Optimal Hedging with Higher Moments

Chris Brooks
ICMA Centre, University of Reading

Aleš Černý
Cass Business School

Joëlle Miffre
Cass Business School

November 2006

CHRIS BROOKS[†], ALEŠ ČERNÝ[‡], AND JOËLLE MIFFRE[§]

November 18, 2006

ABSTRACT. This study proposes a utility-based framework for the determination of optimal hedge ratios that can allow for the impact of higher moments on the hedging decision. The approach is applied to a set of 20 commodities that are hedged with futures contracts. We find that in sample, the performance of hedges constructed allowing for non-zero higher moments is only very slightly better than the performance of the much simpler OLS hedge ratio. When implemented out of sample, utility-based hedge ratios are usually less stable over time, and can make investors worse off for some assets compared to hedging using the traditional methods. We conclude, in common with a growing body of very recent literature, by suggesting that higher moments matter in theory but not in practice.

[†]ICMA Centre, University of Reading, c.brooks@icmacentre.reading.ac.uk.

[‡]Cass Business School, City University London, cerny@martingales.info, http://www.martingales.info.

[§]Cass Business School, City University London, j.miffre@city.ac.uk, http://www.cass.city.ac.uk/faculty/j.miffre.

1. Introduction

There is now indisputable evidence to suggest that the return distributions of risky assets depart from normality. For example, deviations from normality have been observed for emerging stock market indices (Harvey, 1995), hedge fund indices (Agarwal and Naik, 2004), individual hedge funds (Brooks and Kat, 2002), relative-strength strategies (Harvey and Siddique, 2000) and futures contracts (Christie-David and Chaudhry, 2001). Under some fairly weak assumptions concerning the shape of investor utility functions, Scott and Horvarth (1980) show that investors are concerned not just with the mean and variance of asset returns, but also with the distribution's higher moments as well. Scott and Horvarth further demonstrate that investors will have a preference for larger odd moments and smaller even moments. Importantly, Kraus and Litzenberger (1976), Harvey and Siddique (2000) and Chung et al. (2006) have made it clear that systematic risks related to skewness and kurtosis are priced by the market. To put it differently, exposure to systematic "non-normality risks" commands a risk premium.

In parallel to the analysis of Markowitz (1959), Kraus and Litzenberger (1976) have shown that it is not the total skewness of an asset that will be priced, but rather the contribution of the asset to the skewness of a well-diversified portfolio (also called systematic skewness or co-skewness). Similarly, it will only be systematic kurtosis risk, or the contribution of an asset to the kurtosis of a well-diversified portfolio, that commands a risk premium. As in Markowitz (1959), unsystematic skewness or kurtosis risk should be eliminated through diversification. Recent renewed interest in this proposition has led to a number of studies that extend existing asset pricing models to incorporate higher moments, building on the early work of Kraus and Litzenberger (1983). Examples include Chunhachinda et al. (1997) on the incorporation of moments higher than the second into the investor's portfolio decision, and Barone-Adesi et al. (2004) on incorporating co-skewness into asset pricing models. Harvey and Siddique (2000) demonstrate that conditional skewness can help to explain the cross-sectional variation in asset returns, including momentum effects. Similarly, Chung et al. (2006) have shown that the risk premia associated with size and book-to-market value are compensation for systematic exposures to a set of non-normality risks of order 3 to 10. Following this argument, it is possible that failure to incorporate higher moment considerations could help to rationalize several other widely documented asset pricing anomalies.

Another, almost entirely separate strand of finance literature has looked at the hedging decisions of risk-averse investors, with particular reference to hedging with futures contracts. A large number of studies have been

concerned with estimation of the optimal hedge ratio, defined as the optimal number of futures contracts to employ per unit of the spot asset to be hedged (see, for example, Baillie and Myers, 1991; Cecchetti et al., 1988; Kroner and Sultan, 1991; Lien and Luo, 1993; Lin, Najand and Yung, 1994; Myers and Thompson, 1989; Park and Switzer, 1995; Strong and Dickinson, 1994). The simplest way to calculate this number of futures contracts is to employ the OLS hedge ratio of Ederington (1979) and Figlewski (1984), which simply measures the hedge ratio as the slope coefficient of an OLS regression of spot returns on futures returns. This implies a static risk management strategy that involves a one-off decision on the optimal hedge and might therefore yield suboptimal hedging decisions in periods of high basis volatility. To overcome this problem, quite a large literature has developed that models the optimal hedge ratio within a conditional framework, taking into account the dynamics between the spot and futures returns (see, for example, Kroner and Sultan, 1991; Brooks et al., 2002; Alexander and Barbosa, 2006; or Miffre, 2004, to name only a few). These studies have mainly employed models from the multivariate generalized autoregressive conditionally heteroscedastic (MGARCH) family. They have reached conflicting results on the out-of-sample hedging effectiveness of conditional minimum variance hedge ratios, even before taking into account the additional costs involved with continually buying and selling futures contracts so as to rebalance the hedged portfolio when the model suggests. At best, MGARCH models have led to very modest improvements in gross hedging efficiency when evaluated on an out-of-sample basis. Hence the benefits of active risk management strategies ought to be viewed with caution.

Almost without exception, studies on the determination of optimal hedge ratios at best assume that investors have two-moment (quadratic) utility functions or that the distribution of returns on the hedged portfolio is normal, so that the mean and variance alone are sufficient to determine the hedge ratio optimally[1]. In a slight generalization, Levy (1969) shows that a cubic utility function can be employed where investor preferences depend on skewness. However, it is not at all obvious, when one is released from the constraint of the mean-variance framework, why one should stop at skewness, for in addition to an aversion to negative skewness, rational investors should possess an aversion to positive excess kurtosis as well. Even less plausibly, many studies focus on minimum variance hedging, where

---

[1]A slightly weaker assumption than return normality is that the spot and corresponding futures returns are drawn from a multivariate elliptical distribution. In such circumstances, even if the spot returns are skewed and/or leptokurtic, the magnitude or otherwise of these higher moments is not affected by hedging with futures and thus optimally, they should not enter into the hedger's objective function.

the mean, as well as any moments of order higher than the second, are ignored. Such an assumption concerning the mean will only be appropriate if investors are infinitely risk-averse, or if the expected return is zero.

Clearly then, if return distributions depart from normality, hedging strategies that assume normality might lead to sub-optimal hedging decisions. The extant literature concerning the impact of higher moments on hedging is very sparse. Gilbert et al. (2006) derive and apply a partial equilibrium model of hedging that allows for skewness (but not kurtosis) in the hedger's utility function. They show that skewness can be important for undiversified agents, and the overall extent of speculation could either rise or fall depending upon whether there is a price bias in the forward market. However, while they allow for skewness in the utility function, they do not explicitly consider its impact on the determination of optimal hedge ratios.

The only other relevant contribution in this area is by Harris and Shen (2006), who consider cross-hedging with currencies rather than with futures. They show, using a set of daily currency exposures, that minimum variance hedging is likely to reduce the out-of-sample variance of the hedged portfolio, but the skewness and kurtosis are likely to fall and rise respectively. This result indicates that the benefit of hedging may be overstated since these higher moments move in exactly the opposite directions to those preferred by a rational utility maximizer of the form described in the theoretical literature. Similarly, Brooks and Kat (2002) observed that hedge funds, while they demonstrate impressive performance on mean-variance grounds, also typically have less desirable higher moment values than traditional asset classes.

To our knowledge, there are no previous studies that have attempted to estimate optimal hedge ratios within a utility-based framework that allows for investors to have non-zero preferences for higher moments. We measure, for the first time, the loss of welfare that may be incurred if we use OLS hedge ratios in non-quadratic utility functions. We define the optimal hedge ratio as the derivative of the optimal futures position with respect to the change in the spot position. By doing so, we draw together the literatures on hedging with futures, and on utility maximization with higher moments. An important precursor to our work is Kallberg and Ziemba (1983) who study optimal equity portfolios and conclude that mean–variance portfolios differ insignificantly in welfare terms from general utility-based optimal portfolios once risk aversion is taken into account. We extend their work in several directions by allowing i) non-parametric return distributions instead of Gaussian distribution; ii) more comprehensive family of utility functions; and iii) welfare measures adjusted for risk aversion. The main difference, of course, is that we do not study optimal investment in itself but we use it as an intermediate step to formulate optimal hedge ratios. In

summary, we draw the conclusion that out-of-sample, investors maximize the expected utility of their hedged portfolio better if they ignore higher moments. In other words, when it comes to hedging, higher moments do not matter, despite the fact that these higher moments are clearly present in the data and faithfully represented in our analysis.

The remainder of the article is organized as follows. Section 2 presents the theory that underpins our higher moment hedge ratio. Section 3 introduces the dataset and Section 4 presents the empirical results. Finally section 5 concludes.

2. Methodology

An agent who hedges a long spot position at time t using h_t futures contracts will receive the following payoff at time $t + 1$, R_{t+1}, to the hedged position

$$R_{t+1} = C_{t+1} - h_t F_{t+1}, \tag{2.1}$$

where C_{t+1} and F_{t+1} denote the changes in the cash (spot) and futures prices respectively between times t and $t + 1$.

Suppose that the agent has the four-moment utility function given by

$$U_t(R_{t+1}) = E_t(R_{t+1}) - a\mathrm{Var}_t(R_{t+1}) + b\mathrm{Skew}_t(R_{t+1}) - c\mathrm{Kurt}_t(R_{t+1}) \tag{2.2}$$

where $E_t(R_{t+1})$ is the expectation formed at time t for the return during the next period, Var, Skew, and Kurt are the second, third, and fourth moments of the distribution of expected returns respectively, and a, b, c are parameters that represent the relative desirability of the moments of the return distribution in the agent's utility function. Given the signs used to precede the parameters in (2.2), we would usually expect $a, b, c > 0$ (see, for example, Scott and Horvath, 1980). While the literature on determining optimal hedge ratios is now vast, traditionally, academic research has assumed that only the first two moments of the utility function are of concern to the investor, a restriction equivalent to $b = c = 0$ in (2.2). Under this assumption, and provided that the value of the hedged portfolio follows a pure martingale process, it is easy to show that the optimal hedge ratio is simply the ratio of the covariance between the cash and futures returns to the variance of the futures returns, equivalent to the OLS hedge.

Our objective in this study is to implement a rule for determining hedge ratios that

(1) is not limited to the first two moments of the hedged portfolio return distribution,

(2) does not impose parametric distribution of returns,

(3) does not require the martingale assumption,

(4) puts a monetary measure on the welfare gain from following a particular hedging strategy,

(5) permits fast and reliable numerical implementation.

Employing an approach based on jointly optimizing over the mean, variance, skewness and kurtosis is not feasible unless an auxiliary assumption is made concerning the investor's relative preferences for each of these moments. For example, one would have to answer questions such as, "how much extra expected return would be required for the investor to be willing to lose one unit of skewness or to gain one unit of kurtosis in the return distribution?" This is equivalent to attempting to find plausible values of a, b, and c in (2.2) above. It may be possible to obtain such information indirectly from the prices of traded options (for example, based on compound options), but there does not yet exist any firmly established method of achieving this goal. The fact that the mean and standard deviation both scale with the returns, but the higher moments are usually presented in a standardised form, and are therefore unit free, makes the problem even more complex[2].

Thus we are compelled to adopt a more general approach based not on directly optimizing simultaneously across multiple moments, but rather on a utility function. Broadly speaking, we define the optimal hedge ratio as the slope of the optimal futures position with respect to the change in the spot position. This concept coincides with the standard OLS hedge if the mean futures return is zero and preferences are dictated by quadratic utility. Details of the methodology are spelled out in Section 2.1. To test robustness of our results we examine a whole family of utility functions including the logarithmic, exponential, power and quadratic utility (the so called HARA class, see Tsiang 1972) as well as fourth moment polynomial approximations thereof corresponding to different choices of a, b, c in equation (2.2). These utility functions are discussed in Sections 2.3 and 2.4, respectively. The first challenge is to find meaningful comparison of optimal hedging portfolios and of the welfare loss from adopting OLS hedging strategies across *different* utility functions. The key insight is provided by Arrow (1971), who notes that optimal portfolios normalized by local risk tolerance behave robustly across utility functions (see also Samuelson, 1970). We apply the same insight to risk-adjusted performance measurement to

---

[2]Ad-hoc preferences over mean, variance and skewness that do not have interpretation in monetary terms appear, for example, in de Athayde and Flôres (2004), Chunhachinda et al. (1997), Lai (1991) and Prakash et al. (2003). We believe that a monetary measure of performance is crucial to draw meaningful conclusions on the welfare loss of OLS hedging strategies, or more generally, on the welfare loss from following mean-variance investment rules.

obtain a monetary measure of performance which again is robust across utility functions (see Černý 2004, Chapter 3).

The second challenge is to solve the utility maximization problem accurately and quickly. To make the problem tractable, the literature on optimal investment has in the past opted for parametric distributions of returns (normal in Kalberg and Ziemba, 1979; normal combined with a specific skewed distribution in Simaan 1993a,b) or else has restricted its attention to the first four moments in a non-parametric setting (Post et al., 2003; Jondeau and Rockinger, 2006). Our approach, based on Newton's optimization method detailed in Section 2.5, is able to deal with non-parametric distributions of returns and apart from strict concavity it does not impose restrictions on the utility function. The method exhibits quadratic convergence, doubling the number of digits of accuracy in each iteration, and it is thus extremely fast[3].

2.1. **Performance measurement.**

Definition 2.1. *We call $U : \mathbb{R} \to [-\infty, \infty)$ with effective domain $\mathcal{D}_U$ a utility function if*

(1) *U is at least twice differentiable on the interior of $\mathcal{D}_U$,*
(2) *$U'' < 0$ on the interior of $\mathcal{D}_U$,*
(3) *the maximal domain $\bar{\mathcal{D}}_U$ on which U is strictly increasing has non-empty interior,*
(4) *$\lim_{v \to -\infty} U'(v) = -\infty$ or $\lim_{v \to +\infty} U'(v) \leq 0$.*

In cases when $\bar{\mathcal{D}}_U \subsetneq \mathcal{D}_U$ we define the inverse utility U^{-1} as taking values in $\bar{\mathcal{D}}_U$.

Fix a probability space $(\Omega, P, \mathcal{F})$ with $|\Omega| < \infty$. Denote by X, Y two random variables representing the excess returns of the future contract and of the spot asset, respectively[4]. We assume that there is no arbitrage, i.e. there is a measure Q equivalent to P and such that $E^Q(X) = E^Q(Y) = 0$. The next Lemma states that the optimal investment problem is well-defined in the absence of arbitrage.

---

[3]Apart from computing optimal hedge ratios our procedure is also suitable for solving optimal investment problems with a large number of assets because the computational effort grows only quadratically with the number of assets. In contrast, the use of co-skewness and co-kurtosis (see, for example, de Athayde and Flôres, 2004; Harvey et al., 2004; Jondeau and Rockinger, 2006) makes the computational time grow with the third and fourth power of the total number of assets, respectively.

[4]We suppress time subscripts throughout this section. The random variable X corresponds to the change in the futures index F_{t+1} and Y is interpreted as the change in the cash value C_{t+1}. The expectation $E(.)$ is intepreted as the expectation at time t conditional on the information at that time.

Theorem 2.2. *Consider a utility U and initial endowment $v \in \bar{\mathcal{D}}_U$ and define*

$$u(v, \eta, \vartheta) := \mathrm{E}\left(U\left(v + \eta Y + \vartheta X\right)\right) \in [-\infty, \infty)$$

There is an interval I containing an open neighbourhood of zero such that for every $\eta \in I$ the optimal trading strategy in $\sup_{\vartheta \in \mathbb{R}} u(v, \eta, \vartheta)$ exists and is unique; we denote it by $\varphi(v, \eta)$.

Define the certainty equivalent (CE) *wealth increase in the standard way,*

$$\mathrm{CE}(v, \eta, \vartheta) := U^{-1}\left(u(v, \eta, \vartheta)\right) - v,$$

and denote its maximal value by $\widehat{\mathrm{CE}}(v, \eta) := \mathrm{CE}(v, \eta, \varphi(v, \eta))$. For $\eta \in I$ the quantity $\widehat{\mathrm{CE}}(v, \eta)$ is finite whereas for $\eta \notin I$ we have $\widehat{\mathrm{CE}}(v, \eta) = -\infty$.

Proof. See Černý (2003, Theorem 2). □

Suppose the investor goes long η units of the spot asset. If the investor does not hedge, she continues to hold $\varphi(v, 0)$ futures contracts. If she hedges optimally, her position in the futures changes to $\varphi(v, \eta)$. One can now define the optimal hedge per unit of the spot asset

$$\mathrm{opt.hedge}\,(v, \eta) = -\frac{\varphi(v, \eta) - \varphi(v, 0)}{\eta}. \tag{2.3}$$

We use the standard convention whereby the hedge ratio signifies the number of futures contracts the investor *shorts* as a result of being *long* one unit of the spot asset, hence the extra minus sign in equation (2.3).

The welfare gain from a particular (not necessarily optimal) hedge h is defined as follows[5]

$$\mathrm{welfare\ gain}\,(v, \eta, h) = \mathrm{CE}(v, \eta, \varphi(v, 0) - \eta h) - \mathrm{CE}(v, \eta, \varphi(v, 0)). \tag{2.4}$$

The literature on optimal hedging typically assumes that $\mathrm{E}\left(X\right) = 0$, in which case $\varphi(v, 0) = 0$ by Jensen's inequality. When $\mathrm{E}\left(X\right) \neq 0$ the optimal futures position is non-zero even if the agent holds no spot assets, therefore $\varphi(v, 0)$ does not constitute a hedge in itself. In such case only the incremental position over and above $\varphi(v, 0)$ should be interpreted as the hedging position, which is reflected in definitions (2.3) and (2.4).

If one wants to understand and compare optimal investment/hedging dictated by different utility functions then it is important to normalize the resulting portfolio by some measure of risk aversion. This insight goes back to Arrow (1971). The most convenient normalization factor turns out to be the Arrow-Pratt coefficient of risk aversion, cf. Pratt (1964), Kallberg

---

[5]Our measure of the welfare loss arising from using a second-best hedging strategy is based on the certainty equivalent as in Kallberg and Ziemba (1979) and Pulley (1983), in contrast to Simaan (1993b) who uses compensating variation in terminal wealth.

and Ziemba (1983), Samuelson (1970). Černý (2004) shows that a similar normalization can be applied to performance measurement.

Using the coefficient of local absolute risk aversion

$$A(v) := -\frac{U''(v)}{U'(v)},$$

we define the normalized spot and futures positions

$$\lambda := A(v)\eta, \qquad \theta := A(v)\vartheta. \tag{2.5}$$

Similarly we set

$$
\begin{aligned}
\text{normalized opt. hedge}\,(v, \eta) &= A(v)\text{opt. hedge}\,(v, \eta)\,, \\
\text{normalized welfare gain}\,(v, \eta, h) &= A(v)\text{welfare gain}(v, \eta, h).
\end{aligned}
$$

The normalization is performed to make the hedging coefficients and welfare measurements comparable across different utility functions.

To evaluate the normalized quantities mathematically it is convenient to define a normalized utility $f_{v,U} : \mathbb{R} \to [-\infty, \infty)$

$$f_{v,U}(z) := U\left(v + \frac{z}{A(v)}\right).$$

We review specific functional forms of f implied by the CRRA and HARA class of utility functions in Section 2.3 where we also provide further economic interpretation for the normalized utility f. Preferences over the first four moments are examined in Section 2.4.

Theorem 2.3. *Consider a utility U, initial endowment $v \in \bar{\mathcal{D}}_U$ and the corresponding normalized utility f. Define*

$$
\begin{aligned}
a(\lambda, \theta) &= f^{-1}\left(E\left(f\left(\lambda Y + \theta X\right)\right)\right), \\
\alpha(\lambda) &= \arg\max_{\theta \in \mathbb{R}} E\left(f\left(\lambda Y + \theta X\right)\right), \tag{2.6} \\
\hat{h}(\lambda) &= -\frac{\alpha(\lambda) - \alpha(0)}{\lambda}, \\
g(\lambda, h) &= a(\lambda, \alpha(0) - \lambda h) - a(\lambda, \alpha(0)).
\end{aligned}
$$

Then

$$
\begin{aligned}
\text{normalized opt. hedge}\,(v, \eta) &= \hat{h}(\lambda)\,, \tag{2.7} \\
\text{normalized welfare gain}\,(v, \eta, h) &= g(\lambda, h), \tag{2.8}
\end{aligned}
$$

where λ, θ are the risk-normalized spot and futures positions from equation (2.5).

Proof. By a straightforward calculation

$$A(v)\text{CE}\,(v, \eta, \vartheta) = f^{-1}\left(E\left(f\left(\lambda Y + \theta X\right)\right)\right) = a(\lambda, \theta),$$

whereby we obtain (2.8). The existence and uniqueness of the maximizer in (2.6) was shown in Theorem 2.2. ◻

Černý (2004) calls the quantity $a(\lambda, \theta)$ *investment potential*. One interprets $a(\lambda, \theta)$ as the percentage increase in initial wealth per unit of local *relative* risk tolerance. Hence if $a(\lambda, \theta) = 5\%$, the agent's risk-free wealth is USD 1 million and her local relative risk aversion is 5, then the perceived certainty equivalent wealth due to holding $\eta = \frac{\lambda \times 1\text{million}}{5}$ units of the spot asset and $\vartheta = \frac{\theta \times 1\text{million}}{5}$ units of the futures contract equals

$$1 \text{ million} \times \left(1 + \frac{0.05}{5}\right) = 1.01 \text{ million.}$$

If the agent's local risk aversion is 1, the certainty equivalent wealth increases to

$$1 \text{ million} \times \left(1 + \frac{0.05}{1}\right) = 1.05 \text{ million,}$$

provided that the spot and futures positions are scaled up 5 times (so that the exposure normalized by risk aversion remains the same).

The investment potential has a close link to the Sharpe ratio. If we define the Sharpe ratio of X as $\text{SR}(X) := \text{E}(X)/\sqrt{\text{Var}(X)}$ then one can show[6] that for *any utility function* the investment potential from optimal investment in X is approximately equal to $\frac{1}{2}\text{SR}^2(X)$ for $\text{SR}(X)$ small,

$$a(0, \alpha(0)) \approx \frac{1}{2}\text{SR}^2(X) \text{ as } \text{SR}(X) \to 0.$$

For the quadratic utility we have identically[7]

$$a(0, \alpha(0)) = 1 - \sqrt{\left(1 + \text{SR}^2(X)\right)^{-1}}.$$

For exponential utility and normally distributed X we have identically[8]

$$a(0, \alpha(0)) = \frac{1}{2}\text{SR}^2(X).$$

2.2. **Optimal hedging and OLS.** Assuming sufficient smoothness ($f \in \mathcal{C}^2$) the quantity $\alpha(\lambda)$ is differentiable and we can think of the optimal hedge $\hat{h}(\lambda)$ as the *average* value of the marginal hedge ratios $-\alpha'(s)$ with $s \in [0, \lambda]$,

$$\hat{h}(\lambda) = -\frac{\int_0^\lambda \alpha'(s)\,ds}{\lambda}.$$

---

[6]See Černý (2003, equation 25).
[7]See Černý (2004, equation 3.46).
[8]See Černý (2004, Section 3.8.1) and Hodges (1998).

By differentiating the first order condition $E\left(Xf'\left(\alpha\left(\lambda\right)X+\lambda Y\right)\right)=0$ with respect to λ we have

$$E\left(X^2 f''\left(\lambda Y+\alpha\left(\lambda\right)X\right)\right)\alpha'\left(\lambda\right)=-E\left(XY f''\left(\lambda Y+\alpha\left(\lambda\right)X\right)\right),$$
$$\alpha'\left(\lambda\right)=-\frac{E\left(XY f''\left(\lambda Y+\alpha\left(\lambda\right)X\right)\right)}{E\left(X^2 f''\left(\lambda Y+\alpha\left(\lambda\right)X\right)\right)}.$$

In the special case $f''=$const, corresponding to quadratic utility, we obtain

$$\alpha'\left(\lambda\right)=-\frac{E\left(XY\right)}{E\left(X^2\right)},$$

which means that $\hat{h}(\lambda)$ is independent of λ. If, in addition, the mean of X is zero then the quadratic hedge equals the slope coefficient from the OLS regression of Y onto X and intercept. For other utility functions the choice of λ matters to some extent, but our numerical results show that this dependence is extremely weak for $\lambda\in\left[0,1\right]$.

The literature on optimal hedging with non-quadratic utility typically chooses the value of the non-normalized spot position η to be 1. It is clear, however, that this may distort results substantially if the agent's risk aversion $A(v)$ is very high because the initial spot position is then unrealistically high relative to agent's attitude to risk. We therefore opt for $\lambda=1$ across all utility functions.

2.3. **CRRA and HARA utility.** To simplify notation we *extend* the meaning of a power function with real exponent as follows.

Definition 2.4. *For all $z,\delta\in\mathbb{R}$ we define*

$$\text{''}z^{\delta}\text{''}:=\begin{cases} -\infty & \text{for }\delta<0,z\leq 0,\\ -\infty & \text{for }0<\delta<1,z<0,\\ 0 & \text{for }0<\delta,z=0,\\ 1 & \text{for }\delta=0,z\in\mathbb{R},\\ z & \text{for }\delta=1,z\in\mathbb{R},\\ |z|^{\delta} & \text{elsewhere.}\end{cases} \quad\quad (2.9)$$

Definition 2.5. *The utility function*

$$U^{(\gamma)}(V):=\begin{matrix}\frac{V^{1-\gamma}-1}{1/\gamma-1} & \text{for }\gamma\notin\{0,1\}\\ \ln V & \text{for }\gamma=1\end{matrix}$$

is called the CRRA (constant relative risk-aversion) utility[9]. We denote the corresponding effective domain by $\mathcal{D}_\gamma$ and the maximal domain on which $U^{(\gamma)}$ is increasing by $\bar{\mathcal{D}}_\gamma$.

---

[9]Note that CRRA utility is typically defined only for $\gamma>0$. To make the definition meaningful for $\gamma<0$ and thus to prepare the ground for quadratic utility ($\gamma=-1$), one has to extend the power function in the manner indicated in equation (2.9).

The CRRA utility is an infinitely differentiable utility in the sense of Definition 2.1. $U^{(\gamma)}$ is strictly increasing and unbounded from above for $\gamma \in (0,1]$; it is strictly increasing and bounded from above for $\gamma > 1$ and for $\gamma < 0$ it has a bliss point at zero. The coefficient of relative risk aversion at v reads $\gamma \mathrm{sgn}(v\gamma)$ for $v \in \bar{\mathcal{D}}_\gamma$, hence the acronym CRRA. As it stands, for $\gamma < 0$, the utility is increasing only for negative values of wealth but this unpleasant feature is fixed in the HARA class, which we discuss shortly.

Proposition 2.6. *Fix $\gamma \in \mathbb{R} \setminus \{0\}$ and $v \in \bar{\mathcal{D}}_\gamma$. Then Theorem 2.3 applies to the CRRA utility $U^{(\gamma)}$ with*

$$f^{(\gamma)}(z) = \begin{cases} \frac{\left(1+\gamma^{-1}z\right)^{1-\gamma}-1}{1/\gamma-1} & \text{for } \gamma \in \mathbb{R} \setminus \{0,1\}, \\ \ln(1+z) & \text{for } \gamma = 1. \end{cases} \qquad (2.10)$$

In particular, the normalized optimal portfolio $\alpha^{(\gamma)}(\lambda)$, the investment potential $a^{(\gamma)}(\lambda, \theta)$, the optimal hedge $h^{(\gamma)}(\lambda)$ and the normalized welfare gain $g^{(\gamma)}(\lambda, h)$ are independent of $v \in \bar{\mathcal{D}}_\gamma$.

Proof. The normalized utility corresponding to $U^{(\gamma)}$ equals

$$f_{v,U^{(\gamma)}} = = \begin{cases} \frac{v^{1-\gamma}\left(1+\gamma^{-1}z\right)^{1-\gamma}-1}{1/\gamma-1} & \text{for } \gamma \in \mathbb{R} \setminus \{0,1\}, \\ \ln v + \ln(1+z) & \text{for } \gamma = 1. \end{cases} \quad .$$

Since v is fixed we have

$$f_{v,U^{(\gamma)}}^{-1}\left(E\left(f_{v,U^{(\gamma)}}(\lambda Y + \theta X)\right)\right) = \left(f^{(\gamma)}\right)^{-1}\left(E\left(f^{(\gamma)}(\lambda Y + \theta X)\right)\right),$$
$$\arg\max_{\theta \in \mathbb{R}} E\left(f_{v,U^{(\gamma)}}(\lambda Y + \theta X)\right) = \arg\max_{\theta \in \mathbb{R}} E\left(f^{(\gamma)}(\lambda Y + \theta X)\right),$$

for all $\lambda, \theta \in \mathbb{R}$, which proves $\alpha^{(\gamma)}(\lambda)$ and $a^{(\gamma)}(\lambda, \theta)$ are independent of v. The same claim for $\hat{h}^{(\gamma)}(\lambda)$ and $g^{(\gamma)}(\lambda, h)$ follows trivially. $\square$

Note that $f^{(\gamma)}$ converges pointwise to an exponential utility as $\gamma \to \pm\infty$. We define

$$f^{(\infty)}(z) = 1 - e^{-z}. \qquad (2.11)$$

The function $f^{(\gamma)}$, which one can think of as the normalized CRRA utility, is at the heart of the HARA class. Effectively, HARA utility shifts the original domain of definition so that the HARA utility is increasing for some positive levels of wealth even when $\gamma < 0$. It also re-sets the local relative risk aversion to an arbitrary $\tilde{\gamma} > 0$ rather than keeping it fixed at $|\gamma|$. This reflects the motivation in Tsiang (1972) who searches for a modification of power utility with reasonable values of local relative risk aversion and comes up with a pre-cursor of the HARA class.

Definition 2.7. *We say that an agent with initial wealth $v > 0$ has HARA (hyperbolic absolute risk aversion) utility with local relative risk aversion $\tilde{\gamma} > 0$ and baseline risk aversion $\gamma \in (-\infty, \infty] \setminus \{0\}$ if it is of the form*

$$\bar{U}_{\tilde{\gamma},v}^{(\gamma)}(V) := f^{(\gamma)}\left(\tilde{\gamma}\frac{V-v}{\gamma v}\right). \tag{2.12}$$

Remark 2.8.

(1) *The HARA class includes the most popular utility functions: quadratic ($\gamma = -1$), quartic ($\gamma = -3$ with preferences over mean, variance, skewness and kurtosis only), logarithmic ($\tilde{\gamma} = \gamma = 1$), power utility ($\tilde{\gamma} = \gamma > 0$), and exponential ($\gamma = \pm\infty$).*

(2) *For fixed $V, v, \tilde{\gamma} > 0$ the utility is continuous with respect to γ*

$$\bar{U}_{\tilde{\gamma},v}^{(1)}(V) = \lim_{\gamma \to 1} \bar{U}_{\tilde{\gamma},v}^{(\gamma)}(V),$$

$$\bar{U}_{\tilde{\gamma},v}^{(\infty)}(V) = \lim_{\gamma \to \pm\infty} \bar{U}_{\tilde{\gamma},v}^{(\gamma)}(V).$$

(3) *For $\gamma < 0$ the utility function $\bar{U}_{\tilde{\gamma},v}^{(\gamma)}(V)$ has a satiation point at $\bar{V} = (1 - \gamma/\tilde{\gamma})\, v > v$.*

(4) *An optimizing agent will avoid bankruptcy as long as $0 < \gamma < \tilde{\gamma}$.*

(5) *By construction the HARA class of utility functions in (2.12) is parametrized to achieve*

$$-\frac{v\left(\bar{U}_{\tilde{\gamma},v}^{(\gamma)}\right)''(v)}{\left(\bar{U}_{\tilde{\gamma},v}^{(\gamma)}\right)'(v)} = \tilde{\gamma} \text{ for all } \gamma \in (-\infty, \infty] \setminus \{0\}, v, \tilde{\gamma} > 0.$$

Theorem 2.9. *Fix $\gamma \in (-\infty, \infty] \setminus \{0\}$ and $v, \tilde{\gamma} > 0$. Then Theorem 2.3 applies to the HARA utility $\bar{U}^{(\gamma)}$ with $f^{(\gamma)}$ given in equations (2.10) and (2.11).*

Proof. The statement follows directly from Theorems 2.2 and 2.3. □

2.4. **Preferences over skewness and kurtosis.** It is clear that with the exception of quadratic utility ($\gamma = -1$), all other members of the HARA class take into account higher moments. If one wants to consider only mean, variance and skewness[10] one may take $\gamma = -2$ and to add kurtosis $\gamma = -3$. Yet another possibility is to create an ad-hoc fourth power polynomial by taking

---

[10]To be specific $E(f^{(-2)}(z)) = E\left(\left|1 - \frac{z}{2}\right|^3\right)$ so the HARA utility $f^{(-2)}$ generates a preference over mean, variance and skewness provided the excess return z does not exceed 2.

the Taylor expansion[11] of the HARA utility to the fourth order around the risk-free wealth v. This yields

$$
\begin{aligned}
\tilde{U}_{\tilde{\gamma},v}^{(\gamma)}(V) &= \tilde{f}^{(\gamma)}\left(\frac{\tilde{\gamma}}{v}\left(V-v\right)\right) \\
\tilde{f}^{(\gamma)}(z) &= z - \frac{z^2}{2} + \left(1+\frac{1}{\gamma}\right)\frac{z^3}{6} \\
&\quad - \left(1+\frac{1}{\gamma}\right)\left(1+\frac{2}{\gamma}\right)\frac{z^4}{24}.
\end{aligned}
\tag{2.13}
$$

Since the original utility function and its Taylor expansion share the first two (in fact the first four) derivatives at v we retain the identity

$$
-\frac{v\left(\tilde{U}_{\tilde{\gamma},v}^{(\gamma)}\right)''(v)}{\left(\tilde{U}_{\tilde{\gamma},v}^{(\gamma)}\right)'(v)} = \tilde{\gamma}.
$$

$\tilde{f}^{(\gamma)}$ is strictly concave for $\gamma \notin (-3,-1)$; for these values it is also increasing at 0. This implies that $\tilde{U}_{\tilde{\gamma},v}^{(\gamma)}$ is a utility function in the sense of Definition 2.1. For $\gamma = -3$ the polynomial approximation coincides with the fourth-power HARA utility. The following proposition shows that the case $\gamma = -3$ has the best properties among the polynomial approximations.

Theorem 2.10. *For $\gamma \notin (-3,-1) \cup \{0\}$ and $\tilde{\gamma}, v > 0$ the following statements hold*

1) Theorem 2.3 applies to $\tilde{U}_{\tilde{\gamma},v}^{(\gamma)}$ with $f = \tilde{f}^{(\gamma)}$.

2) the maximum investment potential detectable by the polynomial utility $\tilde{U}_{\tilde{\gamma},v}^{(\gamma)}$ equals $\tilde{z}^{(\gamma)}$, given as the unique maximum of $\tilde{f}^{(\gamma)}$ on $\mathbb{R}$,

$$
\begin{aligned}
\tilde{z}^{(\gamma)} &= \frac{\gamma}{\gamma+2} + \sqrt[3]{\frac{\gamma^2\left(\gamma^2+7\gamma+12\right)}{(\gamma+1)(\gamma+2)^3} + \sqrt{2\gamma^4\frac{(\gamma+3)^2}{(\gamma^2+3\gamma+2)^3}}} \\
&\quad + \sqrt[3]{\frac{\gamma^2\left(\gamma^2+7\gamma+12\right)}{(\gamma+1)(\gamma+2)^3} - \sqrt{2\gamma^4\frac{(\gamma+3)^2}{(\gamma^2+3\gamma+2)^3}}}.
\end{aligned}
$$

---

[11]Jondeau and Rockinger (2006) use $\gamma = \infty$ in this context. See also Guidolin and Timmerman (2005).

We have

$$
\begin{aligned}
3 &= \tilde{z}^{(-3)} \geq \tilde{z}^{(\gamma)}, \\
\lim_{\gamma \to 0} \tilde{z}^{(\gamma)} &= 0, \\
\lim_{\gamma \to \pm\infty} \tilde{z}^{(\gamma)} &= 1 + \sqrt[3]{1 + \sqrt{2}} + \sqrt[3]{1 - \sqrt{2}} \approx 1.596.
\end{aligned}
$$

2.5. **Numerical algorithm.** The problem

$$
\begin{aligned}
\alpha &= \arg\max_{\vartheta \in \mathbb{R}^n} E\left(f\left(Y + \vartheta X\right)\right), \\
a &= f^{-1}\left(E\left(f\left(Y + \alpha X\right)\right)\right),
\end{aligned}
$$

can be solved by Newton's iteration method provided that the initial guess ϑ_0 is close to the optimal portfolio α. In practice $\vartheta_0 = 0$ works very well. We define $g : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$

$$
g(\vartheta) = f^{-1}\left(E\left(f\left(Y + \vartheta X\right)\right)\right)
$$

and assume that in each iteration $g(\vartheta) > -\infty$. The target function g corresponds to a percentage increase in certainty equivalent wealth of an agent with unit relative risk aversion. Starting at ϑ_0 we use the iteration

$$
\vartheta_{k+1} = \vartheta_k - \frac{g'(\vartheta_k)}{g''(\vartheta_k)},
$$

where

$$
\begin{aligned}
g'(\vartheta) &= \frac{E\left(X f'\left(Y + \vartheta X\right)\right)}{f'\left(g(\vartheta)\right)}, \\
g''(\vartheta) &= \frac{E\left(X^2 f''\left(Y + \vartheta X\right)\right)}{f'\left(g(\vartheta)\right)} - \frac{f''\left(g(\vartheta)\right)\left(E\left(X f'\left(Y + \vartheta X\right)\right)\right)^2}{\left(f'\left(g(\vartheta)\right)\right)^3}.
\end{aligned}
$$

The first iteration ϑ_1 always corresponds to the optimal investment with quadratic utility. We stop the iteration when

$$
\frac{\left(g'(\vartheta_k)\right)^2}{g''(\vartheta_k)} < 10^{-12},
$$

which implies $|g(\vartheta_k) - g(\alpha)| < 10^{-12}$.

3. DATA

The data, downloaded from Datastream International, comprise end-of-month spot and futures prices on 20 US commodities. This set of series was chosen on the grounds that it has been the subject of an important scrutiny in the literature (Ederington, 1979; Myers and Thompson, 1989; Baillie and Myers, 1991) and covers a wide range of commodities of interest to

investors. The cross-section covers 10 agricultural commodity futures (cocoa, coffee, corn, cotton, oats, soybean meal, soybean oil, soybeans, sugar and wheat), 2 energy futures (heating oil and light crude oil), 5 metal futures (aluminum, copper, gold 100 oz, platinum and silver 1000 oz) and the futures on frozen pork bellies, lean hogs and lumber. To compile the time-series of futures prices, we collect the futures prices on all nearest and second nearest contracts. We hold the first nearby contract up to one month before maturity. At the end of that month, we roll our position over to the second nearest contract and hold that contract up to one month prior to maturity. Returns are then computed as the changes in the logarithms of these settlement prices. The procedure is then rolled forward to the next set of nearest and second nearest contracts when a new sequence of futures returns is compiled. The process is repeated throughout the dataset to generate a sequence of nearby maturity futures returns.

The characteristics of the underlying asset of the contract do not necessarily match those of the commodity that is being hedged. This is to be expected since futures contracts often amalgamate commodities with different grades or countries of origin. As a result, the return correlation between the spot asset and its corresponding futures ranges from a low of 0.27 for aluminum to 0.96 for gold with an average of 0.78.

The dataset covers the period January 31, 1979 to September 30, 2004. Note that we include in our analysis some commodity futures and spot assets that started trading after January 1979 or that were delisted before September 2004. As a result, the sample spans shorter periods for some contracts (aluminum, cocoa, copper, cotton, heating oil, lean hogs, light crude oil, live cattle, lumber, silver and soybeans). Optimal hedge ratios are first constructed using the entire sample for estimation and for in-sample tests of hedging effectiveness. Then subsequently, hedging effectiveness is tested out-of-sample and for this purpose, the whole period is split into two sub-samples. The in-sample period covers approximately two-third of the dataset and is used for estimation. The out-of-sample period, used for forecasting and hedging decisions, covers the remaining one-third.

Table 1 presents some summary statistics for the futures returns, the spot returns, and for the hedged portfolio returns, where a time-invariant OLS hedge is employed. Most spot series are significantly leptokurtic and are positively skewed because events such as hurricanes or wars positively affect commodity prices. Hedging with futures is evidently very successful for the vast majority of the series. Compared with the spot return variance, the hedged portfolio variance is an average of 62% lower, and for gold, the reduction in variance is over 90%. However, interestingly, the skewness falls for the hedged portfolio returns in 13 of the 20 series compared with the spot skewness, falling by an average of 0.64, while the kurtosis rises

for 15 of the series, by an average of 3.74. Thus, if we accept the premise that hedgers are indeed concerned with higher moments, then the effectiveness of the OLS hedge may be overstated by a consideration only of the reduction in variance.

4. Empirical Results

4.1. **In-sample analysis.** We measure the investment potential of the OLS and the utility-based hedge ratios, where the investment potential can be thought of as the percentage increase in certainty equivalent wealth that each hedge ratio generates per unit of hedger's risk aversion. We consider HARA utility functions with baseline risk aversion $\gamma \in \{-3, -1, 1, 5, \infty\}$ and their polynomial approximations[12] for $\gamma \in \{1, 5, \infty\}$, as described in Section 2.4. Our framework allows us to examine a much wider range of parameters but we have found that all utility function with $|\gamma| > 5$ behave essentially like the exponential utility, $\gamma = \infty$. On the other hand, utility functions with $|\gamma| \approx 0$ are undesirable because their polynomial approximations have a low bliss point, as shown in Theorem 2.10.

Table 2 reports i) OHR, the optimal hedge ratios, $\hat{h}(1)$, obtained for each utility function via Theorems 2.3, 2.9 and 2.10, ii) OLS GAIN, $g(1, h_{OLS}) \times 1200$, the normalized welfare gain that results from using the second-best (i.e. the OLS) hedge ratio in each utility function, and iii) OHR GAIN, $g(1, \hat{h}(1)) \times 1200$, the welfare gain of the optimal hedge for each utility. Function g, too, is defined via Theorems 2.3, 2.9 and 2.10. The multiplication by 1200 means we interpret the welfare gain as the percentage points increase in initial wealth per year.

The results warrant two comments. First, the OLS hedge ratios only differ slightly from the utility-based hedge ratios. For example, all utility-based hedge ratios are within a range of 0.1 away from the OLS hedge ratios. This suggests that the OLS hedge ratio might be tractable and convenient first approximation of the utility-based hedge ratios. Second, overall, the welfare gain of the first-best hedge ratio only marginally exceeds, if at all, the investment potential of the OLS hedge ratio. This tells us that per unit of risk aversion, hedgers increase their in-sample certainty equivalent wealth by only a very small amount when using the first-best hedge ratio as opposed to the OLS hedge ratio. In other words, taking higher moments into account does not substantially increase the welfare of the hedger, very much regardless of the utility function.

Take, for example, cocoa and assume HARA utility with $\gamma = 1$. The reward per unit of risk that is obtained from using the first-best hedge ratio exceeds that which is achieved with the standard OLS hedge ratio by a very

---

[12]For $\gamma \in \{-3, -1\}$ the HARA utility coincides with its polynomial approximation.

marginal 0.002% per year. This suggests that, using the OLS hedge ratio (of 0.853) as opposed to the OHR (of 0.873) generates in a logarithmic HARA utility function a welfare loss of only 0.002% per year for an agent with local relative risk aversion of 1. If the risk aversion of the agent is higher (which seems likely), then the welfare loss is proportionately smaller. For the HARA utility with $\gamma = -3$, the welfare of the second-best is identical (to four decimal places) to the first best welfare. In other words, there is no increase in welfare that is achieved by using the utility-based hedge ratio.

Oats stands out as the commodity for which hedgers will get the maximum increase in welfare from using a utility-based hedge ratio for the majority of utility functions that we consider. Using the OLS hedge ratio (0.78) in the logarithmic HARA utility function generates an investment potential of 4.08. On the other hand, using the optimal hedge ratio estimated from the logarithmic HARA utility (0.87) generates an investment potential of 4.14 and thus increases welfare by, roughly, 0.06% a year. This 0.06% is the highest increase in welfare that can be achieved in-sample from using the utility-based hedge ratio as opposed to the OLS hedge ratio. By any standard, the increase in welfare is economically insignificant.

Out of the eight utility functions depicted, the logarithmic HARA utility ($\gamma = 1$) is the one that generates the highest average yearly increase in welfare (0.008% across the 20 commodities). This means that using the optimal hedge ratios, as opposed to the OLS hedge ratios, in the logarithmic utility function increases wealth by less than 0.01% per annum on average. The results are even less economically significant for the other utility functions, where the increase in welfare obtained with the optimal hedge relative to the standard OLS hedge is roughly 0.002%.

Intuitively, one would expect that spot series for which the hedged portfolio returns show significant departures from normality (such as cotton), should show more considerable increases in welfare from the use of a utility-based hedge ratio estimate, and indeed this is the case. By contrast, for the copper series, where neither the skewness nor the kurtosis of the hedged portfolio returns demonstrate any statistically significant departures from normality, the incremental impact on the estimated OLS HR and the welfare benefits of utility-based OHR calculation are negligible. For that series, the gain in investment potential as one moves away from the OLS hedge is a meagre 0.0005% per annum at most.

4.2. **Out-of-sample analysis.** What increase in welfare can be achieved if one uses historical hedge ratios to determine appropriate hedging strategies for future time periods? Hedgers are assumed to update their information set once a month and to re-estimate their optimal hedge ratios accordingly. The new hedge ratios are then used as a basis for risk management over the

following month. We calculate the resulting time series of returns according to equation (2.1). The out-of-sample (ex-post) investment potentials generated from different utility functions are reported in Table 3 for both the case when we inappropriately use the OLS hedge (OLS IP) and the case when we rightly use the utility-based optimal hedge (OHR IP).

There is no tendency ex-post for the investment potential of the optimal hedge to exceed that of the OLS hedge. In other words, modelling the hedge ratios with the true distribution and thus taking into account higher moments does not necessarily increase the welfare of the hedger out-of-sample. To put it differently, there is no systematic loss in wealth that occurs from inappropriately using OLS hedging.

Let us take silver as an example. For all values of γ other than -1, the investment potential of the utility-based hedge ratio exceeds that of the OLS hedge ratio. In other words, adopting a more sophisticated approach to determining the HR in this case helps as it increases welfare by an incremental average return of 0.4% a year compared to the OLS hedge. At the other end of the spectrum, some commodities are better hedged with the OLS hedge ratio. Take, for example, cotton. Irrespective of the hedger's utility function, the ex-post investment potential of the OLS hedge is higher than that of the predictive OHR. In effect, correctly modelling the optimal hedge ratio with a utility function decreases welfare by an average of 0.9%. This suggests that in this case, anything more sophisticated than OLS hedging actually hurts.

Bringing together the evidence of Table 3, it seems that there is no consistent support for the hypothesis that utility-based hedge ratios substantially increase welfare. The welfare benefits of using utility functions sensitive to higher moments are small but positive for 10 commodities (aluminium, cocoa, corn, gold, heating oil, light crude oil, oats, silver, soybean oil, and wheat). For the remaining 10 commodities, utility-based hedging is actually detrimental.

All else equal, a hedge ratio that is stable over time is preferable to one that is highly volatile in order to keep the transactions costs from rebalancing the hedged portfolio to a minimum. In order to investigate the variability of the estimated hedge ratios from the various techniques, Table 3 also reports the means and the standard deviations of the estimated 1-step ahead rolling hedge ratios. The means of the utility-based optimal hedge ratios are bigger than the means of the OLS hedge ratios for 13 of the 20 spot

series hedged, while they are smaller for the remaining 7.[13] Thus, most of the time, switching to a utility-based approach that explicitly incorporates higher moments leads to higher hedge ratios, commensurate with a more precise estimate of the risks associated with systematically leptokurtic return distributions. In almost all cases, OLS-based hedging yields HRs that have lower variances, indicating more stable hedge ratios and therefore a lower cost of hedging. In fact, only for three series (soybean meal, lean hogs, and light crude oil), are the OLS HRs less stable than utility-based HRs out of sample.

In order to examine the relative sizes and stabilities of the estimated hedge ratios, Figures 1 to 3 plot the predictive HRs implied by OLS and various utility functions in the HARA class. The hedge ratios are estimated recursively using all in-sample data, with one observation added at each time step, for the cotton, gold, and soybean meal series respectively.[14] Figure 1 shows that in the case of cotton, the OLS hedge ratio is higher and less variable than those estimated from HARA utility functions, and in particular, logarithmic utility generates a dynamic OHR that has a lower mean but much higher variance than the others. Similarly, for gold (Figure 2), again the OLS hedge ratio is much less volatile than that of the other utility functions (although now the OLS hedge also has a lower average value). This increased variability of the utility-based hedge ratios suggests that more frequent rebalancing of the hedged portfolio would be required, which could have important consequences for the cost of implementing the hedges. Finally, Figure 3 illustrates that for some of the series, there is very little indeed to choose between the different hedge ratios, as indicated by the indistinguishability of the lines in the figure. In such cases, the temporal variation in the HRs is much more significant than the contemporaneous differences across the HRs.

5. CONCLUSIONS

This study has proposed a utility-based framework for the determination of optimal hedge ratios that can allow for the impact of higher moments on the hedging decision. The approach is applied to a set of 20 commodities that are hedged with futures contracts. We find that in sample, the utilities of hedges constructed allowing for non-zero higher moments are only very

---

[13]This part of the analysis leaves aside the case where the utility function has a risk aversion parameter of -1 since these results are quite different from those emanating from the other utility-based measures). γ = -1 almost invariably leads to low hedging coefficients but also very stable hedges.

[14]The three figures are shown for illustration and we do not include plots for all 20 series due to space constraints, but comparable figures for every asset in our dataset are available from the authors on request.

slightly higher than those determined by much simpler OLS. When implemented out of sample, utility-based hedge ratios are usually less stable over time, and can make investors worse off for some assets than hedging using the traditional methods.

To the extent that using a considerably more sophisticated approach does not reliably improve upon simple OLS hedging, our results confirm those of Harris and Shen (2006). They are unable to find any consistent improvement in minimum value at risk with additional complexity. Similarly, Jondeau and Rockinger (2006) show that under many circumstances, incorporating higher moments does not affect asset allocation decisions; Post et al. (2002) reach comparable conclusions concerning the usefulness of coskewness in explaining the cross-sectional variation in asset returns. Thus, in summary, our findings add to a growing body of very recent literature suggesting that higher moments matter in theory but not in practice.

Our research suggests several potentially fruitful avenues for further investigation. The practical implementation of hedging strategies requires a consideration of returns on a net of transactions costs basis. We conjecture that, given the lack of welfare benefits from utility-based hedge ratio estimation even on a gross basis, this approach is likely to be even less attractive once reasonable transactions costs are accounted for. These nonparametric hedge ratios are typically less stable than those estimated using mean-variance analysis, with a consequent need for more frequent and larger rebalancing. It would also be useful to determine whether our broad conclusions also hold for other hedging assets, sample periods and data frequencies.

<div align="center">

REFERENCES

</div>

Alexander, C. and Barbosa, A. (2006) The Impact of Electronic Trading and Exchange Traded Funds on the Effectiveness of Minimum Variance Hedging. *ICMA Centre Discussion Papers in Finance* DP2006-04.

Agarwal, V. and Naik, N. (2004). Risks and Portfolio Decisions Involving Hedge Funds. *Review of Financial Studies* 17, 63–98.

Arrow, K.J. (1971). *Essays in the Theory of Risk-Bearing.* Markham, Chicago.

de Athayde, G.M. and Flôres, Jr, R.G. (2004). Finding a Maximum Skewness Portfolio — A General Solution to Three-Moments Portfolio Choice. *Journal of Economic Dynamics and Control* 28(7), 1335–1352.

Baillie, R.T. and Myers, R.J. (1991). Bivariate GARCH Estimation of the Optimal Commodity Futures Hedge. *Journal of Applied Econometrics* 6, 109–124.

Barone-Adesi, G., Gagliardini, P., and Urga, G. (2004), Testing Asset Pricing Models with Coskewness *Journal of Business and Economic Statistics* 22 (4), 474–485.

Brooks, C., Henry, O.T., and Persand, G. (2002) The Effect of Asymmetries on Optimal Hedge Ratios *Journal of Business*, 75(2), 333–352.

Brooks, C. and Kat, H. (2002) The Statistical Properties of Hedge Fund Index Returns and Their Implications for Investors *Journal of Alternative Investments* 5(2), 26–44.

Cechetti, S.G., Cumby, R.E., Figlewski, S. (1988) Estimation of Optimal Futures Hedge *Review of Economics and Statistics* 70, 623–630

Černý, A. (2003). Generalized Sharpe Ratios and Asset Pricing in Incomplete Markets. *European Finance Review 7*(2), 191–233.

Černý, A. (2004). *Mathematical Techniques in Finance: Tools for Incomplete Markets*. Princeton University Press.

Christie-David, R. and Chaudhry, M. (2001). Coskewness and Cokurtosis in Futures Markets, *Journal of Empirical Finance* 8, 55–81.

Chung, Y., Johnson, H., and Schill, M. (2006) Asset Pricing When Returns Are Nonnormal: Fama-French Factors versus Higher-Order Systematic Comoments, *Journal of Business* 79, 2, 923–940.

Chunhachinda, P., Dandapani, K., Hamid, S., and Prakash, A. (1997). Portfolio Selection and Skewness: Evidence from International Stock Markets *Journal of Banking and Finance* 21, 143–167.

Dennis, Jr., J. E. and R. B. Schnabel (1996). *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*. SIAM.

Ederington, L.H. (1979). The Hedging Performance of the New Futures Market. *Journal of Finance,* 34, 157–170.

Figlewski, S. (1984) Hedging Performance and Basis Risk in Stock Index Futures. *Journal of Finance*, 39, 657–669.

Gilbert, S., Jones, S.K., and Morris, G.H. (2006) The Impact of Skewness in the Hedging Decision *Journal of Futures Markets* 26(5), 503–520.

Guidolin, M., and A. Timmermann (2005), Optimal Portfolio Choice under Regime Switching, Skew and Kurtosis Preferences, Working Paper 2005-006A, Federal Reserve Bank of St. Louis.

Harris, R.D.F. and Shen, J. (2006) Hedging and Value at Risk *Journal of Futures Markets* 26(4), 369–390.

Harvey C. (1995) Predictable Risk and Returns in Emerging Markets *Review of Financial Studies* 8, 773–816.

Harvey, C. R., J. C. Liechty, M.W. Liechty, and P.Müller (2004), Portfolio Selection with Higher Moments, SSRN Working Paper, http://ssrn.com/abstract=634141.

Harvey, C.R. and Siddique, A. (2000). Conditional Skewness in Asset Pricing Tests *Journal of Finance* 55, 1263–1295.

Hodges, S.D. (1998) A Generalization of the Sharpe Ratio and its Application to Valuation Bounds and Risk Measures. FORC Preprint 98/88, University of Warwick.

Jondeau, E. and Rockinger, M. (2006) Optimal Portfolio Allocation under Higher Moments. *European Financial Management* 12(1), 29–55.

Kallberg, J.G. and Ziemba, W.T. (1983). Comparison of alternative utility functions in portfolio selection problems. *Management Science* 29(11), 1257–1276.

Kraus, A. and Litzenberger, R. (1976) Skewness Preference and the Valuation of Risky Assets *Journal of Finance* 31, 1085–1100.

Kraus, A. and Litzenberger, R. (1983) On the Distributional Conditions for a Consumption-Oriented Three Moment CAPM *Journal of Finance* 38(5), 1381–1391.

Kroner, K.F. and Sultan, J. (1991) Exchange Rate Volatility and Time Varying Hedge Ratios in Rhee, S.G. and Chang, R.P. (eds.) *Pacific Basin Capital Markets Research* Elsevier, North Holland.

Lai, T. Y. (1991), Portfolio Selection with Skewness: A Multiple-Objective Approach, *Review of Quantitative Finance and Accounting*, 1(3), 293–305.

Levy, H. (1969) A Utility Function Depending on the First Three Moments *Journal of Finance* 24, 715–719.

Lien, D. and Luo, X. (1993) Estimating Multiperiod Hedge Ratios in Cointegrated Markets *Journal of Futures Markets* 13, 909–920.

Lin, J.W., Najand, M., and Yung, K. (1994) Hedging with Currency Futures: OLS Versus GARCH *Journal of Multinational Financial Management* 4, 45–67.

Markowitz, H. (1959) *Portfolio Selection: Efficient Diversification of Investments*, John Wiley & Sons, New York.

Myers, R.T. and Thompson, S.R. (1989) Generalised Optimal Hedge Ratio Estimation *American Journal of Agricultural Economics* 71, 858–867.

Miffre, J.(2004) Conditional OLS minimum variance hedge ratios *Journal of Futures Markets* 24, 10, 945–964.

Park, T.H. and Switzer, L.N. (1995) Bivariate GARCH Estimation of the Optimal Hedge Ratios for Stock Index Futures: A Note *Journal of Futures Markets* 15, 61–67.

Post, T., Vliet, P., and Levy, H. (2003) Risk Aversion and Skewness Preference: a Comment. ERIM Report No. ERS-2003-009-F&A, Erasmus University Rotterdam, http://ssrn.com/abstract=411660.

Prakash, A., C. Chang, and Pactwa, T. (2003), Selecting a Portfolio with Skewness: Recent Evidence from US, European, and Latin American Equity Markets, *Journal of Banking and Finance*, 27(7), 1375–1390.

Pratt, J.W. (1964). Risk aversion in the small and in the large. *Econometrica* 32, 122–136.

Pulley, L.B. (1983). Mean-variance approximations to expected logarithmic utility. *Operations Research* 31(4), 685–696.

Samuelson, P. (1970). The fundamental approximation theorem of portfolio analysis in terms of means, variances and higher moments. *Review of Economic Studies* 37(4), 537–542.

Simaan, Y. (1993a), Portfolio Selection and Asset Pricing –Three-Parameter Framework, *Management Science*, 39(5), 568–577

Simaan, Y. (1993b), What is the opportunity cost of mean-variance investment strategies?, *Management Science*, 39(5), 578–587

Strong, R.A. and Dickinson, A. (1994) Forecasting Better Hedge Ratios *Financial Analysts Journal* January-February 70–72.

Scott, R. and P. Horvath (1980), On the Direction of Preference for Moments of Higher Order Than the Variance, *Journal of Finance* 35, 915–919.

Tsiang, S.C. (1972) The Rationale of the Mean-Standard Deviation Analysis, Skewness Preference, and the Demand for Money *American Economic Review* 62, 354–371.

TABLE 1. Summary statistics for spot and future commodity returns.

	Spot Commodity					Futures Contract					OLS Hedged Portfolio				
	Mean %	Std %	Skew	Kurt	JB	Mean %	Std %	Skew	Kurt	JB	Mean %	Std %	Skew	Kurt	JB
Aluminum	0.5	10.0	0.25	4.03*	43*	3.1	14.4	-0.12	-0.16	0.2	-0.1	9.6	0.45	5.22*	73*
Cocoa	1.6	26.3	0.72*	1.31*	39*	-8.0	29.0	0.54*	1.42*	33.0*	8.4	8.9	-0.60*	8.19*	713*
Coffee	4.6	38.0	1.15*	4.27*	301*	-4.7	38.3	0.56*	1.57*	47.7*	8.6**	19.3	0.68*	4.38*	270*
Copper	6.8	30.7	0.50*	1.37*	37*	2.4	31.5	0.28	3.65*	74.4*	6.4	19.3	0.07	-0.32	1
Corn	2.1	25.5	0.25***	3.83*	191*	-8.0***	21.8	0.49*	4.21*	240.2*	10.4*	12.0	-1.10*	5.06*	390*
Cotton	2.0	28.4	-0.71*	9.58*	1156*	-3.0	24.8	0.14	1.10*	15.8*	4.6	18.6	-6.48*	81.38*	83762*
Frozen Pork	17.5	58.2	0.74*	2.24*	92*	-9.6	39.1	-0.12	1.16*	17.9*	26.7*	44.9	0.47*	2.96*	124*
Gold	4.0	18.8	1.11*	5.83*	500*	-3.6	18.7	0.30**	5.19*	350.1*	7.5*	5.2	1.15*	10.69*	1535*
Heating Oil	8.4	38.3	0.94*	5.16*	342*	15.8**	34.9	0.53*	2.99*	114.0*	-4.7	25.3	0.14	5.19*	306*
Lean Hogs	3.4	33.7	0.48*	2.30*	55*	3.0	25.7	-0.44*	2.20*	63.0*	0.1	24.7	0.15	1.50*	21*
Light Crude	9.7	38.9	0.93*	5.02*	308*	11.2	33.0	0.02	2.32*	57.9*	-2.3	15.9	1.15*	5.80*	419*
Lumber	10.2	39.2	0.80*	1.46*	23*	-4.4	31.5	0.40***	0.12	3.2	13.3	32.8	0.95*	1.56*	29*
Oats	4.2	29.6	1.79*	14.46*	2850*	-9.6	30.0	1.54*	11.59*	1845.3*	11.7*	18.2	-0.08	2.14*	59*
Platinum	6.4	25.8	0.35*	4.55*	272*	0.1	26.7	-0.57*	6.73*	598.6*	6.3*	7.7	0.97*	4.93*	361*
Silver	5.6	32.9	0.57*	10.88*	1537*	-13.5**	26.1	-0.07	2.39*	61.3*	11.7*	7.9	1.05*	6.39*	484*
Soybean Meal	3.4	23.6	0.47*	4.82*	196*	2.2	23.7	-0.33***	2.85*	69.4*	2.4	21.3	1.15*	9.60*	792*
Soybean Oil	3.4	28.1	0.70*	2.96*	138*	-5.5	25.6	0.36*	2.09*	62.6*	9.0*	10.0	0.40*	1.23*	28*
Soybeans	1.4	23.0	-0.30**	3.20*	136*	-5.0	22.4	-0.27***	2.26*	69.3*	6.3*	7.0	-1.51*	5.66*	528*
Sugar	7.4	37.4	0.99*	2.62*	138*	-9.8	41.9	0.35*	2.01*	58.3*	14.8*	20.4	0.00	2.96*	112*
Wheat	3.1	20.3	0.19	2.00*	53*	-8.5**	21.6	0.14	0.58**	5.4***	8.3*	15.3	0.06	2.24*	65*

Note: One, two or three stars indicate significance at 1%, 5% and 10% level, respectively.

TABLE 2. In-sample hedge ratios and their ex-ante performance.

commodity				γ						
OLS HR		-1	-3	1		5		∞		
				HARA	POLY	HARA	POLY	HARA	POLY	
ALUMINUM	OHR	0.1826	0.1866	0.1973	0.1967	0.1904	0.1903	0.1889	0.1889	
0.1828	OLS gain	0.0348	0.0363	0.0418	0.0415	0.0381	0.0381	0.0374	0.0374	
	OHR gain	0.0348	0.0363	0.0421	0.0417	0.0382	0.0381	0.0374	0.0374	
COCOA	OHR	0.8463	0.8563	0.8732	0.8713	0.8635	0.8633	0.8609	0.8608	
0.8530	OLS gain	3.0529	3.1315	3.3875	3.3689	3.2177	3.2158	3.1828	3.1821	
	OHR gain	3.0531	3.1315	3.3894	3.3705	3.2182	3.2162	3.1831	3.1824	
COFFEE	OHR	0.853	0.8527	0.8484	0.8485	0.8516	0.8516	0.8521	0.8521	
0.8553	OLS gain	5.3606	5.2869	5.2057	5.222	5.2456	5.2477	5.2595	5.2604	
	OHR gain	5.3607	5.287	5.206	5.2224	5.2457	5.2478	5.2596	5.2605	
COPPER	OHR	0.8991	0.8948	0.8878	0.8881	0.8918	0.8918	0.8929	0.8929	
0.8978	OLS gain	3.9998	3.8743	3.8181	3.8388	3.8222	3.8256	3.8371	3.8385	
	OHR gain	3.9999	3.8744	3.8186	3.8392	3.8224	3.8258	3.8372	3.8386	
CORN	OHR	1.0184	1.0306	1.0668	1.052	1.0419	1.0403	1.0374	1.0368	
1.0329	OLS gain	2.5082	2.5881	3.2604	3.007	2.7459	2.7244	2.6744	2.667	
	OHR gain	2.5087	2.5882	3.2642	3.008	2.7461	2.7246	2.6744	2.667	
COTTON	OHR	0.8604	0.8533	0.7674	0.8176	0.8401	0.8424	0.8463	0.847	
0.8623	OLS gain	2.2765	2.2187	1.8207	2.0259	2.1441	2.1539	2.1769	2.1801	
	OHR gain	2.2765	2.2189	1.8494	2.032	2.1456	2.1551	2.1777	2.1808	
FROZEN PORK	OHR	0.9337	0.9518	0.9544	0.9494	0.9585	0.9573	0.9568	0.9563	
0.9477	OLS gain	6.7604	6.7039	6.6696	6.7109	6.6832	6.6885	6.6892	6.6914	
	OHR gain	6.7619	6.704	6.67	6.711	6.6841	6.6892	6.6898	6.6919	
GOLD	OHR	0.9598	0.9742	1.001	0.9977	0.9851	0.9847	0.9811	0.9809	
0.9662	OLS gain	1.6244	1.6579	1.7733	1.7676	1.6964	1.6959	1.6808	1.6806	
	OHR gain	1.6245	1.658	1.7755	1.7694	1.697	1.6965	1.6812	1.681	
HEATING OIL	OHR	0.8197	0.8074	0.8023	0.8165	0.803	0.8057	0.8043	0.8054	
0.8246	OLS gain	4.1521	3.7077	4.4804	4.7656	3.7341	3.8307	3.6915	3.734	
	OHR gain	4.1522	3.7094	4.4836	4.7661	3.7368	3.8328	3.6939	3.7361	
LEAN HOGS	OHR	0.9556	0.9423	0.9302	0.9323	0.9352	0.9356	0.9375	0.9377	
0.9555	OLS gain	2.6371	2.5399	2.5633	2.5928	2.5165	2.5217	2.5202	2.5223	
	OHR gain	2.6371	2.5403	2.5652	2.5943	2.5177	2.5228	2.5211	2.5232	

Continued on next page.

TABLE 2. Continued from previous page.

commodity						γ				
OLS HR		-1	-3	1		5		∞	
				HARA	POLY	HARA	POLY	HARA	POLY
LIGHT CRUDE	OHR	1.0762	1.0505	1.0223	1.0391	1.0363	1.0391	1.0411	1.0422
1.0781	OLS gain	6.3277	5.7828	6.6023	6.9565	5.8014	5.9019	5.7558	5.7999
	OHR gain	6.3277	5.7868	6.6199	6.9653	5.8106	5.9101	5.763	5.8067
LUMBER	OHR	0.6778	0.6726	0.6524	0.6542	0.6662	0.6664	0.6689	0.6689
0.6827	OLS gain	2.2788	2.2512	2.1974	2.208	2.2306	2.2319	2.2383	2.2388
	OHR gain	2.2789	2.2517	2.2023	2.2123	2.232	2.2332	2.2393	2.2398
OATS	OHR	0.7671	0.7922	0.8746	0.8433	0.8185	0.8146	0.8078	0.8063
0.7775	OLS gain	2.6935	2.9648	4.0755	3.7347	3.3014	3.2611	3.1604	3.1455
	OHR gain	2.694	2.9658	4.1356	3.7604	3.3103	3.2683	3.1651	3.1497
PLATINUM	OHR	0.9215	0.921	0.9203	0.9203	0.9207	0.9207	0.9208	0.9208
0.9214	OLS gain	3.0384	3.0122	3.0285	3.0259	3.007	3.0071	3.0073	3.0074
	OHR gain	3.0384	3.0122	3.0285	3.0259	3.007	3.0071	3.0073	3.0074
SILVER	OHR	0.9076	0.9419	0.9947	0.9668	0.9651	0.9601	0.9567	0.9547
0.9265	OLS gain	2.9066	2.9765	3.7111	3.5862	3.167	3.1621	3.0811	3.0804
	OHR gain	2.9079	2.9773	3.73	3.5929	3.1721	3.166	3.0842	3.0831
SOYB. MEAL	OHR	0.4322	0.436	0.449	0.4478	0.4402	0.4401	0.4385	0.4385
0.4314	OLS gain	0.522	0.5309	0.5988	0.5941	0.5492	0.549	0.5412	0.5411
	OHR gain	0.522	0.531	0.5997	0.5949	0.5494	0.5492	0.5413	0.5413
SOYB. OIL	OHR	1.0182	1.0272	1.0459	1.0442	1.0345	1.0343	1.0318	1.0317
1.0245	OLS gain	3.4276	3.4665	3.6222	3.6086	3.5155	3.5142	3.4952	3.4947
	OHR gain	3.4277	3.4665	3.6238	3.6099	3.5158	3.5145	3.4953	3.4949
SOYBEANS	OHR	0.9732	0.9775	0.9868	0.986	0.9811	0.981	0.9797	0.9797
0.9785	OLS gain	2.3901	2.3724	2.4063	2.4079	2.3749	2.3753	2.3722	2.3724
	OHR gain	2.3901	2.3724	2.4065	2.4081	2.3749	2.3754	2.3722	2.3724
SUGAR	OHR	0.7416	0.7573	0.7798	0.7765	0.7677	0.7672	0.764	0.7638
0.7485	OLS gain	4.8832	4.9641	5.1873	5.1688	5.0432	5.041	5.012	5.0112
	OHR gain	4.8836	4.9648	5.1961	5.1759	5.0465	5.0441	5.0141	5.0132
WHEAT	OHR	0.6088	0.6178	0.6297	0.6296	0.624	0.6238	0.6218	0.6218
0.6213	OLS gain	0.8838	0.8977	1.0113	0.9908	0.927	0.9256	0.9139	0.9135
	OHR gain	0.8842	0.8977	1.0115	0.991	0.927	0.9256	0.9139	0.9135

Notes: OLS hedge is computed as $h_{\mathrm{OLS}} = Cov(X,Y)/Var(X)$. Optimal hedge for each utility function is given by $\hat{h}(1)$ from Theorem 2.9. OLS gain shows the welfare gain from using the OLS hedge ratio, $g(1, h_{\mathrm{OLS}})$. The 1st best gives the welfare gain from using the optimal hedge ratio, $g(1, \hat{h}(1))$. For unexplained symbols see Theorem 2.9. For definitions of HARA utility and its polynomial approximation (POLY), see Sections 2.3 and 2.4.

TABLE 3. Out-of-sample hedging performance.

commodity (in, out) mean OLS std OLS		−1	−3	γ 1 HARA	POLY	5 HARA	POLY	∞ HARA	POLY
ALUMINUM	HR mean	0.1803	0.1784	0.1740	0.1742	0.1768	0.1768	0.1774	0.1774
(40, 21)	HR std	0.0317	0.0348	0.0426	0.0423	0.0377	0.0376	0.0366	0.0365
0.1793	OLS IP	2.3640	2.3607	2.3529	2.3530	2.3577	2.3577	2.3589	2.3589
0.03221	OHR IP	2.3461	2.3719	2.4366	2.4313	2.3958	2.3953	2.3866	2.3864
COCOA	HR mean	0.8381	0.8562	0.8847	0.8786	0.8688	0.8678	0.8643	0.8639
(165, 84)	HR std	0.0090	0.0054	0.0083	0.0067	0.0055	0.0054	0.0052	0.0052
0.84979	OLS IP	6.8866	6.8871	6.8879	6.8879	6.8875	6.8875	6.8874	6.8874
0.00637	OHR IP	6.8459	6.9073	6.9666	6.9609	6.9410	6.9395	6.9297	6.9291
COFFEE	HR mean	0.8485	0.8453	0.8378	0.8381	0.8425	0.8425	0.8436	0.8436
(204, 103)	HR std	0.0117	0.0141	0.0176	0.0172	0.0158	0.0157	0.0152	0.0152
0.84934	OLS IP	7.6510	7.6492	7.6294	7.6308	7.6440	7.6442	7.6464	7.6464
0.01224	OHR IP	7.6591	7.6014	7.4914	7.5044	7.5571	7.5587	7.5736	7.5742
COPPER	HR mean	0.9160	0.9371	0.9662	0.9364	0.9499	0.9441	0.9453	0.9429
(86, 44)	HR std	0.0207	0.0467	0.0852	0.0500	0.0632	0.0563	0.0573	0.0544
0.92611	OLS IP	-14.712	-14.723	-14.751	-14.751	-14.733	-14.733	-14.729	-14.729
0.03205	OHR IP	-14.329	-15.138	-16.270	-15.537	-15.646	-15.521	-15.466	-15.416
CORN	HR mean	0.9900	1.0030	1.0393	1.0266	1.0148	1.0134	1.0101	1.0096
(204, 103)	HR std	0.0060	0.0069	0.0160	0.0099	0.0087	0.0082	0.0079	0.0077
1.00139	OLS IP	8.8247	8.8192	8.8044	8.8048	8.8140	8.8140	8.8161	8.8161
0.00837	OHR IP	8.6881	8.8432	9.2558	9.1254	8.9814	8.9672	8.9269	8.9217
COTTON	HR mean	0.8303	0.8104	0.7143	0.7572	0.7884	0.7909	0.7977	0.7985
(196, 99)	HR std	0.0152	0.0201	0.0231	0.0268	0.0238	0.0238	0.0226	0.0225
0.83058	OLS IP	12.700	12.708	12.722	12.722	12.714	12.714	12.712	12.712
0.01559	OHR IP	12.698	12.300	10.400	11.217	11.859	11.907	12.045	12.061
FROZEN PORK	HR mean	0.9325	0.9713	0.9996	0.9744	0.9907	0.9858	0.9845	0.9825
(204, 103)	HR std	0.0213	0.0229	0.0362	0.0256	0.0278	0.0261	0.0258	0.0252
0.95781	OLS IP	-0.4819	-0.3974	-0.6974	-0.7270	-0.4423	-0.4495	-0.4144	-0.4175
0.02039	OHR IP	-0.1383	-0.6171	-1.3030	-0.8916	-0.9469	-0.8624	-0.8292	-0.7947
GOLD	HR mean	0.9568	0.9767	1.0130	1.0060	0.9916	0.9906	0.9861	0.9857
(204, 103)	HR std	0.0030	0.0041	0.0084	0.0062	0.0058	0.0055	0.0051	0.0050
0.96583	OLS IP	4.0846	4.0847	4.0849	4.0849	4.0848	4.0848	4.0847	4.0847
0.00313	OHR IP	4.0632	4.1067	4.1863	4.1809	4.1397	4.1392	4.1275	4.1274
HEATING OIL	HR mean	0.8226	0.8082	0.8061	0.8207	0.8039	0.8066	0.8049	0.8061
(180, 91)	HR std	0.0111	0.0106	0.0111	0.0107	0.0106	0.0105	0.0106	0.0105
0.82848	OLS IP	0.1497	0.1425	0.0779	0.0791	0.1250	0.1249	0.1328	0.1327
0.01065	OHR IP	0.2013	0.5034	0.6121	0.3198	0.6096	0.5610	0.5803	0.5606
LEAN HOGS	HR mean	0.9100	0.9023	0.9044	0.9087	0.9008	0.9015	0.9009	0.9012
(139, 70)	HR std	0.0371	0.0348	0.0310	0.0312	0.0330	0.0330	0.0336	0.0336
0.91373	OLS IP	10.378	10.399	10.364	10.368	10.398	10.398	10.400	10.400
0.03539	OHR IP	10.310	10.303	10.397	10.428	10.327	10.333	10.316	10.318

TABLE 3. Continued from previous page.

commodity (in, out) mean OLS std OLS		-1	-3	γ					
				1		5		∞	
				HARA	POLY	HARA	POLY	HARA	POLY
LIGHT CRUDE	HR mean	1.0623	1.0444	1.0238	1.0315	1.0340	1.0352	1.0376	1.0380
(171, 86)	HR std	0.0167	0.0149	0.0148	0.0147	0.0146	0.0146	0.0147	0.0147
1.06301	OLS IP	-0.5702	-0.5660	-0.5607	-0.5608	-0.5634	-0.5634	-0.5643	-0.5643
0.01675	OHR IP	-0.5641	-0.2879	0.0395	-0.0741	-0.1267	-0.1438	-0.1820	-0.1888
LUMBER	HR mean	0.6102	0.5995	0.5604	0.5662	0.5870	0.5876	0.5921	0.5923
(76, 39)	HR std	0.0401	0.0409	0.0488	0.0469	0.0430	0.0429	0.0421	0.0421
0.61722	OLS IP	11.935	12.129	12.419	12.402	12.259	12.256	12.212	12.211
0.03922	OHR IP	11.993	12.096	12.409	12.356	12.202	12.198	12.160	12.158
OATS	HR mean	0.7767	0.8184	0.9393	0.8896	0.8618	0.8528	0.8444	0.8407
(204, 103)	HR std	0.0158	0.0207	0.0445	0.0313	0.0296	0.0267	0.0257	0.0246
0.79252	OLS IP	2.1613	2.1506	2.1109	2.1129	2.1379	2.1381	2.1431	2.1432
0.01706	OHR IP	2.1163	2.2919	2.5796	2.5255	2.4332	2.4151	2.3818	2.3736
PLATINUM	HR mean	0.9220	0.9315	0.9477	0.9443	0.9384	0.9379	0.9359	0.9357
(204, 103)	HR std	0.0025	0.0057	0.0141	0.0116	0.0092	0.0089	0.0079	0.0078
0.92526	OLS IP	-0.7301	-0.7302	-0.7306	-0.7306	-0.7303	-0.7303	-0.7303	-0.7303
0.00295	OHR IP	-0.7041	-0.7846	-0.9220	-0.8978	-0.8426	-0.8394	-0.8214	-0.8201
SILVER	HR mean	0.9233	0.9561	1.0044	0.9799	0.9777	0.9734	0.9700	0.9682
(170, 86)	HR std	0.0071	0.0097	0.0114	0.0093	0.0110	0.0104	0.0106	0.0103
0.94011	OLS IP	12.001	12.005	12.013	12.013	12.008	12.008	12.007	12.007
0.00735	OHR IP	11.802	12.184	12.767	12.509	12.440	12.398	12.348	12.331
SOYB. MEAL	HR mean	0.2454	0.2567	0.2700	0.2642	0.2629	0.2621	0.2608	0.2605
(129, 65)	HR std	0.0843	0.0722	0.0668	0.0740	0.0676	0.0688	0.0689	0.0694
0.2452	OLS IP	5.5188	5.5467	5.5840	5.5816	5.5643	5.5639	5.5581	5.5580
0.08323	OHR IP	5.5360	5.2208	5.0567	5.2607	5.0951	5.1275	5.1319	5.1449
SOYB. OIL	HR mean	0.9874	0.9970	1.0195	1.0159	1.0053	1.0049	1.0021	1.0020
(204, 103)	HR std	0.0162	0.0169	0.0192	0.0186	0.0176	0.0176	0.0173	0.0173
0.994	OLS IP	12.036	12.044	12.057	12.057	12.050	12.050	12.047	12.047
0.01686	OHR IP	12.002	12.059	12.176	12.172	12.105	12.105	12.088	12.088
SOYBEANS	HR mean	0.9548	0.9685	0.9960	0.9895	0.9792	0.9784	0.9752	0.9749
(204, 103)	HR std	0.0028	0.0046	0.0109	0.0084	0.0069	0.0066	0.0060	0.0059
0.96451	OLS IP	-0.9915	-0.9949	-1.0026	-1.0025	-0.9979	-0.9978	-0.9967	-0.9967
0.00371	OHR IP	-0.9683	-0.9991	-1.0651	-1.0400	-1.0240	-1.0206	-1.0146	-1.0133
SUGAR	HR mean	0.7505	0.7723	0.8048	0.7984	0.7870	0.7860	0.7817	0.7814
(204, 103)	HR std	0.0062	0.0095	0.0157	0.0137	0.0121	0.0118	0.0111	0.0110
0.76002	OLS IP	-1.4177	-1.4234	-1.4414	-1.4409	-1.4295	-1.4294	-1.4270	-1.4270
0.00757	OHR IP	-1.4034	-1.4384	-1.5112	-1.4882	-1.4667	-1.4634	-1.4560	-1.4547
WHEAT	HR mean	0.6294	0.6382	0.6514	0.6509	0.6445	0.6444	0.6423	0.6422
(204, 103)	HR std	0.0176	0.0155	0.0132	0.0133	0.0141	0.0142	0.0146	0.0146
0.63787	OLS IP	10.659	10.663	10.663	10.663	10.664	10.664	10.664	10.664
0.01474	OHR IP	10.523	10.671	10.877	10.871	10.771	10.769	10.736	10.735

Notes: The values in parentheses below the series labels denote the number of in-sample and out-of-sample observations respectively. Below that, means and standard deviations of the OLS hedge ratios are also presented in the first column. The entries for each asset in the remaining columns give first the mean and standard deviations of the hedge ratios for the utility-based hedges, followed by the investment potentials of the OLS and of the utility-based hedges respectively. For definitions of HARA utility and its polynomial approximation (POLY), see Sections 2.3 and 2.4.

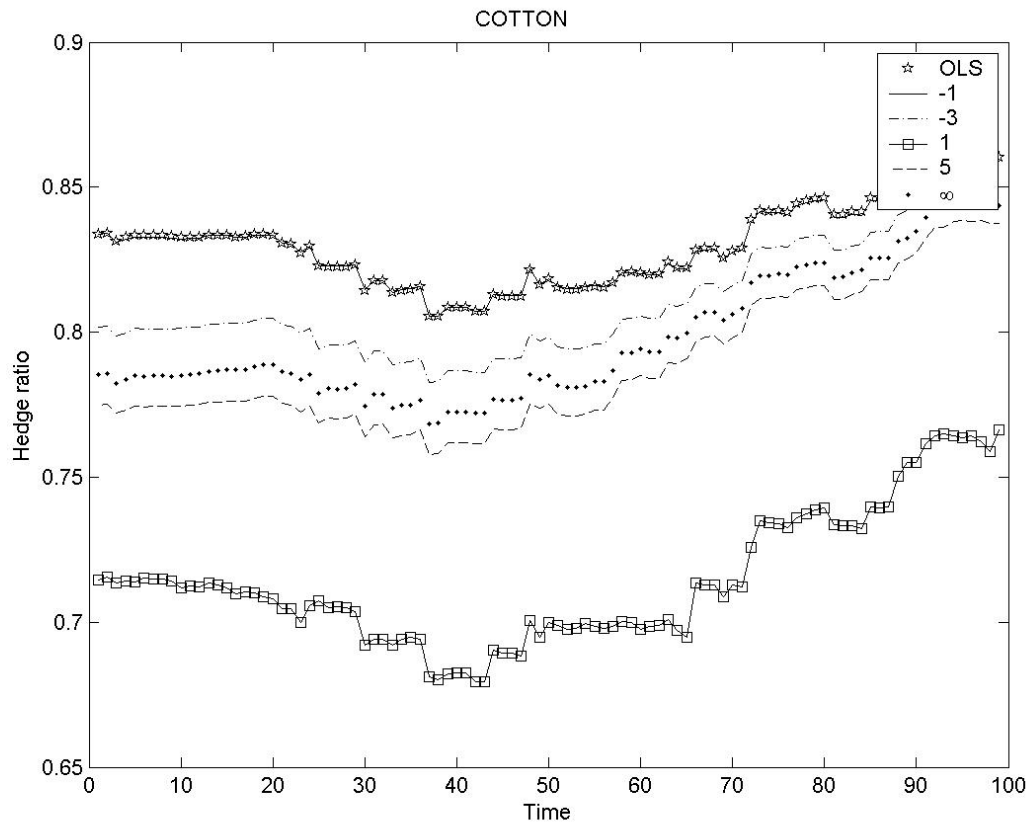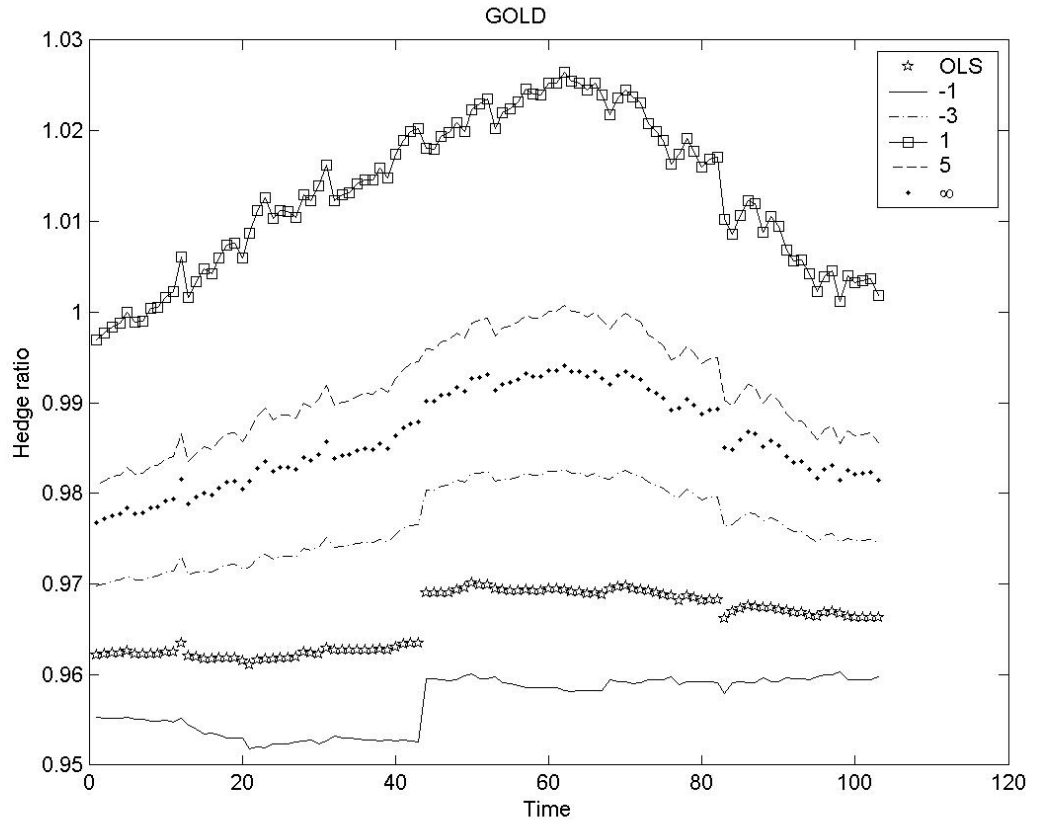FIGURE 1. Out-of-sample hedge ratios for cotton.

FIGURE 2. Out-of-sample hedge ratios for gold.

FIGURE 3. Out-of-sample hedge ratios for soybean meal.