Global convergence for Inverse Problems

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Abstract. A globally convergent numerical method for a multidimensional Coefficient Inverse Problem for a hyperbolic equation is presented. It is shown that this technique provides a good starting point for the finite element adaptive method (adaptivity). This leads to a natural two-stage numerical procedure, which synthesizes both these methods.

Keywords: global convergence, adaptive finite element method, hyperbolic coefficient inverse problem, ill-posed problems
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INTRODUCTION

This paper represents a short outline of our recent publications [5,6,7,9,10], where a globally convergent numerical method for a Coefficient Inverse Problem (CIP) for a hyperbolic PDE was developed analytically and verified numerically on both computationally simulated and experimental data. Applications of this technique are in detection of explosives in e.g. search for plastic land mines and airport security. In [5,6] that globally convergent numerical method was developed, its global convergence was analytically established and verified on computationally simulated data. This technique was independently verified in [10], where blind studies of experimental data were presented. The term âĂŹâĂŹblindâĂŹâĂŹ means here that first images were obtained by the method of [5,6] from time resolved experimental data. No a priori knowledge of refractive indices of imaged dielectric abnormalities was available to the authors of [10]. Next those refractive indices were directly measured a posteriori. Finally, results of those measurements were compared with the results of computations. This comparison has revealed that blindly computed refractive indices differed from measured ones by only a few percent, which was within measurement error. Thus, an excellent accuracy of blind computations was evident. In [6,7] a synthesis of the globally convergent method of [5] with the locally convergent Adaptive Finite Element technique (adaptivity below) was developed. Actually this is a two-stage numerical procedure. On the first stage the globally convergent numerical method provides a good approximation for the solution. On the second stage this approximation is used as a starting point for the locally convergent adaptive algorithm. We point out that a good first approximation is a the truly key point for any locally convergent method. So, we have it from the first stage. The adaptivity refines the first stage solution. Although the adaptivity is a classical tool for forward problems [1], the first application of the adaptivity to a CIP was published in [2] with follow up works [3,4]. We now explain why it is necessary to refine solution obtained on the globally convergent stage. CIPs are both nonlinear and ill-posed. These two factors combined cause tremendous challenges in the development of globally convergent algorithms for these problems. Hence, it was necessary and perhaps even inevitable to make an approximation in the globally convergent method of [5]. Namely, we have truncated a certain integral over an infinite interval at a large value $\bar{s}$ of the parameter $s > 0$ of the Laplace transform. We call s pseudo-frequency. The number $\bar{s}$ is the regularization parameter of the technique of [5]. This truncation is similar to the truncation at high frequencies, which is routinely done in engineering. Details are discussed in the subsection 3.3 of [10] and subsection 6.3 of [6]. It was pointed out in these references that we cannot proof convergence for $\bar{s} \to \infty$. Indeed, if we would do this, then we would be able to proof uniqueness theorem for our CIP, which is a well known long standing open question. It is shown in [6,10] that from the analytical standpoint, the above truncation is neither better or worse than the classical truncation of divergent asymptotic series in the Real Analysis.

The adaptivity consists in minimizing the Tikhonov functional on a sequence of locally refined meshes in FEM. It was shown numerically in [6,7,8,9] that mesh refinements are very important, because the solution obtained on the same mesh where the globally convergent method was applied does not show an improvement, unlike those on refined meshes. Meshes are refined in such subdomains of the original domain, where a posteriori error analysis indicates

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the maximal error of the solution. That error analysis does not use a knowledge of the exact solution. instead, one should know an upper bound of that solution, and such a bound should be imposed a priori, in accordance with the Tikhonov principle [14]. It was shown analytically in [8] that the mesh refinement indeed improves the accuracy of the regularized solution as long as the modulus of the gradient of the Tikhonov functional is not too small. However, as soon as it becomes too small, mesh refinements should be stopped. The latter has been consistently observed in [2-4,6,7].

A to the locally convergent methods alone, they perform poorly for our experimental data (although they might perform well for some other data). Indeed, it was demonstrated in [10] that a modified gradient method has a poor performance for our data. Our numerical studies have also revealed that the adaptivity does not perform well in the case when the solution obtained on the globally convergent stage is not used. On the other hand, it was demonstrated in [6,7,9] that the availability of the solution obtained on the first stage is crucial here: otherwise the adaptivity doe snot work. This points towards the importance of the globally convergent stage.

STATEMENTS OF FORWARD AND INVERSE PROBLEMS

As the forward problem, we consider the following Cauchy problem

$$\begin{aligned} \varepsilon_r(x)u_{tt} &= \Delta u, \text{ in } \mathbb{R}^3 \times (0,\infty), && (1)\\ u(x,0) &= 0, u_t(x,0) = \delta(x-x_0). && (2) \end{aligned}$$

Here $\varepsilon_r(x)$ is the spatially variable dielectric constant (relative dielectric permittivity),

$$\varepsilon_r(x) = \frac{\varepsilon(x)}{\varepsilon_0},\ \sqrt{\varepsilon_r(x)} = n(x) = \frac{c_0}{c(x)} \geq 1, \quad (3)$$

where ε_0 is the dielectric permittivity of the vacuum (which we assume to be the same as one in the air), $\varepsilon(x)$ is the spatially variable dielectric permittivity of the medium of interest, $n(x)$ is the refractive index of the medium of interest, $c(x)$ is the speed of the propagation of the EM field in this medium, and c_0 is the speed of light in the vacuum, which we assume to be the same as one in the air. We point out that it is the refractive index rather than the dielectric constant, which is measured in physics. The assumption $n(x) \geq 1$ means that the speed of the EM field propagation in the medium does not exceed one in the air, which is reasonable.

Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$. We assume that the coefficient $\varepsilon_r(x)$ of equation (1) is such that

$$\begin{aligned} \varepsilon_r(x) &\in (1,d], \varepsilon_r(x) = 1 \text{ for } x \in \mathbb{R}^3 \setminus \Omega, && (4)\\ \varepsilon_r(x) &\in C^2\left(\mathbb{R}^3\right). && (5) \end{aligned}$$

The inequality $\varepsilon_r(x) \geq 1$ follows from (3). An upper estimate for the constant $d > 1$ is assumed to be known, although we do *not* assume that $d-1$ is small.

Inverse Problem. *Suppose that the coefficient $\varepsilon_r(x)$ satisfies (4) and (5). Assume that the function $\varepsilon_r(x)$ is unknown in the domain Ω. Determine the function $\varepsilon_r(x)$ for $x \in \Omega$, assuming that the following function $g(x,t)$ is known for a single source position $x_0 \notin \overline{\Omega}$*

$$u(x,t) = g(x,t), \forall (x,t) \in \partial\Omega \times (0,\infty). \quad (6)$$

The assumption $\varepsilon_r(x) = 1$ for $x \in \mathbb{R}^3 \setminus \Omega$ means that one has air outside of the medium of interest Ω. The question of uniqueness of this Inverse Problem is a well known long standing open question. It is addressed positively only if the function $\delta(x-x_0)$ above is replaced with a function $f(x)$ such that $f(x) \neq 0, \forall x \in \overline{\Omega}$. Corresponding uniqueness theorems were proven via the method of Carleman estimates [11]. Still, due to the applied aspect, numerical methods is worthy to develop, assuming that the uniqueness question is addressed positively.

THE GLOBALLY CONVERGENT STAGE

Consider the Laplace transform of the solution of the problem (1), (2),

$$w(x,s) = \int_0^\infty u(x,t)\, e^{-st} dt, s \geq \underline{s} = const. > 0. \quad (7)$$

Then $w(x,s) > 0$ for sufficiently large s. We consider the function $q(x,s) = \partial_s \left(s^{-2} \ln w(x,s) \right)$. Under certain conditions

$$D_x^\alpha D_s^k \left(\frac{\ln w(x,s)}{s^2} \right) = O\left(\frac{1}{s^{k+1}} \right), s \to \infty, k = 0,1; |\alpha| \leq 2. \tag{8}$$

We obtain a nonlinear integral differential equation for the function q for $x \in \Omega, s \in (\underline{s}, \infty)$ with Volterra integrals in which the s-integration is carried out from an arbitrary $s \geq \underline{s}$ to ∞. One of the key features of this equation is that the unknown coefficient $\varepsilon_r(x)$ is not involved in it. The Dirichlet boundary condition at $\partial\Omega$ is generated by the function g in (6). If one would approximate the function q well, then one would approximate the function $\varepsilon_r(x)$ well via backwards computations. The main difficulty then is to solve the resulting Dirichlet boundary value problem for q. To do this, we first truncate those Volterra integrals at a large value $s := \bar{s} > \underline{s}$. However, we complement that truncation by the so-called "tail function" $V(x,\bar{s}) \approx \bar{s}^{-2} \ln w(x,\bar{s})$. The tail function is unknown. However, it is small for large $\bar{s}$ because of (8). Hence, the resulting equation for q contains two unknown functions: q and V. The reason why we can approximate both of them is that we treat them separately: while we approximate q via inner iterations, we approximate V via outer iterations.

To solve the resulting problem, we divide the interval $[\underline{s}, \bar{s}]$ into N small subintervals. We assume that the function q is constant with respect to s on each of those subintervals. As a result, we obtain N elliptic Dirichlet boundary value problems for functions $q_n(x)$, where n is the number of the subinterval. Because originally we had Volterra integrals with respect to s, we can solve these problems sequentially starting from q_1. Let $q_{n,k}$ be the approximation for q_n obtained on the inner iteration number k and $V_{n,k}(x)$ be the corresponding approximation for the tail. Then we find the corresponding approximation $\varepsilon_r^{(n,k)}(x)$ for the function $\varepsilon_r(x)$, solve the problem (1), (2) with $\varepsilon_r := \varepsilon_r^{(n,k)}(x)$, calculate the Laplace transform $w_{n.k+1}(x,\bar{s})$ via (7) for it and find a new approximation $V_{n,k+1}(x) := \bar{s}^{-2} \ln w_{n,k+1}(x,\bar{s})$ for the tail. Convergence criteria for this algorithm are described in [5, 6, 7, 8, 10]. In particular, in our computations for experimental data we use the criterion described in subsection 7.1 of [10].

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