characterized by the parameters ν and β, respectively), the non-linear evolution of the wave amplitude may exhibit four main regimes near marginal stability: a steady-state, periodic amplitude modulation, and chaotic and explosive regimes, cf. Refs. 4 and 5. This is in contrast to the case when dynamical friction (characterized by the parameter α) is the dominant collision process, where the explosive evolution of the wave amplitude was found to be the only possible behavior, i.e., within the context of the perturbation theory, the mode reaches arbitrarily large amplitude in a finite time.[8] The actual limit of applicability of this explosive solution is when the bounce frequency ω_B of the trapped particles approaches the growth rate in the absence of dissipation, independent of the closeness to marginal stability.

One of the most important problems in the theory of fast ion driven instabilities is the understanding of the wave saturation mechanisms and its implications on the confinement of the fast ions. The existing theory, Berk-Breizman theory, describes the energetic particle excitation and nonlinear evolution of single coherent waves so that bifurcations at single-mode saturation as well as formation of long-lived coherent nonlinear structures can be analyzed. However, in order to investigate scenarios with marginal where the resonance overlap between different modes can lead to a strong nonlinear regime, it is necessary to generalize the theory to a multi-mode case. Previously, the interaction of two beating electrostatic waves with plasma electrons was studied in

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Refs. 13 and 14, and the nonlinear saturation of two plasma modes which are initially only marginally unstable was investigated in Ref. 15, where the time-asymptotic saturation amplitude and frequency shift of each unstable mode are evaluated by the Bogoliubov method. In the present paper, the Berk-Breizman theory is extended to the case of two weakly linearly unstable modes, which are driven resonantly by fast ions near the instability thresholds. We treat the problem as one-dimensional and assume that the two plasma eigenmodes with wave numbers k_i and frequencies ω_i, $i = 1,2$, have phase velocities ω_i/k_i being close to each other and lying on the positive slope of the fast ion velocity distribution function. The discussion is restricted to the isolated, but close, resonances, for which the frequency separation is much less than the corresponding bounce frequencies. Following the single mode theory, we apply a perturbation analysis which is based on the assumption of small deviations of the particles from their unperturbed orbits. Formally, we generate an expansion in the small parameter ω_B/γ.

The structure of the paper is as follows: In Sec. II, we outline derivation of the two coupled integro-differential equations for two mode amplitudes driven resonantly by interaction with fast ions just above the stability threshold. In Sec. III, a numerical analysis of various nonlinear scenarios described by the derived equation system is presented and Sec. IV includes summary and conclusions.

II. BASIC EQUATIONS OF THE MODEL

To establish the proper form of the equations describing nonlinear evolution of slowly varying wave amplitudes, we begin with a brief outline of the basic equations and ideas describing the resonant interaction of fast particles with an electrostatic wave. For a longitudinal electromagnetic wave field, the Maxwell equations yield

$$\frac{\partial E}{\partial t} + \frac{1}{\varepsilon_0} J = 0, \tag{2}$$

where E is the electric wave field and J is the total current density. Since we have

$$J = J^p + J^f, \tag{3}$$

where J^p and J^f are the current densities due to the background plasma and fast ions, respectively, we expect the Landau damping due to the background plasma and the wave growth due to fast ions, which are both much less than the real part of the wave frequency. We assume also that the density of fast particles n_f is much less than the density of background plasma n_p. Thus, the contribution of J^f to the real part of the wave frequency can be neglected. Introducing the small parameter $\epsilon \ll 1$ $(|\gamma_L - \gamma_d/\gamma_d| \approx O(\epsilon))$, we can write

$$J = J^{p_0} + \varepsilon(J^{p_1} + J^f), \tag{4}$$

where J^{p_0} determines the wave frequency, while J^{p_1} and J^f determine $>\gamma_d$ and γ_L, respectively. Let us now write the wave field as a sum of two independent background plasma eigenmodes with the frequencies ω_1, ω_2 and the wave numbers k_1, k_2, respectively,

$$E(x,t) = E_1(x,t) + E_2(x,t), \tag{5}$$

where

$$E_1(x,t) = \hat{A}_1(t)e^{i(k_1 x - \omega_1 t)} + c.c., \quad E_2(x,t) = \hat{A}_2(t)e^{i(k_2 x - \omega_2 t)} + c.c. \tag{6}$$

Here, we assume that

$$\left|\frac{1}{\hat{A}_i}\frac{d\hat{A}_i}{dt}\right| \ll \omega_i. \tag{7}$$

Note that the slowly varying complex wave amplitudes $\hat{A}_1(t)$ and $\hat{A}_2(t)$ can be expressed in terms of some slowly varying real amplitudes $\hat{E}_1(t)$ and $\hat{E}_2(t)$ accompanied with appropriate slowly varying phases $\tilde{\alpha}(t)$ and $\tilde{\beta}(t)$

$$\hat{A}_1(t) = \frac{1}{2}\hat{E}_1(t)e^{i\tilde{\alpha}(t)}, \quad \hat{A}_2(t) = \frac{1}{2}\hat{E}_2(t)e^{i\tilde{\beta}(t)}. \tag{8}$$

In accordance with Eqs. (5) and (6), we can write

$$J(x,t) = J_1(x,t) + J_2(x,t) = \hat{J}_1(t)e^{i(k_1 x - \omega_1 t)} + \hat{J}_2(t)e^{i(k_2 x - \omega_2 t)} + c.c., \tag{9}$$

where it is assumed that

$$\left|\frac{1}{\hat{J}_i}\frac{d\hat{J}_i}{dt}\right| \ll \omega_i, \quad i = 1,2. \tag{10}$$

On the basis of the Eq. (2), a separate equation for each mode amplitude may be written as

$$\epsilon\frac{\partial \hat{A}_i}{\partial t} - i\omega_i\hat{A}_i + \frac{1}{\varepsilon_0}\left(\hat{J}_i^{p_0} + \epsilon(\hat{J}_i^{p_1} + \hat{J}_i^f)\right) = 0, \quad i = 1,2. \tag{11}$$

To order $O(\epsilon^0)$, Eq. (11) gives

$$-i\omega_i\hat{A}_i + \frac{1}{\varepsilon_0}\hat{J}_i^{p_0} = 0, \quad i = 1,2, \tag{12}$$

which determines ω_i. To $O(\epsilon^1)$ one obtains

$$\frac{\partial \hat{A}_i}{\partial t} + \frac{1}{\varepsilon_0}(\hat{J}_i^{p_1} + \hat{J}_i^f) = 0, \quad i = 1,2. \tag{13}$$

Note that

$$\hat{J}_i^f = q_i\int_{-\infty}^{\infty} v f_i^f dv, \quad \hat{J}_i^p = -q_e\int_{-\infty}^{\infty} v f_i^p dv, \quad i = 1,2, \tag{14}$$

where f_i^f and f_i^p are, respectively, the parts of the distribution function of fast ions and distribution function of plasma electrons which oscillate with frequency ω_i.

Using now the linearized Vlasov equation for electrons, one easily gets that

$$\hat{J}_i^p = -q_e\int_{-\infty}^{\infty} v f_i^p dv = \hat{J}_i^{p_0} + \epsilon\hat{J}_i^{p_1}, \quad i = 1,2, \tag{15}$$

where

$$\hat{J}_i^{p_1} = \gamma_d \hat{A}_i, \quad i = 1,2 \tag{16}$$

and

$$\frac{1}{\epsilon_0}\hat{J}_i^{p_0} = i\frac{\omega_{pe}^2}{\omega_i}\hat{A}_i, \quad i = 1,2. \tag{17}$$

Then it follows from Eqs. (17) and (12) that, in the cold plasma limit, the wave frequencies are determined by

$$\omega_i^2 = \omega_{pe}^2 = \frac{n_p q_e^2}{m\varepsilon_0}, \quad i = 1,2. \tag{18}$$

However, beyond the linear approximation, the wave frequencies ω_i, $i = 1, 2$, may slightly differ from the electron plasma frequency ω_{pe}.

Let us now focus our attention on Eq. (13) that can be written in the form of evolution equations for the two mode amplitudes

$$\frac{\partial \hat{A}_i}{\partial t} + \frac{q_e}{\varepsilon_0}\int_{-\infty}^{\infty} \upsilon f_i^f(t,\upsilon)d\upsilon + \gamma_d \hat{A}_i = 0, \quad i = 1,2, \tag{19}$$

which are determined by the two distribution functions of fast ions, f_i^f, $i = 1,2$. In fact, the functions f_i^f involve a nonlinear coupling of the modes. The physical reason of the nonlinear coupling is the resonant interaction between the fast ions and the modes. Following the Berk and Breizman approach,[4–6] we represent the fast ion distribution function $F(t, x, \upsilon)$ as a series

$$\begin{aligned} F(t,x,\upsilon) = F_0(\upsilon) + f_0(t,\upsilon) + [&f_1(t,\upsilon)e^{i\psi_1} + f_2(t,\upsilon)e^{i\psi_2} \\ &+ g_1(t,\upsilon)e^{2i\psi_1} + g_2(t,\upsilon)e^{2i\psi_2} + h_-(t,\upsilon)e^{i(\psi_1-\psi_2)} \\ &+ h_+(t,\upsilon)e^{i(\psi_1+\psi_2)} + \cdots + c.c.], \end{aligned} \tag{20}$$

where $\psi_i = k_i x - \omega_i t$; note also that we have abbreviated the notation of f_i^f to f_i, $i = 1,2$. The kinetic equation for $F(t, x, \upsilon)$ has the form

$$\frac{\partial F}{\partial t} + \upsilon\frac{\partial F}{\partial x} + \frac{q_e}{m}E(t,x)\frac{\partial F}{\partial \upsilon} = S(\upsilon) - \nu F + \alpha^2\frac{\partial F}{\partial \upsilon} + \beta^3\frac{\partial^2 F}{\partial \upsilon^2}, \tag{21}$$

and the equilibrium distribution function $F_0(\upsilon)$ is determined by

$$S(\upsilon) = \nu F_0 - \alpha^2\frac{\partial F_0}{\partial \upsilon} - \beta^3\frac{\partial^2 F_0}{\partial \upsilon^2}, \tag{22}$$

where $S(\upsilon)$ is a constant source of fast particles. In order to solve Eq. (21), we apply a perturbative procedure by assuming that $F_0 \gg f_1, f_2 \gg f_0, g_1, g_2, h_-$, and h_+. Combining then Eqs. (20) and (21) and separating with respect to different harmonics, we obtain

$$\begin{aligned} &\frac{\partial f_0}{\partial t} - \beta^3\frac{\partial^2 f_0}{\partial \upsilon^2} - \alpha^2\frac{\partial f_0}{\partial \upsilon} + \nu f_0 = -\frac{q_e}{m}\left(\hat{A}_1\frac{\partial f_1^*}{\partial \upsilon} + \hat{A}_2\frac{\partial f_2^*}{\partial \upsilon} + c.c.\right), \\ &\frac{\partial f_1}{\partial t} + i(k_1\upsilon - \omega_1)f_1 - \beta^3\frac{\partial^2 f_1}{\partial \upsilon^2} - \alpha^2\frac{\partial f_1}{\partial \upsilon} + \nu f_1 = -\frac{q_e}{m}\left[\hat{A}_1\left(\frac{\partial F_0}{\partial \upsilon} + \frac{\partial f_0}{\partial \upsilon}\right) + \hat{A}_2\frac{\partial h_-}{\partial \upsilon}\right], \\ &\frac{\partial f_2}{\partial t} + i(k_2\upsilon - \omega_2)f_2 - \beta^3\frac{\partial^2 f_2}{\partial \upsilon^2} - \alpha^2\frac{\partial f_2}{\partial \upsilon} + \nu f_2 = -\frac{q_e}{m}\left[\hat{A}_2\left(\frac{\partial F_0}{\partial \upsilon} + \frac{\partial f_0}{\partial \upsilon}\right) + \hat{A}_1\frac{\partial h_-^*}{\partial \upsilon}\right], \\ &\frac{\partial h_-}{\partial t} + i(\upsilon\Delta k - \Delta\omega)h_- - \beta^3\frac{\partial^2 h_-}{\partial \upsilon^2} - \alpha^2\frac{\partial h_-}{\partial \upsilon} + \nu h_- = -\frac{q_e}{m}\left(\hat{A}_1\frac{\partial f_2^*}{\partial \upsilon} + \hat{A}_2^*\frac{\partial f_1}{\partial \upsilon}\right). \end{aligned} \tag{23}$$

It turns out that the terms g_1, g_2, h_+ do not contribute to the final result; therefore, the equations for them have been omitted. Solving Eqs. (23), we can calculate the integrals $\int \upsilon f_i(t,\upsilon)d\upsilon$ of Eq. (19), which, e.g., for $i = 1$ takes the following form:

$$\begin{aligned} \frac{\partial \hat{A}_1}{\partial t} = \gamma\hat{A}_1 - 2\gamma_L\int_0^{t/2}d\eta\int_0^{t-2\eta}d\chi\cdot\Bigg[\eta^2\cdot\Bigg(&(q_e k_1/m)^2\hat{A}_1(t-\eta)\hat{A}_1(t-\eta-\chi)\hat{A}_1^*(t-2\eta-\chi)\cdot e^{i\alpha_k^2\eta(\eta+\chi)} \\ &+ (q_e k_2/m)^2\hat{A}_1(t-\eta)\hat{A}_2(t-\eta-\chi)\hat{A}_2^*(t-2\eta-\chi)\cdot e^{-i\omega_1\eta\left(\frac{\Delta k}{k_1}-\frac{\Delta\omega}{\omega_1}\right)+i\alpha_k^2\eta(\eta+\chi)}\Bigg) \\ &+ (q_e k_2/m)^2\hat{A}_2(t-\eta)\hat{A}_1(t-\eta-\chi)\hat{A}_2^*(t-2\eta-\chi)\cdot e^{-i\omega_1(2\eta+\chi)\left(\frac{\Delta k}{k_1}-\frac{\Delta\omega}{\omega_1}\right)+i\alpha_k^2\eta(\eta+\chi)}\cdot\eta\left(\eta+\frac{\Delta k}{k_1}(\eta+\chi)\right)\Bigg]e^{-\nu(2\eta+\chi)-\beta_k^3\eta^2\left(\frac{2}{3}\eta+\chi\right)}, \end{aligned} \tag{24}$$

where $\beta_k^3 = \beta^3 k_1^2$, $\alpha_k^2 = \alpha^2 k_1$, $\Delta k = k_1 - k_2$, and $\Delta\omega = \omega_1 - \omega_2$. The equation for $\hat{A}_2$ is obtained by exchanging indices according to $1 \leftrightarrows 2$. In deriving Eq. (24), we have assumed that the shifts Δk and $\Delta\omega$ are small, i.e., $\left|\frac{\Delta k}{k_i}\right|, \left|\frac{\Delta\omega}{\omega_i}\right| \ll 1$, $i = 1, 2$. In addition, we have used the approximations $\beta^3 k_1^2 \approx \beta^3 k_2^2$ and similarly $\alpha^2 k_1 \approx \alpha^2 k_2$, which allows to take the same parameters β_k^3 and α_k^2 in both amplitude equations. For numerical purposes, it is convenient to make use of the dimensionless variables and normalized parameters according to $t \rightarrow \gamma t$, $\hat{A}_i \rightarrow [(q_e k_i/m)\hat{A}_i/\gamma^2](\gamma_L/\gamma)^{1/2}$, $\nu \rightarrow \nu/\gamma$, $\beta_k^3 \rightarrow \beta_k^3/\gamma^3$, and $\alpha_k^2 \rightarrow \alpha_k^2/\gamma^2$. Furthermore, in addition to the small parameters $u_i \equiv \frac{\Delta k}{k_i}$, $c_i \equiv \frac{\Delta\omega}{\omega_i}$ $(i = 1,2)$, we introduce a large parameter $R \equiv \frac{\omega_1}{\gamma} \gg 1$. Since only two of the four parameters, u_i and c_i ($i=1$, 2) are independent, one may choose, e.g., u_1 and c_1, that characterize the small shift between the two modes ($|u_1||c_1| \ll 1$). Note that the natural dimensionless quantities appearing in the mode amplitude equations are $p_i \equiv \frac{\omega_i}{\gamma}\left(\frac{\Delta k}{k_i} - \frac{\Delta\omega}{\omega_i}\right)$, $i=1,\ 2$, which can be expressed through u_1, c_1, and R.

The final form of the dimensionless equations for the mode amplitudes is

$$\begin{aligned}\frac{\partial \hat{A}_1}{\partial t} = \hat{A}_1 - 2\int_0^{t/2} d\eta \int_0^{t-2\eta} d\chi \cdot \\ \Big[\eta^2 \cdot \Big(\hat{A}_1(t-\eta)\hat{A}_1(t-\eta-\chi)\hat{A}_1^*(t-2\eta-\chi)\cdot e^{i\alpha_k^2\eta(\eta+\chi)} \\ + \hat{A}_1(t-\eta)\hat{A}_2(t-\eta-\chi)\hat{A}_2^*(t-2\eta-\chi)\cdot e^{-ip_1\eta+i\alpha_k^2\eta(\eta+\chi)}\Big) \\ + \hat{A}_2(t-\eta)\hat{A}_1(t-\eta-\chi)\hat{A}_2^*(t-2\eta-\chi)\cdot e^{-ip_1(2\eta+\chi)+i\alpha_k^2\eta(\eta+\chi)}\cdot\eta(\eta+u_1(\eta+\chi))\Big]e^{-\nu(2\eta+\chi)-\beta_k^3\eta^2\left(\frac{2}{3}\eta+\chi\right)}.\end{aligned} \tag{25}$$

and

$$\begin{aligned}\frac{\partial \hat{A}_2}{\partial t} = \hat{A}_2 - 2\int_0^{t/2} d\eta \int_0^{t-2\eta} d\chi \cdot \\ \Big[\eta^2 \cdot \Big(\hat{A}_2(t-\eta)\hat{A}_2(t-\eta-\chi)\hat{A}_2^*(t-2\eta-\chi)\cdot e^{i\alpha_k^2\eta(\eta+\chi)} \\ + \hat{A}_2(t-\eta)\hat{A}_1(t-\eta-\chi)\hat{A}_1^*(t-2\eta-\chi)\cdot e^{-ip_2\eta+i\alpha_k^2\eta(\eta+\chi)}\Big) \\ + \hat{A}_1(t-\eta)\hat{A}_2(t-\eta-\chi)\hat{A}_1^*(t-2\eta-\chi)\cdot e^{-ip_2(2\eta+\chi)+i\alpha_k^2\eta(\eta+\chi)}\cdot\eta(\eta+u_2(\eta+\chi))\Big]e^{-\nu(2\eta+\chi)-\beta_k^3\eta^2\left(\frac{2}{3}\eta+\chi\right)}.\end{aligned} \tag{26}$$

The system of the two coupled equations (25) and (26) describes evolution of the two complex amplitudes $\hat{A}_1$ and $\hat{A}_2$ and, therefore, in fact it contains four coupled nonlinear equations for two real amplitudes $\hat{E}_1(t)$ and $\hat{E}_2(t)$ and two real phases $\tilde{\alpha}(t)$ and $\tilde{\beta}(t)$. These equations depend on two parameter groups: the collision parameters $(\nu, \beta_k^3, \alpha_k^2)$, and the mode parameters (u_i, c_i, R).

Note that when $u_i = c_i = 0$ ($i=1, 2$), the equation system (25) and (26) reduces to the original single mode equation of Ref. 8 only when the terms describing nonlinear coupling between the two modes are assumed to vanish. This problem needs an explanation. At first glance, it seems that adding the two equations for the amplitudes $\hat{A}_1$ and $\hat{A}_2$ with $u_i = c_i = 0$, i.e., without any shift in frequencies and wave numbers between the modes, one should obtain one mode with well determined frequency and wave number and, therefore, the system should be described by a single mode equation for the amplitude $\hat{A} = \hat{A}_1 + \hat{A}_2$. It follows from Eqs. (25) and (26) that this is not the case, because these include only six nonlinear terms, while the reduction to a single mode equation requires additional two mode coupling terms. To resolve this puzzle, let us first consider the Berk and Breizman model for a single mode, see, e.g., Ref. 4. In this model, fast particles distribution function F is expanded into a Fourier series around the equilibrium function F_0. However, only the terms f_0 and f_1 describing the slow time variation and the oscillations with the electric field frequency ω of the distribution function, respectively, essentially contribute to the final result. Note that in that model the frequency of the electric field is given stiffly and in fact it is the frequency of an eigenmode of the background plasma. A small shift in the mode frequency arises in this model only due to a slow variation of the electric field amplitude. The fast particles modify the plasma mode amplitude rather than the plasma mode frequency. It is also briefly mentioned in Ref. 4 that the component f_2 of the fast particle distribution function that oscillates with frequency 2ω does not contribute to the final model equation and the consideration is limited to a system of equations for components f_0 and f_1 only. Actually, performing calculations including f_2, one obtains delta function in time, which sets the contribution of this component outside the integration domain. However, what is the physical reason that only the two components f_0 and f_1 are sufficient for the Berk and Breizman model? In our opinion, the components f_2, f_3, and so on, introduce multiple harmonics of the basic frequency ω in the distribution function. These harmonics should generate electric field modes with the corresponding frequencies. However, the electric field is given stiffly with the frequency ω and since there are no other frequencies, the contributions from $f_2, f_3, \ldots,$ have to be zero. Only the components f_0 and f_1 are responsible for the modification of the distribution function in the Berk and Breizman model. Note also that even provided that some other harmonics of the electric field could exist, they should be eigenmodes of the background plasma laying on the positive slope of the bump on tail. However, such assumption is not incorporated in the framework of the model. An additional problem is due to the fact that the derivation of the evolution equation for the wave amplitude involves an application of delta functions. On the other hand, the resonant region around the frequency ω is very narrow but, due to small nonlinear corrections, it is not infinitesimally narrow like it is in the linear theory. In the case of nonlinear evolution of two modes, which is the subject of the present paper, we have limitations that are similar to the single mode case as discussed above. The two modes are given stiffly and are assumed to be two different eigenmodes of the background plasma. The important point is that they are different eigenmodes. Thus, the mode fields have to fulfill the orthogonality condition. We have constructed the fast particle distribution function according to similar rules to those of the single mode case. Consequently, because the two mode fields are given stiffly, the components of the fast particle distribution function having multiple and mixed frequencies are excluded. The only exception we have made is for the h_-. This is since the phase difference $\psi_1 - \psi_2$ is so small, that the exponent is almost constant, and the contribution of h_- to the mode evolution equation is similar to the one of f_0. Moreover, it explicitly grasps the frequency and wave number shifts between the two modes. In Sec. III, we numerically examine Eqs. (25) and (26) with $u_i = c_i = 0$. However, the condition $u_i = c_i = 0$ does not mean that the both modes can be treated as the same single mode. We emphasize that the orthogonality condition for the mode fields is supposed to be satisfied and an identification of the two eigenmodes as the same mode would change the orthogonality condition into a normalization condition. Roughly speaking, it would change zero into unity, which cannot be done in a continuous way, a limit which does not exist. Consequently, the case with $u_i = c_i = 0$ should be understood in the following way. Our analysis is based on the assumption that we have two different but laying very close to each other eigenmodes of the background plasma. Nevertheless, if the difference between the two modes is very small, the very small frequency and wave number shifts in the final equations (25) and (26) may be replaced by zeros, i.e., we put $u_i = c_i = 0$. Thus, the modes are treated as very close but still as different two eigenmodes. This explains the apparent discrepancy between the equation for the sum of the two modes and a single mode equation. We hope to give mathematically more rigorous exposition of the problem in a future paper on a multi-mode case.

III. NUMERICAL RESULTS

In order to examine the importance of the different collision models for the mode evolution, let us examine first the case when u_i, $c_i = 0$, and thus $p_i = 0$, $i = 1, 2$. In the absence of drag, $\alpha_k = 0$, it is found that the Krook and the diffusion models, described by the parameters ν and β_k, respectively, generate similar behavior and the common effect sums up so that qualitatively they do not produce new types of behavior in comparison with the Krook model alone. Therefore, to gain a qualitative view, it is enough to consider the dependence of the evolution of the modes on ν. For relatively large values of ν, a competition between the modes for survival is observed, as shown in Fig. 1. As a result of this competition, the growth rate of the disappearing mode is at first reduced to zero and then becomes negative, while the surviving mode reaches a zero growth rate and becomes saturated. Figure 2 is an illustration that the Krook and diffusion models produce qualitatively similar types of behavior. For decreasing

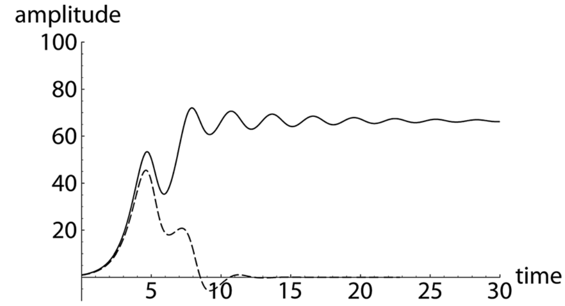

FIG. 1. $\nu = 5$, $\beta_k = 0$, $\alpha_k = 0$, $u_1 = 0$, $c_1 = 0$.

values of ν, both modes can survive in an oscillating regime, Fig. 3, or for still smaller values of ν in a chaotic regime, Fig. 4. Further decrease of ν leads to the "blow-up" behavior, Fig. 5, where the system breaks into oscillations with decreasing periods and increasing amplitudes, which become infinite in a finite time.

Let us now include the effect of drag model into consideration. A nonzero value of α_k gives an interesting result, that has previously been discovered in the single mode case,[8] namely that the solution always blows up regardless of value of α_k if the other collision parameters are vanishing, i.e., ν, $\beta_k = 0$, see Fig. 6.

However, for relatively large values of ν and moderate values of α_k, one may say that the effect of the Krook collision model dominates over the drag, because the mode competition is observed again, where one of the modes disappears and the other reaches saturation, Fig. 7. A somewhat larger value of α_k changes the picture essentially, both mode amplitudes tend to the same steady state value, see Fig. 8. This common steady state turns out to be unstable for greater values of α_k, see Fig. 9 and it breaks into oscillations with infinitely increasing amplitude at a finite time. The blow-up occurs.

A similar qualitative picture is obtained if instead of ν one uses appropriate values of β_k or β_k and ν in some proper combination. Figure 10 with all the three collision parameters being different from zero is equivalent to Fig. 9, where only ν and α_k are involved.

To perform numerical calculations in the presence of the mode shift, we arbitrary set the value of $R = 1000$ Then to have the parameters p_1 and p_2 of the order of the collision parameters ν, α_k, β_k, we take $p_1, p_2 \sim 1$. This implies that the order of the shift parameters must be u_i, $c_i \sim 0.001$. However,

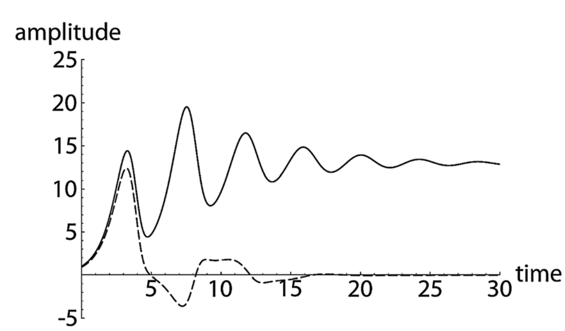

FIG. 2. $\nu = 0$, $\beta_k = 3$, $\alpha_k = 0$, $u_1 = 0$, $c_1 = 0$.

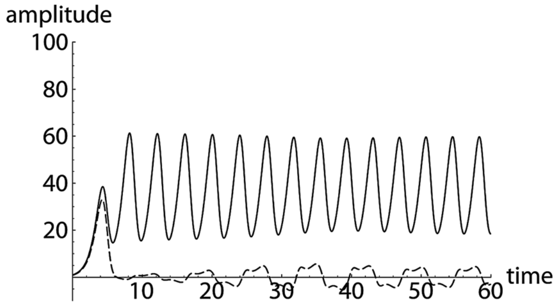

FIG. 3. $\nu=4$, $\beta_k=0$, $\alpha_k=0$, $u_1=0$, $c_1=0$.

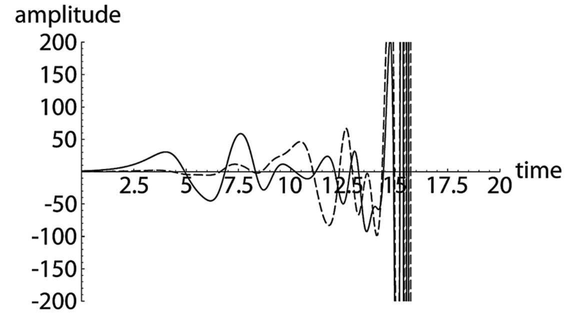

FIG. 5. $\nu=2.3$, $\beta_k=0$, $\alpha_k=0$, $u_1=0$, $c_1=0$.

because $p_1=R\cdot(u_1-c_1)$ one obtains in case when $u_1=c_1$ that $p_i=0$, $i=1,2$. Note also that in Eqs. (25) and (26), the parameters u_i are not only included in p_i but also appear at other places. On the other hand, if the parameters p_i are vanishing, the small values of $u_i\sim 0.001$ make their influence on the mode evolution rather weak, compare Fig. 11 with Fig. 1.

We consider first the effect of the shift in the absence of drag and for $\nu=5$ as well as for various combination of ν and β_k. To assure the nonzero value of u_1-c_1, we put $c_1=0$, then p_1 is simply $R\cdot u_1$. Comparing Figs. 8 and 12, it can be seen that qualitatively the mode behavior for $p_1=1$ is similar to that of drag with $\alpha_k=3.3$. The conclusion that the shift effects are similar to those of drag is only valid for sufficiently small values of $p_1\sim 1$. The coupling between the modes becomes weaker for larger values of p_i. The steady state shown in Fig. 12 becomes unstable and periodic for a slightly larger value of p_1. A remarkable feature of the oscillations is the synchronization of periods (in opposite phase) of the mode amplitudes shown in Fig. 13. For $p_1\sim 10$ ($u_1\sim 0.01$), the synchronization vanishes and the oscillations become chaotic. Another interesting case is shown in Fig. 14 for a relatively weak coupling between the modes. Each of the modes influences rather weakly on the dynamics of the other mode which gives rise to results in beating of the amplitudes. For still larger values of p_1, e.g., $p_1\sim 30$ ($u_1\sim 0.03$), the coupling between the modes becomes so weak that the modes behave as practically independent, Fig. 15. Such behavior can be explained if one notes that p_1 and p_2 may be expressed as $p_1=\frac{k_2}{\gamma}(\upsilon_2-\upsilon_1)$ and $p_2=\frac{k_1}{\gamma}(\upsilon_1-\upsilon_2)$, where υ_1 and υ_2 are the phase velocities of the modes. In the velocity space, each of the modes produces a local plateau in the resonance region (in the vicinity of the phase velocity) of the fast ion distribution function. The local modification of the distribution function caused by one of the modes affects the dynamics of the other mode. Nevertheless, if the difference of the phase velocities $\upsilon_1-\upsilon_2$ is too large, the two local plateaus do not overlap and the modes behave as independent from each other.

Let us now add the effect of drag into the case of a finite shift between the modes. The combination of drag and Krook (or diffusion) operators together with the mode shift gives a new type of the mode amplitude behavior. Although, for small parameter values $\alpha_k=1$ and $p_1=1$ ($u_1=0.001$), the mode competition can be observed, a small change of the mode shift to $p_1=1.3$ ($u_1=0.0013$) reveals the existence of two different steady states for each amplitude, Fig. 16. For larger values of the shift parameters, the two steady states become unstable and start to oscillate. Synchronization of periods appears in a similar way to Fig. 13. For a sufficiently large shift, e.g., $p_1=30$ ($u_1=0.03$), the modes become practically independent, which is illustrated in Fig. 15.

Another scenario takes place for larger values of the drag parameter, e.g., for $\alpha_k=3$ or larger. The previous analysis of such cases with a vanishing mode shift has shown that when the drag dominates over the Krook collisions, a blow-up appears. In the presence of the mode shift, the situation becomes modified in that way that the two steady states appear very quickly, i.e., a picture similar to the one shown in Fig. 16 is observed for a very small value of p_1, e.g., $p_1=0.1$ ($u_1=0.0001$) for $\alpha_k=3$. The distance between the two steady states increases with increasing of the shift parameter. However, due to increasing values of the shift, the two modes become more and more independent and for independent modes, the blow-up occurs if the drag is a

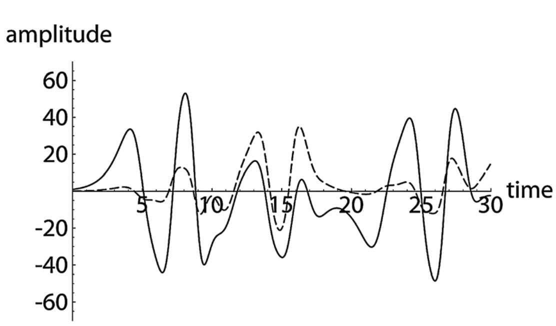

FIG. 4. $\nu=2.5$, $\beta_k=0$, $\alpha_k=0$, $u_1=0$, $c_1=0$.

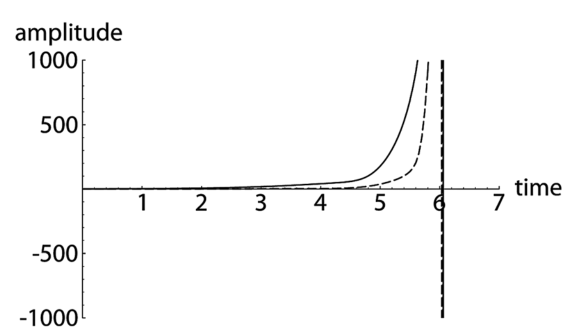

FIG. 6. $\nu=0$, $\beta_k=0$, $\alpha_k=10$, $u_1=0$, $c_1=0$.

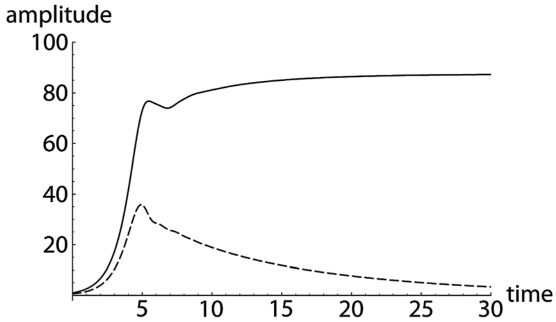

FIG. 7. $\nu = 5$, $\beta_k = 0$, $\alpha_k = 2.5$, $u_1 = 0$, $c_1 = 0$.

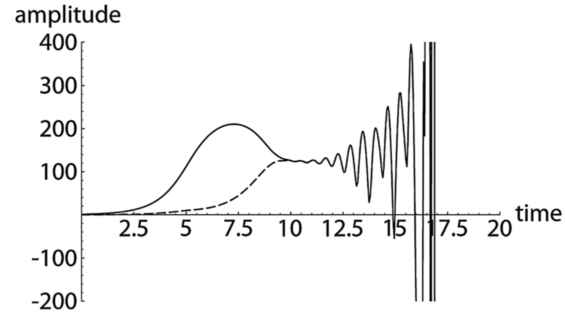

FIG. 9. $\nu = 5$, $\beta_k = 0$, $\alpha_k = 4.1$, $u_1 = 0$, $c_1 = 0$.

dominating collision process. In a matter of fact, in the described case, the mode blow-up takes place far before the modes really become independent. Figure 17 shows a case of mode evolution, where the two *different* steady states become unstable and then blow up.

IV. SUMMARY

In the present paper, we have outlined a basis of the nonlinear theory of two wave modes excited in a plasma by the resonant interaction with fast ions. It is assumed that the mode excitation takes place just above the linear instability threshold, and the model includes the effects of a resonant fast ions source and classical relaxation processes represented by the Krook, diffusion, and dynamical friction collision operators. A system of two coupled integro-differential equations for the mode amplitudes has been derived and examined numerically. The goal of our numerical analysis has been to give the reader a basic insight into the behaviour of two plasma modes coupled to each other through resonant interaction with fast ions. In particular, we have shown how different collision operators may lead to similar (compare, e.g., diffusion and Krook operators, Figs. 1 and 2) or quite different (dynamic friction, Fig. 6) behaviour of the mode amplitudes. However, one should keep in mind that the final state of mode amplitudes strongly depends on the initial parameters, and there are critical parameter values at which there is a bifurcation from one type of final state to another. This takes place in the evolution of a single mode as well as in the case of two modes. Therefore, the effort made in the present paper has been only to choose the cases presenting the most characteristic features of two-mode evolution. This is the reason why the mode dynamics is illustrated by including mainly the Krook operator with relatively large values of the Krook parameter instead of a full combination of the Krook and diffusion operators, as well by discussing the influence of dynamic friction and frequency and wave number shifts between modes on the mode evolution. We believe that in this way we grasp the most basic features of the two-mode dynamics.

It is interesting to observe the strong interaction between the two modes, which has been demonstrated in the first ten figures, where we have assumed that the frequency and wave number shifts between the modes vanish. In this limit, an important observation is that the two-mode evolution equations (25) and (26) reduce to the original single mode equation of Ref. 8 only when the terms describing nonlinear coupling between the two modes are assumed to vanish. It seems that adding the two equations for the amplitudes $\hat{A}_1$ and $\hat{A}_2$ with $u_i = c_i = 0$, i.e., without any shift in frequencies and wave numbers between the modes, one should obtain one mode with well determined frequency and wave number, and therefore, the system should be described by a single mode equation for the amplitude $\hat{A} = \hat{A}_1 + \hat{A}_2$. It follows from Eqs. (25) and (26) that this is not the case, because these include only six nonlinear terms, while the reduction to a single mode equation requires additional two mode coupling terms. However, the condition $u_i = c_i = 0$ does not mean that the both modes can be treated as the same single mode. We emphasize that the orthogonality condition for the mode fields is supposed to be satisfied and an identification of the two eigenmodes as the same mode would change the orthogonality condition into a normalization condition. Roughly speaking, it would change zero into unity, which cannot be done in a continuous way, a limit which does not exist. Consequently, the case with $u_i = c_i = 0$ should be understood in

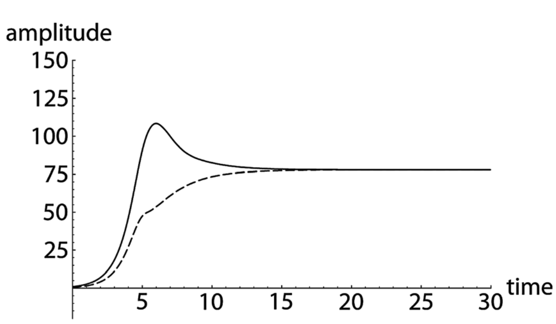

FIG. 8. $\nu = 5$, $\beta_k = 0$, $\alpha_k = 3.3$, $u_1 = 0$, $c_1 = 0$.

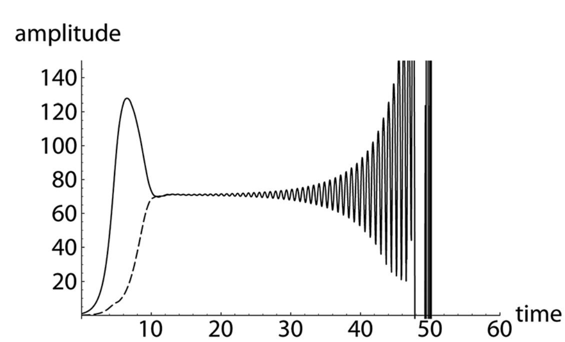

FIG. 10. $\nu = 4$, $\beta_k = 2$, $\alpha_k = 3.4$, $u_1 = 0$, $c_1 = 0$.

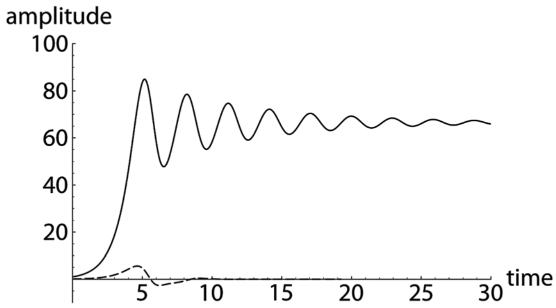

FIG. 11. $\nu = 5$, $\beta_k = 0$, $\alpha_k = 0$, $u_1 = 0.001$, $c_1 = 0.001$.

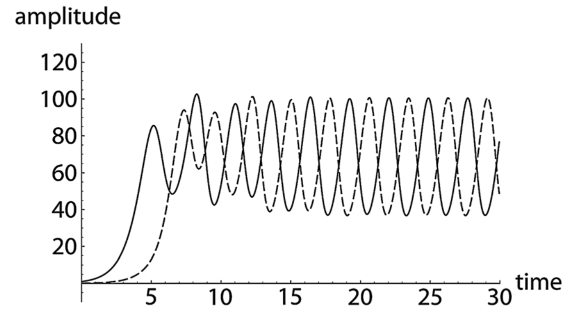

FIG. 13. $\nu = 5$, $\beta_k = 0$, $\alpha_k = 0$, $u_1 = 0.005$, $c_1 = 0$.

the following way. Our analysis is based on the assumption that we have two different but laying very close to each other eigenmodes of the background plasma. Nevertheless, if the difference between the two modes is very small, the very small frequency and wave number shifts in the final equations (25) and (26) may be replaced by zeros, i.e., we put $u_i = c_i = 0$. Thus, the modes are treated as very close but still as different two eigenmodes.

A strong mode interaction may lead to competition between the modes. This behaviour can be especially seen in Figs. 1 and 2, where only one of the modes survives. Since the Berk and Breizman mode evolution model[8] is valid for relatively weak field nonlinearities, the saturation level of the mode amplitude is achieved due to collisions represented by the Krook or diffusion collision operators (not by the drag operator) and not due to phase mixing that appears for strong field nonlinearity and is caused by bouncing of trapped particles in the plasma wave even in the absence of collisions. Therefore, in the Berk and Breizman model extended to the case of two modes, the mode competition and thus saturation of one of the modes can be possible only for relatively large values of the parameters ν or β_k. For smaller values of these parameters, periodic amplitude modulations as well as chaotic and explosive regimes take place, as shown in Figs. 3–5.

Dynamical friction (drag) introduces a new feature into the mode dynamics. Since its presence means that fast ions are subject to the friction force in velocity space, there is a continuous process of new ions with velocities slightly greater than the phase velocity coming into the resonant region and of ions with velocities smaller than the phase velocity leaving the resonant region. As a result, the particle energy is constantly transferred to the wave. If the "annihilation" collision mechanisms like Krook or diffusion are too weak to counteract the drag, then it leads to an explosive evolution of the wave amplitudes, as it is shown in Figs. 6, 9, 10, and 17.

An interesting feature of drag, when is not too weak (because then the mode competition occurs) and at the same time not too strong (because it causes the explosive amplitude evolution), is that it also may lead to the same finite saturation level of each mode amplitude, see Fig. 8 and also compare Figs. 7, 9, and 10. However, adding the frequency and wave number shifts between the modes splits the single saturation into different finite steady states for each of the modes, see Figs. 16 and 17.

Inclusion of the finite frequency and wave number shifts weakens the interaction strength between the modes. For large values of the parameters ν and/or β_k and zero or small values of the drag parameter α_k, as well as in the presence of an slight frequency and wave number shifts between the modes, a periodic exchange of energy between the two modes ("beating") is observed, see, e.g., Figs. 13 and 14. Such periodic exchange of energy between modes is a typical behaviour of many physical systems with not too strong nonlinear mode coupling.

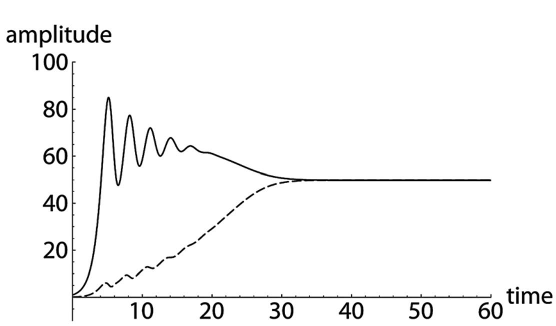

FIG. 12. $\nu = 5$, $\beta_k = 0$, $\alpha_k = 0$, $u_1 = 0.001$, $c_1 = 0$.

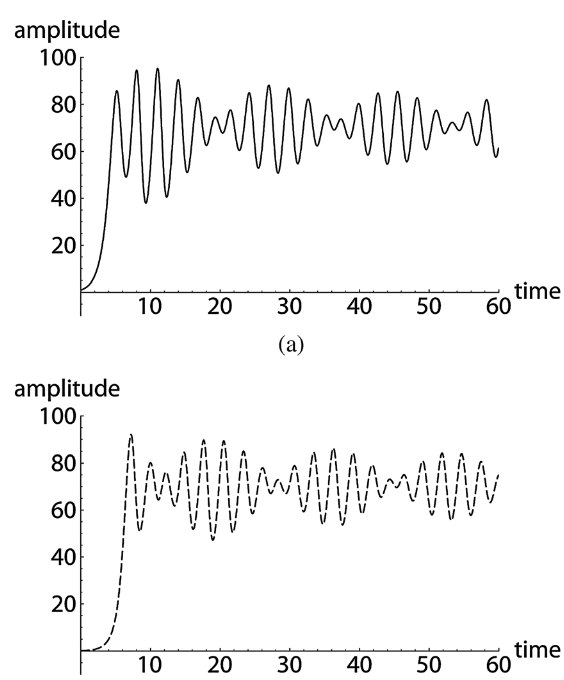

FIG. 14. $\nu = 5$, $\beta_k = 0$, $\alpha_k = 0$, $u_1 = 0.017$, $c_1 = 0.001$, (for clarity the two amplitudes are shown in separate boxes).

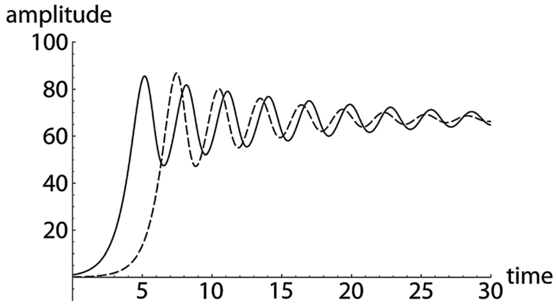

FIG. 15. $\nu = 5$, $\beta_k = 0$, $\alpha_k = 0$, $u_1 = 0.03$, $c_1 = 0.001$.

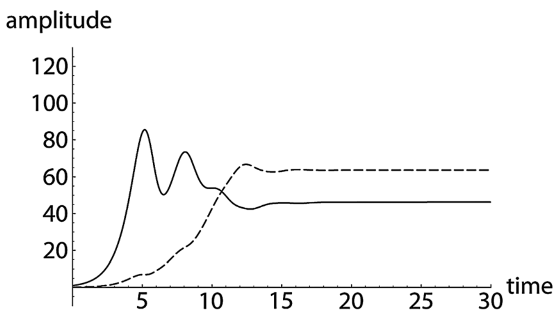

FIG. 16. $\nu = 5$, $\beta_k = 0$, $\alpha_k = 1$, $u_1 = 0.0013$, $c_1 = 0.001$.

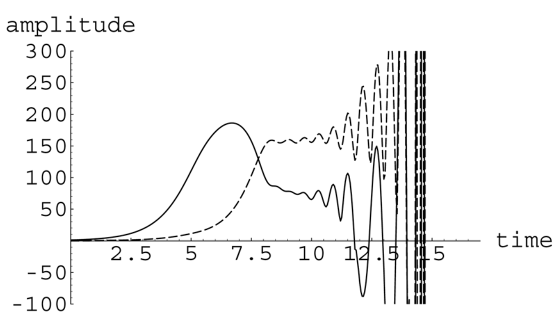

FIG. 17. $\nu = 5$, $\beta_k = 0$, $\alpha_k = 4$, $u_1 = 0.001$, $c_1 = 0$.

A large separation between phase velocities of the two modes may weaken the interaction strength between them to such an extent that they become almost independent, see Fig. 15 and compare it with the evolution of a single mode shown in, e.g., Ref. 4.

In summary, we conclude that depending on the type of relaxation process, on the specific values of the appropriate relaxation parameters, and on initial conditions, different nonlinear regimes in the evolution of two mode amplitudes can be observed, including steady-state, periodic amplitude modulations, and chaotic and explosive regimes. Though such regimes have also been observed in the case of single mode evolution,[8] the two-mode dynamics is richer due to interplay between the modes as it has been described above.

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