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Orthogonal Methods for Generating Large Positive Semi-Definite Covariance Matrices

Discussion Papers in Finance 2000-06

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The ISMA Centre is supported by the International Securities Market Association

Abstract

It is a common problem in risk management today that risk measures and pricing models are being applied to a very large set of scenarios based on movements in all possible risk factors. The dimensions are so large that the computations become extremely slow and cumbersome, so it is quite common that over-simplistic assumptions will be made. In particular, in order to generate the large covariance matrices that are used in Value-at-Risk models, some very strong constraints are imposed on the movements in volatility and correlations in all the standard models. The constant volatility assumption is also imposed, because it has not been possible to generate large GARCH covariance matrices with mean-reverting term structures.

This paper introduces a new method for generating large positive semi-definite covariance matrices. It is based on univariate GARCH volatilities of a few, uncorrelated key risk factors to provide more realistic term structure forecasts in covariance matrices. Alternatively the method can be used with exponentially weighted moving average key risk factor volatilities, where the smoothing constant is automatically determined by the correlation in the system. In addition to implementing multivariate GARCH of arbitrarily large dimension without the need for constrained parameterizations, advantages of this method include: the ability to tailor the amount of noise in the system so that correlation estimates are more stable; and the volatility and correlation forecasting of new issues or illiquid markets in the system.

Introduction

During the last few years there have been many changes in the way that financial institutions model risk. New risk capital regulations have motivated a need for vertically integrated risk systems based on a unified framework throughout the whole office. If the risk exposures in all locations of a large institution are to be aggregated, the risk system must also be horizontally integrated. Internationally, regulators are pushing towards an environment where traders, quants and risk managers from all offices are referring to risk measures generated by the same models. This is a huge task which remains a challenge for many financial institutions, but the result should be useful to manage risks for allocation of capital between different areas of the firm, and to set traders limits as well as levels of capital reserves.

Following the Basle Accord Amendment in 1996 for the calculation of market risk capital using internal models, the Basle Committee on Banking Supervision (1995) have recommended two methods for generating a unified set of risk measures on a daily basis. These methods have become industry standards for measuring risk not only for external regulatory purposes, but also for internal risk management. The first approach is to calculate a Value-at-Risk (VaR) measure, which is a lower percentile of an unrealized profit and loss distribution. This distribution is based on movements of the market risk factors over a fixed risk horizon. The second approach is to quantify the maximum loss over a large set of scenarios for movements in the risk factors.

Given the huge number of market risk factors affecting the positions of a large financial institution, the VaR models and scenario-based loss models may become very complex indeed. In fact their implementation becomes extraordinarily cumbersome, if not impossible, without making assumptions that restrict the possibilities for movements in the risk factors. For example, at the heart of most risk models there is a covariance matrix that captures the volatilities and correlations between the risk factors. Typically hundreds of risk factors, such as all yield curves, interest rates, equity indices, foreign exchange rates and commodity prices, need to be encompassed by a very large dimensional covariance matrix. It is not easy to generate this matrix and so simplifying assumptions may be necessary. For example the RiskMetrics methodologies designed by JP Morgan use either simple equally weighted moving averages, or exponentially weighted moving averages with the same smoothing constant for all volatilities and correlations of returns. There are substantial limitations with both of these methods, described in Alexander (1996).

Another example of how the standard methods necessitate simplifying assumptions is in maximum loss calculations. The applicability of maximum loss measures depends on portfolio revaluation over all possible scenarios, including movements in both prices and implied volatilities of all risk factors. In complex portfolios the computational burden of full revaluation over thousands of scenarios would be absolutely enormous, and certainly not possible to achieve within an acceptable time frame unless analytic price approximations and advanced sampling techniques are employed in conjunction with a restriction of the possibility set for scenarios.

The problems outlined in both of the above examples have a common root: the computations, be they volatility and correlation calculations for a covariance matrix, or portfolio revaluation for the calculation of maximum loss, are being applied to the full set of risk factors. So the dimensions of the problem become too large to manage and the problem is intractable.

But there is an alternative: to apply computations to only a few key market risk factors that capture the most important uncorrelated sources of information in the data. Such an approach is computationally efficient because it allows an enormous reduction in the dimension of the problem whilst retaining a very high degree of accuracy. Because the risk factors are uncorrelated it does not significantly increase the computational complexity even if a large number of key risk factors are employed. Normally a sufficient number of key risk factors will be generated so that any movements that are not captured by these factors are deemed to be insignificant 'noise' in the system, and by cutting out this noise the risk measures will become more stable and robust over time. Also, being able to quantify how much risk is associated with each key factor is an enormous advantage for risk managers, because their attention is more easily directed towards the most important sources of risk.

The method used here to identify key uncorrelated sources of risk within a large system is principal component analysis. Jamshidian and Zhu (1996) have shown how principal components may be used to improve computational efficiency for scenario based risk measures in large multi-currency portfolios. This paper extends these ideas to the efficient computation of the large positive semi-definite covariance matrices that are necessary for many internal models for measuring market risk.

Identification of the Key Risk Factors

Suppose a set of data with T observations on k asset or risk factor returns is summarized in a Txk matrix **Y**. Principal component analysis will give up to k uncorrelated stationary variables, called the principal components of **Y**, each component being a simple linear combination of the original returns as in (1) below. At the same time it is stated exactly how much of the total variation in the original system of risk factors is explained by each principal component, and the components are ordered according to the amount of variation they explain.

The first step in principal component analysis is to normalize the data in a Txk matrix **X** that represents the same variables as **Y**, but in **X** each column is standardized to have mean zero and variance 1. So if the ith risk factor or asset return in the system is $\mathbf{y}_i$, then the normalized variables are $\mathbf{x}_i = (\mathbf{y}_i - \mu_i)/\sigma_i$ where μ_i and σ_i are the mean and standard deviation of $\mathbf{y}_i$ for i = 1, … k. Now let **W** be the matrix of eigenvectors of **X'X**, and **Λ** be the associated diagonal matrix of eigenvalues, ordered according to decreasing magnitude of eigenvalue.[1] The principal components of **Y** are given by the Txk matrix

$$\mathbf{P} = \mathbf{XW} \quad (1)$$

Thus a linear transformation of the original risk factor returns has been made in such a way that the transformed risk factors are orthogonal, that is, they have zero correlation.[2]

The new risk factors are ordered by the amount of the variation they explain.[3] Hence only the first few, the most important factors may be chosen to represent the system as follows: Since **W** is orthogonal (1) is equivalent to **X = PW'**, that is

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[1] Thus $\mathbf{X'X\,W} = \mathbf{W}\,\mathbf{\Lambda}$.

[2] Note that $\mathbf{P'P} = \mathbf{W'X'XW} = \mathbf{W'W\Lambda}$, but **W** is an orthogonal matrix so $\mathbf{P'P} = \mathbf{\Lambda}$, a diagonal matrix.

$$\mathbf{x}_i = w_{i1}\,\mathbf{p}_1 + w_{i2}\,\mathbf{p}_2 + \ldots\ldots + w_{ik}\,\mathbf{p}_k \tag{2}$$

so the matrix **W** is called the matrix of 'factor weights'. In terms of the original variables **Y** the representation (2) is equivalent to

$$\mathbf{y}_i = \mu_i + \omega^*_{i1}\,\mathbf{p}_1 + \omega^*_{i2}\,\mathbf{p}_2 + \ldots\ldots + \omega^*_{im}\,\mathbf{p}_m + \boldsymbol{\varepsilon}_i \tag{3}$$

where $\omega^*_{ij} = w_{ij}\sigma_i$ and the error term in (3) picks up the approximation from using only the first m of the k principal components. These m principal components are the 'key' risk factors of the system, and the rest of the variation is ascribed to 'noise' in the error term. The representation (3) indicates how, when covariance or scenario calculations are based only on the most important principal components, the effect may be easily translated back to the original system through a simple linear transformation.

Efficient Computation of Positive Semi-Definite Covariance Matrices

This section outlines the theory and methodology for using a few key market risk factors that represent only the most important uncorrelated sources of information to generate large covariance matrices. These matrices will be positive semi-definite, relatively stable over time, and may be computed easily using sophisticated models that have many advantages, but that are too complex for a direct application to large systems.

Since principal components are orthogonal their covariance matrix is simply the diagonal matrix of their variances. These variances can be quickly transformed into a covariance matrix of the original system using the factor weights as follows: Taking variances of (3) gives

$$\mathbf{V} = \mathbf{ADA'} + \mathbf{V}_\varepsilon \tag{4}$$

where $\mathbf{A} = (\omega^*_{ij})$ is the kxm matrix of normalized factor weights, $\mathbf{D} = \text{diag}(V(P_1), \ldots V(P_m))$ is the diagonal matrix of variances of principal components and $\mathbf{V}_\varepsilon$ is the covariance matrix of the errors. Ignoring $\mathbf{V}_\varepsilon$ gives the approximation

$$\mathbf{V} \approx \mathbf{ADA'} \tag{5}$$

[3] The proportion of the total variation in **X** that is explained by the mth principal component is λ_m/k, where the eigenvalue λ_m of **X'X** corresponds to the mth principal component and the column labeling in **W** has been chosen so that $\lambda_1 > \lambda_2 > \ldots\ldots > \lambda_k$.

with an accuracy that is controlled by choosing more or less components to represent the system. This shows how the full kxk covariance matrix of asset or risk factor returns **V** is obtained from a just a few estimates of the variances of the principal components.

Note that **V** will be positive semi-definite, but it may not be strictly positive definite unless m = k.[4] Although **D** is positive definite because it is a diagonal matrix with positive elements, there is nothing to guarantee that **ADA'** will be positive definite when m < k. To see this write

$$\mathbf{x'ADA'x = y'Dy}$$

where **A'x = y**. Since **y** can be zero for some non-zero **x** , **x'ADA'x** will not be strictly positive for all non-zero **x**. It may be zero, and so **ADA'** is only positive semi-definite. When covariance matrices are based on (5) with m < k, they should be run through an eigenvalue check to ensure strict positive definiteness. However it is reasonable to expect that the approximation (5) will give a strictly positive definite covariance matrix if the representation (3) is made with a high degree of accuracy.

Advantages of the Orthogonal Method, Limitations of Direct Methods

The first advantage of using this type of orthogonal transformation to generate risk factor covariance matrices is clear. There is a very high degree of computational efficiency in calculating only m variances instead of the k(k+1)/2 variances and covariances of the original system. For example in a single yield curve with, say, 15 maturities, only the variances of the first 2 or 3 principal components need to be computed, instead of the 120 variances and covariances of the yields of 15 different maturities.[5]

Exponentially weighted moving averages of the squares and cross products of returns are a standard method for generating covariance matrices. But a limitation of this type of direct

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[4] A symmetric matrix **A** is positive definite if **x'Ax** > 0 for all non-zero **x**. If **w** is a vector of portfolio weights and **V** is the covariance matrix of asset returns, then the portfolio variance is **w'Vw**. So covariance matrices must always be positive definite, otherwise some portfolios may have non-positive variance.

[5] In highly correlated systems the first principal component, which represents a common trend in the variables, will explain a large part of the variation. In term structures and other ordered systems the second principal component represents a 'tilt' from shorter to longer maturities. Often the majority of the variation in a term structure may be explained when the system is represented by these two components alone. It is common for over 90% of the variation to be explained when a third component, the 'curvature' is added, so the considerable dimension reduction achieved by using 2 or 3 principal components results in little loss of accuracy. More details and examples may be found in Alexander (2000).

application of exponentially weighted moving averages is that the covariance matrix is only guaranteed to be positive semi-definite if the same smoothing constant is used for all the data.[6] That is, the reaction of volatility to market events and the persistence in volatility must be assumed to be the same in all the assets or risk factors that are represented in the covariance matrix. A major advantage of the orthogonal factor method described here is that it allows exponentially weighted moving average methods to be used without this unrealistic constraint. Each principal component exponentially weighted moving average variance would normally be applied with a different smoothing constant. So the degree of smoothing in the variance of any particular asset or risk factor that is calculated by the orthogonal method will depend on the factor weights in the principal component representation. Since the factor weights of an asset are determined by its correlation with other variables in the system, so also is the degree of smoothing. That is, the market reaction and volatility persistence of a given asset will not be the same at the other assets in the system, but instead it will be related to its correlation with the other assets.

The univariate generalised autoregressive conditional heteroscedasticity (GARCH) models that were introduced by Engle (1982) and Bollerslev (1986) have been very successful for short term volatility estimation and forecasting in financial markets. The mathematical foundation of GARCH models compares favourably with some of the alternatives used by financial practitioners, and this mathematical coherency makes GARCH models easy to adapt to new financial applications. There is also evidence that GARCH models generate more realistic long-term forecasts than exponentially weighted moving averages. This is because the GARCH volatility and correlation term structure forecasts will converge to the long-term average level, which may be imposed on the model, whereas the exponentially weighted moving average model forecasts average volatility to be same for all risk horizons (see Alexander, 1998). As for short-term volatility forecasts, statistical results are mixed (see for example Brailsford and Faff, 1996, Dimson and Marsh, 1990, Figlewski, 1994, Alexander and Leigh, 1997). This is not surprising since the whole area of statistical evaluation of volatility forecasts is fraught with difficulty. Another test of volatility forecasting models is in their hedging performance. There is much to be said for using the GARCH volatility

[6] See the RiskMetrics Technical Document, 3rd Edition, 1996 (www.riskmetrics.com)

framework for pricing and hedging options (see Duan 1995, 1996). Engle and Rosenberg (1995) provide an operational evaluation of GARCH models in option pricing and hedging, demonstrating a clear superiority to the Black-Scholes methods with an extensive empirical study. The beauty of the GARCH approach stems from the fact that a stochastic volatility is built into the model, which is closer to the real world, yet it does not introduce an additional source of uncertainty and therefore delta hedging is still sufficient.

Large covariance matrices that are based on GARCH models would, therefore, have clear advantages over those generated by exponentially or equally weighted moving averages. But previous research in this area has met with rather limited success. It is straightforward to generalize the univariate GARCH models to multivariate parameterizations, as in Engle and Kroner (1993). But the actual implementation of these models is extremely difficult. With so many parameters, the likelihood function becomes very flat, and so convergence problems are very common in the optimization routine. If the modeler also needs to 'nurse' the model for systems with only a few variables, there is little hope of a fully functional implementation of a direct multivariate GARCH model to work on large risk systems.

The idea of using factor models with GARCH is not new. Engle, Ng and Rothschild (1990) use the capital asset pricing model to show how the volatilities and correlations between individual equities can be generated from the univariate GARCH variance of the market risk factor. Their results have a straightforward extension to multi-factor models, but unless the factors are orthogonal a multi-variate GARCH model will be required, with all the associated problems.

A principal component representation is a multi-factor model. In fact the orthogonal GARCH model introduced in Alexander (2000) is a generalization of the factor GARCH model introduced by Engle, Ng and Rothschild (1990) to a multi-factor model with orthogonal factors. The orthogonal GARCH model allows kxk GARCH covariance matrices to be generated from just m univariate GARCH models. It may be that m, the number of principal components can be much less than k, the number of variables in the system - and quite often one would wish m to be less than k so that extraneous 'noise' is excluded from the data. But since only univariate GARCH models are used it does not really matter: there no dimensional restrictions as there are with the direct parameterizations of multivariate GARCH.

Of course, the principal components are only unconditionally uncorrelated, so a GARCH covariance matrix of principal components is not necessarily diagonal. However the assumption of zero conditional correlations has to be made, otherwise it misses the whole point of the model, which is to generate large GARCH covariance matrices from GARCH volatilities alone. The degree of accuracy that is lost by making this assumption is investigated by a thorough calibration of the model, comparing the variances and covariances produced with those from other models such as exponentially weighted moving averages or, for small systems, with multivariate GARCH. Care needs to be taken with the initial calibration, in terms of the number of components used and the time period used to estimate them, but once calibrated the orthogonal GARCH model may be run very quickly and efficiently on a daily basis.

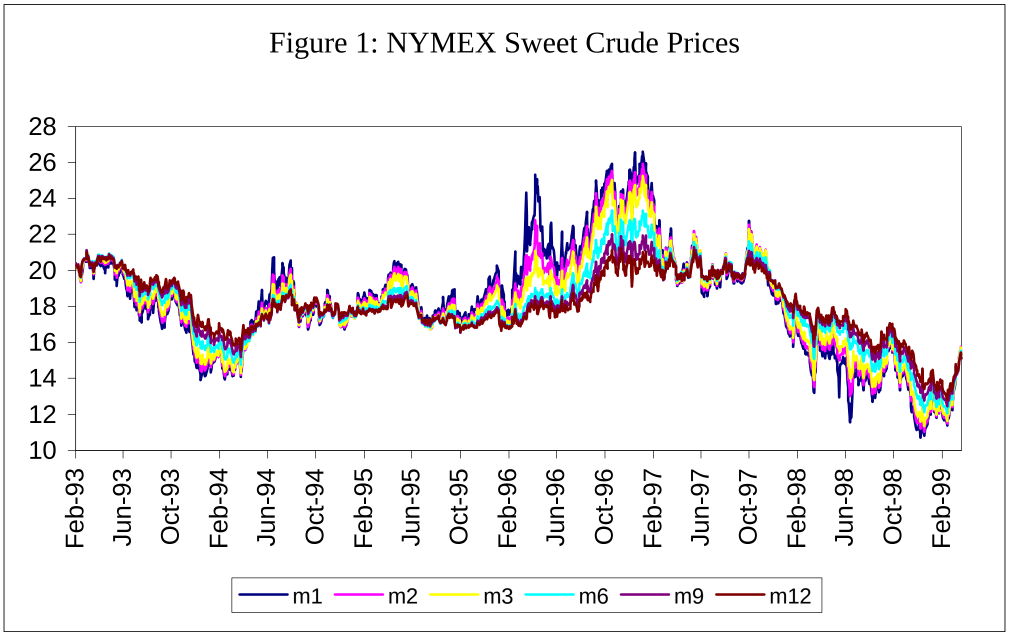
Another advantage is that the orthogonal method, applied with either GARCH or exponentially weighted moving average variances, allows one to generate estimates for volatilities and correlations of variables in the system even when data are sparse and unreliable, for example in illiquid markets. For example, the direct estimation of a time-varying variance of a 12-year bond may be difficult, but the orthogonal method allows its variance to be calculated from the variances of the key risk factors used in its representation.

Some Examples [7]

The orthogonal method is ideally suited to highly correlated ordered systems such as a term structure. The first example uses (a) exponentially weighted moving average variances and (b) GARCH(1,1) variances of just two principal components for the WTI crude oil futures from 1 month to 12 months, sampled daily between 4th February 1993 and 24th March 1999.

[7] The results in this section are reported in more detail, along with several other examples, in Alexander (2000).

The 1, 2, 3, 6, 9 and 12-month maturity futures prices are shown in figure 1[8] and the results of a principal component analysis on daily returns are given in table 1.



Of course the factor weights show that, as with any term structure, the interpretations of the first three principal components are the trend, tilt and curvature components respectively. In fact this particular system is so highly correlated that over 99% of its variation may be explained by just two principal components and the first principal component alone explains almost 96% of the variation over the period.

Table 1a: Eigenvalue Analysis

Component	Eigenvalue	Cumulative R^2
P1	11.51	0.9592
P2	0.397	0.9923
P3	0.069	0.9981

[8] See Alexander (1999) for a full discussion of these data and of correlations in energy markets in general. Many thanks to Enron for providing these data.

Table 1b: Factor Weights

	P1	P2	P3
1mth	0.89609	0.40495	0.18027
2mth	0.96522	0.24255	-0.063052
3mth	0.98275	0.15984	-0.085002
4mth	0.99252	0.087091	-0.080116
5mth	0.99676	0.026339	-0.065143
6mth	0.99783	-0.020895	-0.046369
7mth	0.99702	-0.062206	-0.023588
8mth	0.99451	-0.098582	0.000183
9mth	0.99061	-0.13183	0.020876
10mth	0.98567	-0.16123	0.040270
11mth	0.97699	-0.19269	0.064930
12mth	0.97241	-0.21399	0.075176

The GARCH(1,1) model defines the conditional variance at time t as

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \quad (6)$$

where $\omega > 0$, $\alpha, \beta \geq 0$. This simple GARCH model effectively captures volatility clustering and provides convergent term structure forecasts to the long-term average level of volatility $100\sqrt{250}\omega/(1-\alpha-\beta)$. The coefficient α measures the intensity of reaction of volatility to yesterday's unexpected market return ε^2_{t-1}, and the coefficient β measures the persistence in volatility.[9]

Applying (6) to the first two principal components of these data gives the parameter estimates reported in table 2. Note that the first component has low market reaction but high persistence, and the opposite is true for the second component. This reflects much of what is already known about the data from the principal component analysis: the system is very highly correlated indeed, in fact price decoupling occurs for only very short periods of time.

[9] Note that these are determined independently in the GARCH(1,1) model, subject only to the constraint that $\alpha+\beta<1$. In the exponentially weighted moving average model these parameters are not independent because they always sum to 1, and the constant is zero, so there is not long-term average level in the model and volatility terms structures are constant.

Now in the orthogonal model all the variation in correlations will come from the second or higher principal components because with only one component all variables are assumed to be perfectly correlated. The second component here has a 'spiky' volatility, and this gives rise to orthogonal GARCH correlations that also have only temporary deviations from normal levels. Thus the orthogonal GARCH model is capturing the true nature of crude oil futures markets. Unfortunately the exponentially or equally weighted moving average correlations that are in standard use have a substantial bias that arises from only very temporary price decoupling.

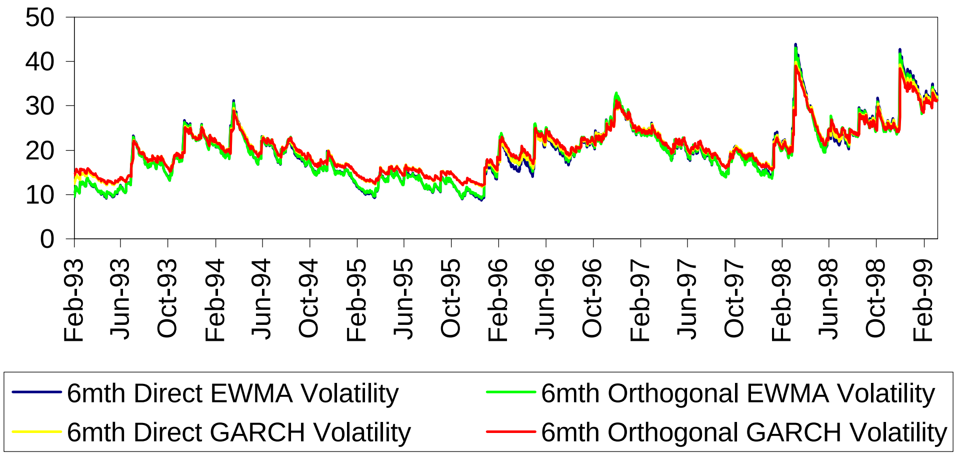
Table 2: GARCH(1,1) models of the first two principal components

	1st Principal Component		*2nd Principal Component*	
	Coefficient	*t-stat*	*Coefficient*	*t-stat*
constant	.650847E-02	.304468	.122938E-02	.066431
ω	.644458E-02	3.16614	.110818	7.34255
α	.037769	8.46392	.224810	9.64432
β	.957769	169.198	.665654	21.5793

Figure 2 shows how closely the volatilities that are obtained using the orthogonal method compare with those obtained by the direct application of (a) exponentially weighted moving averages and (b) GARCH(1,1) models.[10] Of course there is no space here to graph all 78 volatilities and correlations from the 12x12 covariance matrix. But interested readers may use the programs provided with Alexander (2000) to verify that all volatilities, not just those shown in figure 2, are very similar. But there is a difference in correlations. Not depending on whether a direct or an orthogonal approach is used, but depending on whether exponentially weighted moving averages or GARCH(1,1) models are used. As mentioned above, the GARCH correlations more accurately reflect the true nature of the data.

[10] There is no optimal method for choosing a value for the smoothing in these exponentially weighted moving averages. A value of 0.95 has been used throughout, but the reader may experiment with different values by adjusting the programs that are provided with Alexander (2000).

Figure 2: Direct and Orthogonal Volatilities



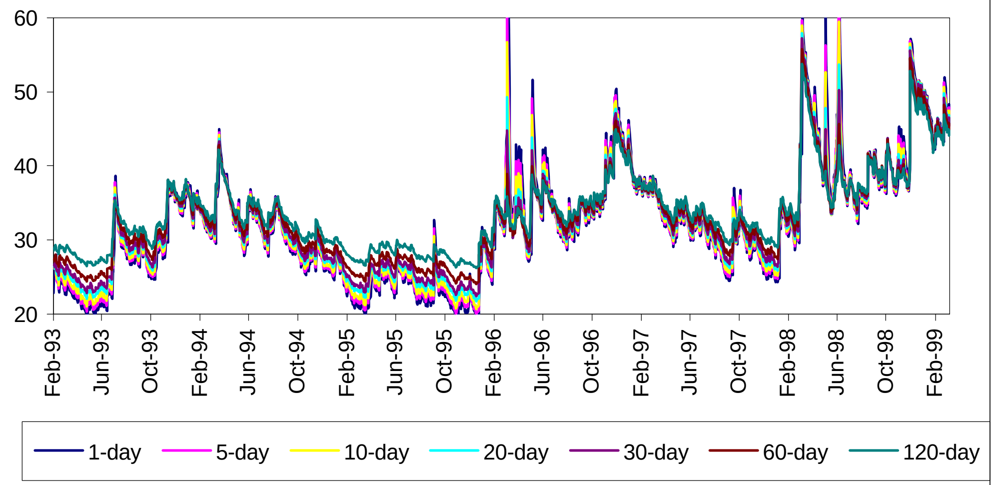
The main disadvantage of the direct method is that it requires estimating 78 volatilities and correlations, using (a) the same value of the smoothing constant for the exponentially weighted moving average model, or (b) a 12-dimensional multivariate GARCH model. Both of these approaches have substantial limitations as described above. However using the orthogonal method only two moving average variances, or two univariate GARCH(1,1) variances, of the trend and tilt principal components need to be generated. The entire 12x12 covariance matrix of the original system is simply a transformation of these two variances, as defined in (5) above, and it may be recovered in this way with negligible loss of precision.

Several good reasons to prefer GARCH models to exponentially weighted moving averages have already been mentioned, and one of the most attractive reasons is that only the GARCH approach will give convergent term structure forecasts. In the orthogonal GARCH model these forecasts, for volatilities and correlations of all maturities, are obtained from the simple transformations (5) where the diagonal matrix **D** contains the n-period GARCH(1,1) variance forecasts of the principal components.[11] Some of these are illustrated for volatilities of the 1-mth oil future in figure 3.

[11] The n-period GARCH(1,1) variance forecast is the sum of n forward variances for j = 1, ….n:

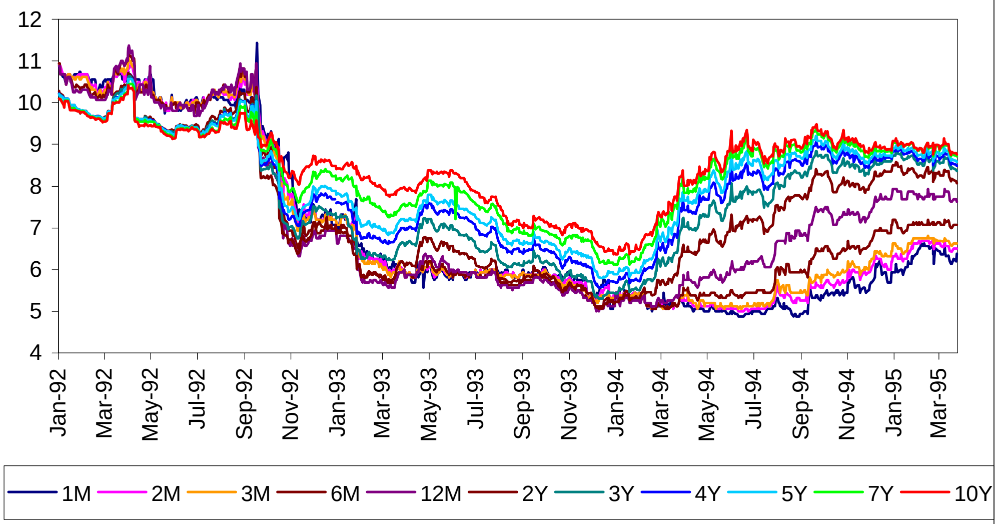
$$\hat{s}^2_{t+j} = \hat{w} + (\hat{a} + \hat{b})\hat{s}^2_{t+j-1}$$

Figure 3: Orthogonal GARCH Term Structure Volatility Forecasts for 1mth Crude Oil Futures



The next example applies the orthogonal GARCH(1,1) model to another term structure, but a rather difficult one. Daily zero coupon yield data in the UK with 11 different maturities between 1mth and 10 years from 1st Jan 1992 to 24th Mar 1995 are shown in figure 4. It is not an easy task to estimate univariate GARCH models on these data directly because the yields may remain relatively fixed for a number of days. Particularly on the more illiquid maturities, there is insufficient conditional heteroscedasticity for univariate GARCH models to converge well. So an 11-dimensional multivariate GARCH model is completely out of the question.

Figure 4: UK Zero-Coupon Yields



Again two principal components were used in the orthogonal GARCH, but the principal component analysis reported in table 3 shows that these two components only account for 72% of the total variation. Also the 10yr yield has a very low correlation with the rest of the system, as reflected by its factor weight on the 1st principal component, which is quite out of line with the rest of the factor weights on this component. So the fit of the orthogonal model could be improved if the 10yr bond were excluded from the system. Despite these difficulties the volatilities obtained using the orthogonal GARCH model are very similar to those obtained by direct estimation of exponentially weighted moving averages.[12]

<u>Table 3a: Eigenvalue Analysis</u>

Component	Eigenvalue	Cumulative R^2
P1	5.9284117	0.53894652
P2	1.9899323	0.71984946
P3	0.97903180	0.80885235

[12] The smoothing constant for all exponentially weighted moving averages was again set at 0.95.

Table 3b: Factor Weights

	P1	P2	P3
1mth	0.50916	0.60370	0.12757
2mth	0.63635	0.62136	-0.048183
3mth	0.68721	0.57266	-0.10112
6mth	0.67638	0.47617	-0.10112
12mth	0.83575	0.088099	-0.019350
2yr	0.88733	-0.21379	0.033486
3yr	0.87788	-0.30805	-0.033217
4yr	0.89648	-0.36430	0.054061
5yr	0.79420	-0.37981	0.14267
7yr	0.78346	-0.47448	0.069182
10yr	0.17250	-0.18508	-0.95497

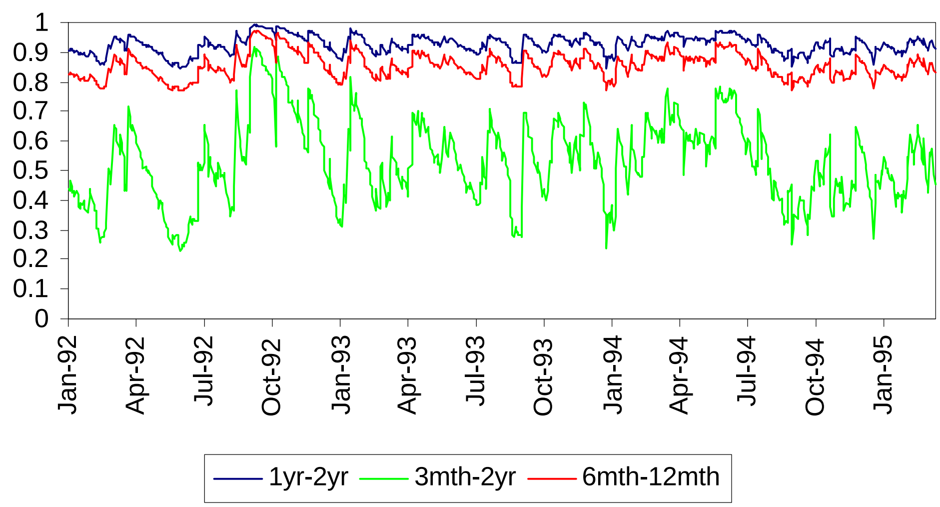
The GARCH (1,1) parameter estimates of the principal components are given in table 4. This time both components have fairly persistent volatilities, and both are less reactive than the volatility models reported in table 2. Combine this with the fact that almost 28% of the variation has been ascribed to 'noise' by using only these first two principal components, and it is not unsurprising that the orthogonal GARCH model produces quite stable correlation estimates: more stable than those obtained by direct application of exponentially weighted moving averages.

Table 4: GARCH(1,1) models of the first two principal components

	1st Principal Component		*2nd Principal Component*	
	Coefficient	*t-stat*	*Coefficient*	*t-stat*
constant	.769758E-02	.249734	.033682	1.09064
ω	.024124	4.50366	.046368	6.46634
α	.124735	6.46634	.061022	9.64432
β	.866025	135.440	.895787	50.8779

Figure 5 shows some of the orthogonal GARCH correlations for the UK zero coupon yields. So not only does the orthogonal method provide a way of estimating GARCH volatilities and volatility term structures that may be difficult to obtain by direct univariate GARCH estimation. They also give very sensible GARCH correlations, which would be very difficult indeed to estimate using direct multivariate GARCH. And all these are obtained from just two principal components, the key market risk factors that are representing the most important sources of information - all the rest of the variation is ascribed to 'noise' and is not included in the model.

Figure 5: Orthogonal GARCH Correlations
UK Zero-Coupon Yields



Generating a Large Covariance Matrix across All Risk Factor Categories

The risk factors - equity market indices, exchange rates, commodities, government bond and money market rates and so on - are first divided into reasonably highly correlated categories, according to geographic locations and instrument types. Principal component analysis is then used to extract the key risk factors from each sub-system and their diagonal covariance matrix is obtained using one of the methods outlined above. Then the factor weights from the principal component analysis are used to ‘splice’ together a large covariance matrix for the original system.

The method is explained for just two categories, then the generalization to any number of categories is straightforward. Suppose there are n variables in the first system, say it is European equity indices, and m variables in the second system, European exchange rates say. It is not the dimensions that matter. What does matter is that each system of risk factors is suitably co-dependent, so that it justifies the categorization as a separate and coherent sub-system. The first step is to find the principal components of each system, $\mathbf{P} = (P_1, \ldots P_r)$, and separately $\mathbf{Q} = (Q_1, \ldots Q_s)$ where r and s are number of principal components that are used in the representation of each system. Denote by **A** (nxr) and **B** (mxs) the normalized factor weights matrices obtained in the principal component analysis of the European equity and exchange rate systems respectively. Then the 'within factor' covariances, i.e. the covariance matrix for the equity system, and for the exchange rate system separately, are given by $\mathbf{AD}_1\mathbf{A}'$ and $\mathbf{B}\,\mathbf{D}_2\mathbf{B}'$ respectively. Here $\mathbf{D}_1$ and $\mathbf{D}_2$ are the diagonal matrices of the variances of the principal components of each system. The cross factor covariances are $\mathbf{ACB}'$ where **C** denotes the rxs matrix of covariances of principal components across the two systems, that is $\mathbf{C} = \{COV(P_i, Q_j)$. Then the full covariance matrix of the system of European equity and exchange rate risk factors is:

$$\begin{pmatrix} \mathbf{AD}_1\mathbf{A}' & \mathbf{ACB}' \\ (\mathbf{ACB}')' & \mathbf{BD}_2\mathbf{B}' \end{pmatrix}$$

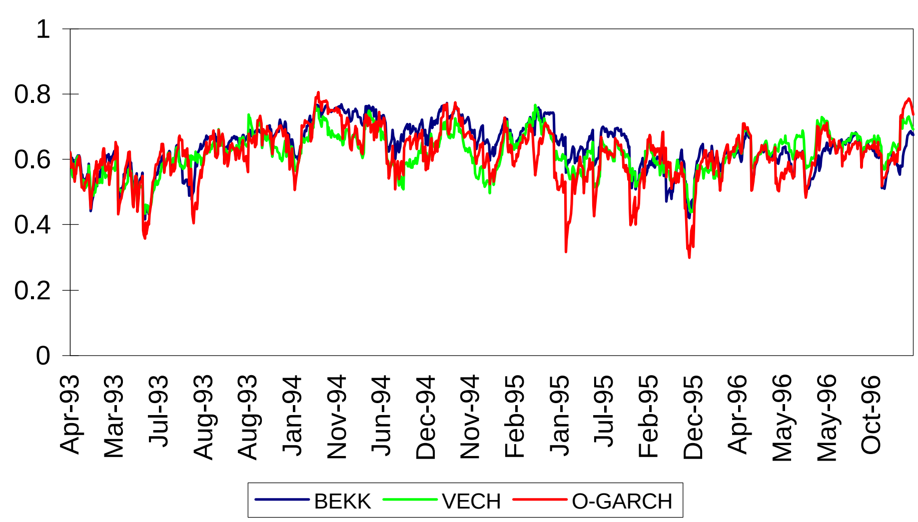
The within factor covariance matrices $\mathbf{AD}_1\mathbf{A}'$ and $\mathbf{BD}_2\mathbf{B}'$ will always be positive semi-definite. But it is not always possible to guarantee positive semi-definiteness of the full covariance matrix of the original system, unless the off diagonal blocks $\mathbf{ACB}'$ are set to zero. This is not necessarily a silly thing to do; in fact it may be quite sensible in the light of the huge instabilities often observed in cross-factor covariances.[13]

The method is illustrated using four European equity indices and their associated sterling foreign exchange rates. The graphs in figure 6 are based on daily return data from 1st April 1993 to 31st December 1996 on France (CAC40), Germany (DAX30), Holland (AEX), and the UK (FTSE100). In this 7-dimensional system of equity indices and foreign exchange rates

[13] For non-zero cross-factor covariances it is possible to estimate the covariance between principal components of different risk factor sub-systems using exponentially weighted moving averages or bivariate GARCH, giving the required estimate for **C**.

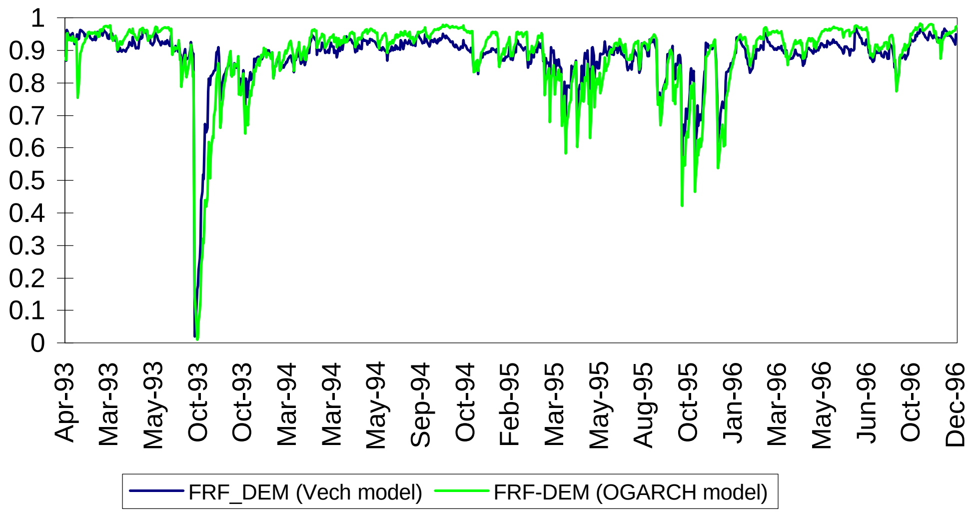
there are 28 volatilities and correlations in total. Figure 6 shows just two of the correlations from an orthogonal GARCH(1,1) model of the system compared with those obtained from two different direct parameterizations of a multivariate GARCH(1,1) model: (a) the Vech model, and (b) the BEKK model. These multivariate GARCH models were only possible to estimate on each sub-system separately. In fact convergence problems with the BEKK model for the foreign exchange system were encountered, so only the Vech model correlations, which have severe cross equation restrictions[14] are shown in figure 6b. These two graphs, which indicate a close similarity between the correlations, were chosen at random from the correlations for which multivariate GARCH models also produce results. Principal component analysis and orthogonal GARCH, Vech and BEKK model parameter estimates are not reported here due to lack of space, but full details of these models and the results are given in Alexander (2000).

Figure 6a: Equity Correlation Comparison (CAC-FTSE)



[14] In the Vech model all variances and covariances depend only on their own lag, and not the lags of other variances and covariances in the system.

Figure 6b: FX Correlation Comparison



The example has been mentioned here to illustrate the scope and flexibility of the approach to all types of asset class. It shows that it is possible to estimate these covariance matrices when direct methods are not possible, or require unrealistic restrictions. Provided the assets are first divided into reasonably highly correlated categories, principal component analysis provides a way to extract the important uncorrelated sources of information in each category. The covariance matrices for each category are generated from the variances of these key risk factors, and then a large covariance matrix that encompasses all categories is spliced together.

Summary and Conclusions

It is a common problem in risk management today that risk measures and pricing models are being applied to a very large set of scenarios based on movements in all possible risk factors. The dimensions are so large that the computations become extremely slow and cumbersome, so it is quite common that over-simplistic assumptions will be made. This paper presents an alternative. Large covariance matrices are generated from only a few key market risk factors that capture the most important uncorrelated sources of information in the data.

Orthogonal methods for generating covariance matrices have been applied to several different types of asset class: commodity futures prices, yield curves, equity indices and foreign exchange rates. Large covariance matrices that are based on the volatility of a few, uncorrelated key market risk factors alone are calculated, and are shown to have many advantages over the other methods in standard use:

- Positive semi-definiteness is assured, without severe constraints such as using the same model parameters for all assets and all markets;
- Stochastic volatility and correlation models such as multivariate GARCH, that have many advantages but that are usually difficult to apply in higher dimensions, may be employed;
- Correlations are more stable because the 'noise' in the system may be measured and, if required, be ignored;
- The method should conform to the standard regulatory requirements on historic data if at least one year of data is used in the principal component analysis;
- Periods of sparse trading on some (but not all) assets do not present a problem because their current volatilities and correlations will be inferred from their historic relationship with the other variables in the system.

To conclude, the method outlined in this paper is computationally efficient because it allows an enormous reduction in the dimension of the scenario set, whilst retaining a very high degree of accuracy in the risk measures and prices obtained. Since the key risk factors are uncorrelated, the method is computational efficient even when many factors are used to represent the system. In most cases only a few key factors are necessary, and any movements that are not captured by these factors are ascribed to 'noise' in the system. In fact, by cutting out this noise the model produces risk measures and prices that are more robust. Finally, it is quite straightforward to quantify how much risk is associated with each key factor. So risk managers will be able to focus their attention on the most important sources of risk.

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