DYNAMIC QUANTILE MODELS

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Abstract

This paper introduces new dynamic quantile models called the Dynamic Additive Quantile (DAQ) model and the Quantile Factor Model (QFM) for univariate time series and panel data, respectively. The Dynamic Additive Quantile (DAQ) model is suitable for applications to financial data such as univariate returns, and can be used for computation and updating of the Value-at-Risk. The Quantile Factor Model (QFM) is a multivariate model that can represent the dynamics of cross-sectional distributions of returns, individual incomes, and corporate ratings. The estimation method proposed in the paper relies on an optimization criterion based on the inverse KLIC measure. Goodness of fit tests and diagnostic tools for fit assessment are also provided. For illustration, the models are estimated on stock return data from the Toronto Stock Exchange (TSX).

Keywords: Value-at-Risk, Factor Model, Information Criterion, Income Inequality, Panel Data, Loss-Given-Default.

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1 Introduction

In 1995, the Governors of the Central Banks met in Basle (Switzerland) to establish common rules for banking supervision. An important decision that was taken at that occasion was mandatory computation of a risk measure, called the Value-at-Risk (VaR). A few years later, the regulators added the requirement of compulsory use of internal models for more frequent and detailed risk monitoring in banks. Since then, banks have been routinely computing daily VaR's for each category of assets and for each line of the balance sheet. The VaR has also been used by banks to determine the minimum capital reserve that is required to be set aside for coverage of potential losses due to extreme risks [see e.g. Basle Committee (1995), Jorion (1997), and Gourieroux, Jasiak (2003) for a survey). Technically, the VaR is a conditional quantile of the return distribution of a portfolio, and varies in time according to the dynamics of its returns. Therefore, a natural approach to VaR modelling leads to a path dependent conditional quantile specification.

The aim of this paper is to implement this idea by introducing the following two types of models: the Dynamic Additive Quantile (DAQ) model, for univariate time series of portfolio returns and the Quantile Factor Model (QFM), for applications to time series of cross-sectional asset return distributions. The QFM model can also be used in studies on income inequality, corporate rating dynamics, recovery rates, or for the analysis of mutual fund performances [1].

The paper is organized as follows. Sections 2 and 3 discuss the specification and estimation of dynamic quantiles in univariate time series. In particular, Section 2 presents examples of dynamic quantile models that already exist in the literature and introduces the new Dynamic Additive Quantile (DAQ) model. The maximum likelihood estimation of a parametric dynamic

[1] These applications concern panel data with a high number of cross-sectional data at each point in time (see Section 5).

quantile model can be numerically cumbersome. Therefore, Section 3 introduces an information based estimation method for DAQ models, along the lines of Kitamura, Stutzer (1997). Sections 4 and 5 concern the estimation of quantiles from panel data. More precisely, Section 4 considers quantile models for panel data driven by some unobservable macrofactors. This section introduces the dynamic Quantile Factor Model (QFM) and describes its estimation procedure. Section 5 presents two applications of dynamic quantile models to stock return data from the Toronto Stock Exchange (TSX). The first one examines the performance of DAQ in a study of the dynamics of the conditional quantiles of market returns. The second one investigates the use of QFM for modeling the dynamics of quantiles of cross-sectional asset return distributions and provides a more detailed summary of market risk. Section 6 concludes the paper. Proofs are gathered in Appendices.

2 Dynamic Quantile Models

The first part of this section reviews the main properties of a quantile function. It is followed by examples of dynamic quantile models existing in the literature. The third part introduces the new Dynamic Additive Quantile (DAQ) model.

2.1 Quantile Functions

A quantile function of a random variable is defined as the inverse of the cumulative distribution function. Let us consider a continuous random variable that takes values in $(-\infty, +\infty)$ [resp.$(0, \infty)$], and suppose that its quantile function is denoted by Q. The domain of Q is the interval $[0, 1]$. Q is increasing, takes value $-\infty$ [resp.0] at 0, and value $+\infty$ at 1. Examples of quantile functions for some selected continuous random variables are given in Table 1.

Table 1 : Quantile functions

distribution	range	quantile
Cauchy	$(-\infty, +\infty)$	$Q(u) = tg[\pi(u - 1/2)]$
Logistic	$(-\infty, +\infty)$	$Q(u) = \log[u/(1-u)]$
Exponential	$(0, \infty)$	$Q(u) = -\log(1-u)$
Pareto	$(0, \infty)$	$Q(u, a) = \dfrac{1}{(1-u)^a} - 1, a > 0$

A quantile function can be transformed into a different quantile function using some specific transformations. Those transformations can be determined by taking into consideration the following properties of quantile functions [see e.g. Joe (1997)].

i) If Q is a quantile function with range $(-\infty, +\infty)$, then $Q^*(u) = -Q(1-u)$ is also a quantile function.

ii) If Q is a quantile function with range $(0, \infty)$, then $Q^*(u) = Q(u^\gamma)$, where $\gamma > 0$, is also a quantile function.

iii) If $Q_k, k = 1, \ldots, K$ are quantile functions with identical range which is either $(0, \infty)$, or $(-\infty, +\infty)$, then $Q^*(u) = \sum_{k=1}^{K} a_k Q_k(u)$, where the coefficients a_k are positive, is a quantile function.

iv) If Q_1, Q_2 are quantile functions with range $(0, \infty)$, then :

$Q_1^*(u) = Q_1(u)Q_2(u)$ and $Q_3^*(u) = Q_1(u)^\gamma$, where $\gamma > 0$, are also quantile functions. If Q_1, Q_2 are quantile functions with range $(0, 1)$, then $Q_2^*(u) = \dfrac{Q_1(u)}{1 - Q_2(u)}$, is a quantile function too.

Note that in some cases $Q_1(u) = \sum_{k=1}^{K} a_k Q_k(u)$ can behave like a quantile function, even if some a_k coefficients are negative. This is because the difference of increasing functions can be an increasing function as well.

2.2 Dynamic Quantile Models : Definition and Examples

A common approach to dynamic quantile analysis is the following. The conditional quantile at risk level α is specified as a function of some conditioning variables known at time t : $Q_t(\alpha) = g(x_t; \theta_\alpha)$, say. Then, parameter θ_α is estimated by minimizing the following criterion [see Koenker, Basset (1978), Koenker (2005)] [2] [3]:

$$\begin{aligned}\hat{\theta}_\alpha &= \arg\min_\theta \sum_{t=1}^{T} \{\alpha[y_t - g(x_t;\theta)]^+ + (1-\alpha)[y_t - g(x_t;\theta)]^-\} \\ &= \arg\min_\theta \sum_{t=1}^{T} \rho_\alpha[y_t - g(x_t;\theta)],\end{aligned}$$

where $y^+ = \max(y, 0), y^- = \max(-y, 0)$, and $\rho_\alpha(u) = u[\alpha - \mathbb{1}_{u<0}]$.
This approach has drawbacks that become apparent when θ_α is estimated for a set of different risk levels α. For example, let us consider a linear index model with $Q_t(\alpha) = g(x_t; \theta_\alpha) = x_t'\theta_\alpha$. One would expect that $g(x_t, \theta_\alpha)$ is an increasing function of risk level α. However, according to the above specification, this is not necessarily the case. In particular, $x_t'\theta_\alpha$ may not be an increasing function of α for **all** admissible values of the conditioning variables. Especially, this problem may occur when some explanatory variables take both positive and negative values. Also, even when the explanatory variables are nonnegative, separate estimations of θ_α for different levels α do not necessarily produce outcomes that are increasing in α. This is an important limitation, since a well-specified quantile model is expected to provide

[2] and Portnoy (1991), Weiss (1991), Kould, Saleh (1995), Koenker, Zhao (1996), Mukherjee (1999) for the properties of this estimation method in nonlinear dynamic framework.

[3] An alternative estimation method with similar drawback is the Quasi Maximum Likelihood for conditional quantile [see e.g. Gourieroux, Monfort, Renault (1987), Gourieroux Monfort (2001), Proposition 8.23, Komunjer (2005)]

estimators that behave like true quantiles (that is, increase with α), and can be interpreted as such for any values of the parameters and conditioning variables. The monotonicity of quantile estimators with respect to α is crucial for practical applications. For example, when the monotonicity condition is violated, there is nothing that prevents the (estimated) Value-at-Risk from becoming a decreasing function of risk level α. This, in turn, suggests lowering the (estimated) minimum required capital for balancing risky portfolio when the loss probability increases!

In general, the specification of a dynamic quantile model is based on a conditional quantile function. The conditional quantile function Q_t can be parametric, or semi-parametric, provided that it is tractable and in a closed form. It depends on the selected conditioning information set. Given the dynamic quantile model, the conditional pdf at time t,

$$f_t(y) = 1/\frac{dQ_t}{du}[Q_t^{-1}(y)],$$

has, in general, a complicated form, and requires inversion of the conditional quantile function Q_t.

Let us now present a brief overview of dynamic quantile models that have already appeared earlier in the econometric literature (often with specific terminologies).

Example 1 : Path dependent location and scale parameters

Let us assume that the conditional distribution of Y_t is such that :
$Y_{t+1} = m_t + \sigma_t \varepsilon_{t+1}$, where m_t, σ_t are functions of the information available at time t and ε_t are i.i.d. variables, with p.d.f. f_0 and quantile function Q_0. The forms of the conditional pdf and the conditional quantile function are both tractable. We get :

$$f_t(y) = \frac{1}{\sigma_t} f_0 \left(\frac{y - m_t}{\sigma_t} \right), \text{ and } Q_t(u) = m_t + \sigma_t Q_0(u).$$

The conditional quantile is simply a path dependent linear affine transformation of the path independent baseline quantile function Q_0. This specification includes the accelerated hazard model [see Kalbfleisch, Prentice (1980), Lancaster (1990)], and the autoregressive conditional duration (ACD) model [Engle, Russell (1998)] introduced for time series of intertrade durations. Both these models involve only a scale effect, that is, $m_t = 0$. The conditional scale σ_t is generally parametrized, while the baseline distribution f_0, or Q_0 can be either parametrized, or left unconstrained.

The path-dependent location and scale parameters model is not sufficiently flexible to describe the dynamics of risk for a portfolio of assets, as it includes a single "scale" parameter. This eliminates the possibility of distinguishing between the dynamics of negative and positive extreme risks, or of standard and extreme risks. For this purpose, at least three path dependent "scale" parameters are necessary.

Example 2 : Conditional Autoregressive Value-at-Risk (CAViaR)

In risk management, it is important to determine some conditional quantiles of a portfolio return distribution, called the Value-at-Risk (VaR). Engle, Manganelli (2001), (2004) directly defined the dynamics of risk by means of an autoregression involving the lagged Values-at-Risk (that is, lagged conditional quantiles) and the lagged value of the endogenous variable. Since the Value-at-Risk at level u is equal to the conditional quantile $Q_t(u)$, the symbol $Q_t(u)$ replaces $VaR_t(u)$ from now on until the end of the paper.

Let us first discuss models without the autoregressive component [Engle, Manganelli (2004), p7]. To this group belong the CAViaR model with a symmetric absolute value :

$$Q_t(u) = \beta_0(u) + \beta_2(u)|y_{t-1}|,$$

and the CAViaR model with an asymmetric slope :

$$Q_t(u) = \beta_0(u) + \beta_2(u)(y_{t-1})^+ + \beta_3(u)(y_{t-1})^-.$$

The conditional quantile function Q_t is well-defined, provided that functions $\beta_0, \beta_2, \beta_3$ are quantile functions too [see Property iii) in Section 2.1]. In brief, the CAViaR model assigns weights to different baseline quantile functions, depending on the observed history.

When an autoregressive component of VaR is included, the CaViaR model becomes:

$$Q_t(u) = \beta_0(u) + \beta_1(u)Q_{t-1}(u) + \beta_2(u)|y_{t-1}|,$$

and :

$$Q_t(u) = \beta_0(u) + \beta_1(u)Q_{t-1}(u) + \beta_2(u)(y_{t-1})^+ + \beta_3(u)(y_{t-1})^-.$$

The conditional quantile function Q_t is well-defined, if functions $\beta_0, \beta_1, \beta_2, \beta_3$ are quantile functions as well, and if Q_{t-1} is nonnegative. Indeed, if Q_t takes values in $(0,1)$, the autoregressive specification can be written as follows:

$$Q_t(u) = \frac{\beta_0(u)}{1-\beta_1(u)} + \sum_{h=1}^{\infty} \beta_2(u)\beta_1(u)^h |y_{t-h}|.$$

$Q_t(u)$ is now a linear combination of quantile functions $\frac{\beta_0}{1-\beta_1}, \beta_2\beta_1^h, h$ varying, with nonnegative coefficients (see Section 2.1, Property iv)). Therefore, it satisfies the properties of a quantile function [4].

[4] However, other CAViaR specifications do not satisfy the monotonicity property. This is the case of the indirect GARCH (1,1) : $Q_t(u) = (\beta_0(u) + \beta_1(u)Q_{t-1}(u)^2 + \beta_2(u)y_{t-1}^2)^{1/2}$, or of the adaptive CAViaR models :

$$Q_t(u) = Q_{t-1}(u) + \beta_0(u)\{[1 + \exp(max(T, K(u))[y_{t-1} + Q_{t-1}(u)])]^{-1} - \beta_1(u)\}$$

[see Engle, Manganelli (2001)].

In practice, the conditional upper quantiles of portfolio return distributions are not always nonnegative. In contrast, the nonnegative range constraint holds for the conditional quantiles of intertrade durations (the so-called Time-at-Risk (TaR)) [See Ghysels, Gourieroux, Jasiak (2003)] in the analysis of liquidity risk, and the (0,1) range constraint holds the Loss-Given-Default involved in the computation of the VaR for credit portfolios.

The models presented above can be extended to a simple class of parametric dynamic quantile models that combines different baseline quantile functions with path dependent coefficients. This idea is explored in the following section that introduces the DAQ Model.

2.3 Dynamic Additive Quantile (DAQ) Model

The new class of dynamic quantile models is defined as follows.

Definition 1 : A Dynamic Additive Quantile (DAQ) model is :

$$Q_t(u;\theta) = \Sigma_{k=1}^{K} a_k(\underline{y_{t-1}};\alpha_k) Q_k(u;\beta_k) + a_0(\underline{y_{t-1}};\alpha_0), \tag{1.1}$$

where Q_k are path-independent baseline quantile functions and $a_k(\underline{y_{t-1}};\alpha_k)$ are nonnegative functions of the past.

According to the above definition, the model with path-dependent location and scale parameters is written as a sum of $K+1$ different components. The DAQ specification is quite flexible because it can combine baseline quantile functions with tails of different thickness. From a practical point of view, the additive quantile specification is very convenient for computing quantiles for a set of different levels u, since it does not require inversion of the conditional cdf. It is also suitable for simulation of future paths of the process.

Indeed, if a process is observed up to time t, the future simulated values are defined recursively as:

$$y^s_\tau(\theta) = \sum_{k=1}^{K} a_k[\underline{y^s_{\tau-1}}(\theta), \alpha_k] Q_k(u^s_\tau; \beta_k) + a_0[\underline{y^s_{\tau-1}}(\theta), \alpha_0],$$

where : $y^s_\tau(\theta) = y_\tau$, if $\tau \leq t$, and $u^s_\tau, \tau \geq t+1$ are drawn independently in the uniform distribution on $[0, 1]$.

The formulas of models in the class of Dynamic Additive Quantile models are quite simple. For instance, let us consider the following specification:

$$\begin{aligned} Q_t(u; \theta) &= a_{0,0} + a_{0,1} y_{t-1} + a_{0,2}|y_{t-1}| + (a_{1,0} + a_{1,1}|y_{t-1}|) Q_1(u) \\ &+ (a_{2,0} + a_{2,1}|y_{t-1}|) Q_2(u), \end{aligned}$$

where $a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}$ are nonnegative. This model is a parametrized conditional quantile function, that is a linear function of the parameters. Up to a conditional drift effect, it arises as a combination of the following baseline (possibly path-dependent) conditional quantile functions : $Q_1(u)$, $|y_{t-1}|Q_1(u)$,$Q_2(u)$, $|y_{t-1}|Q_2(u)$.

In particular, to the class of Dynamic Additive Quantile models belongs the linear DAQ model defined below.

Definition 2 : A linear Dynamic Additive Quantile (DAQ) model is :

$$Q_t(u; \theta) = \sum_{j=1}^{p} \theta_j \tilde{Q}_{j,t}(u) = \theta' \tilde{Q}_t(u), \text{ say.}$$

The above conditional quantile function is linear in the parameters [5].

[5] This specification is different from the encompassing quantile model introduced in Giacomini, Komunjer (2005). That encompassing model is $\theta^*_{1t} Q_{1t}(u; \beta_1) + \theta^*_{2t} Q_{2t}(u; \beta_2)$ where $\theta^*_{1t}, \theta^*_{2t}$ are solutions of $Min_{\theta_1,\theta_2} E_t[\rho_\alpha[y_{t+1} - \theta_1 Q_{1t}(\alpha; \beta_1) - \theta_2 Q_{2t}(\alpha; \beta_2)]]$. Indeed, the solutions of this optimization depend on the past but also on the critical level α.

2.4 Conditional Moment Conditions

The DAQ model for nonnegative variables implies restrictions on all conditional moments. The conditional mean and variance restrictions are discussed below.

In general, the moments about 0 of a nonnegative random variable are easily written in terms of its quantile function as :

$$E(Y^p) = \int_0^1 Q(u)^p du,$$

provided that they exist. It follows, that the DAQ model implies a conditional mean of the form:

$$E_{t-1}(Y_t) = \sum_{k=1}^{K} a_k(\underline{y_{t-1}}) m_k + a_0(\underline{y_{t-1}}),$$

where m_k is the mean of distribution Q_k. Thus, the conditional mean depends on $K+1$ functions of the past $a_k(\underline{y_{t-1}}),\ k = 0, ..., K$.

The conditional variance is:

$$\begin{aligned} V_{t-1}(Y_t) &= E_{t-1}(Y_t^2) - [E_{t-1}(Y_t)]^2 \\ &= \sum_{k=1}^{K} \sum_{l=1}^{K} a_k(\underline{y_{t-1}}) a_l(\underline{y_{t-1}}) [\mu_{kl} - m_k m_l], \end{aligned}$$

where : $\mu_{kl} - m_k m_l = Cov\ [Q_k(U), Q_l(U)]$ and $U \sim U_{[0,1]}$. The conditional variance depends on $K(K+1)/2$ functions of the past $a_k(\underline{y_{t-1}}) a_l(\underline{y_{t-1}})$.

3 Statistical Inference for Parametric Dynamic Quantile Models

In this section, we consider observations $y_1, \ldots, y_T$, on a univariate time series, such as portfolio returns. Since estimation of a dynamic quantile model by maximizing the log-likelihood can be time consuming, we propose a modified objective criterion that yields estimators faster, while preserving their asymptotic efficiency. In the second part of this section, we discuss specification tests.

3.1 Maximum Likelihood

Let us consider a parametric conditional quantile model, defined by : $Q_t(u;\theta) = Q(u|\underline{y_{t-1}};\theta)$, where $\underline{y_{t-1}} = \{y_{t-1}, y_{t-2}, \ldots\}$, and θ is a finite dimensional parameter. For instance, the parameter of a Dynamic Additive Quantile model is : $\theta = [\alpha_0', \alpha_1', \ldots, \alpha_K', \beta_1', \ldots, \beta_K']'$, and it can be estimated by the maximum likelihood. The m.l. estimator is:

$$\tilde{\theta}_T = \arg\max_\theta - \sum_{t=1}^{T} \log q_t[Q_t^{-1}(y_t;\theta);\theta], \tag{3.1}$$

where $q_t(u,\theta) = \dfrac{\partial Q_t}{\partial u}(u;\theta)$ and Q_t^{-1} denotes the inverse of Q_t with respect to the argument u. In the Dynamic Additive Quantile model, the expression of the conditional quantile Q_t is simple when it is written as a function of baseline quantile functions $Q_k(\ ;\beta_k)$. In contrast, the conditional cumulative distribution function (cdf) Q_t^{-1} is a complicated function of the baseline cdf $Q_k^{-1}(\ ;\beta_k)$. In practice, the inversion has to be performed numerically for each observation, at each step of the optimization algorithm used to maximize the log-likelihood function. This explains why the standard maximum likelihood approach can be computationally cumbersome due to T inversions required at each step of the optimization algorithm.

In the remainder of this section, we follow the approach of Amari (1990), Kitamura, Stutzer (1997), and introduce an alternative Kullback-Leibler information criterion (KLIC). The advantage of this approach is that KLIC can be written directly as a function of the quantile function and provides asymptotically efficient estimators when used as an objective function.

3.2 Information Criterion

There exist several measures of proximity between two pdf, f_0 and f, say, such as the α-divergence measures [see Csiszar (1975), Amari (1990)] These include, as special cases, two Kullback-Leibler Information Criteria given below. Let f_0 denote the true p.d.f. and let f be the p.d.f. of the approximating parametric model.

$$I_1(f_0, f) = \int f_0(y) \log \frac{f_0(y)}{f(y)} dy, \tag{3.2}$$

and

$$I_{-1}(f_0, f) = \int f(y) \log \frac{f(y)}{f_0(y)} dy. \tag{3.3}$$

The KLIC measure I_1 underlies the standard maximum likelihood estimation procedure, while the KLIC measure I_{-1} provides the empirical likelihood interpretation of the GMM estimator [Kitamura, Stutzer (1997)]. In particular, the KLIC measure I_{-1} is more suitable for a quantile model. Indeed, let us denote by $Q(u)$ and $q(u) = \frac{dQ(u)}{du}$ the quantile and quantile density functions, respectively. We get :

$$I_{-1}(f_0, f) = \int \log \frac{f(y)}{f_0(y)} dF(y) = \int_0^1 \log \frac{f[Q(u)]}{f_0[Q(u)]} du,$$

and the KLIC measure I_{-1} written in terms of the quantile function:

$$I_{-1}(f_0, f) = -\int_0^1 \log q(u) du - \int_0^1 \log f_0[Q(u)] du. \tag{3.4}$$

3.3 The Information Based Estimation Method

Let us consider observations on a stationary process $(y_t, x_t), t = 1, \ldots, T$, where the variables x include exogenous or lagged endogenous variables. For example, let us assume that $x_t = (y_{t-1}, y_{t-2}..)$ is a series of portfolio returns. The conditional quantile function given the information x_t is denoted by

$$Q_t(u;\theta) = Q(u|x_t;\theta), \tag{3.5}$$

and the associated conditional quantile density is

$$q_t(u;\theta) = q(u|x_t;\theta). \tag{3.6}$$

Next, consider a kernel estimator $\hat{f}_{0T}$ of the conditional p.d.f. of y given x :

$$\hat{f}_{0T}(y|x) = \frac{1}{h_T} \frac{\sum_{\tau=1}^{T} K\left(\frac{y_\tau - y}{h_T}\right) K^*\left(\frac{x_\tau - x}{h_T^*}\right)}{\sum_{\tau=1}^{T} K^*\left(\frac{x_\tau - x}{h_T^*}\right)}, \tag{3.7}$$

where K, K^* are kernels and h_T, h_T^* are bandwidths, which tend to zero at an appropriate rate, when T tends to infinity. Then, an information based estimator can be obtained as a solution of the optimization of the following sample-based KLIC measure I_{-1} :

$$\hat{\theta}_T = \arg\max_\theta \sum_{t=1}^{T} \left\{ \int_0^1 \log q(u|x_t;\theta)du + \int_0^1 \log \hat{f}_{0T}[Q(u|x_t;\theta)|x_t]du \right\}. \tag{3.8}$$

Proposition 1 : Under standard regularity conditions, the information based estimator $\hat{\theta}_T$ is consistent, asymptotically normal and asymptotically efficient.

Proof : See Appendix 1.

Under the maximum likelihood approach based on KLIC I_1, the true distribution is approximated by the sample distribution, and no smoothing is required. In contrast, smoothing becomes necessary, when the KLIC measure I_{-1} is used, due to the logarithmic transform of density that appears in the expression of $\hat{\theta}_T$ [see Kitamura, Stutzer (1997), or Gagliardini, Gourieroux, Renault (2005)].

Optimization (3.8) is performed numerically. The procedure can be accelerated by setting the initial values of the parameters equal to some consistent, but inefficient estimators produced either by 1) a method of moments, or 2) a simulation based method. Earlier in the text, we have shown that closed form expressions of the first- and second-order moments for dynamic quantile models exist (for 1) and that these models can be easily simulated (for 2).

3.4 Specification Tests

Commonly used specification tests for a parametric dynamic model with conditional cdf $F_t(y;\theta_0)$ are based on the estimated ranks : $\hat{u}_{t+1} = F_t(y_{t+1};\hat{\theta}_T)$, where $\hat{\theta}_T$ is a consistent estimator of parameter θ_0. If the model is well-specified, $\hat{\theta}_T$ is close to θ_0 and the estimated ranks are close to the ranks $u_{t+1} = F_t(y_{t+1};\theta_0)$, which appear as a sequence of iid uniform variables. In a quantile model, the expression of the conditional cdf is too complicated for the estimated ranks approach to be followed. Instead, let us consider the stochastic function :

$$\alpha \longrightarrow z_{t+1}(\alpha) = \mathbb{1}_{y_{t+1} < Q_t(\alpha;\theta_0)} - \alpha,$$

which is defined on $[0,1]$. $z_{t+1}(\alpha)$ is called a **hit variable** [Christoffersen (1998), Engle, Manganelli (2001)]. We know that :

$$\begin{aligned}u_{t+1} &= F_t(y_{t+1};\theta_0)\\ &= inf\ \{\alpha; y_{t+1} < Q_t(\alpha,\theta_0)\}\\ &= inf\ \{\alpha : z_{t+1}(\alpha) > 0\}.\end{aligned}$$

There exists a one-to-one relationship between the rank u_{t+1} and the stochastic function $z_{t+1}(.)$. Thus, instead of using the estimated ranks for testing the specification of the nonlinear dynamic model, one can equivalently use the functionals z_{t+1}, which are iid with mean zero. In particular, their conditional moments satisfy the following equality:

$$E_t z_{t+1}(\alpha) = 0, \forall \alpha.$$

These conditional moment conditions can be used in several overidentification tests [Szroeter (1983)]. Among useful diagnostic tools are for instance :

i) the plot of : $\alpha \longrightarrow \frac{1}{T}\sum_{t=1}^{T} \hat{z}_t(\alpha) \pm \frac{2}{\sqrt{T}}[\alpha(1-\alpha)]^{1/2}$, where $z_{t+1}(\alpha) = \mathbb{1}_{y_{t+1}<Q_t(\alpha;\theta)} - \alpha$ verifies the marginal moment condition $Ez_{t+1}(\alpha) = 0$;

This idea has been initially suggested by Christoffersen (1998), Engle, Manganelli (2001), Section 5, and applied to a fixed value of the critical level α.

However, as noted by Giacomini, White (2006), the test based on the marginal moment is not sufficiently powerful, and it is preferable to use a larger set of instruments.

ii) For example, one can plot the cross-autocorrelations at lag 1 :

$$(\alpha,\beta) \longrightarrow Corr[\hat{z}_t(\alpha), \hat{z}_{t-1}(\beta)] \pm 2/\sqrt{T},$$

which are based on the unconditional moment condition $E[z_t(\alpha)z_{t-1}(\beta)] = 0$, that is use the set of instruments $z_{t-1}(\beta)$, β varying.

At this point, it is important to discuss the relationship between the hit variables and the objective function that is commonly used in quantile regression. It is easy to see that

$$z_{t+1}(\alpha) = \left[\frac{\partial \rho_\alpha}{\partial \mu}(y_{t+1} - \mu)\right]_{\mu = Q_t(\alpha;\theta_0)},$$

where $\frac{\partial \rho_\alpha}{\partial \mu}$ denotes the right derivative of function ρ_α. In some sense, the specification tests given above are score tests associated with a quantile regression criterion (Koenker (2005)). This explains why the hit variable is called a α-**quantile score** or **rank score** in the statistical literature [Kould, Saleh (1995)].

4 Factor Quantile Model for Panel Data

The previous section discussed the estimation of the conditional quantile function from one observed trajectory (y_t). This approach allows for computation of the VaR, when y_t is a univariate series of returns on a portfolio for which the VaR needs to be determined. However, there exist other potential applications of the model, such as panel data with a fairly high number of individuals or assets. These data consist, at any date t, of individual observations $y_{1,t}, \ldots, y_{n_{t},t}$, with large n_t. Given large n_t's, it is possible to determine the sample distribution at each date t and to compute the deciles $\hat{Q}_t(u_l)$ at each date t, where $u_l = 1/10, 2/10, \ldots, 9/10$.

Examples of panel data of interest are the following:

i) Data on household income, where the individual is a household, y is the income of that household and Q_t is the income distribution at date t.

ii) Data on corporate default, where the individual is a firm defaulted at date

t, y is the loss-given-default (LGD) and Q_t is the distribution of loss-given-default at date t.
iii) Data on mutual funds, where the individual is a mutual fund, y is the Sharpe performance of that fund over the last month, and Q_t is the distribution of Sharpe performances.
iv) Data on corporate credit ratings, where the individual is a firm, y is the current value of the quantitative score assigned by the credit rating agency (an approximation of its expected probability of default in the next 3 years), and Q_t is the distribution of scores.

Aggregation of panel data accross individuals allows for investigation of the global evolution of the population of interest. For instance, in Example i) the aggregate data show the evolution of global inequality in the population, rather than the transitions of individual households between the categories of rich and poor. In Example iii), the aggregate data provide information on global performance of the market, instead of showing the performance of each fund manager in time. In Example iv), the aggregate data reveal the cross-sectional distribution of scores that constitutes a global measure of risk, while the individual data also depict the credit rating migration of individual firms.

4.1 The Quantile Factor Model (QFM)

This section introduces the dynamic conditional Quantile Factor Model (QFM) for panel data. The analysis is performed under the following assumption:

Assumption 1: At any date t, the cross-sectional sample quantile $\hat{Q}_t(u_k)$, $k = 1, \ldots, K$, is a consistent, asymptotically normal estimator of the true quantile $Q_t(u_k)$, $k = 1, \ldots, K$, say.

The true distribution Q_t is the cross-sectional distribution of individual

realizations of variable $y_{i,t}$ at time t. For ease of exposition, let us assume that individual histories $(y_{i,t}), i = 1, \ldots, n$ are observed, and that each variable $y_{i,t}$ is driven by some macrofactors and an idiosyncratic component. The joint evolution of ($y_{i,t}$, i, t varying) can be specified as follows:
(*) Conditional on the past, current and future values of macrofactor (Z_t), say, the individual histories are independent, identically distributed.
(**) Conditional on (Z_t), the transition pdf of $y_{i,t}$ depends on the past through $y_{i,t-1}$ and Z_t. The conditional transition is $f(y_{i,t}|y_{i,t-1}, Z_t)$, say.
(***) The factor has its own dynamics represented, for instance, by a transition pdf $g(z_t|z_{t-1})$.
This panel data model accommodates both common macroeffects (Z_t) that, in part, determine the aggregate distribution Q_t, and individual mobility reflected by the presence of $y_{i,t-1}$. According to the above specification, any individual observation $y_{i,t}$ can be written in the final form [6] :

$$y_{i,t} = h(Z_t, Z_{t-1}, Z_{t-2}, \ldots, \varepsilon_{i,t}, \varepsilon_{i,t-1}, \ldots),$$

where $\varepsilon_{i,t}$ are independent standard Gaussian shocks. Therefore, the cross-sectional sample distribution at date t converges to the distribution of
$h(Z_t, Z_{t-1}, Z_{t-2}, \ldots, \varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2} \ldots)$, where ε_t are $IIN(0, 1)$, and Z_t, Z_{t-1}, $Z_{t-2} \ldots$ are fixed and equal to their observed values. Thus, the distribution Q_t is a complicated function of macrofactor dynamics and individual mobility. If the factor value were constant in time $Z_t = Z, \forall t$, the true quantile function Q_t would be constant in time too, and it would depend on the factor level Z. Such a quantile function could be be interpreted as the stationary distribution of $(y_{i,t})$ associated with a constant level of the factor.
Note also that, for panel data, the theoretical cross-sectional quantile func-

[6] It is known that the conditional distribution is $f(y_{i,t}|y_{i,t-1}, Z_t)$, iff, $y_{i,t} = c(y_{i,t-1}, Z_t, \varepsilon_{i,t})$, where $\varepsilon_{i,t}$ are i.i.d. standard Gaussian variables. The final form is deduced by recursive substitution under regularity conditions for function g.

tion Q_t has a different interpretation than the conditional quantile examined in Section 3.

4.2 Linear Quantile Factor Model

A special case of the dynamic Quantile Factor Model is the linear QFM defined as [7]:

$$Q_t(u) = \sum_{k=1}^{K} Q_k(u;\theta) Z_{k,t} + Z_{0,t}, \tag{4.1}$$

where $Z_{k,t}, k = 0, \ldots, K$ are factors, with $Z_{k,t} > 0, k = 1, \ldots, K$, and $Q_k(u;\theta)$ parametrized quantile functions. The model is completed by specifying the dynamics of the $K+1$ factors, where the conditional pdf of the factor process is denoted by $g(z_t|\underline{z_{t-1}};\theta)$.

Several specifications of factor dynamics can be considered. For instance, one can assume that either a) variables $\log Z_{k,t}, k = 0, \ldots, K$ follow a Gaussian Vector Autoregressive (VAR) model, or b) variables $Z_{k,t}, k = 0, \ldots, K$ follow independent gamma autoregressive processes [Gourieroux, Jasiak (2006)], which are time discretized Cox-Ingersoll-Ross processes. Models a) and b) can be used in dynamic analysis of the term structure of interest rates [see e.g. Duffie, Kan (1996)]. Alternatively, one can write $Z_{k,t} = Tr(D_k Z_t^*)$, where (Z_t^*) is a (L, L) Wishart process and D_k are deterministic symmetric positive semi-definite matrices to ensure the nonnegativity of $Z_{k,t}$ [Gourieroux, Jasiak, Sufana (2004)].

The linear factor representation (4.1) is convenient for empirical applications, as it implies an (approximately) linear relationship between the observed

[7] In application to income distribution, the basic tool is a Lorenz (concentration) curve defined by $L_t(v) = (\int_0^v Q_t(u)du)/(\int_0^1 Q_t(u)du)$. It is equal to the following ratio of linear functions of factor values $L_t(v) = (\sum_{k=1}^K L_k(v;\theta) Z_{k,t} + Z_{0,t})/(\sum_{k=1}^K \mu_k Z_{k,t} + Z_{0,t})$, where L_k denotes a baseline Lorenz curve and μ_k a baseline expectation.

sample cross-sectional quantiles and the underlying factors. Indeed, we get :

$$\hat{Q}_t(u_l) \simeq \sum_{k=1}^{K} Q_k(u_l;\theta) Z_{k,t} + Z_{0,t}, l = 1, \ldots, L, \tag{4.2}$$

or :

$$\hat{Q}_t \simeq B(\theta) Z_t, \tag{4.3}$$

$$\text{where :} \quad \hat{Q}_t = \left[\hat{Q}_t(u_1), \ldots, \hat{Q}_t(u_L)\right]',$$

$$B(\theta) = \begin{bmatrix} Q_1(u_1;\theta) \ldots & Q_K(u_1;\theta) \ 1 \\ \vdots & \vdots \\ Q_1(u_L;\theta) \ldots & Q_K(u_L;\theta) \ 1 \end{bmatrix}.$$

The advantage of this approach, as compared to the standard one that consists in representing the set of log-quantiles by a Gaussian VAR model, is that it preserves the ordering (ascending or descending) of successive quantiles in a systematic manner.

4.3 Maximum likelihood estimation

4.3.1 Just-identified factors

When the number of factors $K + 1$ is equal to the number L of observed sample cross-sectional quantiles, relation (4.3) can be inverted, yielding factor approximations written as functions of the parameters and the observed quantiles : $Z_t \simeq B(\theta)^{-1}\hat{Q}_t$. The factor approximations are consistent, when the cross-sectional dimensions n_t tend to infinity at any date t. In such a case, the parameters can be estimated by maximizing the log-likelihood computed for $Z_t = B(\theta)^{-1}\hat{Q}_t$. This approximated log-likelihood is given by :

$$\log L_T(\theta)$$
$$= \left\{ -T \log \ det \ B(\theta) + \sum_{t=1}^{T} \log g[B(\theta)^{-1}\hat{Q}_t | B(\theta)^{-1}\hat{Q}_{t-1}; \theta] \right\} \Pi_{t=1}^{T} \mathbb{1}_{\mathcal{D}}[B(\theta)^{-1}\hat{Q}_t], .$$

where D is the domain of the factor. When the factors are left unconstrained, the domain restrictions are not relevant. In contrast, when the factors are constrained, some domain restrictions need to be imposed [see Gourieroux, Monfort (2005)b].

4.4 Overidentified Factors

In practice, we can expect to find evidence of a rather small number of factors. When $L > K + 1$, the factors are overidentified. Then, the model can be written as follows :

$$\hat{Q}_t(u_l) = \sum_{k=1}^{K} Q_k(u_l; \theta) Z_{k,t} + Z_{0,t} + u_{l,t}, \ l = 1, \ldots, L.$$

where $u_{l,t}$ are cross-sectional errors, which are approximately Gaussian by Assumption 1, with a variance-covariance matrix derived from standard results on quantile estimation. The expression given above can be seen as a standard linear measurement equation in a state space representation, which needs to be completed by a transition equation for factor dynamics.
The transition equation can be written in the form of a Gaussian VAR model :

$$Z_t = \mu + \Phi Z_{t-1} + \varepsilon_t,$$

where (ε_t) is independent of (u_t) such that $\varepsilon_t \sim N(0, \Omega)$. In that case, we obtain a standard linear Gaussian state space representation, which can be estimated by the Kalman filter. It has been proved in Gourieroux, Monfort

(2005a) that the Kalman filter estimator is consistent and asymptotically efficient, provided that $\sqrt{n}/T \to \infty$.

The approximate Kalman filter approach is more difficult to implement when the factors display nonlinear dynamics, and are modelled as gamma Autoregressive or Wishart processes. In that case, it is possible to use a linear approximation of the Kalman filter, and to correct for asymptotic bias by indirect inference.

5 Applications

The estimation of dynamic quantile models is illustrated by two applications to series of returns on stocks traded on the Toronto Stock Exchange (TSX). The first application examines the dynamics of the conditional quantile of market returns. The second one investigates the dynamics of the cross-sectional asset return distribution for a more detailed description of market risk.

5.1 Conditional Quantile of Market Return

Our series of daily market returns of the TSX index is computed as the difference of log market index values on consecutive days. The sample contains $T = 247$ observations recorded between October, 1^{st}, 2002, and October, 1^{st}, 2003. The series is plotted in Figure 1. The average daily return is 0.00011, with a daily standard deviation equal to 0.007. The series is right skewed with a skewness equal to 0.12, and features fat marginal tails with a kurtosis equal to 5.14. The autocorrelations of market index returns are not significant, while the squared market returns feature slight persistence. In particular, the first-order correlation of y_t^2 is equal to 0.36.

[Insert Figure 1 : Daily Market Returns].

Let us consider the following DAQ model:

$$\begin{aligned} Q_t(u;\theta) &= m_0 + m_1|y_{t-1}-\mu| \\ &+ (\sigma_{0,0}+\sigma_{0,1}|y_{t-1}-\mu|)^{1/2}\Phi^{-1}(u) \\ &+ (\sigma_{1,0}+\sigma_{1,1}|y_{t-1}-\mu|)^{1/2} tg[\pi(u-1/2)], \end{aligned}$$

where μ denotes the median of returns.

This model includes drift $m_0 + m_1|y_{t-1}-\mu|$, with a risk premium, and accommodates endogenous tail behaviour that arises from mixing Gaussian and Cauchy tails. In particular, if $\sigma_{1,0} = \sigma_{1,1} = 0$ [resp. $\sigma_{0,0} = \sigma_{0,1} = 0$], we get a kind of an ARCH-in-Mean model with Gaussian [resp. Cauchy] standardized errors. Note that, due to the presence of Cauchy tails, the volatilities do not exist.

The estimation of $\theta = (m_0, m_1, \sigma_{0,0}, \sigma_{0,1}, \sigma_{1,0}, \sigma_{1,1})$ is performed by applying the method described in Section 3.3, with $x_t = y_{t-1}$. The kernels K, K^* are standard Gaussian kernels with fixed bandwidth $h_T = h_T^* = 0.01$. The estimated parameters and their standard errors are reported in Table 2.

Table 2 : Estimated DAQ model

parameter $\times 10^2$	m_0	m_1	$\sigma_{0,0}$	$\sigma_{0,1}$	$\sigma_{1,0}$	$\sigma_{1,1}$
estimates	18.50	3.72	-17.31	+6.92	31.91	-8.70
std.error	(0.71)	(0.15)	(0.12)	(2.01)	(1.71)	(0.04)

The estimation was performed without imposing any restrictions on the parameters. Therefore, ex-post, it is important to check, if the quantities $\sigma_{0,0} + \sigma_{0,1}|y_{t-1}-\mu|$, and $\sigma_{1,0} + \sigma_{1,1}|y_{t-1}-\mu|$ are nonnegative [8]. We found that only 3 values of these expressions out of the set of computed 592 values were negative.

[8]This is a sufficient (but not necessary) condition for a quantile function interpretation.

We observe that significantly different ratios $\sigma_{0,0}/\sigma_{1,0} = 0.54$ and $\sigma_{0,1}/\sigma_{1,1} = 0.80$, pertain to different mixtures of Gaussian and Cauchy tails determined by the lagged market returns. The estimated model provides the 10 % and 90 % daily VaR that allow us to analyse the required capital for either a short, or a long position in the index. The estimated daily 10 % and 90 % VaR are plotted in Figure 2.

[Insert Figure 2 : 10 % and 90% VaR for TSX Return].

Figure 3 shows all nine series of estimated deciles and illustrates the fact that the decile estimators satisfy the monotonicity property with respect to risk level α. As expected, the extreme deciles feature much more variability than the middle range deciles [9].

[Insert Figure 3 : Dynamic Decile Estimation].

Let us now discuss the specification tests based on the hit variables. Figure 4 displays the confidence interval for the mean of hit residuals, at different risk levels: 0.1, 0.2, ..., 0.9. The upper bound (resp. lower bound) of the confidence interval is plotted by the dotted line (resp. solid line). When the model is well-specified, the mean of the hit variables is close to the theoretical value zero. We observe that zero lies inside the confidence intervals, for all risk levels, except for the extreme risk levels of 0.1 and 0.9. This indicates a misspecification of the DAQ model that will be corrected by rolling estimations described later in the text.

[Insert Figure 4 : Mean of the Hit Variables]

The first-order auto- and cross-correlations of hit variables are given in Table 3. They are nonsignificant for absolute values less than 0.14. We

[9] Note that according to the regulations, the required capital is derived by smoothing the VaR to reduce the variability (see, Gourieroux, Jasiak (2001) Chapter 16)

observe that all first-order correlations are nonsignificant, except the first-order autocorrelation of the hit variable at $\alpha = 0.1$.

Table 3: First-Order Correlation of Hit Variables

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	0.206	0.092	0.122	0.064	0.018	0.102	0.069	0.044	0.043
0.2	0.046	0.102	0.093	0.047	0.021	0.032	0.034	-0.022	0.044
0.3	0.122	0.138	0.129	0.062	0.040	0.038	0.052	0.033	0.021
0.4	0.142	0.126	0.113	0.036	0.026	0.042	0.059	0.076	0.061
0.5	0.096	0.078	0.052	0.028	0.036	0.058	0.065	0.131	0.096
0.6	0.100	0.048	0.051	0.008	0.003	0.035	0.055	0.101	0.022
0.7	0.154	0.074	0.086	0.024	0.009	0.015	0.010	0.062	-0.003
0.8	0.097	0.000	-0.002	-0.066	-0.014	0.001	-0.044	0.007	-0.073
0.9	0.037	-0.089	-0.052	-0.067	-0.037	-0.012	-0.045	-0.065	-0.028

Higher order auto- and cross-correlations of hit variables were all found nonsignificant. At this point, it is interesting to mention that the series of estimated conditional deciles have large and significant first-order autocorrelations. In particular, for $Q_t(u)$, $u = 1, ..., 9$, their values are: 0.47, 0.35, 0.27, 0.25, 0.19, 0.19, 0.18, 0.20, 0.29, respectively. As expected, the model reveals a U-shape pattern of first-order autocorrelations of dynamic deciles plotted as a function of risk level α.

Let us now proceed to correcting the slight misspecification of the model suggested by Figure 4 and Table 3. By estimating the model on various subsamples, we find that the specification error is due to the variation of parameters in time. This finding indicates that the fit can be improved by using a rolling estimator [10]. The rolling estimation procedure is as follows. The dynamic quantile model is estimated on a window of 60 observations, equivalent to a 3-month period, and applied to the following set of 20 observations, equivalent to one month. The estimates obtained from the consecutive subsamples are reported in Table 4.

[10] which, in particular, is compatible with the suggestion of the Basle Committee.

Table 4: Estimated DAQ model,by rolling

period parameter $\times 10^2$	m_0	m_1	$\sigma_{0,0}$	$\sigma_{0,1}$	$\sigma_{1,0}$	$\sigma_{1,1}$
20-80	-12.02	21.85	-0.56	26.04	27.86	-12.87
40-100	-4.24	-5.61	6.79	-7.69	22.21	-3.81
60-120	12.27	-1.56	0.00	0.00	41.03	-24.95
80-140	-3.94	11.73	0.00	0.32	35.80	-17.31
100-160	0.62	11.28	40.80	-13.09	17.72	-8.42
120-180	9.52	11.43	33.69	-15.05	18.45	-8.37
140-200	13.08	13.45	28.45	-23.56	16.82	1.43
160 -220	15.00	14.32	22.95	-20.89	23.41	-2.91
180-240	15.50	-7.60	18.37	29.33	16.24	-19.72

As before, the nonnegativity condition for $\sigma_{0,0} + \sigma_{0,1}|y_{t-1} - \mu|$, and $\sigma_{1,0} + \sigma_{1,1}|y_{t-1} - \mu|$ has to be checked. The proportion of dates for which it was found satisfied is given in Table 5.

Table 5: Nonnegativity Condition in %

period	Gaussian	Cauchy
20-80	90	100
40-100	90	100
60-120	100	100
80-140	100	98.5
100-160	100	98.5
120-180	100	100
140-200	98.5	100
160-220	95	100
180-240	100	97.5

Finally, it is interesting to investigate the effect of time varying parameters on the Value-at-Risk estimator. The rolling VaR's are displayed in Figure 5 (daily returns $\times$ 100).

[Insert Figure 5: 10% and 90% VaR for TSX Returns, by Rolling]

The specification tests performed on the rolling hit variables do not reject the rolling DAQ models.

5.2 Dynamic Analysis of Cross-Sectional Return Distribution

5.2.1 Factor Pricing Models

Multifactor pricing models in Finance are based on either the Arbitrage Pricing Theory developed by Ross (1976), or the Intertemporal Capital Asset Pricing Model developed by Merton (1973). These are usually written under a linear factor representation in which the return of asset i for period $(t, t+1)$ is :

$$y_{i,t} = r_{f,t} + \beta_i' Z_t + \varepsilon_{i,t}, i = 1, \ldots, n_t, \tag{5.1}$$

where $r_{f,t}$ denotes the riskfree rate and the idiosyncratic errors $(\varepsilon_{i,t})$ are assumed iid, with mean zero and variance σ^2.

Let us assume that the number of assets is large $(n_t \to \infty)$ and that the idiosyncratic error and the heterogenous sensitivity coefficients β_i have a joint Gaussian distribution. The cross-sectional distribution at time t is such that :

$$\begin{aligned} & P_t[y_{i,t} - r_{f,t} > x] \\ = \; & P_t[\beta_i' Z_t + \varepsilon_{i,t} > x] \\ = \; & \Phi[(\sigma^2 + Z_t' \Omega_\beta Z_t)^{-1/2} (\mu_\beta' Z_t - x)], \end{aligned}$$

where : $\mu_\beta = E(\beta_i), \Omega_\beta = V(\beta_i)$. We deduce :

$$Q_t(u) = \mu_\beta' Z_t - r_{f,t} - (\sigma^2 + Z_t' \Omega_\beta Z_t)^{1/2} \Phi^{-1}(u). \tag{5.2}$$

Thus, the dependence between the cross-sectional conditional quantile function and the factor is nonlinear, due to the effect of risk and the assumption of normality. This remark leads to a model of the type $Q_t(u) = Z_{o,t} + Z_{1,t} \Phi^{-1}(u)$, in which the factor components $Z_{o,t}$, $Z_{1,t}$ satisfy a nonlinear deterministic relationship. This model is a benchmark for the following application, involving the stock returns on TSX.

5.2.2 The return data

On each trading day t, we consider the set of 40 largest stocks traded on the TSX. Their daily returns are ordered to construct the series of cross-sectional deciles. The dynamics of the cross-sectional deciles are displayed in Figure 6.

[Insert Figure 6 : Deciles for the 40 Largest TSX Companies]

The nine decile series are constrained by the monotonicity property that affects their distributional properties. As expected, the extreme deciles are much more volatile than the intermediate deciles. Moreover, the large deciles [resp. small deciles] have right skewed [resp. left skewed] stationary distributions. As an illustration, the medians and quartiles of the marginal distributions for the decile series are displayed in Table 5.

Table 5: Median and Quartiles of the Decile Series

decile	1/10	2/10	3/10	4/10	5/10	6/10	7/10	8/10	9/10
1Q	−0.0041	−0.0041	−0.0038	−0.0035	−0.0030	−0.0027	−0.0028	−0.0030	−0.0032
median	−0.0002	−0.0003	−0.0005	−0.0001	−0.0000	−0.0002	0.0002	0.0003	0.0007
3Q	0.0036	0.0033	0.0036	0.0031	0.0030	0.0025	0.0029	0.0034	0.0035
interquartile value	0.0077	0.0074	0.0074	0.0066	0.0060	0.0052	0.0057	0.0064	0.0067

We observe that the medians of the decile series are close to zero.

5.3 Estimation of the QFM factor model

Prior to estimation, a factor analysis is performed to determine how many factors K have to be introduced in addition to the factor that represents the cross-sectional intercept. The eigenvalues ranked in a descending order are :

$$0.601, 0.013, 0.004, 0.003, 0.0011, 0.001, 0.000, \ldots$$

The large gap between the first eigenvalue and the following ones suggests that $K = 1$. Thus, we proceed to estimation of a model of the type :

$$Q_t(u) \sim Q(u) Z_{1,t} + Z_{o,t}.$$

This model is estimated by an approximated Kalman filter under the assumption that factors $Z_{0,t}, Z_{1,t}$ have an autoregressive representation. The estimated baseline quantile function is given in Table 6.

Table 6: Estimated Baseline Quantile Function

Q(1/10)	Q(2/10)	Q(3/10)	Q(4/10)	Q(5/10)	Q(6/10)	Q(7/10)	Q(8/10)	Q(9/10)
-0.7085	-0.4057	-0.2310	-0.0971	0.0004	0.1031	0.2317	0.4015	0.7051

The Q-Q plot is given in Figure 7 below. The distribution is almost symmetric, and quite close to a Gaussian distribution, although with fatter tails.

[Insert Figure 7: Comparison with the Gaussian Distribution].

The underlying factors feature weakly dependent dynamics. The first factor $Z_{0,t}$ is interpreted as a conditional centrality parameter and follows an AR(2) model:

$$Z_{0,t} = \text{const} + \underset{(0.06)}{0.17} Z_{0,t-1} + \underset{(0.06)}{0.16} Z_{0,t-2} + v_{0,t}.$$

The second factor $Z_{1,t}$ is interpreted as a conditional dispersion parameter. As expected, this factor features long memory (the so-called dispersion clustering). It follows an ARIMA model:

$$\Delta Z_{1,t} = \text{const} - \underset{(0.08)}{0.11} \Delta Z_{1,t-1} + v_{1,t} + \underset{(0.05)}{0.74} v_{1,t-1}.$$

The filtered values of factors are plotted in Figure 8.

[Insert Figure 8 a,b: Filtered Factor Values]

Finally, Figure 9 is a plot of $Z_{1,t}^2$ against $Z_{0,t}$. As expected, we observe a parabolic form of relationship between Z_1^2 and Z_0, which is compatible with the factor pricing model (5.1), (5.2).

[Insert Figure 9 : Relation Between Factors]

6 Concluding Remarks

For empirical applications, the standard specification based on the conditional transition density is inconvenient for dynamic modelling of the Value-at-Risk. Following the approach of Engle, Manganelli (2001), (2004), this paper introduces the Dynamic Additive Quantile (DAQ) model, which ensures the monotonicity of conditional quantile functions. This paper shows that asymptotically efficient estimators of a DAQ model can be derived by using an information based estimation approach, and discusses the specification tests of the DAQ. A quantile factor model (QFM) for panel data is also proposed. It can be used for the analysis of the behaviour of cross-sectional distributions over time. The flexibility of both DAQ and QFM specifications is illustrated by applications to returns of stocks traded on the Toronto Stock Exchange.

Appendix 1
Proof of Proposition 1

The asymptotic properties of the KLIC estimators are based on standard results on empirical processes of stationary mixing sequences [see e.g. Arcones, Yu (1994), VanderVaart, Wellner (1996)]. A detailed description of the set of regularity conditions is beyond the scope of this paper. We provide the asymptotic expansions to justify the asymptotic normality and efficiency of the estimators.

The estimator is solution of the optimization problem :

$$\begin{aligned}\hat{\theta}_T &= \arg\min_\theta \sum_{t=1}^{T} \left\{ \int f(y|x_t;\theta) \log f(y|x_t;\theta) dy \right.\\ &\qquad \left. - \int f(y|x_t;\theta) \log \hat{f}_{0T}(y|x_t) dy \right\} \\ &= \arg\min_\theta \sum_{t=1}^{T} \left\{ \int f_t(y;\theta) \log f_t(y;\theta) dy - \int f_t(y;\theta) \log \hat{f}_{0t}(y) dy \right\},\end{aligned}$$

say.

i) First-order conditions

They are given by :

$$\sum_{t=1}^{T} \left\{ \int \frac{\partial f_t}{\partial \theta}(y;\hat{\theta}_T) \log f_t(y;\hat{\theta}_T) dy + \int \frac{\partial f_t}{\partial \theta}(y;\hat{\theta}_T) dy - \int \frac{\partial f_t}{\partial \theta}(y;\hat{\theta}_T) \log \hat{f}_{0t}(y) dy \right\}$$

$= 0,$

or :

$$\sum_{t=1}^{T} \left\{ \int \frac{\partial f_t}{\partial \theta}(y;\hat{\theta}_T) \log f_t(y;\hat{\theta}_T) dy - \int \frac{\partial f_t}{\partial \theta}(y;\hat{\theta}_T) \log \hat{f}_{0t}(y) dy \right\} = 0,$$

since : $\int \frac{\partial f_t}{\partial \theta}(y;\hat{\theta}_T) dy = 0.$

ii) Expansion of the first-order conditions

Let us consider the expansion when $\hat{\theta}_T$ is in a neighbourhood of θ_0. We get :

$$\sum_{t=1}^{T} \left\{ \int \left[\frac{\partial f_t}{\partial \theta}(y;\theta_0) + \frac{\partial^2 f_t}{\partial \theta \partial \theta'}(y;\theta_0)(\hat{\theta}_T - \theta_0) \right] \log[f_t(y;\theta_0) + \frac{\partial f_t}{\partial \theta'}(y;\theta_0)(\hat{\theta}_T - \theta_0)]dy \right\}$$

$$- \int \left[\frac{\partial f_t}{\partial \theta}(y;\theta_0) + \frac{\partial^2 f_t}{\partial \theta \partial \theta'}(y;\theta_0)(\hat{\theta}_T - \theta_0) \right] \log \left[f_t(y;\theta_0) + \hat{f}_{0t}(y) - f_t(y;\theta_0)]dy \right\} \simeq$$
0,

or :

$$\sum_{t=1}^{T} \int \frac{1}{f_t(y;\theta_0)} \frac{\partial f_t}{\partial \theta}(y;\theta_0) \frac{\partial f_t}{\partial \theta'}(y;\theta_0) dy (\hat{\theta}_T - \theta_0)$$

$$- \sum_{t=1}^{T} \int \frac{\partial f_t}{\partial \theta}(y;\theta_0) \frac{1}{f_t(y;\theta_0)} [\hat{f}_{0t}(y) - f_t(y;\theta_0)]dy \simeq 0.$$

We deduce that :

$$\begin{aligned} \sqrt{T}(\hat{\theta}_T - \theta_0) &\simeq \left[\frac{1}{T} \sum_{t=1}^{T} E_t \left[\frac{\partial \log f_t(y;\theta_0)}{\partial \theta} \frac{\partial \log f_t(y;\theta_0)}{\partial \theta'} \right] \right]^{-1} \\ & \sqrt{T} \sum_{t=1}^{T} \int \frac{\partial \log f_t(y;\theta_0)}{\partial \theta} \left[\hat{f}_{0t}(y) - f_t(y;\theta_0) \right] dy \\ &\simeq \left(E \left[\frac{\partial \log f_t(y;\theta_0)}{\partial \theta} \frac{\partial \log f_t(y;\theta_0)}{\partial \theta'} \right] \right)^{-1} \\ & \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \int \frac{\partial \log f_t(y;\theta_0)}{\partial \theta} [\hat{f}_{0t}(y) - f_t(y;\theta_0)]dy. \end{aligned}$$

The result follows from the asymptotic properties of the kernel estimator of the conditional density. Indeed, we get :

$$
1/\sqrt{T}\sum_{t=1}^{T}\int \frac{\partial \log f_t(y;\theta_0)}{\partial\theta}\left[\hat{f}_{0t}(y)-f_t(y;\theta_0)\right]dy
$$

$$
\rightsquigarrow N\left(0, E\left[\frac{\partial \log f_t(y;\theta_0)}{\partial\theta}\frac{\partial \log f_t(y;\theta_0)}{\partial\theta'}\right]\right).
$$

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Figure 1: Daily Market Returns

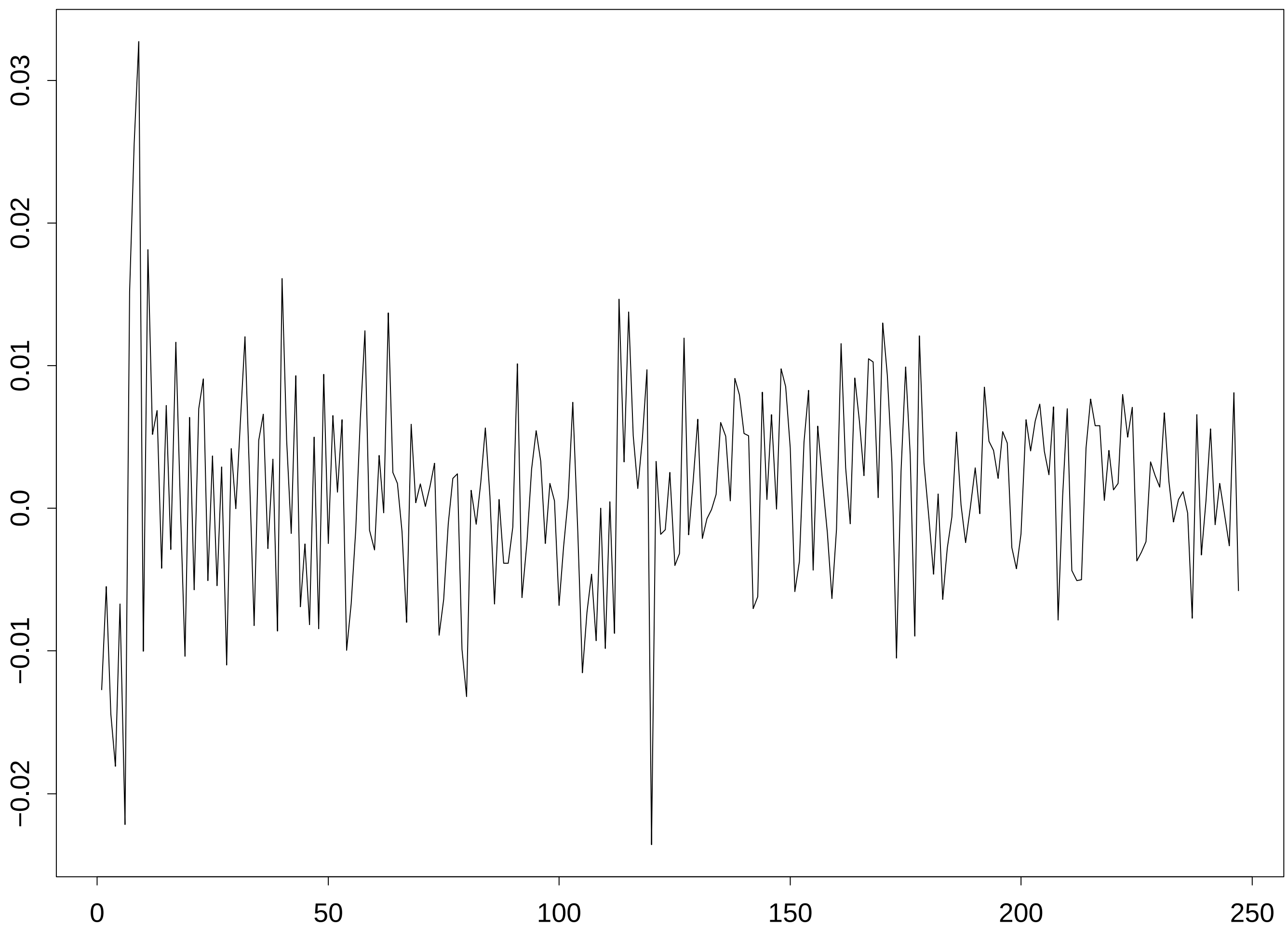


Figure 2: 10% and 90% VaR for TSX Returns

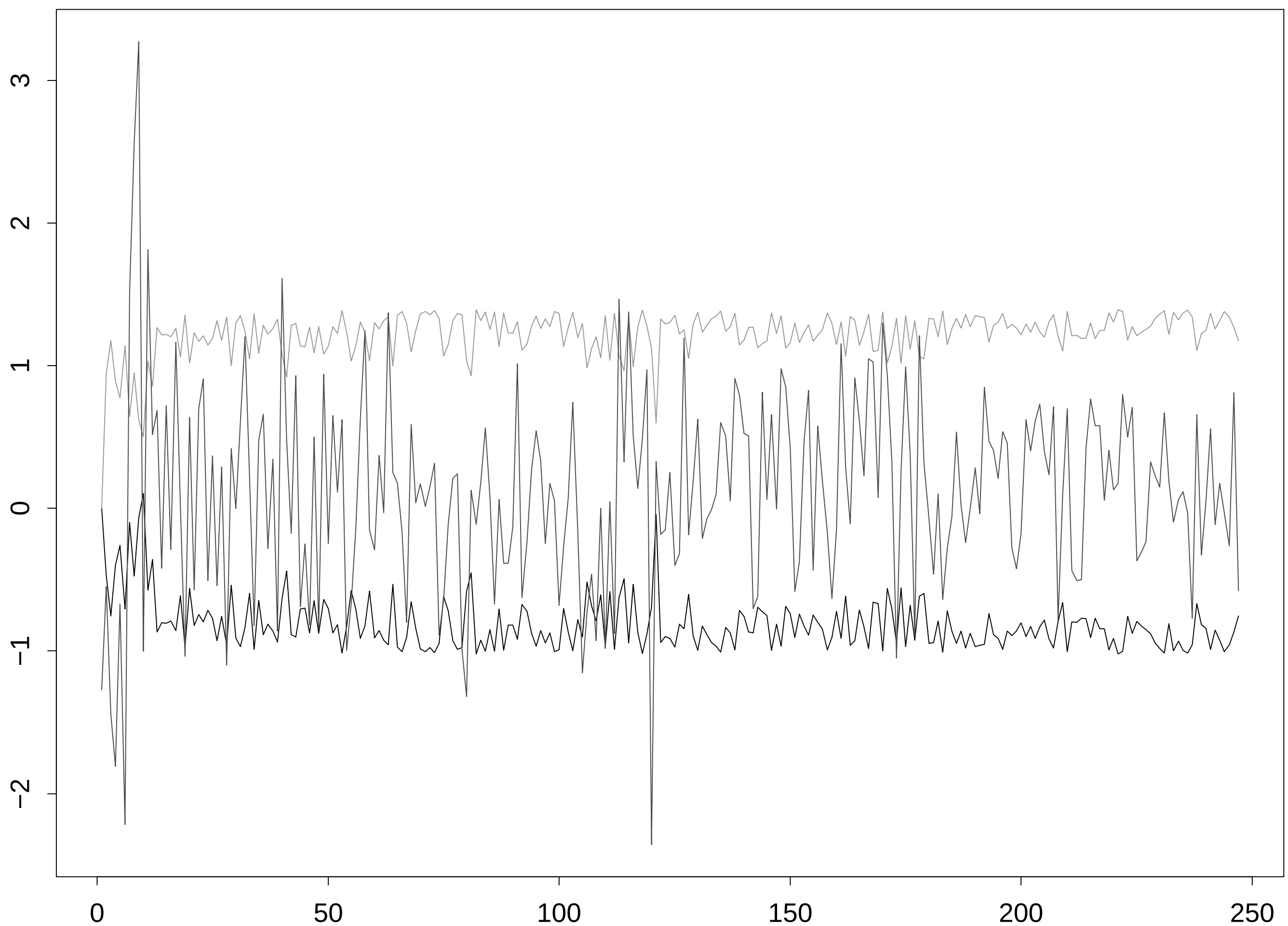


Figure 3: Dynamic Decile Estimation

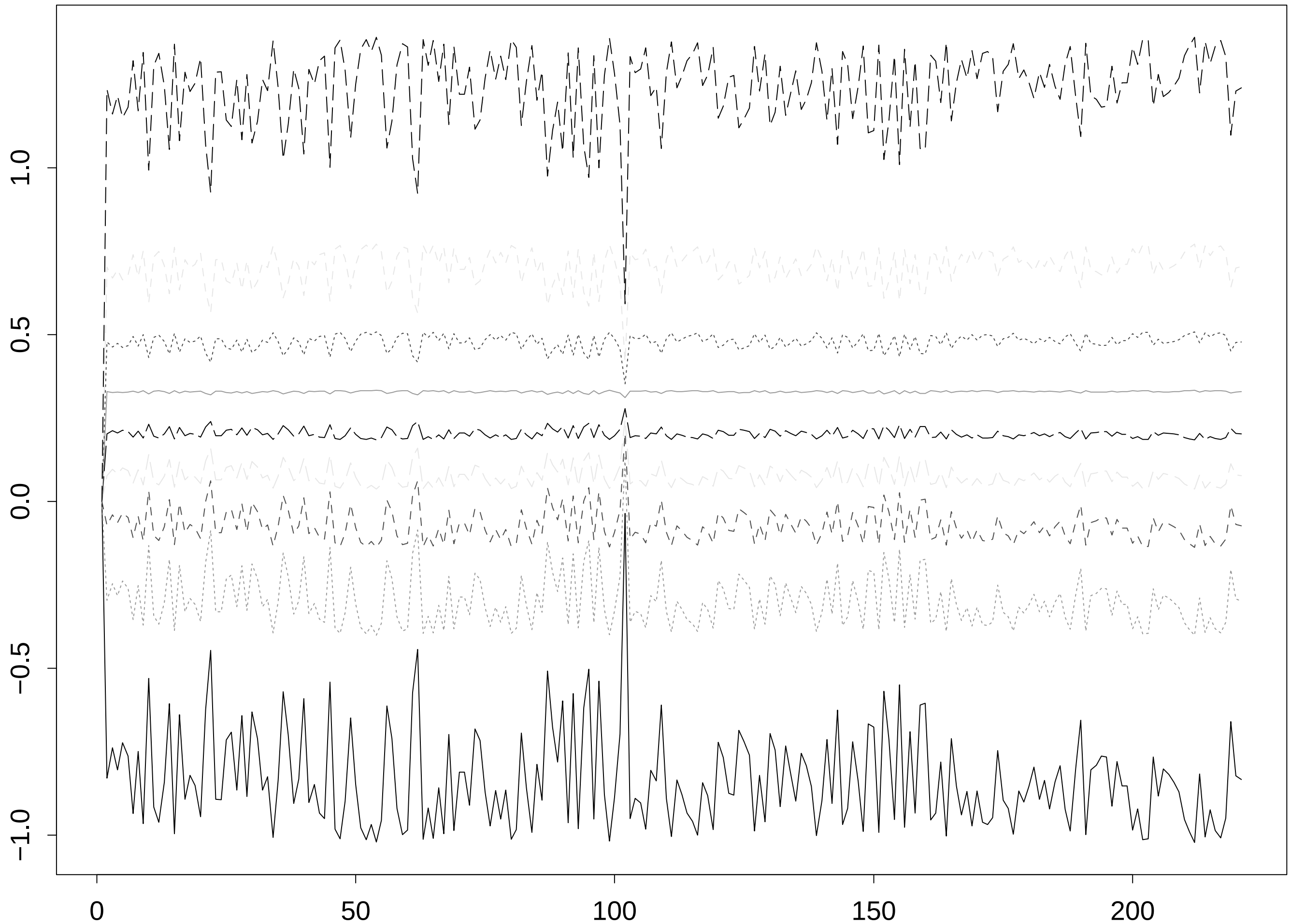


Figure 4: Mean of the Hit Variables

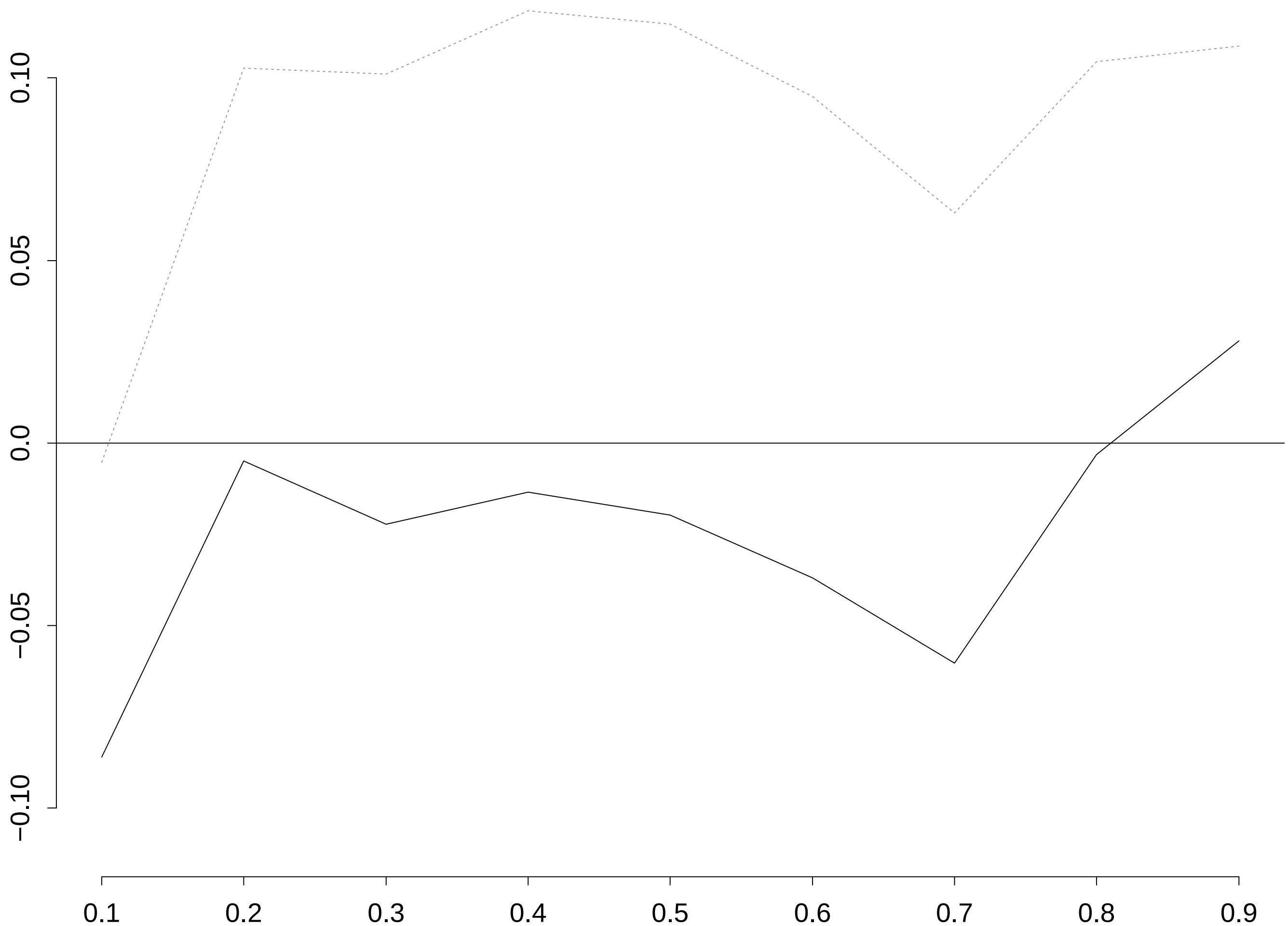


Figure 5: 10% and 90% VaR for TSX Returns, by Rolling

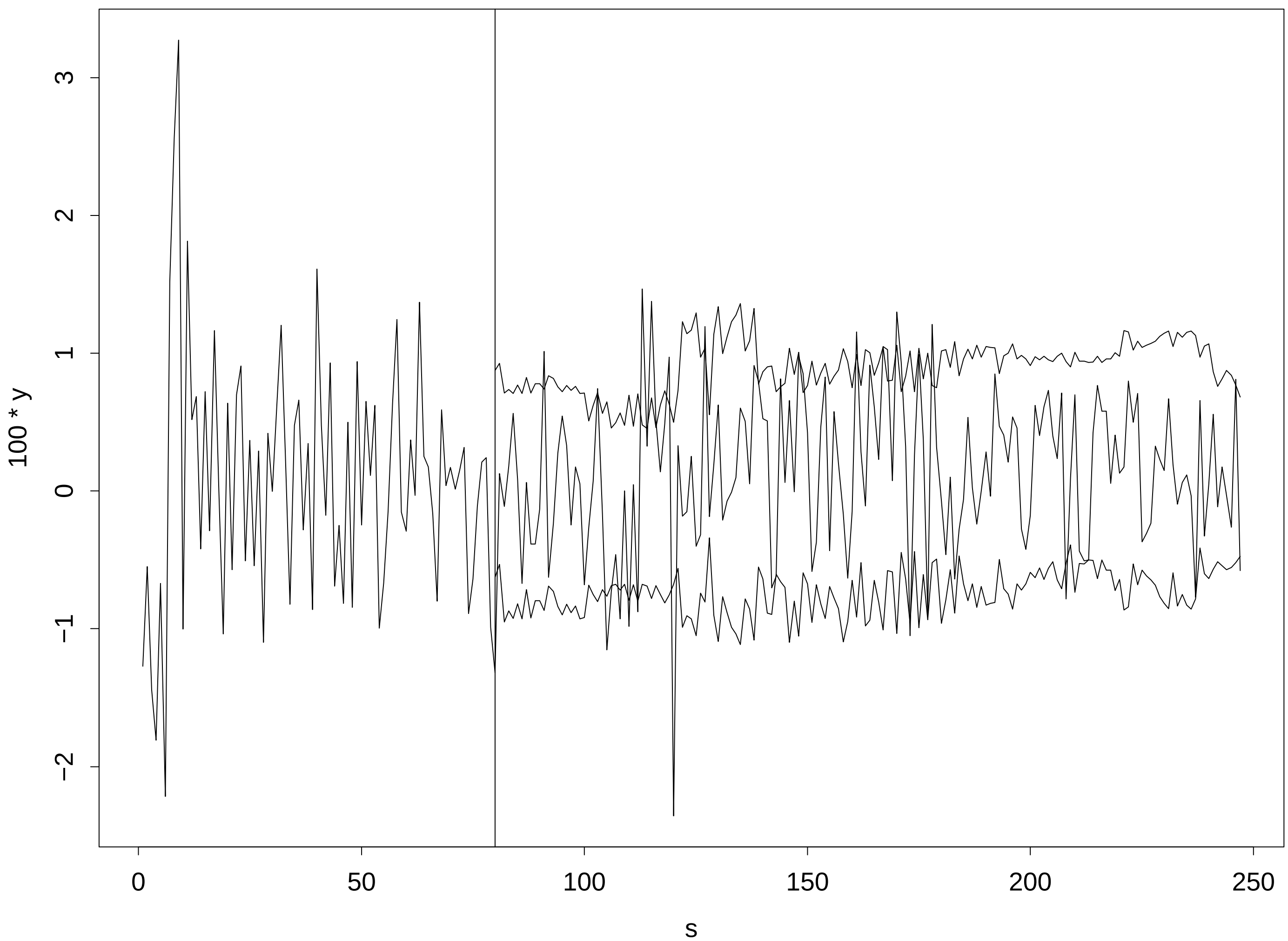
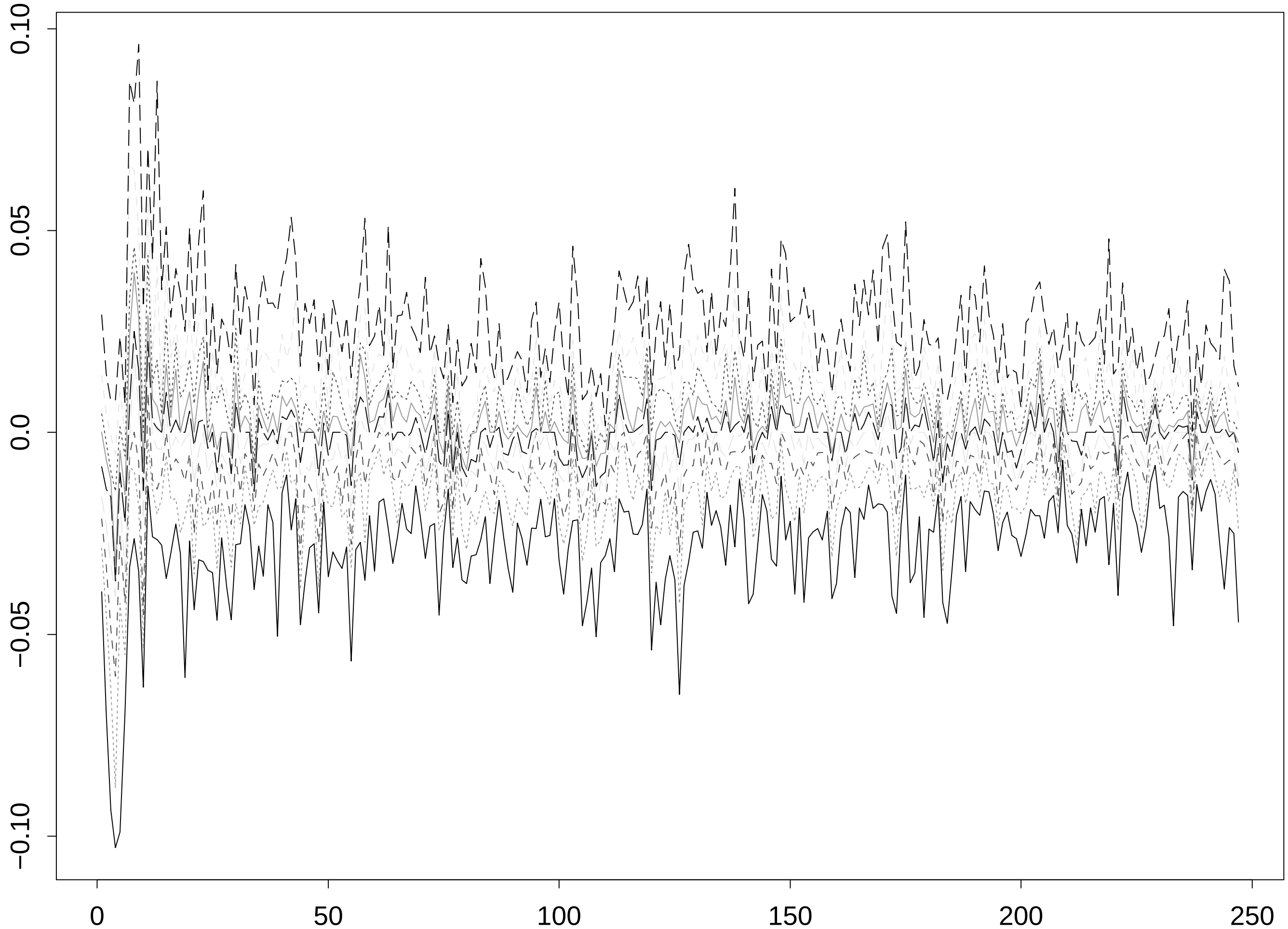


Figure 6: Deciles for the 40 Largest TSX Companies



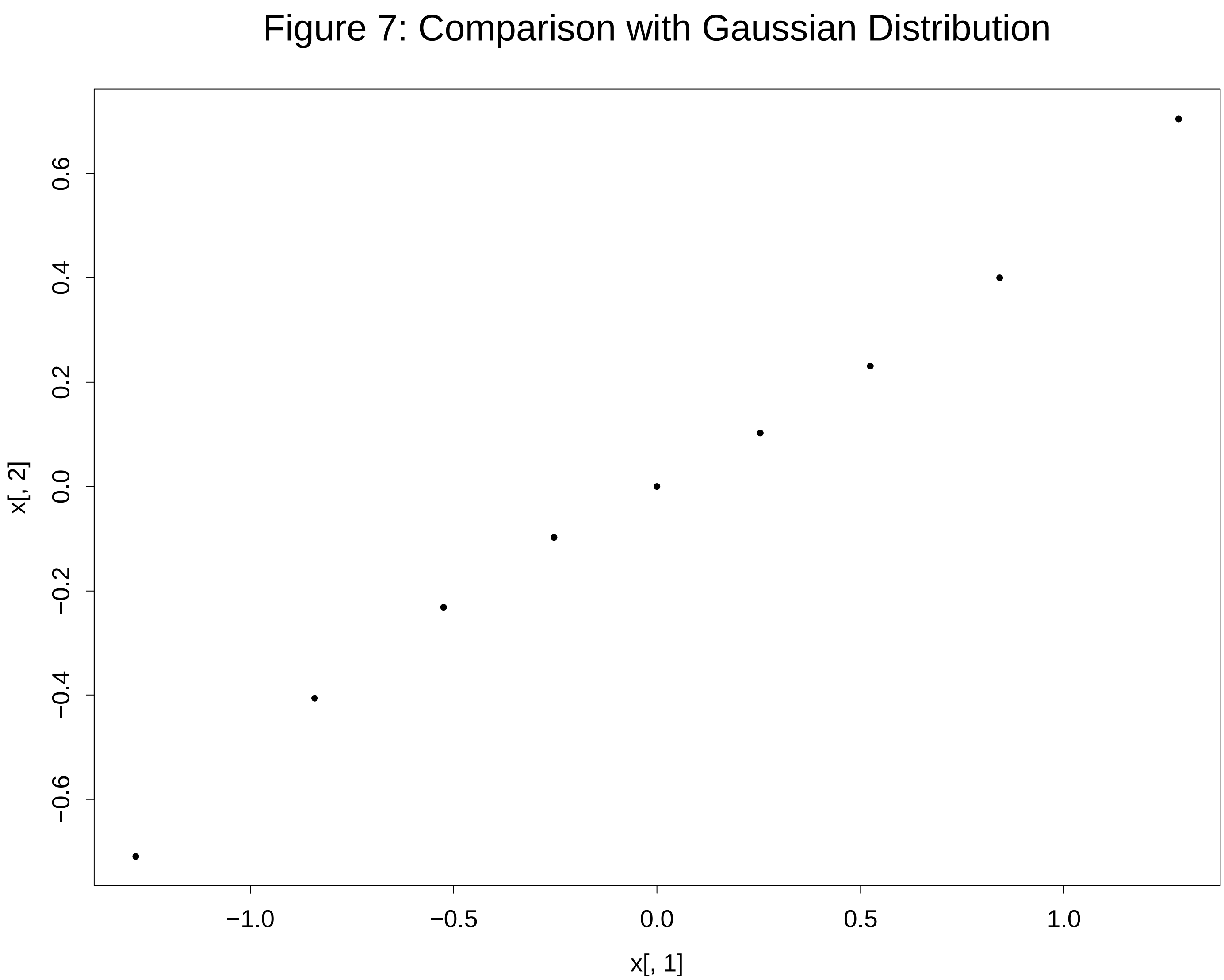


Figure 7: Comparison with Gaussian Distribution
x[, 2]
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−1.0
−0.5
0.0
0.5
1.0
x[, 1]

Figure 8a: Filtered Factor Values Z_0

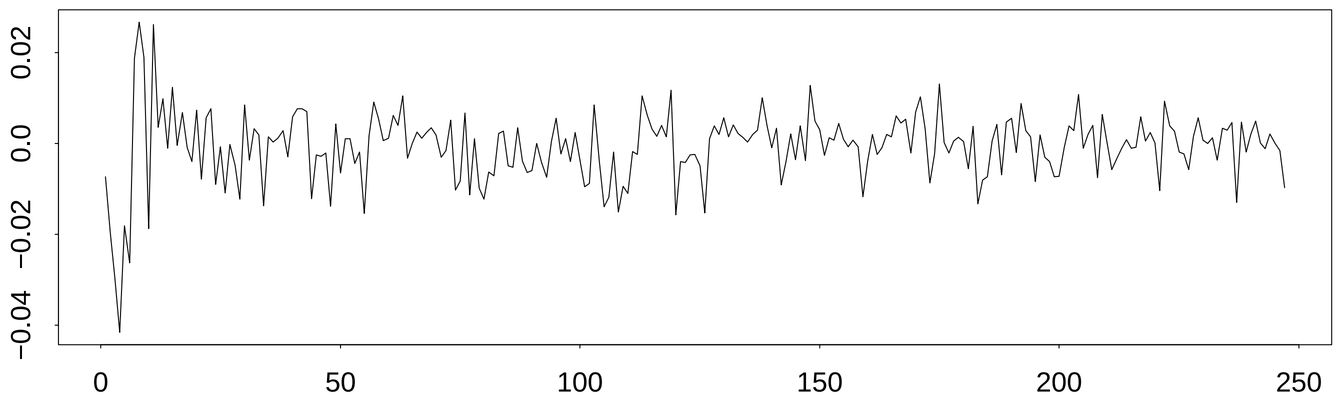


Figure 8b:Filtered Factor Values Z_1

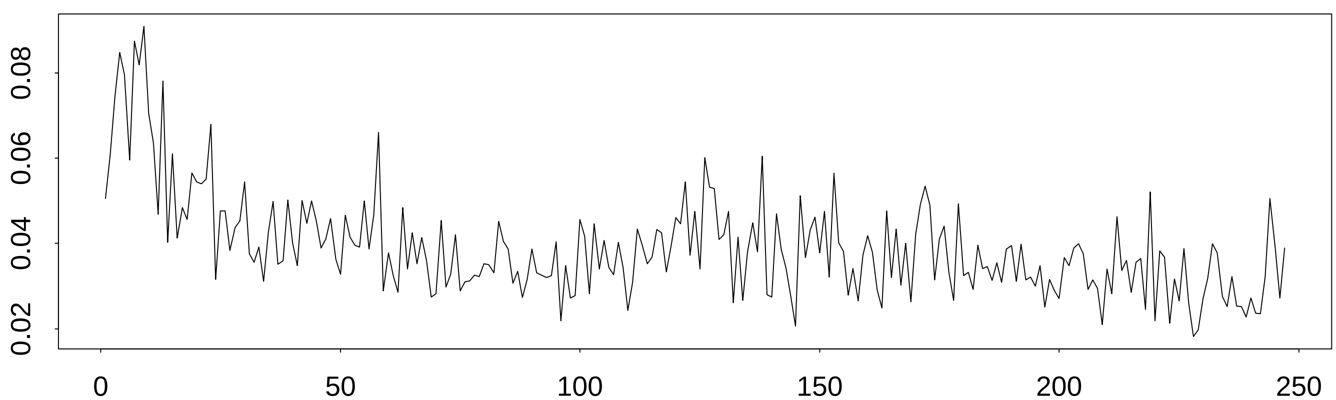


Figure 9: Relation Between Factors