Common Correlation and Calibrating the Lognormal Forward Rate Model

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Carol Alexander

ISMA Centre, University of Reading, UK

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Abstract
This paper examines some challenging problems for calibrating the lognormal forward rate (LFR) model. For interest rate options, the calibration of forward rate correlations to the swaption implied volatility surface requires careful selection of a parametric form, otherwise over-fitting can produce prices that are unstable over time. This paper introduces some useful new full-rank and reduced-rank correlation parameterizations that are very parsimonious. First, two full-rank parameterizations that connect semi-annual with annual forward rate correlations are derived, and then the main focus of the paper is on a reduced rank parameterization which is based on the common eigenvector model of Flury (1988).

In the common eigenvector framework the same eigenvectors are used for all correlation matrices of the same dimension. Although the eigenvalues are fully time-varying and are specific to each tenor and maturity, the eigenvectors are constant and will be common to all swaptions of the same tenor. The 'trend', 'tilt', 'curvature' interpretation is shown to be a very accurate interpretation to place on common eigenvectors. Therefore only four parameters are required, for three orthogonal eigenvectors which can be calibrated to the market using, simultaneously, implied volatilities of different maturity swaptions.

Several academics and practitioners advocate using historical forward rate data when calibrating reduced rank correlations, and use the longer term swaption data for calibrating volatilities. We discuss the model risk that arises from using historical forward rate data, and the reasons why one should seek to use market data alone. However, lack of market data, particularly at the long end, means that a very parsimonious parameterization of correlation is required. Since the common eigenvector parameterization is much parsimonious than individual eigenvector parameterizations, and is based simultaneously on swaptions of several maturities, it is likely to give more stable results. Common eigenvector parameterizations also have more general applications to multi-factor pricing models where forward correlations play an important role, e.g., for pricing and hedging OTC options in the power market.

Author Contacts:
Prof. Carol Alexander
ISMA Centre
The University of Reading
Whiteknights Park
Reading RG6 6BA
United Kingdom
Tel. +44-1189-316431
E-mail: c.alexander@ismacentre.rdg.ac.uk

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1. Introduction

In 1997 three papers that introduced very similar lognormal diffusion processes for interest rates appeared virtually simultaneously. These models, now commonly called the 'Libor models' are based on either lognormal diffusions of forward rates as in Brace, Gatarek & Musiela (1997) and Miltersen, Sandmann & Sondermann (1997) or lognormal diffusions of swap rates, as in Jamshidian (1997). The consequent research interest in the calibration of these models has engendered a growing empirical literature, and in particular there are very many papers by Brigo and Mercurio, and Riccardo Rebonato. Further details are given in Brigo and Mercurio (2001, 2002a), Rebonato (1999b, 2002) and many of their papers are downloadable from www.fabiomercurio.it and www.damianobrigo.it and www.rebonato.com.

The Lognormal forward rate version of the Libor model is very useful for pricing and hedging exotic OTC products such as auto-caps and in-arrear swaps. However, the model is not easy to calibrate because its parameters include forward rate correlations that are difficult to forecast, particularly at the short end. To gain some idea of how these correlations change, Figure 1 shows the average daily correlations between semi-annual forward rates in the UK during four three month periods between June 2001 and June 2002. It is clear that changes in correlation are not as significant at the long end as they are at the short end. Figure 2 shows the correlation between the semi-annual forward rate starting in six months, and several longer maturity semi-annual forward rates starting in one year, 18 months, 2 years and so forth. Each line in Figure 2 represents the average daily correlation during a quarterly period. The correlations between the six month forward rate and the medium maturity forward rates are particularly unstable, and in the third quarter of 2001, when the terrorist attacks in the US occurred, they even became negative.

Based on the variability of the historical correlations observed in Figures 1 and 2, correlation forecasts must be difficult to obtain, particularly for short maturity forward rates. The forecasts have a great deal of uncertainty. This is one of many reasons why we should seek to use market data rather than historical data for calibrating correlations. The current swaption implied volatility surface captures market expectations of future correlation, and forecast uncertainty can be ignored if one uses instruments that are linear in the correlation. An at-the-money swaption is
approximately linear in the forward rate volatility which is itself approximately linear in the factors of the forward rate correlation matrix. This correlation matrix should be calibrated to the swaption implied volatility surface, but one must be very careful how the factors of the correlation matrix are specified.

Many authors, including Rebonato (1999a), Rebonato and Joshi (2001) Hull and White (1999, 2000) and Longstaff, Santa-Clara and Schwartz (1999) advocate the use of historical data on forward rates for calibrating correlations. Long term swaptions are used for calibrating forward rate volatilities, since the cap market is typically only short to medium term, and so there is a lack of reliable data at the long end for calibrating also the correlations.

But historical correlations are not stable over time, and even if they were, there is another rather important problem with the use of historical data on forward rates. Long dated forward rates are unobservable. The market convention for stripping out long dated forward rates is straightforward, and simply assumes a flat semi-annual rate between observed swap rates at annual maturities. Although this convention is necessary, for example to obtain cap prices from the Black cap volatilities that are quoted in the market, it is not informative of the instantaneous forward rate correlations in the Libor model.

If historical data on forward rates were used to calibrate these correlations, the forward rates would have to be obtained via a yield curve fitting model. Those in Figures 1 and 2 were obtained using the Svensson (1994) model, but there are many alternative yield curve models: for example see McCulloch (1975), Nelson and Siegel (1987) and Steely (1991). Alexander and Lvo (2003) show that the forward rate correlations that are estimated from historical data can be quite different depending on the yield curve model chosen. Therefore, the model risk arising from (a) the choice of yield curve model from which the forward rates are derived, and (b) the choice of historical period over which observations are taken, may be considerable.

1 When estimating volatilities and correlations from historical data, one has to consider whether the first difference of the forward rates, or the first difference of the log of forward rates, is the variable. The choice should be consistent with the assumed diffusion process. If a normal geometric Brownian process were assumed, then the appropriate stationarity transform would be the difference in the forward rates. But a lognormal geometric Brownian process is assumed in the LFR model, so the appropriate stationarity transform is the difference in the log forward rates.
On the other hand, when trying to calibrate forward rate correlations to the swaption implied volatility surface, one encounters other problems. These are similar in spirit to the problems of calibrating local volatility surfaces to option implied volatilities. It is well-known that, if no parameterization for local volatility is used, as advocated by Dupire (1994), Derman, Kani and Chriss (1996) and Andersen and Brotherton-Ratcliffe (1997) amongst others, the local volatilities will be very sensitive to extrapolation and interpolation of implied volatilities and the volatility surface will lack robustness. Anyone who has tried to calibrate local volatilities directly from the implied surface will know that exotic path dependent option pricing models which depend directly on local volatilities do not always give stable results, even when regularization is used to introduce smoothness to the local volatility function as in Avellaneda et. al. (1997) and Bouchouev and Isakov (1997, 1999). So a functional form for local volatility can be chosen in order to calibrate local volatility to only the observed market data, which may be quite sparse. Many different parsimonious functional forms for local volatility have been suggested in the literature, by Coleman, Li and Verma (1998), Dumas, Fleming and Whaley (1998), Brown and Randall (1999), McIntyre (2001), Brigo and Mercurio (2002b) and others. But the subjective choice of functional form is open to question: how should we determine which is the best choice?

The same type of problems arise when attempting to calibrate forward rate correlations to market data. Direct calibration can lead to unstable results, where correlations are also counter-intuitive, or even lie outside [-1, 1]. Therefore one should seek to impose a parametric form that extends to all the correlation matrices underlying all swaptions, which is parsimonious enough to give stable results but at the same time sufficiently general to capture the 'typical' behaviour of forward rate correlations – for example, where the correlation between adjacent forward rates increases with their maturity. Many different functional forms have been proposed and a useful review of these is given in Brigo (2001).

The aim of this paper is to derive some new parsimonious parameterizations of forward rate correlation matrices that enable the lognormal forward rate model to be calibrated to market implied swaption volatilities, without reverting to historical data for calibrating the correlations. These include full-rank parameterizations of the correlation matrices for semi-annual and annual
forward rates, and reduced rank parameterizations that are based on the common principal components for the correlation matrices that relate to all swaptions.

The outline of this paper is as follows: Section two provides a brief outline of the lognormal forward rate version of the Libor model and the calibration of forward rate volatilities. Section three proposes some full rank parameterizations of a semi-annual correlation matrix that are very parsimonious, involving only one or two parameters, and that yield approximate correlations between annual forward rates which are determined by the same parameters. These are useful because they link the semi-annual cap market forward rate correlations with swap market forward rate correlations which, when the fixed leg has annual payments, are normally derived from annual rates. Section four explains the orthogonal transformation of the LFR model that is commonly used for pricing path dependent products. Rebonato (1999a), Rebonato and Joshi (2001) Hull and White (1999, 2000) and Longstaff, Santa-Clara and Schwartz (1999) reduce the rank of forward rate covariance matrices by setting all but the three largest eigenvalues to zero. The implication of zeroing eigenvalues is a transformation of the lognormal forward rate model, where each forward rate is driven by three orthogonal factors that are derived from a principal component analysis (PCA). Section five introduces a new method for calibrating reduced rank correlation matrices to market data. Instead of using historical data on forward rates to estimate the three eigenvectors in the PCA, calibration to the market can be based on the common principal components model of Flury (1988). That is, the same eigenvectors, with only four parameters, are calibrated to all swaptions of the same tenor. The number of correlation parameters is very substantially reduced and the simultaneous use of market data on swaptions with a range of maturities should ensure that calibrated correlation matrices are stable. Section six summarizes the main results and concludes.

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2 Most Sterling and Euro swaps have semi-annual fixed leg payments, but in the US the fixed leg sometimes has annual payments.
2. The Lognormal Forward Rate Model

To ease notation we assume that day counts are constant. That is, year-fractions between payment (or reset) dates are constant for all forward rates, and the basic forward rate is a semi-annual rate. Denote by $f_i$ the semi-annual forward rate that is fixed at time $t_i$ but stochastic up to that point in time. Each forward rate has its own 'natural' measure, which is the measure with numeraire $P_{i+1}$ where $P_i$ is the value of a zero coupon bond maturing at date $t_i$. Under it's natural measure each forward rate is a martingale and therefore has zero drift in its dynamics. The log normal forward rate (LFR) model is thus:

$$\frac{df_i(t)}{f_i(t)} = \sigma_i(t) \, dW_i \quad [i = 1, \ldots, m; \ 0 < t < t_i] \quad (1)$$

where $dW_1, \ldots, dW_m$ are Brownian motions with correlations $\rho_{ij}(t)$. That is,

$$\mathbb{E}[dW_i dW_j] = \rho_{ij}(t) \, dt \quad (2)$$

Calibration of the model requires estimation of the parameters of the instantaneous volatilities $\sigma_i(t)$ and instantaneous correlations $\rho_{ij}(t)$ for $i, j = 1, \ldots, m$.

Now consider a T maturity cap with strike $K$ as a set of caplets from $t_i$ to $t_{i+1}$ [$i = 1, \ldots, m-1$ and $t_m = T$]. Each caplet pay-off = max ($L_i - K, 0$) where $L_i$ is the LIBOR rate revealed at time $t_i$ and at $t_i$ we have $L_i = f_i$. Assume for the moment that $\sigma_i(t) = \sigma_i$. Then the Black-Scholes type formula for each caplet value at time 0 is:

$$\text{B-S (Caplet, } t_i, K, \sigma_i) = P_{i+1} [f_i(0)\Phi(x) - K\Phi(y)] = C_i(\sigma_i), \quad \text{say}$$

and

$$x = [\ln \left( \frac{f_i(0)}{K} \right) / \sigma_i \sqrt{t_i}] + \sigma_i \sqrt{t_i} / 2; \quad y = x - \sigma_i \sqrt{t_i}.$$

The B-S cap price is the sum $\sum C_i(\sigma_i)$ of all prices of the caplets.
If we further assume that $\sigma_i = \sigma$ for every $i$ then we can ‘back-out’ a single implied ‘flat volatility’ $\theta(K, T)$ from the observed market price of a cap with strike $K$ and maturity $T$. Similarly, we can ‘back-out’ the implied caplet volatilities from the observed market prices of several caps of different maturities, using some interpolation between prices because each fixed strike caplet in a cap with strike $K$ has a different moneyness. We normally define the ‘at-the-money’ (ATM) cap strike as the current value of the swap rate for the period of the cap, so the different caplets in an ATM cap are only approximately ATM.

Figure 3 shows the typical 'humped shaped' term structure for flat volatilities of caps. In the simple version of the LFR model, with $\sigma_i(t) = \sigma_i \forall \ i$, the implied forward rate volatilities are the caplet volatilities that are backed-out from interpolated caplet prices. These volatilities will display a more pronounced humped shape than the cap volatilities – an example is given in the Appendix, which re-examines the relationship between cap and caplet volatilities.

Caplet implied volatilities may be used for calibrating the LFR model (1) with time-varying but deterministic volatilities $\sigma_i(t)$ for the forward rates. It is standard to assume that $\sigma_i(t) = \eta_i \cdot h(t)$ where the time-varying parametric form $h(t)$ that is common to all volatilities is a simple ‘hump’ introduced by Rebonato (1999a) and since used by many others, given by:

$$h(t) = [(a + b(T - t)) \exp(-c(T - t)) + d]$$

Note that $h(t)$ – and therefore also $\sigma_i(t)$ – depends on maturity $T$ and the parameters $a, b, c, d$ which define a common volatility structure. The individual parameters $\eta_i$ are there so that instantaneous volatilities can be adjusted upwards or downwards according to the prices in the cap market. The parameters may be calibrated to implied caplet volatilities, $\theta_i$ and a simple calibration objective is:

$$\min \sum_i \omega_i \left( \theta_i \sqrt{t_i} - \eta_i \int_0^{t_i} h(t)^2 \ dt \right)^2$$

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The weights \( \omega_i \) are there to account for the uncertainty that surrounds different option prices and vegas. To place more weight on the more certain volatilities, the weights should be directly proportional to vega and inversely proportional to the bid-offer spread.\(^3\) The weights should also be a decreasing function of maturity (and tenor, in the case of a swaption), and since gamma is directly proportional to vega and decreases with maturity, we suggest taking the weights as gamma divided by the bid-ask spread.

With so many parameters, of course the model could fit the market prices of caps perfectly, and there is a danger in over-fitting as shown by De Jong, Dreissen and Pelsser (1999). Moreover, since the main use of the LFR lies in pricing exotic OTC products, we shall also need to calibrate forward rate correlations, and this cannot be done using cap market data alone – unless one is happy to use historical forward rate data, which is prone to the difficulties outlined in section one.

3. Calibration to the Swaption Market

Unlike caps, a swaption pay-off [max (value of swap, 0)] cannot be written as a simple sum of options, so forward rate correlations as well as their volatilities will affect the value of a swaption. We need to work in a single measure, but forward rates are only drift-less under their natural measure. Hull and White (1999, 2000) take as numeraire the discretely re-invested money market account. This gives the ‘spot libor measure’, where the appropriate rate for discounting an expected cash flow at time \( t_{i+1} \) to time \( t_i \) is the forward rate \( f_i \). This is intuitive and it leads to a (relatively) tractable specification of the drift terms, so we shall adopt this here.\(^4\) Under the spot libor measure the forward rate \( f_i \) has dynamics given by

\[
\begin{align*}
\frac{df_i(t)}{f_i(t)} &= \mu_i(t) dt + \sigma_i(t) dW_i \\
\mu_i(t) &= \sigma_i(t) \sum_{j=m(t)}^{i} \rho_{ij}(t) \frac{f_j(t)}{1 + f_j(t)} \\
\end{align*}
\]

\(^3\) The bid-offer spread is a good indication of the uncertainty in pricing; it normally increases with maturity. Uncertainty in pricing also translates to uncertainty in volatility through the caplet vega: high vega means that large price errors will induce small volatility errors and low vega means that small price errors will induce large volatility errors.

\(^4\) See Brigo and Mercurio (2001) for the specification of the drifts under other choices of numeraire.
where \( m(t) \) is the ‘number’ of the accrual period. The drift becomes important when using the LFR model to price path dependent options, where the resolution method will, most likely, be Monte Carlo simulation, and the drift will need updating every time step. We shall return to this in the next section, but in this section we are considering the calibration of forward rate volatilities and correlations to swaption volatilities, where the drift is not important.

We now derive some new approximations for the instantaneous volatility of an annual swap rate in terms of the instantaneous volatilities and correlations of the semi-annual forward rates underlying the swap. These approximations allow the calibration of instantaneous volatilities and correlations for semi-annual rates to the average swap rate volatilities that are implied from the market, so that historical data on forward rates are not required. The results are particularly useful in the US market when the fixed leg is based on annual forward rates.

Following Rebonato (1998) we write the swap rate as an approximate linear function of annual forward rates. For an \( m \)-year swap starting at \( t_n \) (that is, an "n into m year" swap) we have the swap rate:

\[
SR_{n,m} = w_1F_{n} + \ldots + w_mF_{n+m-1}
\]

where \( F_i \) is the annual forward rate starting at year \( i \) and the weights \( w_i = P_{n+i}/[P_{n+1} + \ldots + P_{n+m}] \) are assumed constant at their current value.\(^5\) Consequently the variance of the swap rate may be written as a quadratic form in the 3x3 covariance matrix of annual forward rates and thus the swap rate volatilities are linked to annual forward rate volatilities and correlations.

\(^5\) Other relationships between swap and forward rates have been developed by several authors. Hull and White (1999) write the log swap rate as a difference in logs of products of forward rates to derive an exact expression for the volatility of the swap rate. Longstaff, Santa-Clara and Schwartz (1999) use a least square regression technique to write the swap rate as an approximate linear function of forward rates. Jäckel and Rebonato (2000) express the swap rate variance as a weighted sum of forward rate covariances and thus derive an approximation for the volatility of the swap rate. Andersen and Andreasen (2000) refine the relationship between forward rates and swap rates in the presence of a volatility skew.
To see this, write \( \mathbf{w} = (w_1, \ldots, w_n)' \) and let \( \mathbf{V}_{n,m}(t) \) be the instantaneous covariance matrix of the forward rates \( F_{n}, \ldots, F_{n+m-1} \) for \( 0 < t < t_n \). Then \( \text{SR}_{n,m} \) has instantaneous variance given by

\[
\sigma_{n,m}(t)^2 = \mathbf{w}' \mathbf{V}_{n,m}(t) \mathbf{w}
\]

and the average swap volatility during the interval \([0, t_n]\) is

\[
\sigma_{n,m}^{\text{model}} = \sqrt{\frac{1}{t_n} \int_0^{t_n} \sigma_{n,m}(t)^2 \, dt}
\]

We know the weights \( \mathbf{w} \) from current values of discount bonds and we want to infer the parameters of \( \mathbf{V}_{n,m}(t) \) by equating the model volatilities that are defined by equations (6) to the smoothed\(^6\) implied swaption volatilities \( \theta_{n,m} \). The covariance matrix of annual forward rates \( \mathbf{V}_{n,m}(t) \) contains instantaneous volatilities and correlations of annual rates. Indeed it may be factored as

\[
\mathbf{V}_{n,m}(t) = \mathbf{D}_{n,m}(t) \Sigma_{n,m}(t) \mathbf{D}_{n,m}(t)
\]

where \( \mathbf{D}_{n,m}(t) \) is the diagonal matrix of the instantaneous volatilities and \( \Sigma_{n,m}(t) \) is the correlation matrix of the \( m \) annual forward rates \( F_{n}, \ldots, F_{n+m-1} \) underlying the \( n \) into \( m \) year swap.

It is simple to express the \( i \)th annual forward rate variance \( \sigma_{i,ann}(t)^2 \) in terms of the two underlying semi-annual rate variances \( \sigma_{2i-1}(t)^2 \) and \( \sigma_{2i}(t)^2 \). Since

\[
(1 + F_i(t)) = (1 + f_{2i-1}(t)/2)(1 + f_{2i}(t)/2)
\]

\(^6\) If reliable (B-S) swaption prices are available in the market, from these we can immediately imply the average swaption volatility \( \theta_{n,m} \). Otherwise we shall need to smooth the volatility surface using a 2-dimensional smoothing algorithm. It is well known that calibration results will be sensitive to the choice of smoothing algorithm (see for example Brigo and Mercurio, 2001).
we have $F_i(t) = (f_{2i-1}(t) + f_{2i}(t))/2 + f_{2i-1}(t)f_{2i}(t)/4$. With this identity, and assuming lognormal dynamics for the semi-annual rates, it is shown in Brigo and Mercurio (2001) that

$$\sigma_{i,\text{ann}}(t)^2 = u_{2i-1}(t)^2 \sigma_{2i-1}(t)^2 + u_{2i}(t)^2 \sigma_{2i}(t)^2 + 2 u_{2i-1}(t) u_{2i}(t) \rho_{2i-1,2i} \sigma_{2i-1}(t) \sigma_{2i}(t)$$

(8)

where $\rho_{2i-1,2i}$ is the correlation between the two semi-annual rates and $u_{2i}(t) = 1 - [f_{2i}(t)/2F_i(t)]$ and $u_{2i-1}(t) = 1 - [f_{2i-1}(t)/2F_i(t)]$.

Brigo and Mercurio (2001) assume $u_{2i}(t) = u_{2i}(0)$ and $u_{2i-1}(t) = u_{2i-1}(0)$ for all time $t$, and use the consequent approximation based on (8) to connect annual (swaption) volatilities to semi-annual (caplet) volatilities.

The point to note is that the correlation of adjacent semi-annual forward rates also enters the approximation based on (8). Therefore, if such an approximation is used with (7) for calibration to the swaption market, the semi-annual correlation parameters will also enter $D_{n,m}(t)$. However we can use some further assumptions to express $\Sigma_{n,m}(t)$ in terms of semi-annual rate correlations only. In that case all the parameters in $V_{n,m}(t)$ will relate only to semi-annual rates and not to annual rates.

Some researchers, including Brigo and Mercurio (2001) and Rebonato and Joshi (2001), assume that the annual rate correlation matrix is $\Sigma_{n,m} = \{\rho_{i,j}\}$ where

$$\rho_{i,j} = \exp(-\varphi \|i - j\|)$$

(9)

Variants of this, such as $\rho_{i,j} = \exp(-\varphi \|i' - j'\|)$ are discussed in Brigo (2001) but these parameterizations do not lend themselves to a straightforward relationship between the semi-annual rate correlations and the annual correlations. We write $\rho = \exp(-\varphi)$, so that the assumption (9) may be written $\rho_{i,j} = \rho^{|i-j|}$ and, if we use the parameterization (9) for semi-annual
forward rates, we can derive a simple approximate relationship between semi-annual correlations and annual correlations.

Let $\rho$ be the correlation between any two adjacent semi-annual forward rates, and assume that the correlation between two semi-annual rates $f_i$ and $f_j$ is $\rho^{|i-j|}$ so that the $2m \times 2m$ correlation matrix of semi-annual forward rates underlying an $n$ into $m$ year swaption, denoted $\Sigma_{n,m \_semi}$, is given by the circulant correlation matrix:

\[
\begin{bmatrix}
1 & \rho & \rho^2 & \ldots & \rho^{2m-1} \\
\rho & 1 & \rho & \ldots & \rho^{2m-2} \\
\rho^2 & \rho & 1 & \ldots & \rho^{2m-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho^{2m-1} & \ldots & 1
\end{bmatrix}
\]

Then, if the two semi-annual forward rates $f_{2i-1}$ and $f_{2i}$ in the annual forward rate $F_i$ have equal volatilities (though not necessarily the same volatilities for each $i$) the $mxm$ correlation matrix of the annual forward rates underlying the same swaption, denoted $\Sigma_{n,m \_ann}$, is approximated by the single parameter full rank correlation matrix:

\[
\begin{bmatrix}
1 & \phi & \phi^2 & \ldots & \phi^{2m-2} \\
\phi & 1 & \phi & \ldots & \phi^{2m-3} \\
\phi^2 & \phi & 1 & \ldots & \phi^{2m-4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\phi^{2m-2} & \ldots & 1
\end{bmatrix}
\]

where $\phi = \rho(1+\rho)/2$. That is, the correlation between two annual forward rates $F_i$ and $F_j$ is approximately $\rho^{2|s-j|}(1+\rho)/2$. 

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To prove this we must assume that the two semi-annual forward rates $f_{2i-1}^i$ and $f_{2i}$ in the annual forward rate $F_i$ have equal volatilities, though not necessarily the same volatilities for each $i$. Denote the (equal) volatility of $f_{2i-1}^i$ and $f_{2i}$ by $\sigma_{2i}(t)$. We must also introduce a further approximation, that $u_j(t) = u$, a constant, for all time $t$ and for all maturities $j$. Then the annual forward rate dynamics are approximately

$$\frac{dF_i(t)}{F_i(t)} \approx (\ldots)dt + u\sigma_{2i}(t)(dW_{2i-1}(t) + dW_{2i}(t))$$

where the two Brownian motions driving the semi-annual rates are correlated:

$$E[dW_{2i-1}(t)dW_{2i}(t)] = \rho_{2i-1,2i}dt$$

This gives an approximation for the annual forward rate variance:

$$\sigma_{i,\text{ann}}^2(t) = 2u^2\sigma_{2i}(t)(1 + \rho_{2i-1,2i})$$

and an approximation for the covariance between the $i$th and the $j$th annual rates:

$$\sigma_{ij,\text{ann}}(t) = u^2\sigma_{2i}(t)\sigma_{2j}(t)\rho_{2i-1,2j-1} + \rho_{2i-1,2j} + \rho_{2i,2j-1} + \rho_{2i,2j}$$

Now, when the semi-annual rate correlation matrix $\Sigma_{n,m,\text{semi}}$ takes the circulant form given above we have:

$$\sigma_{i,\text{ann}}^2(t) = 2u^2\sigma_{2i}(t)(1 + \rho)$$

and

$$\sigma_{ij,\text{ann}}(t) = u^2\sigma_{2i}(t)\sigma_{2j}(t)\rho^{\frac{i-j}{2}}(1 + \rho)^2$$

so the result follows.

Figure 4 illustrates the correlations obtained using the circulant structure with $\rho = 0.9$. Although this leads to a simple form for annual rate correlations, it is a little inflexible. In particular it does not allow for the correlation of adjacent forward rates to increase with maturity, as has been observed recently in some markets (see Alexander and Lvov, 2003). Schoenmakers and Coffey (2000) point out the importance of allowing for this, and consequently propose a full-rank semi-parametric form based on an increasing sequence of $M$ real numbers. In fact the increasing correlation of adjacent forward rates with maturity may be more parsimoniously captured by a two parameter correlation matrix for semi-annual rates, as shown in Figure 5.
Suppose the $2m \times 2m$ correlation matrix $\Sigma_{n, m, \text{semi}}$ of semi-annual forward rates underlying an $n$ into $m$ year swaption is given by the two parameter correlation matrix $(\rho_{ij})$ where:

$$\rho_{ij} = \psi^{j-i} \rho^{|i-j|} \quad i < j, \quad 0 < \rho < 1 \quad \text{and} \quad \rho < \psi^{2m-1} \rho < 1.$$ 

Then the $mxm$ correlation matrix of the annual forward rates underlying the same swaption, denoted $\Sigma_{n, m}$ is approximated by the correlation matrix $(\rho_{ij})$ where:

$$\rho_{ij} = \frac{\rho^{2|i-j|-1} \psi^{2j-1} (1+\rho)/(\psi^2+\rho)}{2 \sqrt{(1+\psi^{2(i-1)} \rho)(1+\psi^{2(j-1)} \rho)}}$$

This result follow after some algebra using similar assumptions and methods as for the circulant parameterization. In Figure 5, the case $m = 10$, $\rho = 0.9$, $\psi = 1.007$ is illustrated.

In a common correlation structure, each correlation matrix $\Sigma_{n,m}(t)$ will be defined in terms of the same two parameters, $\rho$ and $\psi$. Also, by (8), the diagonal matrix of the annual forward rate volatilities $D_{n,m}(t)$ is expressed in terms of the semi-annual forward rate volatilities and these same parameters. Thus from (7), the only parameters in the annual rate covariance matrix $V_{n,m}(t)$ are: $a$, $b$, $c$, $d$, $\eta$, $\rho$ and $\psi$. Note that we may suppose that $\eta$ is itself parameterized in terms of $a$, $b$, $c$, and $d$, serving only to equate model and market prices for caps, as advocated by Brigo and Mercurio (2001). In that case only five or six parameters need to be determined by the swaption data: these are $a$, $b$, $c$ and $d$, for the volatilities; $\rho$ and possibly also $\psi$ for the correlations. If there are sufficient market data to calibrate more parameters, one could assume time-varying parametric forms for $\rho$ and $\psi$, or that $\rho$ and $\psi$ depend on maturity and tenor so they are different for each $V_{n,m}(t)$. However this should be attempted with caution – too many correlation parameters may lead to results that are not sufficiently stable as the implied volatility surface changes from day to day.
4. Calibration for Pricing Path Dependent Interest Rate Options

The LFR model may be used to price path dependent options, but no analytic pricing formulae exist and some sophisticated resolution methods, often Monte Carlo simulation, are used. Whichever method is chosen the drifts that are induced by the change of numeraire will be important, and they will need updating with the current value of the instantaneous volatilities and correlations at every time step. The instantaneous drift term in the ith forward rate \( f_i \) depends on the forward rates of lower maturity \( f_{i-1}, f_{i-2}, \) etc., but only if these are still random variables at the time that the drift is estimated. The correlation matrix that one needs to consider for the calibration of the drift term therefore decreases in dimension (by 1) as every payment/reset date passes. This is not the only complication for the simulation. If \( m \) is large the dimension of the simulation will be large and computationally burdensome. However it is possible to reduce dimensions by reducing the rank of the correlation matrix. A review of rank reduction methods is given in Brigo (2001).

A standard method for rank reduction is advocated by Rebonato (1999c) and Hull and White (1999, 2000) and many others. They use an orthogonal transformation of the correlated Brownian motions in (5). The forward rate dynamics are express in terms of three uncorrelated stochastic processes that are common to all forward rates:

\[
df_i(t)/f_i(t) = \mu_i(t) \, dt + \lambda_{i,1}(t) \, dZ_1 + \lambda_{i,2}(t) \, dZ_2 + \lambda_{i,3}(t) \, dZ_3
\]  
\( (10) \)

where \( dZ_1, dZ_2, dZ_3 \) are uncorrelated Brownian motions and:

\[
\sigma_i(t) \, dW_i = \lambda_{i,1}(t) \, dZ_1 + \lambda_{i,2}(t) \, dZ_2 + \lambda_{i,3}(t) \, dZ_3
\]  
\( (11) \)

From this it follows that:

\[
\sigma_i(t) = \sqrt{\lambda_{i,1}(t)^2 + \lambda_{i,2}(t)^2 + \lambda_{i,3}(t)^2}
\]  
\( (12) \)

and

\[
\rho_{ij}(t) = \frac{[\lambda_{i,1}(t) \lambda_{j,1}(t) + \lambda_{i,2}(t) \lambda_{j,2}(t) + \lambda_{i,3}(t) \lambda_{j,3}(t)]}{\sigma_i(t) \sigma_j(t)}
\]  
\( (13) \)
Thus the forward rate volatilities and correlations are completely determined by three volatility ‘components’ for each forward rate, which are \( \lambda_{i,1}(t) \), \( \lambda_{i,2}(t) \) and \( \lambda_{i,3}(t) \) for the \( i \)th forward rate. Denote by \( \sigma_i(t) \) the instantaneous \( i \)th annual forward rate volatility. Given these, the volatility components may then be determined from the spectral decomposition of the covariance matrix of the first difference in the logarithm of the \( m \) annual forward rates \( F_n, \ldots, F_{n+m-1} \) underlying the \( n \) into \( m \) year swap.\(^7\) The spectral decomposition of the covariance matrix is

\[
V_{n,m}(t) = A_{n,m} \Lambda_{n,m}(t) A_{n,m}^	op
\]

where \( A_{n,m}(t) \) is the diagonal matrix of eigenvalues and \( A_{n,m} \) is the \( m \times m \) matrix of eigenvectors of \( V_{n,m}(t) \). Note that the eigenvalues are assumed to be time-varying whereas the eigenvectors are assumed constant. To derive the volatility components, denote by \( \Lambda_1(t) \), \( \Lambda_2(t) \), \( \Lambda_3(t) \), the three largest eigenvalues of \( V_{n,m}(t) \) and denote their eigenvectors by \( \alpha_1 \), \( \alpha_2 \), \( \alpha_3 \). Set \( M(t) = \sigma_i(t)/\sqrt{V_i(t)} \) where

\[
V_i(t) = \alpha_{i,1}^2 \Lambda_1(t) + \alpha_{i,2}^2 \Lambda_2(t) + \alpha_{i,3}^2 \Lambda_3(t)
\]

Then, to satisfy (12), simply set the volatility components to be:

\[
\begin{align*}
\lambda_{i,1}(t) &= M(t) \alpha_{i,1} \sqrt{\Lambda_1(t)} \\
\lambda_{i,2}(t) &= M(t) \alpha_{i,2} \sqrt{\Lambda_2(t)} \\
\lambda_{i,3}(t) &= M(t) \alpha_{i,3} \sqrt{\Lambda_3(t)}
\end{align*}
\]

5. A Common Principal Components Model

A number of researchers, notably Rebonato (1999a), Rebonato and Joshi (2001) Hull and White (1999, 2000) and Logstaff, Santa-Clara and Schwartz (1999) have advocated the use of volatility components derived from eigenvectors that are estimated using historical forward rate data, and eigenvalues that are calibrated to the market. That is, calibration of the volatility components in (11) is based on a principal component analysis (PCA) of historical data on forward rates.

\(^7\) Alternatively, the \( 2m \times 2m \) covariance matrix of semi-annual forward rates can be used. Note that the correlation or covariance matrix should be calculated on the first differences in the logarithms of the forward rates, as is standard for lognormal diffusions.
However, following the remarks made in section one, a LFR model calibration that is based on PCA of historical forward rates will give prices that depend on (a) which forward rates are used, and (b) the length of the historical period chosen for the PCA.

A preferred approach would be to parameterize the eigenvectors of the correlation matrices in the natural way: the first eigenvector should be constant and have only one parameter which represents a parallel shift in the forward curve, the second eigenvector will have two parameters and represent a tilt in the forward curve, and the third eigenvector will have three parameters and represents the curvature of the forward curve. The orthogonality of these vectors induces certain restrictions and reduces the number of parameters required. For example, the orthogonality between the constant and tilt (parameterized as $c_1 + c_2 k$, say, for $k = 1, 2, \ldots, m$) implies that $c_2 = -2c_1 / (m+1)$. In fact three orthogonal trend-tilt-curvature eigenvectors can be parameterized by only four parameters.

Nevertheless, this construction still gives rather too many parameters to be calibrated: four parameters for the eigenvectors of every covariance matrix $V_{n,m}(t)^8$ and there are $nm$ of these matrices! With so many parameters, the current swaption implied volatility surface will of course be fit very closely — but over fitting will lead to prices that are too sensitive and unstable over time.

How, then, should the parameters be reduced? One possibility is to deduce a parsimonious representation of the eigenvectors and eigenvalues from a full-rank parameterization such as those given in section 2. Another possibility, described here, is to use the same eigenvectors for all correlation matrices of the same dimension. This is a special case of the common principal components model introduced by Flury (1988).

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8 The eigenvectors $\mathbf{A}$ of a covariance matrix $\mathbf{V}$ can be obtained from the eigenvectors $\mathbf{M}$ of the associated correlation matrix $\mathbf{C}$. In fact $\mathbf{A} = \mathbf{D} \mathbf{M} \mathbf{D}^{-1}$ where $\mathbf{D}$ is the diagonal matrix of volatilities. We may therefore parameterize the eigenvectors of the correlation matrix using four parameters, and then scale using the time-varying forward rate volatilities.
Alexander and Lvov (2003) show that the implementation of the common eigenvector tests introduced by Flury (1988) on historical forward rate data yields results that are very robust indeed: to different sets of forward rates (i.e. those that are obtained from several different yield curve models), and to the choice of historic data period. Their results are that there is very strong evidence for common eigenvectors in correlation matrices of the same dimension, for both annual and semi-annual forward rates, and moreover these common eigenvectors are very close indeed to the archetypal 'trend, tilt, curvature' form. They also show that the evidence for common eigenvectors of all covariance matrices of the same dimension is equally strong, but that these common eigenvectors are not as stable over time as the common eigenvectors of the correlation matrices.

The common eigenvector parameterization of the covariance matrix is

\[ V_{n,m}(t) = A_m \Lambda_{n,m}(t) A'_{m} \]

where \( \Lambda_{n,m}(t) \) is the 3 x 3 matrix of the three largest eigenvalues of \( V_{n,m}(t) \) and \( A_m \) is a matrix of eigenvectors. Alternatively, a common eigenvector parameterisation of the forward rate correlation matrix may be chosen, where each common eigenvector of the correlation matrices has only four parameters. Moreover these parameters are the same for all \( n \). The strength of this approach is that it substantially reduces the number of correlation parameters. Without common eigenvectors, one needs to calibrate at least four parameters (for three orthogonal eigenvectors) to each swaption smile of maturity \( n \) and tenor \( m \). For longer maturity, not enough smile data will be available, and this may be one reason why several authors favour using historical data to estimate these eigenvectors, instead of calibrating them to the market. However, with common correlation eigenvectors there are only four correlation parameters applying to all swaptions of the same tenor. The total number of eigenvector parameters is therefore reduced by a factor of \( n \), where \( n \) is the number of different maturities of swaptions in the market.

The great advantage of using common PCA to parameterize forward rate correlations is that the common eigenvectors can be calibrated to the swaption market with constant parameters, whilst the forward rate volatilities and the eigenvalues in the spectral decomposition can be fully time
varying and specific to each tenor and maturity, and these can be calibrated to the cap and/or swaptions market. Parameterizing common eigenvectors of the correlation matrix gives more robust and intuitive results than parameterizing common eigenvectors of the covariance matrix. The common eigenvectors shown in Figure 6 justify the orthogonal 'trend-tilt-curvature' common eigenvector parameterization described above. The eigenvectors of the covariance matrix may also be given this interpretation – although the first eigenvector is not so constant because it contains forward rate volatilities which are not constant with respect to maturity – and for the same reason the eigenvectors of the covariance matrix are more sensitive to the historical estimation period. See Alexander and Lvov (2003) for further details.

6. Summary and Conclusion

In the lognormal forward rate option pricing model, forward rate volatilities and correlations may be calibrated to both cap volatility term structures and a smoothed swaption volatility surface. Often this requires linking annual forward rate correlations with semi-annual forward rate correlations, for example in the US swap market when the fixed leg is paid annually. The first part of this paper has introduced some parsimonious parametric forms for the semi-annual forward rate correlations which imply approximations for the annual forward rate correlations that depend on the same parameter(s).

Pricing of exotic path dependent interest rate options normally involves numerical resolution (e.g. simulation) for the forward rate dynamics, where the drift terms that are induced by the change to a single measure will become important. Since these change at every time step, the computations are burdensome and it is desirable to reduce dimensions in the numerical method. To achieve this, several authors have used principal component analysis in conjunction with historical time series on forward rate data for calibrating the eigenvectors of the forward rate covariance matrix.

However historical data on the unobservable long maturity forward rates are subject to substantial model risk arising from the choice of yield curve model and the choice of historical observation period. The final part of this paper introduces an orthogonal transformation based on common eigenvectors of the forward rate correlation matrices which allows calibration to swaption market
data alone. The parametric form for common eigenvectors is based on the common principal component model of Flury (1988), and the resultant parsimony allows calibration of these common eigenvectors simultaneously to all swaptions of the same tenor. Thus historical data is neither necessary nor desirable for the calibration of the LFR model.

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Appendix

Relationship Between Cap and Caplet Volatilities

It is interesting to consider the relationship between the flat volatilities quoted in the market for caps of different maturities and strikes and the implied volatilities of the caplets that make up these caps. In option markets where the underlying is a single asset whose dynamics are governed by a standard lognormal geometric Brownian motion, the variance of log returns over a period $T = T_1 + T_2$ is the sum of the variance of log returns over period $T_1$ and the variance of log returns over period $T_2$. The additivity of variances is a consequence of the independent increments in the stochastic process for a single underlying asset, and because variances are additive implied volatilities $\{\theta_1, \theta_2, \ldots, \theta_T\}$ can be derived from a term structure $\theta(T)$ using the iterative method:

Set $\theta_1 = \theta(1)$; then solve for $\theta_2$ from $[\theta_1^2 + \theta_2^2]/2 = \theta(2)^2$ and so forth.

However, for a cap there is no single stochastic process for the underlying. The underlying forward rate changes for every caplet in the cap and therefore an iteration on variances is not appropriate. In fact the iteration should be performed on the cap volatilities. Denote by $\theta(K, T)$ the market quote of a 'flat' implied volatility for a cap of strike $K$ and maturity $T$. Denote by $\theta_i(K, T)$ the Black-Scholes implied volatility and by $\nu_i(K, T)$ the vega of the $i$th caplet in the cap of strike $K$ and maturity $T$. Thus $\nu_i(K, T) = \partial C_i(\sigma)/\partial \sigma_i$ evaluated at $\sigma_i = \theta_i(K, T)$. Then

$$\theta(K, T) \approx \Sigma \left[ \nu_i(K,T)\theta_i(K,T) \right] / \Sigma \nu_i(K,T) \quad (A)$$

That is, the flat cap implied volatility $\theta(K, T)$ is approximately equal to the vega weighted sum of the caplet implied volatilities.

For a fixed maturity $T$ and strike $K$, the flat volatility $\theta(K, T)$ is defined as the volatility such that the B-S cap price, (denoted $C(\sigma)$ and written as a function just of volatility) is equal to the sum of the caplets priced at the caplet constant volatilities, $\sigma_i$. That is, at $\sigma = \theta(K, T)$, we have $C(\sigma) = \Sigma C_i(\sigma_i)$. Now expanding each $C_i(\sigma_i)$ using a first order Taylor approximation about $\sigma$ gives:
\[ C(\sigma) = \Sigma C_i(\sigma_i) = \Sigma \left[ C_i(\sigma) + (\sigma_i - \sigma) \frac{\partial C_i(\sigma_i)}{\partial \sigma_i} \right] \]

Where the derivative \( \frac{\partial C_i(\sigma_i)}{\partial \sigma_i} = \nu_i \) is evaluated at \( \sigma_i = \sigma \). This implies

\[ \Sigma (\sigma_i - \sigma) \frac{\partial C_i(\sigma_i)}{\partial \sigma_i} = 0 \]

and so it follows that the flat volatility is approximately a vega weighted sum of each caplet implied volatility \( \theta_i \). That is, \( \theta = \Sigma \omega_i \theta_i \), where \( \omega_i = \nu_i / \Sigma \nu_i \) which proves (A).

The cap vega is the sum of the caplet vegas, so caplet vegas can be backed out from estimates of the cap vegas and they will normally decrease with maturity. Given these, and the relationship (A) between caplet and cap volatilities, caplet implied volatilities can be approximated as follows: Set \( \theta_1 = \theta(1) \); then solve for \( \theta_2 \) from \( \left[ \nu_1 \theta_1 + \nu_2 \theta_2 \right] / \left[ \nu_1 + \nu_2 \right] = \theta(2) \), and so forth. Caplet volatilities that are approximated by a variance iteration may be negative, or even complex. However there will be no problem with imaginary volatilities when a volatility iteration is used and, although they could be negative, negative forward rate volatilities are unlikely, as illustrated by the following simple example.

Suppose that a flat cap volatility term structure is given by \( AT[\exp(-BT)] + C \), which is a simple hump shaped curve, and \( T = 0.5, 1, 1.5, \ldots, 10 \) years. We consider the two cases (i) \( A = 0.2; B = 0.6; C = 0.2 \) and (ii) \( A = 0.2; B = 0.6; C = 0.1 \). Figure A illustrates the cap volatility curve and the caplet volatilities that are obtained by three different methods: the variance iteration, the volatility iteration (which is based on (A) with all vegas equal) and the vega iteration based on (A). The cap volatilities are uniformly greater in case (i), and all three methods lead to real, non-negative caplet volatilities; however in case (ii) – with lower cap volatilities which are perhaps more realistic of current market conditions in the US and Europe – the variance weighted method gives imaginary volatilities from 4 years onwards. Note that the effect of vega weighting is to reduce the longer maturity caplet volatilities in both cases.
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21/01/03 C.Alexander@ismacentre.rdg.ac.uk


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Figure 1: Semi-Annual Forward Rate Correlations

Semi-annual forward rates obtained by fitting the Svensson (1994) model to the UK yield curve. 
Unconditional correlation estimates for the quarter were based on daily first difference in log rates. 
Figure 2: Correlations of Semi-Annual Forward Rates with Six Month Forward Rate

![Graph showing correlations of semi-annual forward rates with six month forward rate.]

Figure 3: Typically 'humped shaped' term structure of flat cap implied volatility: $\theta(T)$ vs $T$

![Graph showing the 'humped shaped' term structure of flat cap implied volatility.]

Volatility

Maturity
Figure 4: Semi-Annual Correlations with the Circulant Form [$\rho = 0.9$]
Figure 5: Semi-annual Correlations with a Two Parameter Form \( \rho = 0.88, \ \psi = 1.007 \)

Figure 6: Parameterizing Common Eigenvectors

The figure shows the three largest common eigenvectors for all semi-annual forward rate correlation matrices of dimension 12; the data are based on Svensson yield curve fitting for UK rates between Jan 1st and Dec 31st 2002 and a total of eight matrices were diagonalized simultaneously in the common principal component algorithm. The eigenvalues are different for each matrix, but they indicate that in every case these three common principal component explain over 99% of the variation of forward rates during 2002. See Alexander and Lvov (2003) for further details.
Figure A: Relationship between Caplet and Cap Implied Volatilities

(i) Cap Vol: A = 0.2; B = 0.6; C = 0.2

(ii) Cap Vol: A = 0.2; B = 0.6; C = 0.1