A NEW PROOF OF AN OLD RESULT BY PICKANDS

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Abstract

Let $\{\xi(t)\}_{t \in [0,h]}$ be a stationary Gaussian process with covariance function r such that $r(t) = 1 - C|t|^\alpha + o(|t|^\alpha)$ as $t \to 0$. We give a new and direct proof of a result originally obtained by Pickands, on the asymptotic behaviour as $u \to \infty$ of the probability $P\{\sup_{t \in [0,h]} \xi(t) > u\}$ that the process ξ exceeds the level u. As a by-product, we obtain a new expression for Pickands constant H_α.

1 Introduction and main result

Let $\{\xi(t)\}_{t \in [0,h]}$ be a continuous centered stationary Gaussian process with covariance function $r(t) = \text{Cov}\{\xi(s), \xi(s+t)\}$ that satisfies

$$r(t) < 1 \quad \text{for } t \in (0,h] \qquad \text{and} \qquad r(t) = 1 - C|t|^\alpha + o(|t|^\alpha) \quad \text{as } t \to 0, \tag{1}$$

where $h > 0$, $\alpha \in (0,2]$ and $C > 0$ are constants. Note that (1) includes, for example, cov- ariance functions of the form $e^{-|t|^\alpha}$, where the case when $\alpha = 1$ corresponds to an Ornstein-Uhlenbeck process. Further, the case when $\alpha = 2$ in (1) corresponds to mean-square differentiable processes, while processes with $0 < \alpha < 2$ are non-differentiable.

The tail distribution of $\sup_{t \in [0,h]} \xi(t)$ was originally obtained in [6] by means of a long and complicated proof involving so called ϵ-upcrossings, although his proof was not quite complete. The Pickands result is as follows:

Theorem. *If* (1) *holds, then we have*

$$\lim_{u \to \infty} \frac{q(u)}{\Phi(u)} P\Big\{ \sup_{t \in [0,h]} \xi(t) > u \Big\} = h\, C^{1/\alpha} H_\alpha. \tag{2}$$

Here $\overline{\Phi}(u) = \int_u^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ is the standard Gaussian tail distribution function, $q(u) = u^{-2/\alpha}$, and H_α is strictly positive and finite constant, the value of which depends on α only and that is given by

$$H_\alpha = \lim_{T \to \infty} T^{-1} \int_0^\infty e^s \, P\left\{ \sup_{t \in [0,T]} \zeta(t) > s \right\} ds, \tag{3}$$

where $\{\zeta(t)\}_{t>0}$ is a nonstationary Gaussian process with mean and covariance function;

$$m_\zeta(t) = E\{\zeta(t)\} = -t^\alpha \quad and \quad r_\zeta(s,t) = \text{Cov}\{\zeta(s), \zeta(t)\} = s^\alpha + t^\alpha - |s-t|^\alpha. \tag{4}$$

Qualls and Watanabe [8], Leadbetter et al. [4] and others gave clarifications of Pickands' original proof. The proof as it stands today in the literature is long and complicated, but is also important, for example, as a model proof in extreme value theory.

In this paper we give a new and direct proof of (2), that is in part inspired by [1, 2, 9, 10]. Given a constant $a > 0$, a key step in our proof is to find the tail behaviour of the distribution of the discretized maximum $\max_{k \in \{0, \ldots, \lfloor h/(aq(u)) \rfloor\}} \xi(aq(u)k)$. Knowing that tail in turn, we can move on to find the tail behaviour of the continuous supremum $\sup_{t \in [0,h]} \xi(t)$ featuring in (2). The technical details that we employ to this end are much simpler and more direct than those in the literature.

As a by-product, our new proof produces the following new formula for the constant H_α (cf. (3)):

Theorem 1. *If* (1) *holds, then* (2) *holds with the constant H_α given by*

$$H_\alpha = \lim_{a \downarrow 0} \frac{1}{a} P\left\{ \max_{k \geq 1} \zeta(ak) + \eta \leq 0 \right\},$$

where $\{\zeta(t)\}_{t>0}$ is the Gaussian process with means and covariances given in (4)*, and η is a unit mean exponentially distributed random variable that is independent of the process ζ.*

The exact values of Pickands constants H_α are known for $\alpha = 1, 2$, i.e. $H_1 = 1$, $H_2 = 1/\sqrt{\pi}$ but unknown for other values of α, although many efforts to find numerical approximations of H_α have been made (see for example [3]). Possibly, our new formula for H_α given in Theorem 1 might be utilized in future work to compute that constant numerically.

2   New proof of (2)

Throughout the proof we use $\overline{\Phi}$ and q as short-hand notation for $\overline{\Phi}(u)$ and $q(u)$, respectively.
A key step in the proof is to show that, given constants $a, h > 0$, we have

$$\lim_{u \to \infty} \frac{q}{\overline{\Phi}} P\left\{ \max_{k \in \{0, \ldots, \lfloor h/(aq) \rfloor\}} \xi(aqk) > u \right\} = \frac{h}{a} P\left\{ \bigcap_{k=1}^\infty \{\zeta(C^{1/\alpha}ak) + \eta \leq 0\} \right\}. \tag{5}$$

Lemma 1. *If* (1) *holds, then* (5) *holds.*

The proof of Lemma 1 requires two preparatory results, Lemmas 2 and 3. In the first of these two preparatory lemmas, with the notation $\xi_u(t) = u(\xi(qt) - u)$, we establish weak convergence of the finite dimensional distributions of the process $\{(\xi_u(t) | \xi_u(0) > 0\}_{t>0}$ as $u \to \infty$:

Lemma 2. *If* (1) *holds, then the finite dimensional distributions of the process $\{(\xi_u(t) | \xi_u(0) > 0)\}_{t>0}$ converge weakly to those of the process $\{\zeta(C^{1/\alpha}t) + \eta\}_{t>0}$ as $u \to \infty$.*

Proof. As $\xi_u(t)$ is the sum of the two independent Gaussian random variables $\xi_u(t) - r(qt)\xi_u(0)$ and $r(qt)\xi_u(0)$, it is sufficient to show that (a) the finite dimensional distributions of $\{\xi_u(t) - r(qt)\xi_u(0)\}_{t>0}$ converge weakly to those of $\{\zeta(C^{1/\alpha}t)\}_{t>0}$, and that (b) the conditional distribution $(r(qt)\xi_u(0)|\xi_u(0) > 0)$ converges weakly to η.

To establish (a), it is enough to observe that, by (1), for $s, t > 0$, we have

$$E\{\xi_u(t) - r(qt)\xi_u(0)\} = -u^2(1 - r(qt)) \rightarrow m_\zeta(C^{1/\alpha}t)$$

$$\mathrm{Cov}\{\xi_u(s) - r(qs)\xi_u(0), \xi_u(t) - r(qt)\xi_u(0)\} = u^2(r(q(t-s)) - r(qt)r(qs)) \rightarrow r_\zeta(C^{1/\alpha}s, C^{1/\alpha}t)$$

as $u \to \infty$. Further, (b) follows from noting that, by elementary calculations, we have

$$P\{\xi_u(0) > x \,|\, \xi_u(0) > 0\} = \frac{\overline{\Phi}(u + x/u)}{\overline{\Phi}(u)} \rightarrow e^{-x} = P\{\eta > x\} \quad \text{as } u \to \infty \quad \text{for } x > 0. \quad \square \qquad (6)$$

In the second preparatory lemma required to prove Lemma 1, we find the asymptotic behaviour of the probability $P\{\max_{k\in\{0,\dots,N\}} \xi(aqk) > u\}$ as $u \to \infty$ and $N \to \infty$:

Lemma 3. *If* (1) *holds, then given any constant $a > 0$, we have*

$$\lim_{N\to\infty} \lim_{u\to\infty} \frac{1}{N\overline{\Phi}} P\Big\{ \max_{k\in\{0,\dots,N\}} \xi(aqk) > u \Big\} = P\Big\{ \bigcap_{\ell=1}^{\infty} \{\zeta(C^{1/\alpha}a\ell) + \eta \leq 0\} \Big\}.$$

Proof. By the inclusion exclusion formula together with stationarity of ξ and Lemma 2, we have

$$\frac{1}{N\overline{\Phi}} P\Big\{ \max_{k\in\{0,\dots,N\}} \xi(aqk) > u \Big\} = \frac{P\{\xi(aqN) > u\}}{N\overline{\Phi}} + \frac{1}{N\overline{\Phi}} \sum_{k=0}^{N-1} P\Big\{ \bigcap_{\ell=k+1}^{N} \{\xi(aq\ell) \leq u\}, \xi(aqk) > u \Big\}$$

$$= \frac{1}{N} + \frac{1}{N} \sum_{k=1}^{N} P\Big\{ \bigcap_{\ell=1}^{k} \{\xi_u(a\ell) \leq 0\} \,\Big|\, \xi_u(0) > 0 \Big\}$$

$$\rightarrow \frac{1}{N} + \frac{1}{N} \sum_{k=1}^{N} P\Big\{ \bigcap_{\ell=1}^{k} \{\zeta(C^{1/\alpha}a\ell) + \eta \leq 0\} \Big\} \quad \text{as } u \to \infty.$$

Sending $N \to \infty$ on the right-hand side, the lemma follows by an elementary argument. $\square$

We are now prepared to prove Lemma 1:

Proof of Lemma 1. In order to find an upper estimate in (5) note that, by Boole's inequality together with stationarity and Lemma 3, we have

$$\varlimsup_{u\to\infty} \frac{q}{\overline{\Phi}} P\Big\{ \max_{k\in\{0,\dots,\lfloor h/(aq)\rfloor\}} \xi(aqk) > u \Big\} \leq \varlimsup_{N\to\infty} \lim_{u\to\infty} \frac{q}{\overline{\Phi}} \Big(\Big\lfloor \frac{h}{aqN} \Big\rfloor + 1 \Big) P\Big\{ \max_{k\in\{0,\dots,N-1\}} \xi(aqk) > u \Big\}$$

$$= \frac{h}{a} P\Big\{ \bigcap_{\ell=1}^{\infty} \{\zeta(C^{1/\alpha}a\ell) + \eta \leq 0\} \Big\}. \qquad (7)$$

To find a lower estimate in (5) we make two preparatory observations: Firstly, by stationarity and Lemma 3, we have

$$\frac{1}{N\overline{\Phi}} P\Big\{ \max_{k\in\{0,\dots,N-1\}} \xi(aqk) > u, \max_{\ell\in\{N,\dots,2N-1\}} \xi(aq\ell) > u \Big\}$$

$$= \frac{2}{N\overline{\Phi}} P\Big\{ \max_{k\in\{0,\dots,N-1\}} \xi(aqk) > u \Big\} - \frac{2}{2N\overline{\Phi}} P\Big\{ \max_{k\in\{0,\dots,2N-1\}} \xi(aqk) > u \Big\} \qquad (8)$$

$$\rightarrow 0 \quad \text{as } u \to \infty \text{ and } N \to \infty \text{ (in that order).}$$

Secondly, by the elementary inequality $2/\sqrt{2+2x} \geq 1 + (1-x)/4$ for $x \in [0,1]$, we have

$$P\{\xi(0)>u, \xi(aqk)>u\} \leq P\{\xi(0)+\xi(aqk) > 2u\} = \overline{\Phi}\left(\frac{2u}{\sqrt{2+2r(aqk)}}\right) \leq \overline{\Phi}\left(u + u\frac{1-r(aqk)}{4}\right).$$

Using the elementary inequality $\overline{\Phi}(u + x/u)/\overline{\Phi}(u) \leq e^{-x}$ for any $u > 0$ and $x > 0$, the right-hand side of the above equation in turn does not exceed

$$\max\left\{\overline{\Phi}\left(u + \frac{C(ak)^\alpha}{8u}\right), \overline{\Phi}\left(u + u\frac{1-\delta(\epsilon)}{4}\right)\right\} \leq \overline{\Phi}\exp\left\{-\frac{C(ak)^\alpha}{8}\right\} + \overline{\Phi}\exp\left\{-u^2\frac{1-\delta(\epsilon)}{4}\right\}$$

for u sufficiently large, where $\epsilon \in (0,h]$ is chosen such that $1 - r(t) \geq C|t|^\alpha/2$ for $t \in [0,\epsilon]$, and where $\delta(\epsilon) = \sup_{t \in [\epsilon,h]} r(t)$ [which is less that 1 by (1)]. It follows that

$$\varlimsup_{u\to\infty} \sum_{k=N}^{\lfloor h/(aq)\rfloor} \frac{P\{\xi(0)>u, \xi(aqk)>u\}}{\overline{\Phi}} \leq \varlimsup_{u\to\infty}\left(\sum_{k=N}^{\lfloor h/(aq)\rfloor} \exp\left\{-\frac{C(ak)^\alpha}{8}\right\} + \frac{h}{aq}\exp\left\{-u^2\frac{1-\delta(\epsilon)}{4}\right\}\right)$$

$$= \sum_{k=N}^{\infty} \exp\left\{-\frac{C(ak)^\alpha}{8}\right\} \qquad (9)$$

$$\to 0 \quad \text{as } N \to \infty.$$

By Bonferroni's inequality and stationarity, together with Lemma 3, (8) and (9), we have

$$\frac{q}{\overline{\Phi}}P\left\{\max_{k\in\{0,\dots,\lfloor h/(aq)\rfloor\}} \xi(aqk) > u\right\}$$

$$\geq \frac{q}{\overline{\Phi}}P\left\{\bigcup_{m=1}^{\lfloor h/(aqN)\rfloor}\left\{\max_{k\in\{(m-1)N,\dots,mN-1\}} \xi(aqk) > u\right\}\right\}$$

$$\geq \frac{q}{\overline{\Phi}}\sum_{m=1}^{\lfloor h/(aqN)\rfloor} P\left\{\max_{k\in\{(m-1)N,\dots,mN-1\}} \xi(aqk) > u\right\}$$

$$- \frac{q}{\overline{\Phi}}\sum_{m=1}^{\lfloor h/(aqN)\rfloor-1}\sum_{n=m+1}^{\lfloor h/(aqN)\rfloor} P\left\{\max_{k\in\{(m-1)N,\dots,mN-1\}} \xi(aqk)>u, \max_{\ell\in\{(n-1)N,\dots,nN-1\}} \xi(aq\ell)>u\right\}$$

$$\geq \frac{q}{\overline{\Phi}}\left\lfloor\frac{h}{aqN}\right\rfloor P\left\{\max_{k\in\{0,\dots,N-1\}} \xi(aqk) > u\right\}$$

$$- \frac{h}{aN\overline{\Phi}}P\left\{\max_{k\in\{0,\dots,N-1\}} \xi(aqk) > u, \max_{\ell\in\{N,\dots,2N-1\}} \xi(aq\ell) > u\right\}$$

$$- \frac{h}{a}\sum_{k=N}^{\lfloor h/(aq)\rfloor} \frac{P\{\xi(0) > u, \xi(aqk) > u\}}{\overline{\Phi}}$$

$$\to \frac{h}{a}P\left\{\bigcap_{\ell=1}^{\infty}\{\zeta(C^{1/\alpha}a\ell) + \eta \leq 0\}\right\} - 0 - 0 \quad \text{as } u \to \infty \text{ and } N \to \infty \text{ (in that order)}.$$

Putting this lower estimate together with the upper estimate (7), we arrive at (5).   □

The next lemma relates the continuous supremum $\sup_{t\in[0,h]} \xi(t)$ to the previously studied discretized maximum $\max_{k\in\{0,\dots,\lfloor h/(aq)\rfloor\}} \xi(aqk)$:

Lemma 4. *If* (1) *holds, then we have*

$$\overline{\lim_{u\to\infty}} \frac{q}{\overline{\Phi}} P\left\{\sup_{t\in[0,h]} \xi(t) > u + \frac{a^{\alpha/4}}{u}, \max_{k\in\{0,\dots,\lfloor h/(aq)\rfloor\}} \xi(aqk) \le u\right\} \to 0 \quad \text{as } a \downarrow 0.$$

Proof. Put $u_n = u + a^{\alpha/4}(1 - 2^{-n\alpha/4})/u$ and note that $u_n > u_0 = u, n = 1, 2, \dots$ By Boole's inequality and stationarity together with continuity of ξ, we have

$$P\left\{\sup_{t\in[0,h]} \xi(t) > u + \frac{a^{\alpha/4}}{u}, \max_{k\in\{0,\dots,\lfloor h/(aq)\rfloor\}} \xi(aqk) \le u\right\}$$

$$\le \left(\left\lfloor \frac{h}{aq}\right\rfloor + 1\right) P\left\{\sup_{t\in[0,aq]} \xi(t) > u + \frac{a^{\alpha/4}}{u}, \xi(0) \le u\right\}$$

$$= \left(\left\lfloor \frac{h}{aq}\right\rfloor + 1\right) P\left\{\bigcup_{n=0}^{\infty}\bigcup_{k=0}^{2^n-1}\{\xi(aqk2^{-n}) > u + \frac{a^{\alpha/4}}{u}\}, \xi(0) \le u\right\}$$

$$\le \frac{2h}{aq} P\left\{\bigcup_{n=0}^{\infty}\bigcup_{k=0}^{2^n-1}\{\xi(aqk2^{-n}) > u_n\}, \xi(0) \le u\right\}$$

$$= \frac{2h}{aq} P\left\{\bigcup_{n=1}^{\infty}\left[\left(\bigcup_{k=1}^{2^n-1}\{\xi(aqk2^{-n}) > u_n\}\right) \cap \left(\bigcap_{\ell=0}^{2^{n-1}-1}\{\xi(aq\ell 2^{-n+1}) \le u_{n-1}\}\right)\right], \xi(0) \le u\right\}$$

$$\le \frac{2h}{aq} P\left\{\bigcup_{n=1}^{\infty}\left[\left(\bigcup_{k=1}^{2^n-1}\{\xi(aqk2^{-n}) > u_n\}\right) \cap \left(\bigcap_{\ell=0}^{2^{n-1}-1}\{\xi(aq\ell 2^{-n+1}) \le u_{n-1}\}\right) \cap \{\xi(0) \le u_{n-1}\}\right]\right\}$$

$$= \frac{2h}{aq} P\left\{\bigcup_{n=1}^{\infty}\left[\left(\bigcup_{k=1}^{2^n-1}\{\xi(aqk2^{-n}) > u_n\}\right) \cap \left(\bigcap_{\ell=0}^{2^{n-1}-1}\{\xi(aq\ell 2^{-n+1}) \le u_{n-1}\}\right)\right]\right\}$$

$$\le \frac{2h}{aq} \sum_{n=1}^{\infty}\sum_{k=1}^{2^n-1} P\left\{\{\xi(aqk2^{-n}) > u_n\} \cap \bigcap_{\ell=0}^{2^{n-1}-1}\{\xi(aq\ell 2^{-n+1}) \le u_{n-1}\}\right\} \tag{10}$$

$$\le \frac{2h}{aq} \sum_{n=1}^{\infty} 2^{n-1} P\left\{\xi(aq2^{-n}) > u_n, \xi(0) \le u_{n-1}\right\} \quad \text{for } u > 0 \text{ sufficiently large.}$$

By (1), on the event $\{\xi(aq2^{-n}) > u_n, \xi(0) \le u_{n-1}\}$, we further have

$$r(aq2^{-n})\xi(aq2^{-n}) - \xi(0) \ge a^{\alpha/4}(2^{\alpha/4}-1)2^{-n\alpha/4}/u - (1 - r(aq2^{-n}))(u + a^{\alpha/4}(1-2^{-n\alpha/4})/u)$$

$$\ge a^{\alpha/4}(2^{\alpha/4}-1)2^{-n\alpha/4}/u - 2Ca^{\alpha}2^{-n\alpha}u^{-1}(1 + a^{\alpha/4}(1-2^{-n\alpha/4})/u^2)$$

$$\ge a^{\alpha/4}(2^{\alpha/4}-1)2^{-n\alpha/4-1}/u$$

for $u > 0$ sufficiently large and $a > 0$ sufficiently small. Putting this together with (10) and using the fact that the random variables $r(aq2^{-n})\xi(aq2^{-n}) - \xi(0)$ and $\xi(aq2^{-n})$ are independent together with (1), it follows that the limit superior in Lemma 4 is at most

$$\overline{\lim_{u\to\infty}} \frac{2h}{a\overline{\Phi}} \sum_{n=1}^{\infty} 2^{n-1} P\left\{r(aq2^{-n})\xi(aq2^{-n}) - \xi(0) > a^{\alpha/4}(2^{\alpha/4}-1)2^{-n\alpha/4-1}/u, \xi(aq2^{-n}) > u\right\}$$

$$= \overline{\lim_{u\to\infty}} \frac{2h}{a} \sum_{n=1}^{\infty} 2^{n-1} \overline{\Phi}\left(\frac{a^{\alpha/4}(2^{\alpha/4}-1)2^{-n\alpha/4-1}}{u\sqrt{1 - r(aq2^{-n})^2}}\right)$$

$$\leq \varlimsup_{u\to\infty} \frac{2h}{a} \sum_{n=1}^{\infty} 2^{n-1} \overline{\Phi}\left(\frac{a^{\alpha/4}(2^{\alpha/4}-1)2^{-n\alpha/4-1}}{u\sqrt{2(1-r(aq2^{-n}))}} \right)$$

$$\leq \frac{2h}{a} \sum_{n=1}^{\infty} 2^{n-1} \overline{\Phi}\left(\frac{a^{\alpha/4}(2^{\alpha/4}-1)2^{-n\alpha/4-1}}{2\sqrt{C}\,a^{\alpha/2}2^{-n\alpha/2}} \right).$$

As the right-hand side goes to zero as $a \downarrow 0$ by elementary arguments, the lemma is proved.   □

We now have done all preparations required to prove (2). Finally, we prove Theorem 1, which provides a new expression for Pickands constant H_α:

Proof of Theorem 1. By (6), given any constant $a > 0$, we have

$$\varlimsup_{u\to\infty} \frac{q}{\overline{\Phi}} \mathrm{P}\left\{ \sup_{t\in[0,h]} \xi(t) > u \right\} = \varlimsup_{u\to\infty} \frac{q(u+a^{\alpha/4}/u)}{\overline{\Phi}(u+a^{\alpha/4}/u)} \mathrm{P}\left\{ \sup_{t\in[0,h]} \xi(t) > u + \frac{a^{\alpha/4}}{u} \right\}$$

$$\leq \varlimsup_{u\to\infty} \frac{q}{e^{-a^{\alpha/4}}\overline{\Phi}} \mathrm{P}\left\{ \sup_{t\in[0,h]} \xi(t) > u + \frac{a^{\alpha/4}}{u}, \max_{k\in\{0,\dots,\lfloor h/(aq)\rfloor\}} \xi(aqk) \leq u \right\}$$

$$+ \varlimsup_{u\to\infty} \frac{q}{e^{-a^{\alpha/4}}\overline{\Phi}} \mathrm{P}\left\{ \max_{k\in\{0,\dots,\lfloor h/(aq)\rfloor\}} \xi(aqk) > u \right\}. \qquad (11)$$

Sending $a \downarrow 0$ in (11) and using Lemmas 1 and 4, we get

$$\varlimsup_{u\to\infty} \frac{q}{\overline{\Phi}} \mathrm{P}\left\{ \sup_{t\in[0,h]} \xi(t) > u \right\} \leq 0 + \lim_{a\downarrow 0} \frac{h}{a} \mathrm{P}\left\{ \bigcap_{k=1}^{\infty} \{\zeta(C^{1/\alpha}ak) + \eta \leq 0\} \right\}$$

$$= \lim_{a\downarrow 0} \frac{h\,C^{1/\alpha}}{a} \mathrm{P}\left\{ \bigcap_{k=1}^{\infty} \{\zeta(ak) + \eta \leq 0\} \right\}.$$

On the other hand, by Lemma 1 alone, we have

$$\lim_{u\to\infty} \frac{q}{\overline{\Phi}} \mathrm{P}\left\{ \sup_{t\in[0,h]} \xi(t) > u \right\} \geq \varlimsup_{a\downarrow 0} \lim_{u\to\infty} \frac{q}{\overline{\Phi}} \mathrm{P}\left\{ \max_{k\in\{0,\dots,\lfloor h/(aq)\rfloor\}} \xi(aqk) > u \right\}$$

$$= \varlimsup_{a\downarrow 0} \frac{h\,C^{1/\alpha}}{a} \mathrm{P}\left\{ \bigcap_{k=1}^{\infty} \{\zeta(ak) + \eta \leq 0\} \right\}.$$

Combining the above two estimates upwards and downwards, it follows that the limit H_α and the limit on the left-hand side in (2) both exist, and that the identity (2) holds.

To complete the proof, we must show that H_α is finite and strictly positive. To that end, note that, by Lemmas 1 and 4, the right-hand side of (11) is finite for $a > 0$ sufficiently small. Hence the left-hand side (that does not depend on a) is finite, so that H_α is finite by (2). On the other hand, by Bonferroni's inequality and stationarity together with (9), we have

$$\lim_{u\to\infty} \frac{q}{\overline{\Phi}} \mathrm{P}\left\{ \sup_{t\in[0,h]} \xi(t) > u \right\} \geq \lim_{u\to\infty} \frac{q}{\overline{\Phi}} \mathrm{P}\left\{ \bigcup_{k=0}^{\lfloor h/(aqN)\rfloor} \{\xi(aqNk) > u\} \right\}$$

$$\geq \varlimsup_{u\to\infty} \frac{h}{aN\overline{\Phi}} \left(\overline{\Phi} - \sum_{k=N}^{\lfloor h/(aq)\rfloor} \mathrm{P}\{\xi(0) > u, \xi(aqk) > u\} \right)$$

$$\geq \frac{h}{2aN} \quad \text{for } N \in \mathbb{N} \text{ sufficiently large.}$$

Here the right-hand side is strictly positive, so that (2) shows that H_α is strictly positive.   □

3 Discussion

In [5] Michna has recently published a proof of the Pickands result. However, Michna's proof is not really a new proof but rather a version of a proof due to Piterbarg [7] of a more general theorem on Gaussian extremes, which has been edited to address only the particular case of the Pickands result as it is also stated by the author himself.

The proof of Michna and Piterbarg are very different from ours in that they make crucial use of a technique to relate the original problem to prove the Pickands result to problems for extremes of Gaussian random fields. Also, Michna and Piterbarg do their calculations directly on the continuous suprema which lead to more complicated arguments, while we all the time make use of appropriate discrete approximations to continuous suprema. Virtually all calculations of Michna and Piterbarg as well as their expression for Pickands constant are completely different from ours. The one exception is the use of the elementary Bonferroni's inequality for a lower bound, which we also employ (and as did Pickands), but where again the technical details of that usage are quite different from ours because they work with continuous suprema while we work with discrete approximations.

Also, our proof is entirely self-contained and only uses basic graduate probability theory. On the other hand, the proof by Michna and Piterbarg make crucial use of additional sophisticated Gaussian theory such as Borel's inequality, Slepian's lemma and weak convergence on space C. This is again because they chose to work with continuous suprema.

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