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Statistical Properties of Forward Libor Rates

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Carol Alexander
and
Dmitri Lvov

ISMA Centre, University of Reading, UK

The University of Reading • ISMA Centre • Whiteknights • PO Box 242 • Reading RG6 6BA • UK
Tel: +44 (0)118 931 8239 • Fax: +44 (0)118 931 4741
Email: research@ismacentre.rdg.ac.uk • Web: www.ismacentre.rdg.ac.uk

Abstract

If historical forward rates are used to calibrate the lognormal forward rate model – as advocated by Hull and White (1999, 2000), Longstaff, Santa Clara and Schwartz (1999), Rebonato (1999a,b,c), Rebonato and Joshi (2001) and many others – a Libor yield curve needs to be fit to the available data on spot libor rates, forward rate agreements (FRAs) or futures, and swap rates. This paper compares the statistical properties of the time series of forward rates that are obtained using three different yield curve fitting techniques. Introduced by McCulloch (1975), Steely (1991) and Svensson (1994), each of the three techniques is well known for its application to the construction of bond yield curves.

Our work focuses on the eigenstructure of estimated forward rate correlation matrices. These are shown to be dominated by the semi-parametric or parametric form that is used in the yield curve model. The spectral decomposition of forward rate correlation – and covariance – matrices is considered in some detail, and in particular we test the common principal component hypothesis of Flury (1988), which has been applied to the lognormal forward rate model by Alexander (2003). We conclude that, if historical data are used to calibrate the lognormal forward rate model, it is best to use Svensson forward rate correlation matrices. However, the empirical evidence is strongly in favour of the common principal component hypothesis, where the three principal eigenvectors in all correlation matrices of the same dimension are identical. Hence we further conclude that a parsimonious parameterisation of forward rate correlations is possible, and this allows for direct calibration of forward rate correlations to market data, so historical data are not necessary.

Author Contacts:
Prof. Carol Alexander
Chair of Risk Management and Director of Research
ISMA Centre, School of Business,
University of Reading, Reading RG6 6AA
c.alexander@ismacentre.rdg.ac.uk

Dmitri Lvov
PhD Student, ISMA Centre, School of Business,
University of Reading, Reading RG6 6AA
d.lvov@ismacentre.rdg.ac.uk

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1.Introduction

A general multi-factor lognormal forward rate model, known as the Libor Market Model, was introduced, independently and virtually simultaneously, by Brace, Gatarek and Musiela (1997), and Miltersen, Sandermann and Sondermann (1997). This model is now widely used for pricing and hedging exotic interest rate derivative products.[1] Pricing is typically done by Monte Carlo simulation taking into account the volatility and correlation structure of forward libor rates. Since the forward rates' volatilities and correlations are not directly observable they need to be recovered using a model calibration procedure. Calibration of forward rate volatilities is comparatively straightforward, for example they could be calibrated so that model prices exactly match market prices of at the money caps and floors. However the calibration of forward rate correlations is a more challenging problem.

Both historical and market data have been used to calibrate forward rate correlations. Hull and White (1999, 2000), Longstaff, Santa Clara and Schwartz (1999), Rebonato (1999a,b,c), Rebonato and Joshi (2001) have used historical data on forward rates to forecast the eigenvectors of a reduced rank covariance matrix. De Jong, Dreissen and Pelsser (2001) have demonstrated the benefits of combining historical data with market data, and have shown that it is important to include measurement error in the historical data on forward rates if they alone are used to calibrate the model parameters. Other authors such as Schoenmakers and Coffey (2000), Brigo (2001), Brigo and Mercurio (2001), Rapisarda, Brigo and Mercurio (2002) and Alexander (2003) formulate semi-parametric or parametric forms for forward rate correlation matrices that allow calibration to market data alone, using implied swaption volatility surfaces.

When historical data on forward rates are used, there are many sources of model risk. Long dated forward rates are not directly observable. The market convention is to use liquid instruments such as futures, FRAs and swap rates to obtain forward libor rates by a bootstrap procedure (see Miron and Swannel, 1991). Although this convention is necessary, for example to obtain cap prices from the Black cap volatilities that are quoted in the market, it is not necessarily informative of the instantaneous forward rate correlations. The time series of forward libor rates obtained by a bootstrap method from daily observations on FRAs and swap rates, for example, often contain a significant amount of noise. This is due to the fact that bootstrap method over fits the yield curve and magnifies the errors in the original data. Therefore, if historical data on forward rates is used to calibrate forward

[1] A similar multi-factor lognormal swap rate model, introduced by Jamshidian (1997), is often used to price and hedge exotic swap products, such as Bermudan swaps.

rate correlation matrices, the forward rates should be obtained via a yield curve fitting model – but which model should be used?

Another decision that will affect historical forward rate correlations is the length of historical observation period. It should be short enough so that correlations represent current market conditions, but long enough so that correlations at the short end are not too unstable. Finally, which statistical model is to be used to forecast the correlations: an equally weighted moving average, and exponentially weighted moving average, or a GARCH model? In summary, when forward rate correlation forecasts are based on historical data they have a great deal of uncertainty. Model risk arises from three choices: the yield curve fitting model; the historical observation period; and the statistical model for correlation.

In view of this model risk, the alternative approach, where parametric or semi-parametric forms are imposed on the correlation matrices in order to use market data in the form of a swaption implied volatility surface, may be preferred. The current swaption implied volatility surface captures market expectations of future correlation, and forecast uncertainty can be ignored if one uses instruments that are linear in the correlation. An at-the-money swaption is approximately linear in the forward rate volatility which is itself approximately linear in the factors of the forward rate correlation matrix. Thus one should seek representations of the correlation matrices so that they can be calibrated to the swaption implied volatility surface. However one must be very careful how the factors of the correlation matrix are specified, via parametric or semi-parametric forms.

The aim of this paper is to use historical data on forward rates to investigate the validity of both approaches to calibrating correlation matrices in the lognormal forward rate model. We investigate the statistical properties of historical forward rates that are derived from different yield curve fitting techniques, during different historical observation periods. The main focus of this work will be the spectral decomposition of the forward rate covariance matrices, since this is a fundamental tool used in the reduced rank parametric or semi-parametric forms that are calibrated to the market. We shall place particular emphasis on the common principal component tests introduced by Flury (1988) because the use of common eigenvectors for all covariance matrices of the same dimension greatly simplifies the model calibration under either approach. We shall show that common eigenvectors should be applied, whether calibration is based on historical forward rate data, or whether it is based on the swaption implied volatility surface.

The remainder of this paper takes the following form: section two describes the lognormal forward rate model and examines different methods for including forward rate correlations in the calibration problem; section three describes the data used in this study; section four covers the three yield curve fitting techniques that are to be compared, and section five applies these methods to fit the libor curve on every day of the sample. This section also relates the goodness of fit of the alternative models to some statistical properties of the forward rates that are obtained from the fitted libor curves; section six examines the spectral decomposition of covariance or correlation matrices that are used in the lognormal forward rate model calibration. In this framework, we investigate the model arising from using historical data on semi-annual forward rates to estimate the eigenvectors in the spectral decomposition; section seven introduces the common principal component framework and provides strong evidence based on historical data that justifies the use of this framework to calibrate correlations to market data alone. Statistical tests for common eigenvectors are performed, and the results are shown to have important implications for the calibration of the lognormal forward rate model; section eight summarizes the results and draws some conclusions for future research.

2. The Lognormal Forward Rate Model

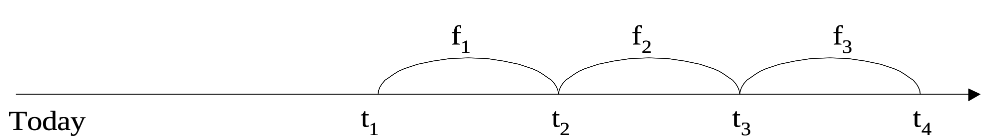
The lognormal forward rate model is constructed given a tenor structure composed of a finite set of dates

$$0 = t_o < t_1 < \ldots . < t_N < t_{N+1}$$

For ease of notation we assume that year-fractions between dates $\delta_i = t_{i+1} - t_i$, i = 0, …N, are all equal to a constant δ. The forward libor rate at time t for the accrual period $[t_i, t_{i+1}]$, $t < t_i$ is given by

$$f_i(t) = \frac{1}{\delta}\left(\frac{B_i(t)}{B_{i+1}(t)} - 1\right), \qquad i = 1, \ldots, N.$$

where $B_i(t)$ stands for the price at time t of a unit par value discount bond maturing at time t_i. Thus the forward rate $f_i(t)$ is a simply compounded rate spanning the period $[t_i, t_{i+1}]$, as observed at time t.



In the lognormal forward rate model the dynamics of the forward libor processes are determined by a stochastic differential equation of the form

$$\frac{df_i(t)}{f_i(t)} = \mu_i(t)dt + \sigma_i(t)dW_i(t)\ ,\ \ i = 1,....,N;\quad t < t_i\ ,$$

where $W_i(t)$, $i = 1, \dots , N$ are correlated Brownian motions, i.e. $E[dW_i dW_j] = \rho_{ij}(t)\, dt$, and the drifts m_i are measure dependent. In particular, choosing a strictly positive semi-martingale M as numeraire and using the following relationship between the bond prices and the forward libor rates

$$B_n(t_i) = \prod_{j=i}^{n-1} \frac{1}{1+df_j(t_i)}\ ,\qquad n > i,$$

it is possible to specify the drifts m_i of forward libor rates in the martingale measure associated with the numeraire M so that the deflated bond prices are martingales.

$$\frac{B_n(t)}{M(t)} = \mathrm{E}_M\left[\frac{B_n(T)}{M(T)}\middle|\Im_t\right],\ t \le T \le t_n\ ,$$

where the expectation is taken under the martingale measure associated with the numeraire M. This ensures that bond markets are arbitrage free.

As a particular choice of numeraire, Musiela and Rutkowski (1997) use the bond maturing at time t_i, $B_i(t)$, to derive the dynamics of forward libor rates under the associated "t_i forward" measure. Each forward rate $f_i(t)$ has its own 'natural' measure, which is the measure associated with numeraire B_{i+1}. Under it's natural measure each forward rate is a martingale and therefore has zero drift in its dynamics. As an alternative choice of numeraire, Jamshidian (1997) uses a discretely compounded analog of the money market account and derives the dynamics of forward libor rates under the associated "spot libor" measure.

Calibration of the model requires estimating the parameters of the instantaneous volatilities $\sigma_i(t)$ and instantaneous correlations $\rho_{ij}(t)$ [$i, j = 1, ..., N$]. De Jong, Dreissen and Plesser (2000) show that, to avoid over fitting the model, a time homogenous form for the volatility functions is preferable. A number of time-varying but deterministic functional forms for the instantaneous volatilities have been proposed in the literature. For example, Rebonato (1999a) introduced the homogenous form

$$\sigma_i(t) = \eta_i \, h(t)$$

where the time-varying parametric form $h(t)$ that is common to all volatilities is a hump-shaped function given by:

$$h(t) = [(a + b(T - t)) \, exp(-c(T - t)) + d]$$

Note that $h(t)$ – and therefore also $\sigma_i(t)$ – depends on maturity T and the parameters a, b, c, d which define a common volatility structure. The individual parameters η_i allow instantaneous volatilities to be adjusted upwards or downwards according to the prices in the market. For calibration to implied caplet volatilities, θ_i, a simple calibration objective proposed by Alexander (2003) is:

$$min \sum_i \omega_i \left(\theta_i \sqrt{t_i} - \eta_i \sqrt{\int_0^{t_i} h(t)^2 \, dt} \right)^2$$

where the weights ω_i are equal to gamma divided by the bid-ask spread.

Forward rate correlations are calibrated using swaption data. Given the fact that a swap is a nonlinear function of spanning forward rates, there is no exact analytical swaption pricing formula consistent with the libor market model. However, several approximations that lead to analytical formulae for swaption prices have been developed, which considerably facilitates the calibration procedure (see Brace (1996, 1997). In addition, Rebonato (1998) and Hull and White (1999) - have developed approximations to swaptions implied volatilities that are expressed as a function of spanning forward rates volatilities and correlations.

The calibrated libor market model is used to price exotic interest rate derivatives, typically by means of Monte Carlo simulation. In particular, the forward rates are simulated using their dynamics under the chosen measure and the derivative price is obtained as a discounted average payoff. Problems that involve simulating a large number of forward rates are computationally burdensome. Therefore it is desirable to reduce dimensions by reducing the rank of the correlation matrix. A review of rank reduction methods is given in Brigo (2001).

A method that is advocated by Rebonato (1999c), Hull and White (1999, 2000), Alexander (2003) and many others, uses an orthogonal transformation of the correlated Brownian motions in (3). The forward rate dynamics are express in terms of three uncorrelated stochastic processes that are common to all forward rates:

$$df_i(t)/\, f_i(t) = \mu_i(t)\, dt + \lambda_{i,1}(t)\, dZ_1 + \lambda_{i,2}(t)\, dZ_2 + \lambda_{i,3}(t)\, dZ_3$$

where dZ_1, dZ_2, dZ_3 are uncorrelated Brownian motions and:

$$\sigma_i(t)\, dW_i = \lambda_{i,1}(t)\, dZ_1 + \lambda_{i,2}(t)\, dZ_2 + \lambda_{i,3}(t)\, dZ_3$$

From this it follows that:

$$\sigma_i(t) = \sqrt{[\, \lambda_{i,1}(t)^2 + \lambda_{i,2}(t)^2 + \lambda_{i,3}(t)^2\,]} \quad (4)$$

and

$$\rho_{ij}(t) = [\lambda_{i,1}(t)\, \lambda_{j,1}(t) + \lambda_{i,2}(t)\, \lambda_{j,2}(t) + \lambda_{i,3}(t)\, \lambda_{j,3}(t)] \,/\, \sigma_i(t)\, \sigma_j(t) \quad (5)$$

Thus the forward rate volatilities and correlations are completely determined by three volatility 'components' for each forward rate, which are $\lambda_{i,1}(t)$, $\lambda_{i,2}(t)$ and $\lambda_{i,3}(t)$ for the *ith* forward rate. Given these, Hull and White (2000) show how the volatility components are obtained from the spectral decomposition of the covariance matrix of the m forward rates f_n , ….f_{n+m-1} underlying the n into m period swap. In the following we shall denote this spectral decomposition by:

$$\mathbf{V}_{n,m}(t) = \mathbf{A}_{n,m}\, \mathbf{\Lambda}_{n,m}(t)\, \mathbf{A}'_{n,m} \quad (6)$$

where $\mathbf{\Lambda}_{n,m}(t)$ is the (time-varying) diagonal matrix of eigenvalues and $\mathbf{A}_{n,m}$ is the (constant) $m \times m$ matrix of eigenvectors of the forward rate covariance matrix $\mathbf{V}_{n,m}(t)$. A number of researchers, notably Rebonato (1999a), Rebonato and Joshi (2001) Hull and White (1999, 2000) and Logstaff, Santa-Clara and Schwartz (1999) use volatility components derived from eigenvectors that are estimated from covariance matrices of historical forward rates, and time-varying eigenvalues with a functional form that is calibrated to the market.

Note that the volatility components may equally well be obtained from the spectral decomposition of the forward rate correlation matrix, given by:

$$\mathbf{\Sigma}_{n,m}(t) = \mathbf{D}_{n,m}(t)^{-1}\mathbf{V}_{n,m}(t)\mathbf{D}_{n,m}(t)^{-1}$$

where $\mathbf{D}_{n,m}(t)$ is the diagonal matrix of the instantaneous volatilities. To see this, denote by $\sigma_i(t)$ the instantaneous *ith* annual forward rate volatility, and by:

$$\mathbf{\Sigma}_{n,m}(t) = \mathbf{B}_{n,m}\ \mathbf{\Theta}_{n,m}(t)\ \mathbf{B'}_{n,m} \tag{7}$$

the spectral decomposition of the correlation matrix, where $\mathbf{\Theta}_{n,m}(t)$ is the diagonal matrix of eigenvalues and $\mathbf{B}_{n,m}$ is the $m \times m$ matrix of eigenvectors of the forward rate correlation matrix $\mathbf{\Sigma}_{n,m}(t)$. Again, the eigenvalues are assumed time-varying whereas the eigenvectors are assumed constant. To derive the three volatility components from (7), denote by $\theta_1(t)$, $\theta_2(t)$, $\theta_3(t)$, the three largest eigenvalues of $\mathbf{\Sigma}_{n,m}(t)$ and by $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, $\boldsymbol{\beta}_3$ their corresponding eigenvectors. Set $M(t) = \sigma_i(t)/\sqrt{V_i}\,(t)$ where

$$V_i(t) = \beta_{i,1}{}^2\theta_1(t) + \ + \beta_{i,2}{}^2\theta_2(t) + \ \beta_{i,3}{}^2\theta_3(t)$$

Then setting $\lambda_{i,j}(t) = M(t)\ \beta_{i,j}\sqrt{\ }\theta_j(t)$ for $j = 1, 2, 3$ satisfies (4) and (5), as required.

In the following we shall examine the spectral decompositions of both covariance and correlation matrices of historical forward rates, aiming to identify a structure that could be translated into the lognormal forward rate model used to price interest rate derivative products. Using a statistical analysis on time series of UK discrete forward libor rates that are constructed using alternative methods, we comment on the impact of the choice of yield curve fitting method on the historical forward rates, and the pitfalls from using historical forward rates to calibrate the eigenvectors of forward correlation or covariance matrices. We shall also make recommendations about alternative parametric specifications of forward rate correlation and covariance matrices, when these are to be calibrated to market data on swaption implied volatilities.

3. Data

The libor yield curve is typically constructed using a combination of spot libor rates, forward rate agreements (FRAs), futures and swap rates. As short term instruments one would typically use the libor rates covering the period of 12 months. In particular, 1- to 12-month libor rates have been used to estimate the short end of the UK libor yield curve. Since libor rates are not available for maturities longer than 12 months some other instruments, with the same credit risk and liquidity as the libor rates, need to be used for building the longer end of the libor curve. There are several instruments that can provide information about the libor rates in the medium and long range; these are interest rate futures, FRAs and swaps.

Interest rate futures and FRAs typically cover a period of two years. Both kinds of contracts enable an investor to lock in a rate of return between two dates within the 2-year period and therefore provide information about the corresponding forward libor rates. The main difference between these contracts is related to the fact that the settlement on a FRA occurs at the contract maturity, while futures position are marked to market on a daily basis resulting in a stream of cash flows between the parties along the whole life of the contract. Since these cash flows are a function of the prevailing level of the libor rate, futures provide biased information about forward rates. The bias result from the fact that the short party experiences a cash outflow in the event of the interest rate increase and a cash inflow in the event of the fall in the interest rates. Therefore, the short party will systematically seek financing when interest rate rise and invest the cash flows from marking to market as interest rates fall. The opposite is true about the long party on the futures contract. Thus the equilibrium futures price will be biased upwards in order to compensate the short party. The importance of the bias depends on the length of the contract and the libor rate volatility expectations. Strictly speaking, one would need to use an interest derivative pricing model to value a futures contract and from this perspective the use of futures as an input to a yield curve fitting model should be avoided.

Unlike futures, FRAs could be seen as instruments providing unbiased information about the forward libor rates. However, the liquidity of FRAs is typically lower than that of the libor rates and futures. Therefore, FRA quotes may be stale and fail to reflect the changes in the yield curve. Consequently, neither futures nor FRAs are ideal instruments to use for libor yield curve estimation. Taking into account the above considerations and the fact that some yield curve fitting techniques allow one to discriminate between more and less liquid instruments, the choice was made to use FRAs in the yield curve fitting procedure. In particular, to estimate the UK libor yield curve, some or all of the following contracts have been used: 3-month FRAs starting in 2, 3, …, 9 months, 6-month FRAs starting in 1, 2, …, 6, 12 and 18 months, 9-month FRA starting in 1 month and 12-month FRAs starting in 2, 3, 6, 9 and 12 months.

The information about the long end of the libor yield curve can be obtained from swaps. Swaps are liquid instruments that cover the period of up to 30 years. For the purposes of yield curve building, a swap can be regarded as an exchange of a fixed coupon bond for a floating bond paying prevailing libor rates. At the initiation the value of a swap is zero and the value of a floater is one, which means that the value of the fixed rate bond paying the swap rate must be one at the initiation of the swap. In other words, the swap rate can be regarded as a coupon rate on a par-coupon bond of a corresponding maturity. Swap contracts maturing in 2, 3, .., 10 years have been used as an input to build the UK libor yield curve.

The daily quotes on the above securities have been obtained from Reuters for the period between 15/10/1999 and 25/06/2002.

4. Fitting the Libor Curve

In this section we review different techniques that can be used to obtain the forward libor rates and, more generally, the yield curve for the libor credit rating. The market convention is to use a bootstrap method on a combination of spot libor rates, FRAs and swap rates, and the procedure is described in detail in Miron and Swannel (1991). From the prospective of statistical analysis however, there are two problems with this approach. First, different combinations of securities can be used to bootstrap the yield curve. In particular, there are many combinations of spot libor rates, FRAs and futures that can be used to construct the yield curve for maturities of up to 2 years. Of course, if the data were perfect and markets are arbitrage free, the choice of instruments should not matter. However, given the inevitable noise in historical data, the choice of securities does have an impact on the shape of short end of the libor curve. This relates to the second problem with the bootstrap method, which is a tendency to over fit the yield curve. Since the securities used in the bootstrap procedure are fitted exactly, any noise in the original data is translated into the bootstrapped forward rates. This often results in spikes in the forward curve, especially and longer maturities, which upsets the correlation structure of historical forward rates obtained via the bootstrap technique. Therefore we argue that alternative methodology should be used to construct time series of forward libor rates from historic observations on spot libor rates, FRAs and swap rates.

For this study we have used three techniques to estimate the libor yield curve, introduced by McCulloch (1975), Steely (1991) and Svensson (1994). The first two techniques essentially model the discount curve as a weighted average of basis spline functions. These techniques make no explicit reference to the stylised shape of a yield curve. The third model, originally introduced by Nelson and Siegel (1987) and later extended by Svensson (1994), imposes a functional form on the instantaneous forward rate curve that captures its typical 'humped' shape. All of the above techniques are widely used to build yield curves based on the data from bond markets. Their application to the libor market is not new. For example, De Jong, Driessen and Plesser (2001) use exponential spline functions to fit a libor curves to US money market and swap rates and the more general application of yield curve fitting techniques to the libor curve is discussed in James and Webber (2000).

The calibration of yield curve models is based on a minimization algorithm. In the case of the techniques introduced by McCulloch and Steely, the model parameters are the weights on basis functions that form the discount curve. Thus the discount factors are linear in the parameters and the calibration can be done by a simple regression where the objective is to minimize the weighted squared differences between the model prices of securities and their market prices[2]. Given the heteroscedastic nature of errors in yields when the fit is done to prices, price error weights were specified as inversely related to the securities' time to maturity.

In the Svensson model the discount factors are non-linear in the parameters and the model calibration requires a non-linear least squares algorithm. In this study we have used the Levenberg-Marquardt algorithm (REF) with the objective to minimize the sum of the squared differences between the model and the market rates.

4. Results

These three yield curve models were applied to 703 daily observation on UK libor rates, FRAs and swap rates between 15/10/1999 and 25/06/2002. For each day all the data available were used to build the discount curve and the associated spot curve, which were then used to obtain model libor rates, FRAs and swap rates. The model rates were then compared to the market data to obtain the root mean squares error (RMSE) expressed in basis points.

[2] In the case of the instruments used to fit the libor yield curve we use theoretical prices, as described in James and Webber (2000)

The average RMSE errors over the entire sample period were: 2.68 (McCulloch); 2.59 (B-Spline); and 2.55 (Svensson); so there is some evidence that the Svensson model provideda slightly better fit , than the spline methods.[3] The quality of fit of spline methods could potentially be improved by adding more knot points. This however, would deteriorate the smoothness of the spot curve which is undesirable. The time series plots of this goodness of fit statistic revealed some interesting characteristics. Figure 1 shows the RMSE for every day during the sample. The two spline procedures produced virtually identical RMSE on every day during the sample. The lowest fitting errors occurred during the year 2000 and greater fitting errors were experienced during the latter half of 2001. The Svensson model produced errors with broadly similar characteristics, except that errors were even smaller during year 2000 and larger and more variable during the latter part of 2001.

In view of this time variation it is difficult to draw general conclusions about the relative performance of in-sample fit from the three techniques. However, further information can be obtained from the historical volatilities and correlations of daily changes in log forward libor rates, as these differ considerably depending on the choice of the technique used to fit the yield curve. In Figure 2 we show the annual volatility estimates of the one- five- and nine-year discrete semi-annual forward rates that are obtained from the spot rates in each of the fitted yield curves. Volatilities were estimated as exponentially weighted moving averages of squared daily differences in the log of the forward rate, with a smoothing constant of 0.92.[4]

Forward rates of all maturities and from all three models were not exceptionally volatile during the year 2000, and this corresponds to all models fitting relatively well during that period. Since the short maturity forward rates are observable as FRAs, the forward rates obtained from the fitted spot rates should be virtually identical, and this is verified by the volatilities shown in Figure 2(a). Short rates became more variable during the summer of 2001, with volatilities exceeding 20% most of the time from June 2001 until April 2002. This period is associated with all three models performing less well, as shown in Figure 1.

[3] Similar characteristics are evident in the chi-squared statistics for goodness of fit: Over the whole period the average Chi-squared statistics were: 62.93 (McCulloch); 59.83 (B-spline); and 57.54 (Svensson)

[4] We use the difference in the log forward rate, rather than the difference in the rate itself, since this is consistent with the lognormal forward rate model. For comparative purposes, the same smoothing constant is used for all series, and the value 0.92 was chosen as being generally closest to the corresponding coefficients estimated from normal GARCH(1,1) models over the sample period.

As we move up the yield curve, the discrete forward rates (which are now unobservable in the market) have different volatility characteristics depending on the yield curve model used. From Figure 2(b) we see that medium maturity (5-year) forward rates are relatively volatile at the very beginning of the sample period. At this time the in-sample fit errors were as high as they were during the summer of 2001 (see Figure 1), but now the lack of fit is associated with increased variability in the middle of the curve, rather than the short end. Both of the spline based forward rate volatilities are greater than the Svensson forward rate volatilities at the beginning of the sample, and notably the fit errors from both of the spline models were greater than those from the Svensson models at this time.

The longest maturity forward rates have volatilities shown in Figure 2(c) and here we see the most difference between the three models. The Svensson model produced highly volatile long rates at the end of 2001, when the goodness of fit statistics implied that the Svensson model was giving the worst in-sample fit. A comparison of Figures 2(b) and 2(c) also shows that the two spline methods, which have virtually identical goodness of fit statistics through the entire sample, are giving forward rates with quite different volatility characteristics. Notably both medium and long maturity forward rates are less volatile when the yield curve is fitted by the B-spline method, and more volatile when fitted by the McCulloch method, except at the very beginning of the sample period. This can be related to the fact that the goodness-of-fit statistics generally indicate a marginally better from the B-spline, rather than McCulloch method.

The short maturity forward rates tend to be less volatile than long maturity rates in all three fitted curves. To a large extent this is due to the presence of a measurement errors in the spot rates that we obtained from the yield curve models. The measurement errors are greater for longer maturity rates, where one observes relatively few data points, and these measurement errors are compounded when spot rates are translated into discrete forward rates. The net result is an increasing volatility of forward rates with respect to maturity. Figure 3 plots some average annual volatilities of the Svensson semi-annual forward rates with respect to maturity, where each line in the figure is observed at the end of every quarter in the sample. Although we have shown the Svensson forward rates here, the generally upwards sloping forward rate term structure volatilities are shared by the forward rates obtained from other yield curve fitting methods. Whilst forward rate volatilities clearly change over time, the empirical evidence reveals little information about a possible functional form for the volatilities of a term structure of forward rates.

To summarize so far, we have related the volatility of the forward rates to the goodness of fit of a model used to build the term structure. The clear relationship between measurement errors and volatilities shown in figure 3, and already noted by De Jong, Driessen and Plesser (2001), is also evident in the in-sample fit statistics shown in figures 1 and 2. These degenerate at the same time that the yield curve model produces more volatile forward rates, compared to the other models. All three models give long maturity forward rates that have volatilities that are generally greater than short maturity forward rate volatilities. The volatility of forward rates generally increases with the maturity of the forward rate, but there is no obvious functional form (e.g., linear or 'humped shaped') that captures the empirical behaviour of volatilities of a term structure of forward rates.

The correlation structure of the semi-annual forward rates obtained from the Svensson model is related to the function form imposed on the instantaneous forward rates. The combination of negative exponentials given in (9) imposes a 'hump' shape of the forward rate curve, and consequently correlations between short and long maturity forward rates can be higher than correlations between medium and long maturity forward rates. Forward rate correlations do not decrease monotonically with the maturity spread, nevertheless the model (9) does impose a smooth pattern in the forward rate correlations. The empirical correlations between semi-annual Svensson forward rates of different maturities are shown in Figure 4(a), based on equal weighting over the entire sample period. As expected from any term structure, the correlation between forward rates decreases with as the maturity spread widens.

The empirical correlations have a less structured pattern when the yield curve is fitted by either of the spline models, and these are shown in Figures 4(b) and 4(c). Here, larger correlations are obtained between forward rates coinciding with the various knot points chosen. In order to maximize the goodness of fit with these models, several knots were placed at both short and long maturities, so both models give higher correlations between the short and the long maturity forward rates than is obtained with the Svensson model. However the medium – long maturity correlations are much lower when estimated using spline fitting to the yield curve, in fact some are even negative![5]

Therefore, although the quality of fit has been related to the volatility of forward rates, no such comparison may be drawn from their correlation structure. In fact forward rate correlations inherit a structure from the form of the model used: the Svensson forward rate correlations are the most well-behaved and their structure is related to the exponential functional form of the model; in the spline methods, correlations are related to the choice of basis functions and knot points.

If historical forward rate correlations are used to calibrate the lognormal forward rate model, it is important to consider how the choice of yield curve model will affect these correlations, and hence also the lognormal forward rate model prices. Although it has been difficult to distinguish between the yield curve models using in-sample goodness of fit diagnostics, an alternative criteria that should be considered is whether the yield curve model gives forward rates that have an intuitive correlation structure.

6. Principal Component Analysis

In this section we take a closer look at the covariances and correlations of the historical series of semi-annual forward libor rates. In particular, we show that the choice of sample period has a substantial effect on the estimates of the eigenvectors of the covariance matrix given by (6), but it has much less of an effect on the eigenvectors of the correlation matrix given by (7).

In section two we linked the spectral properties of forward rates to a reduced rank calibration of the lognormal forward rate model. In fact a reduced rank calibration may be based on either:

(i) the eigenvectors $\mathbf{A}_{n,m}$ of all forward rate covariance matrices $\mathbf{V}_{n,m}(t)$; or

(ii) the eigenvectors $\mathbf{B}_{n,m}$ of all forward rate correlation matrices $\mathbf{\Sigma}_{n,m}(t)$

of the m semi-annual forward rates f_n , ….f_{n+m-1} underlying the n into m period swap. Whichever eigenvector matrix is chosen, it is assume constant, whilst the corresponding eigenvalues are allowed to be time-varying.

Do our historical data give any indication whether (i) or (ii) is the better approach? Consider the problem of calibrating the lognormal forward rate model to UK forward libor rates on 1st April 2002. We have historical data on daily forward rates, going back one year. In order to produce robust prices, we should not use too short an historical data period to forecast the correlation matrices on which we apply PCA – but on the other hand, if the data period is too long, the model prices will not accurate reflect current and future market conditions.

[5] A time series analysis of this will show that negative correlations only occurred during the period following the terrorist attacks on September 11th 2001.

Figure 5 depicts two semi-annual forward rate correlation matrices: Matrix (a) is estimated using three months of daily data from 1st January to 31st March 2002; and matrix (b) is estimated using six months of daily data from 1st October 2001 to 31st March 2002.[6] Figure 6(a) shows the first three eigenvectors of the correlation matrix, as a function of maturity, when the matrix is estimated over different sample periods.[7] Whilst considerable differences between these matrices are apparent in figure 5, it is interesting to note that the first two eigenvectors of the correlation matrices, shown in figure 6 are very similar indeed. Therefore the observed differences in the correlation matrices are mainly due to changes in the third and higher principal components over the two sample periods. On the other hand, figure 6(b) shows the first three eigenvectors of the covariance matrix, estimated over the same two sample periods. Here it is evident that all three eigenvectors are sensitive to the sample data used to estimate them. The reason for this is that, as shown in figure 4, the forward rate volatilities change considerably between the two sample periods.

This example illustrates that the choice of historical sample period is likely to have a significant effect on the eigenvalues of the covariance matrix, due to the fact that forward rate volatilities change considerably over time. However, the sample period has much less effect on the eigenvalues of the correlation matrix. Therefore, if historical data are used at all, the data should be used to estimate the eigenvectors of the correlation matrix, as in (i) above, and not those of the covariance matrix, as in (ii).

Another approach to lognormal forward rate model calibration that has been advocated by Alexander (2003) is to parameterize the first three eigenvectors and calibrate them to option market data alone, thus avoiding the use of historical data entirely. The problem with this approach is that without constraining the parameterization in some way there will be too many parameters to fit to the available market data, particularly at the long end, where data are sparse and unreliable: it will be very difficult to calibrate the parameters of the eigenvectors when either n or m is large. However, the next section introduces some natural constraints that could be imposed on the parameters of eigenvectors and, if appropriate, this will greatly facilitate their calibration to market data alone.

[6] Forward rates were extracted using the Svensson yield curve model. Figure 5(a) indicates that the correlation between adjacent forward rates of long maturities was larger than the correlation between adjacent forward rates of shorter maturities; however in the matrix shown in Figure 5(b), a pattern where adjacent correlations do not increase with maturity is displayed, as indeed it was during most of 2001. Correlations are also generally higher in matrix 5(a) than in matrix 5(b), and in particular the correlations between mid maturity and long forward rates is very low in matrix 5(b), as it was during most of 2001.

[7] Together the first three eigenvectors explain 99.5% of the variation in the in the three month period for figure 5(a) and 98.2% of the variation in the six month period for figure 5(b).

7. Common Principal Component Analysis

The common eigenvector forms of (6) and (7) are now derived. Under the assumption that all covariance matrices of the same dimension have the same eigenvectors the spectral decomposition (6) becomes

$$\mathbf{V}_{n,m}(t) = \mathbf{A}_m \mathbf{\Lambda}_{n,m}(t) \mathbf{A}'_m \tag{6'}$$

where $\mathbf{A}_m$ is a matrix of eigenvectors of $\mathbf{V}_{n,m}(t)$, which is assumed both constant and the same for all n. Similarly, the common eigenvector form of (7) is

$$\mathbf{\Sigma}_{n,m}(t) = \mathbf{B}_m \mathbf{\Theta}_{n,m}(t) \mathbf{B}'_m \tag{7'}$$

and $\mathbf{B}_m$ is a matrix of eigenvectors of $\mathbf{\Sigma}_{n,m}(t)$, which is assumed both constant and the same for all n.

To test whether either (or both) of these parameterizations are appropriate, we use our historical data on forward rates, employing the *common principal component* (CPC) tests introduced by Flury (1988). Consider a set of p covariance matrices,[8] all of the same dimension m, with spectral decompositions $\mathbf{V}_i = \mathbf{M}_i \mathbf{\Lambda}_i \mathbf{M}_i'$ [$i = 1, 2, \ldots, p$]. The null hypothesis of CPC is:

$$H_0: \mathbf{M}_1 = \mathbf{M}_2 = \ldots\ldots = \mathbf{M}_p = \mathbf{M}$$

The test statistic for H_0 is based on sample covariance matrices $\mathbf{S}_i \sim W_m(n_i, \mathbf{V}_i/n_i)$ where n_i is the number of degrees of freedom in the sample used to estimate $\mathbf{S}_i$ and W_m denotes the Wishart distribution. We first compute the average sample covariance matrix

$$\mathbf{S} = \sum_{i=1}^{p} n_i \mathbf{S}_i \Big/ \sum_{i=1}^{p} n_i$$

[8] The analysis here does, of course, apply equally well to correlation matrices. One has only to observe that if the sample data are standardized to have unit standard deviation the standardized sample covariance matrix is equal to the correlation matrix.

Then maximum likelihood estimates of the covariance matrices under the null hypothesis are obtained, as outlined in Flury (1988), and these matrices are denoted $\hat{\mathbf{V}}_i = \hat{\mathbf{M}}\hat{\mathbf{\Lambda}}_i\hat{\mathbf{M}}$ for $i = 1, 2, \dots p$. The likelihood ratio test statistic for H_0 is then

$$X^2 = \sum_{i=1}^{p} n_i (ln\ det\ \hat{\mathbf{V}}_i - ln\ det\ \mathbf{S}) \qquad (8)$$

and under H_0 the asymptotic distribution of X^2 is $\chi^2_{m(p-1)}$. Intuitively each constrained matrix $\hat{\mathbf{V}}_i$ is compared to the average sample covariance matrix, and the greater the difference between them, the less likely it is that the constraints are appropriate. Flury (1988, page 129) also proves that the test statistic (8) can also be used to test a partial hypothesis, that only the first three components are each covariance matrix are equal, and this is the hypothesis of interest in this case.

We now apply Flury's CPC test to covariance matrices estimated from historical forward rate data, to test whether the assumption (6') is appropriate, that is, whether all semi-annual forward rate covariance matrices of the same dimension (greater than 3) can be assumed to share the same eigenvectors. We then repeat the analysis for the associated correlation matrices, to investigate whether the assumption (7') is appropriate. Table 1 reports the results. Not reported are the results of the CPC hypothesis tests for different sub-periods during the sample since no significant time-variability in the CPC test results were found. Our results indicate that:

- The evidence for common principal components in all correlation matrices of the same dimension is very strong indeed when forward rates are obtained using the Svensson model;
- The evidence for common principal components is rather weak when forward rates are obtained using either of the spline methods; indeed many CPC tests on the correlation matrices are rejected, often at 1%
- The evidence for common principal components in all covariance matrices of the same dimension is also strong when forward rates are obtained using the Svensson model;

Figure 7 shows the first three common eigenvectors of all correlation and covariance matrices of Svensson semi-annual forward rates underlying *m* into-8-year swaptions for *m* = 0.5, 1, 1.5 and 2 (so the CPC test in this case was on four matrices). In both figure 7(a) and 7(b) one would have no difficulty to parameterise these eigenvectors parsimoniously. The effect of larger volatility in the longer maturity forward rates is clear in the first eigenvector, which slopes upwards for the covariance matrices in figure 7(a) but not for the correlation matrices in figure 7(b).

Since we have found significant differences in volatilities of forward rates of different maturities, it is clearly not appropriate to place the 'trend-tilt-curvature' interpretation upon the common eigenvectors of the covariance matrix. However, it seems quite appropriate to use this interpretation for the eigenvectors of the correlation matrix. Then, using the fact that they are orthogonal leads to a very parsimonious parameterization with only four parameters for the first three common eigenvectors of all the correlation matrices of the same dimension. The number of correlation parameters to be calibrated is therefore $4N$, where N is the number of different tenors of swaptions in the market. On the other hand, if common eigenvectors of the covariance matrices were used directly in the parameterization, we should need at least one more parameter for every maturity, that is at least $5N$ parameters to be calibrated to market data, in addition to the forward rate volatilities.

We end this section by adding further weight to our case for using a parameterization of common eigenvectors of the forward rate correlation matrices, calibrated to market data, rather than historical data. Figure 8 compares the common eigenvectors of the correlation matrices of 8 year swaptions, across the three different yield curve fitting methods. The Svensson model yields the smoothest common eigenvectors (as expected given the functional form imposed) and the B-Spline eigenvectors are more similar to the Svensson eigenvectors than the McCulloch eigenvectors. This is in line with the (marginally) better in-sample fit of the B-spline method over the McCulloch method, and the best fit overall from the Svensson model, as discussed in section three. If the differences between these common eigenvectors are substantial, particularly in the third common eigenvector, the differences between the individual eigenvectors are even more problematic. This leads us to conclude that if, historical forward rate data were used to estimate the eigenvectors of forward rate correlation or covariance matrices in the lognormal forward rate model, very different model prices will obtain depending on which yield curve fitting method has been employed. This is a subject that is open to further research.

8. Summary and Conclusions

The aim of this paper was to investigate the time series and cross sectional properties of forward rates that were obtained from fitting the libor yield curve. An understanding of the statistical properties of these data is of the utmost importance if – as advocated by Hull and White (1999, 2000), Longstaff, Santa Clara and Schwartz (1999), Rebonato (1999a,b,c), Rebonato and Joshi (2001) and many others – historical time series of forward rates are used in the calibration of the lognormal forward rate model.

Noting the model risk that arises from the choice of yield curve model and the size of the sample, we have provided a detailed analysis of the statistical properties of forward rates that are obtained using three different yield curve fitting techniques. Introduced by McCulloch (1975), Steely (1991) and Svensson (1994), each of the three techniques has been extensively applied to the construction of bond yield curves. Based on approximately two years of daily data on UK spot libor rates, FRAs and swaps, we have shown that the Svensson model provides the closest fit over the whole sample. However, when the in-sample fit statistics are considered in a time series context there is no clear view on which model provides the closest fit at any particular time. However, a clear relationship has been found, in both a time series and a cross-sectional context, between the in-sample fit of a libor yield curve and the historical volatilities of the forward rates so obtained. In line with the results of De Jong, Dreissen and Pelsser (2001), all forward rate volatilities were higher during times when measurement errors increased and, because measurement errors increase with maturity, volatilities were generally increasing with the maturity of the forward rate.

The structure of estimated forward rate unconditional correlation matrices was shown to be dominated by the semi-parametric or parametric form that is used in the yield curve model, rather than the dates of the sample. Not surprisingly therefore, the Svensson forward rates were found to have the most well-behaved correlation matrix. We then considered the spectral decomposition of forward rate correlation – and covariance – matrices. The principal eigenvectors of forward rate correlation matrices were shown to be surprisingly stable over time. However, the principal eigenvectors of the forward rate covariance matrices were far more sensitive to the sample period used to estimate the matrix. We therefore conclude that, if historical forward rates are to be used at all in the calibration of the lognormal forward rate model, it is best to use these data in the form of the eigenvectors of the Svensson correlation matrix. Other yield curve fitting techniques, and/or the use of the covariance matrix eigenvectors, will be subject to a substantially greater degree of model risk. The forward rate volatilities should be calibrated to market data on swaption volatilities.

A major focus of this paper has been to test the common principal component hypothesis of Flury (1988), which has been applied to the lognormal forward rate model by Alexander (2003). Since there is strong empirical evidence to uphold the common principal component hypothesis, a very parsimonious parameterisation of forward rate correlations is possible. Under this hypothesis, for every $m > 2$, the three principal eigenvectors of all forward rate correlation matrices of dimension m will be identical. In other words, the correlation matrices underlying all swaps of the same tenor will have a common eigenvector structure. Under this hypoethesis, market data on short maturity swaption volatilities, which are more reliable and far less sparse than market data on long maturity swaption volatilities, can be used in the calibration of correlations between longer maturity forward rates. The number of correlation parameters to be calibrated is only $4N$, where N is the number of different tenors of swaptions in the market. Using this common eigenvector form for forward rate correlations, direct calibration to market data is eminently feasible and historical forward rate data are not necessary.

Future research should investigate the behaviour of lognormal forward rate model prices that are based on the common principal component hypothesis. This hypothesis can be implemented on either historical or market data, and using either correlation or covariance matrices. A comparison of the out-of-sample accuracy and stability of the model prices so obtained is expected to provide important insights to 'best practice' when calibrating the Libor Market Model.

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Table 1: Common Principal Component Tests

Tenor of Swap (years)	2	3	4	5	6	7	8
Dimension of Matrices	4	6	8	10	12	14	16
Number of Matrices	16	14	12	10	8	6	4
Degrees of Freedom	90	195	308	405	462	455	360
Correlation							
Svensson	3.60	9.15	13.73	17.74	19.10	16.30	9.83
McCulloch	48.31	101.12	145.78**	164.98**	169.42**	129.70*	69.01
B-Spline	50.19	108.16	149.64**	162.61**	154.47**	129.15*	82.84
Covariance							
Svensson	5.56	9.24	11.96	12.49	10.66	8.44	4.78
McCulloch	26.28	45.95	70.37	82.06	76.85	55.29	30.82
B-Spline	23.85	40.79	60.71	65.58	63.23	48.44	31.82

** = CPC Hypothesis rejected at 1%;   * = CPC Hypothesis rejected at 5%

Figure 1: RMSE from Different Models

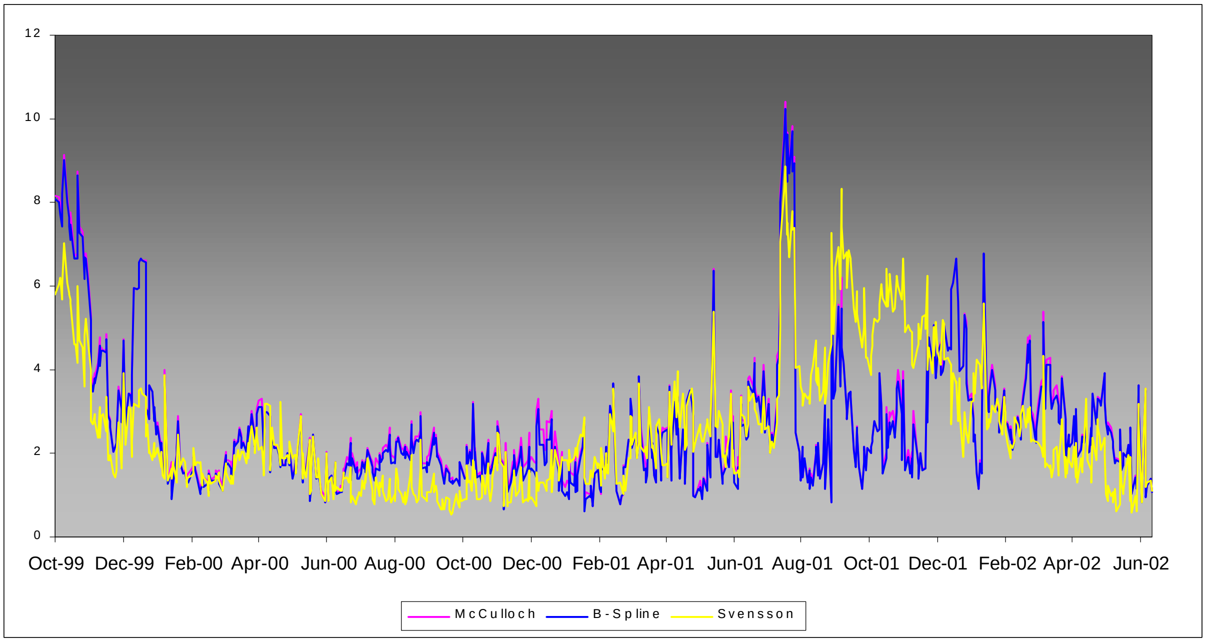

Figure 2: Comparison of volatilities of:
(a) 1 year; (b) 5-year; and (c) 9-year
semi-annual forward rates from each model

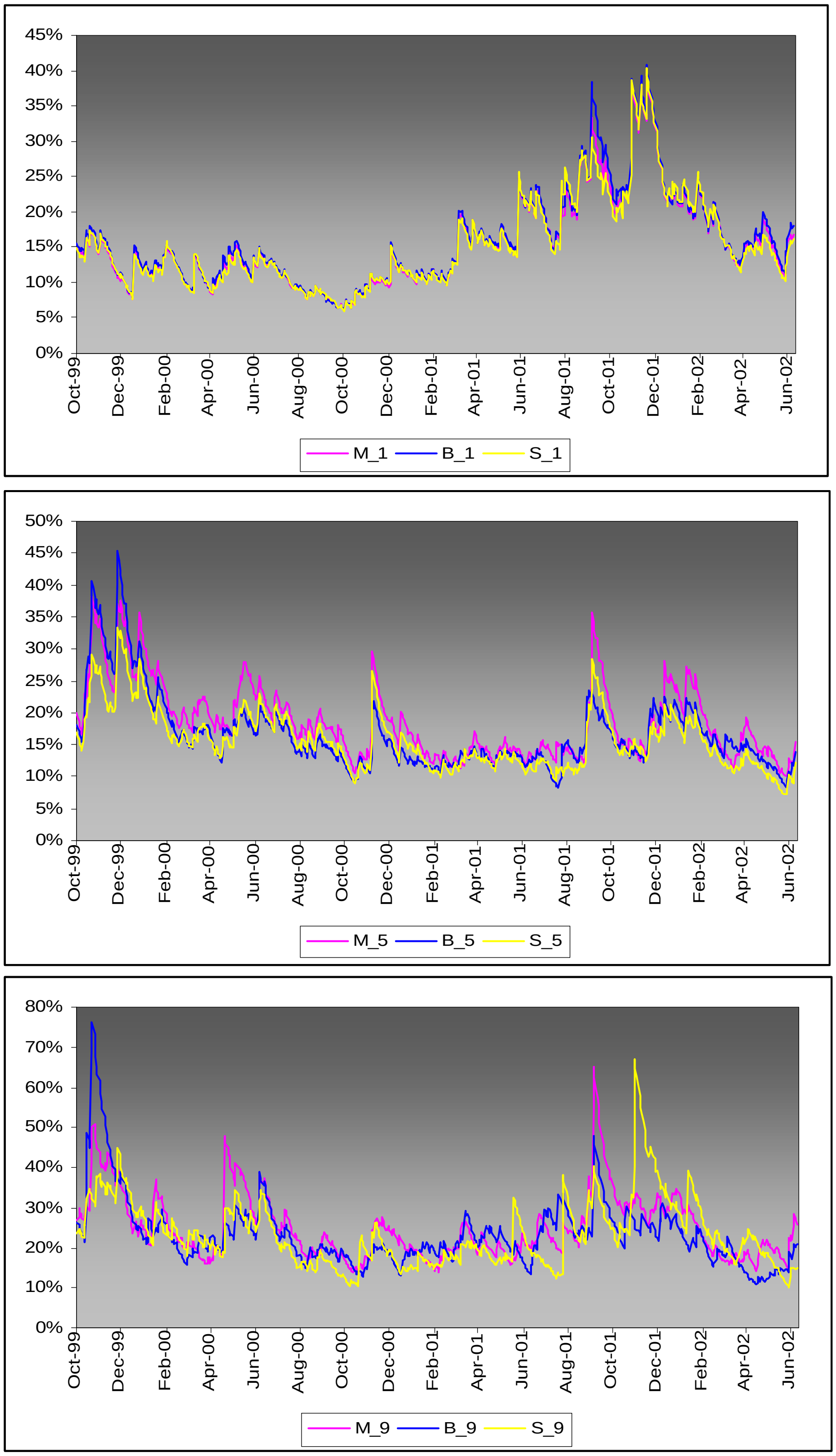

Figure 3: Svensson semi-annual forward rate volatilities observed at the end of each quarter in the sample

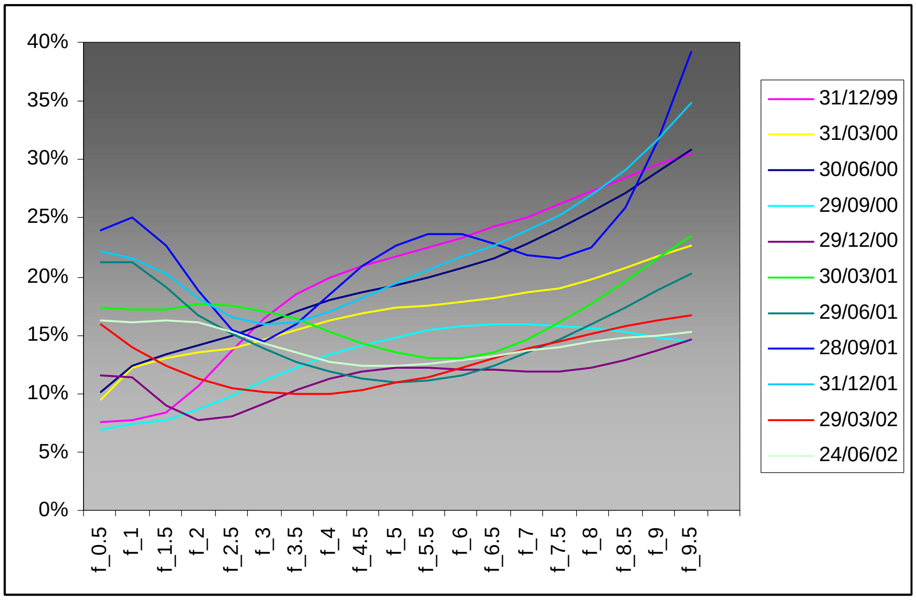

Figure 4: Semi-annual forward rate correlations obtained from yield curves fitted by the three different methods (a) Svensson; (b) B-Spline; and (c) McCulloch

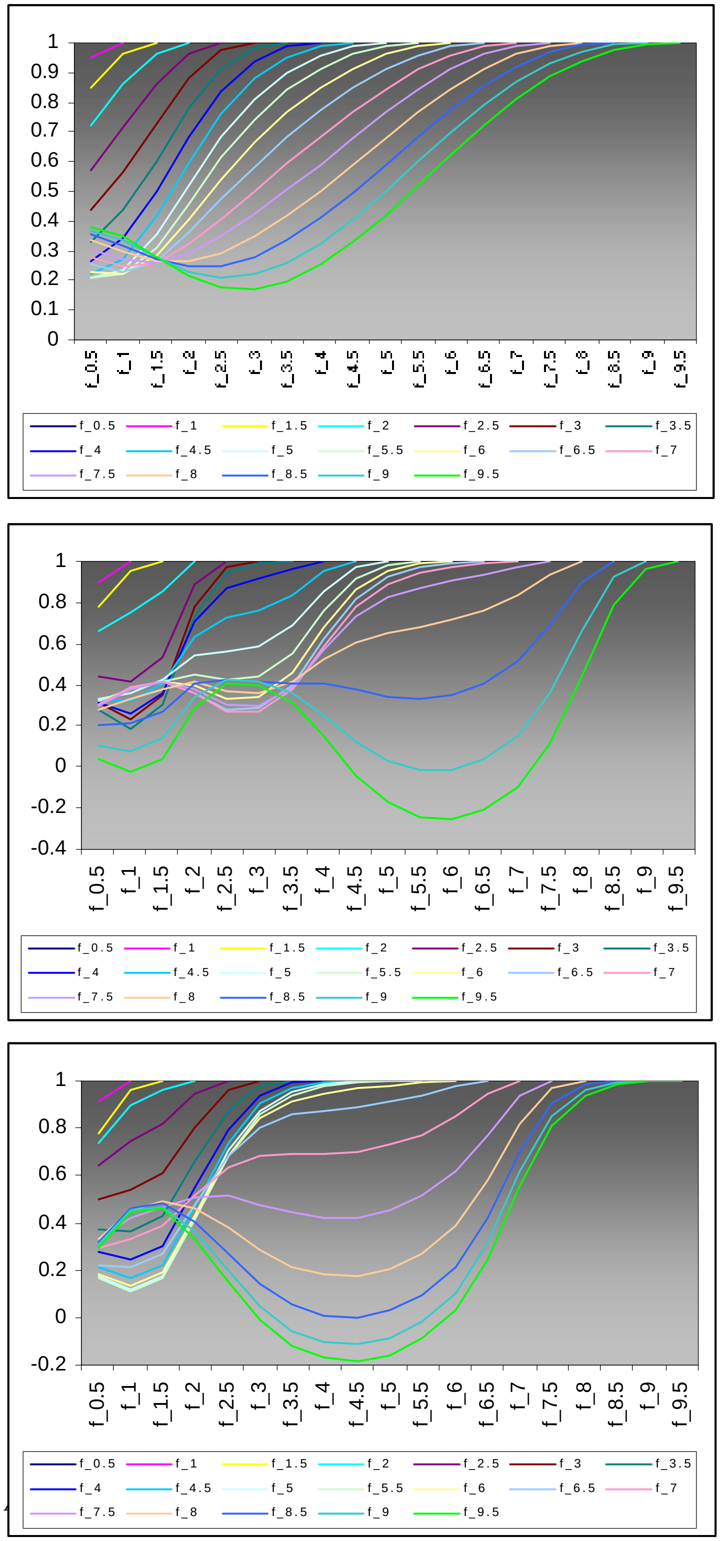

Figure 5: Correlation Matrices of Svensson Forward Rates based on Two Different Historical Observation Periods

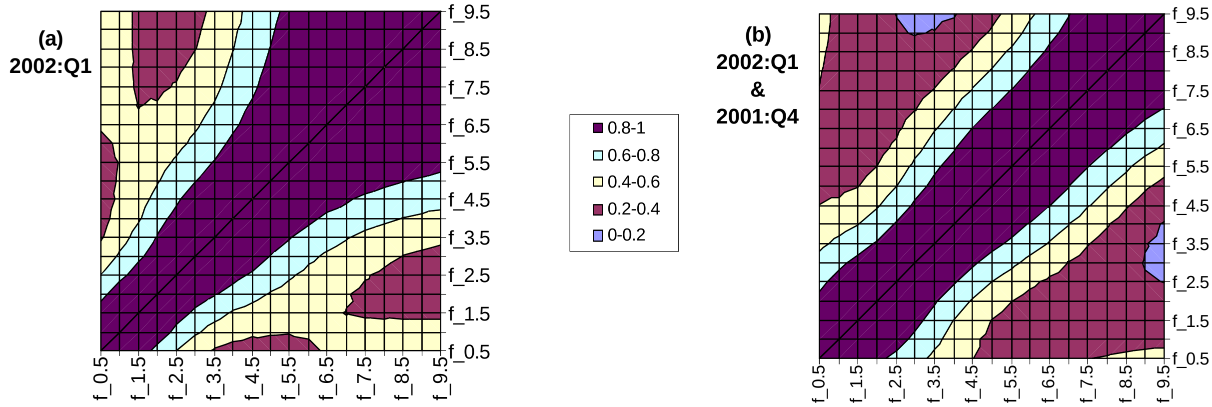

Figure 6a: First three eigenvectors of historical correlation matrices estimated as of 4th January 2002 from Svensson forward rates based on a short sample period (3 months of daily data) and a longer sample period (6 months of daily data).

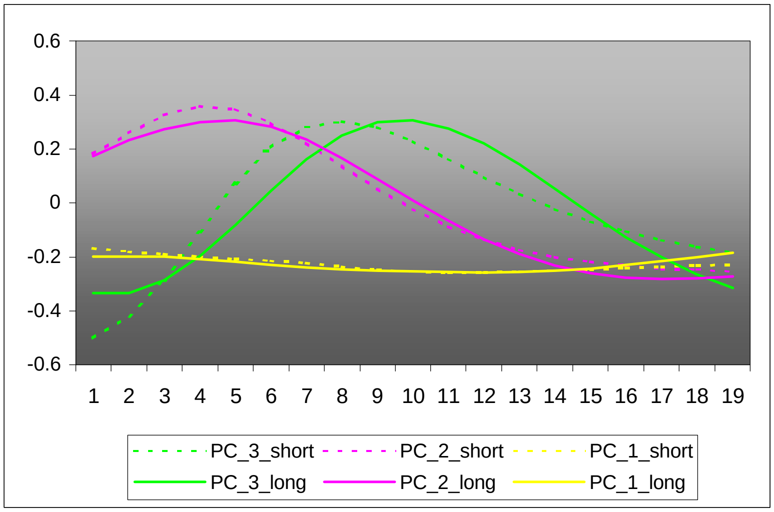

Figure 6b: First three eigenvectors of historical covariance matrices estimated as of 4th January 2002 from Svensson forward rates based on a short sample period (3 months of daily data) and a longer sample period (6 months of daily data).

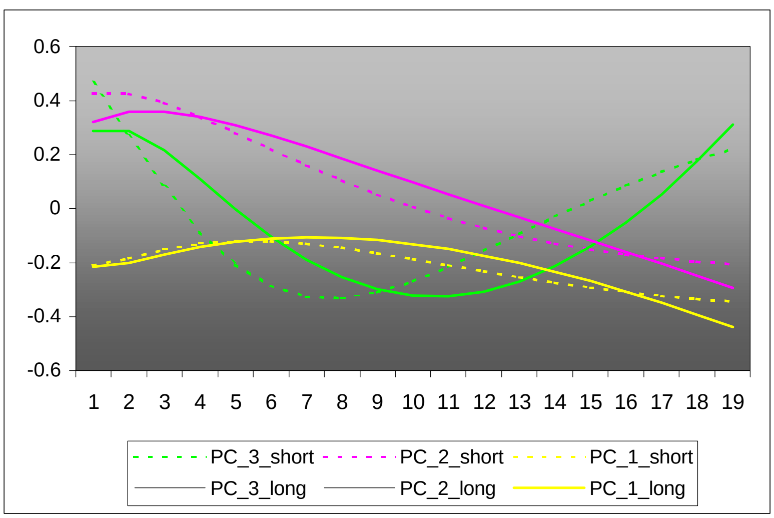

Figure 7(a): Common eigenvectors of the forward rate covariance matrices underlying an 8-year swaption with different maturities

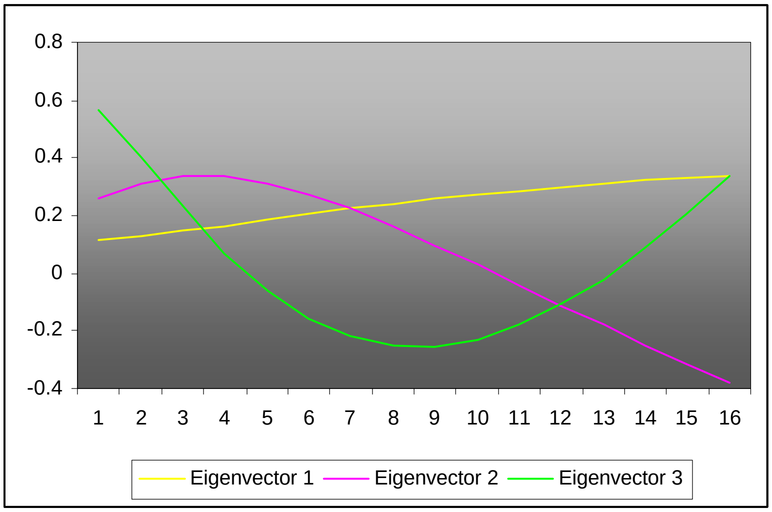

Figure 7(b): Common eigenvectors of the forward rate correlation matrices underlying an 8-year swaption with different maturities

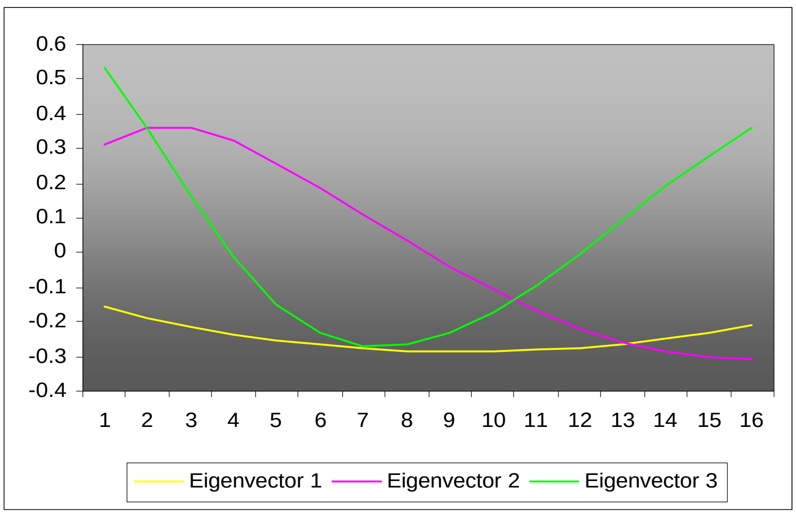

Figure 8: *Comparison of common eigenvectors from the forward rate correlation matrices underlying an 8-year swaption with different maturities obtained from different yield curve models*