Flexible Functional Forms: Bernstein Polynomials

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November 12, 2001§

Abstract

Motivated by the economic theory of cost functions, bivariate Bernstein polynomials are considered for approximating shape-restricted functions that are continuous, non-negative, monotone non-decreasing, concave, and homogeneous of degree one. We show the explicit rates of convergence of our approximating polynomials for general functions. We prove some interesting properties of bivariate Bernstein polynomials, including bimonotonicity for concave functions. Moreover, using the classical results, global approximations for shape-restricted functions can be achieved. We also note that concavity violation by the bivariate Bernstein polynomials occurs when the underlying true function is homogeneous of degree one. However, this violation diminishes as indices get large.

Keywords and Phrases: bivariate Bernstein polynomials, rate of convergence, functional forms, flexibility, cost functions.

Mathematics Subject Classification: 41A10, 41A25, 41A63, 26B25.

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§The authors would like to thank P. Rilstone, T. Stengos and A. Wong for their helpful comments.

1 Introduction

In this paper, we examine the approximation problem for shape-restricted functions that are continuous, non-negative, monotone non-decreasing, concave, and homogeneous of degree one, the so-called regularity conditions, conditions which are imposed by the economic theory of cost functions. Our approach to this approximation problem incorporates a probability result which can be found in Feller (1968). The probability result is a one-dimensional one: for each bounded continuous function f and any distribution function $G_{n,\theta}$, n=1,2,..., whose variance goes to zero uniformly as n goes to infinity, we can construct a functional form which approximates f uniformly in every finite interval. We extend this result to two variables and call it the *bivariate functional form generator*. One functional form that comes out of this generator is the sequence of two-variable Bernstein polynomials, which first appeared in Kingsley (1951).

We choose Bernstein polynomials to study the approximation problem. Bernstein polynomials have many interesting properties in approximation. The properties which are important here include the following. If the true function f is concave, then f majorizes the approximating sequence of polynomials in both univariate and bivariate cases. The Bernstein sequence inherits concavity and monotonicity from the function it is approximating in the univariate case, but does not necessarily preserve concavity in the bivariate case. In general, a Bernstein sequence converges asymptotically. Exceptions occur when the function is either affine or biaffine, in which case, the Bernstein polynomials will be identical with f. The most important property of Bernstein polynomials for our purpose is the following: for any bounded true function f on a closed unit square domain with derivatives of order m, there is a sequence of Bernstein polynomials (unique by definition) such that this sequence and the associated sequences of its derivatives converge uniformly to f and its derivatives. This property guarantees global flexibility,[1] a desirable property in shape-preserving approximation.

Our motivation for this study comes from approximation problems that arise in econometrics. In this context, one frequently wants to estimate a function of one or more variables which is known a priori to be continuous, non-negative, monotone non-decreasing and concave. Examples are production functions or cost functions. Typically we have data that gives us estimates of the values of the function at some discrete set of points, and we wish to find the best approximating function within some class. Unfortunately, many natural classes of functions such as Fourier functions yield approximations that are not necessarily monotone non-decreasing and concave. This motivated Chak (2001) to study the class of non-negative, monotone non-decreasing and concave Bernstein polynomials [see also Chak, Madras, and Smith (2001)]. These properties are relatively easy to enforce by means of restrictions on the Bernstein coefficients. Moreover, under some mild assumptions, they show that the corresponding sequence of constrained least squares estimators and its first and second derivatives converge to the true function and its associated derivatives (as the amount of data increases). These results require knowledge of approximation properties of Bernstein polynomials, and this led to the present work.

The rest of the paper is organized as follows. In section 2 we describe an

[1] A sequence of approximating functions is globally flexible or Sobolev-flexible if the sequences of functions and all relevant derivative functions converge to those of the true function in Sobolev norm. This definition of global flexibility appears to have been suggested by Gallant in 1981.

approach to generating approximating functional forms and extend the concept to the bivariate case. The functional forms that we consider for our approximation problem are the univariate and the bivariate Bernstein functional forms. In section 3 we look at the rates of convergence of the bivariate Bernstein polynomials. The approximation properties of the bivariate Bernstein polynomials are presented in section 4. In section 5 we examine what regularity conditions are satisfied by the univariate and bivariate Bernstein polynomials. In section 6, functions with Hessians having zero determinants and their Bernstein polynomials are discussed. Some graphic and numerical analysis are presented in section 7.

2 Generating Flexible Functional Forms

In this section, we consider an approach to the problem of approximation using probabilistic and statistical ideas. The approach is based on a lemma mentioned in Feller (p.218, 1968) which gives a variety of choices for one-dimensional functional forms. We extend the concept to the bivariate case to obtain the following result.

Theorem 1: *For* $n_1 = 1, 2, \ldots$ *and* $n_2 = 1, 2, \ldots$, *let* $G_{n_1,n_2,\theta_1,\theta_2}$ *be any joint distribution function with expectations* θ_1, θ_2 *and variances* $\sigma^2_{n_1}(\theta_1)$, $\sigma^2_{n_2}(\theta_2)$, *where* θ_1 *and* θ_2 *are parameters belonging to a finite or infinite interval, and* $\sigma^2_{n_i}(\theta_i)$ *is the variance of the marginal distribution of* X_i, $i = 1, 2$. *If*

$$E_{n_1,n_2,\theta_1,\theta_2}\Big[f(X_1, X_2)\Big] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x_1, x_2)\, G_{n_1,n_2,\theta_1,\theta_2}\,(dx_1, dx_2)$$

is the expectation of $f(X_1, X_2)$, *then we have the following:*

(a) *Let* f *be any bivariate bounded continuous function. For each* θ_1 *and* θ_2, *if*

$\sigma^2_{n_1}(\theta_1) \to 0$, $\sigma^2_{n_2}(\theta_2) \to 0$ *as* $n_1 \to \infty$, $n_2 \to \infty$, *then*

$$\lim_{n_1 \to \infty, n_2 \to \infty} E_{n_1,n_2,\theta_1,\theta_2}\Big[f(X_1, X_2) \Big] \;=\; f(\theta_1, \theta_2) \,.$$

(b) *If* f *is uniformly continuous, then the convergence is uniform in every bounded set in which* $\sigma^2_{n_1}(\theta_1) \to 0$ *and* $\sigma^2_{n_2}(\theta_2) \to 0$ *uniformly.*

Proof: We first prove part (b) and note that the proof of part (a) is similar.

$$\begin{aligned} & \Big| E_{n_1,n_2,\theta_1,\theta_2}\Big[f(X_1, X_2) \Big] \;-\; f(\theta_1, \theta_2) \Big| \\ \leq \; & \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Big| f(x_1, x_2) \;-\; f(\theta_1, \theta_2) \Big| G_{n_1,n_2,\theta_1,\theta_2}\,(dx_1, dx_2). \end{aligned}$$

If f is uniformly continuous on its domain, it follows that for $\epsilon > 0$ there exists a δ not depending on x_1, x_2, θ_1 and θ_2 such that for $|x_1 - \theta_1| < \delta$ and $|x_2 - \theta_2| < \delta$, we have $|f(x_1, x_2) - f(\theta_1, \theta_2)| < \epsilon$. This implies that

$$\int_{|x_1-\theta_1|<\delta} \int_{|x_2-\theta_2|<\delta} \Big| f(x_1, x_2) \;-\; f(\theta_1, \theta_2) \Big| G_{n_1,n_2,\theta_1,\theta_2}\,(dx_1, dx_2) \;<\; \epsilon \,.$$

Now consider the region outside the square. Since $f(x_1, x_2)$ is bounded, there exists a constant M such that $|f(x_1, x_2) - f(\theta_1, \theta_2)| < M$. Hence

$$\begin{aligned} & \int_{|x_1-\theta_1|>\delta \text{ or }} \int_{|x_2-\theta_2|>\delta} \Big| f(x_1, x_2) \;-\; f(\theta_1, \theta_2) \Big| G_{n_1,n_2,\theta_1,\theta_2}\,(dx_1, dx_2) \\ < \; & M \int_{|x_1-\theta_1|>\delta \text{ or }} \int_{|x_2-\theta_2|>\delta} G_{n_1,n_2,\theta_1,\theta_2}\,(dx_1, dx_2) \\ \leq \; & M \Bigg[\sum_{i=1}^{2} Pr\Big(|X_i - \theta_i| > \delta \Big) \Bigg], \;\; \forall \; \theta_1 \text{ and } \theta_2 \\ \leq \; & M \Bigg[\frac{\sigma^2_{n_1}(\theta_1)}{\delta^2} + \frac{\sigma^2_{n_2}(\theta_2)}{\delta^2} \Bigg], \;\; \forall \; \theta_1 \text{ and } \theta_2 \,, \end{aligned}$$

where the last expression is obtained by using Chebyshev's inequality. Thus it comes to

$$\Big| E_{n_1,n_2,\theta_1,\theta_2}\Big[f(X_1, X_2) \Big] - f(\theta_1, \theta_2) \Big| < \epsilon + M \Bigg[\frac{\sigma^2_{n_1}(\theta_1)}{\delta^2} + \frac{\sigma^2_{n_2}(\theta_2)}{\delta^2} \Bigg].$$

Note that if $\sigma^2_{n_1}(\theta_1) \to 0$ and $\sigma^2_{n_2}(\theta_2) \to 0$ uniformly in θ_1 and θ_2, then there exists an N not depending on θ_1 and θ_2 such that for $n_1,\, n_2 > N$, we get

$$M\frac{\sigma^2_{n_1}(\theta_1)}{\delta^2} < \epsilon \quad \text{and} \quad M\frac{\sigma^2_{n_2}(\theta_2)}{\delta^2} < \epsilon\,.$$

So for sufficiently large n_1 and n_2

$$\left| E_{n_1,n_2,\theta_1,\theta_2}\left[\, f(X_1,X_2)\,\right] - f(\theta_1,\theta_2) \right| < \epsilon + \epsilon + \epsilon = 3\epsilon\,.$$

Therefore

$$\lim_{n_1\to\infty,\, n_2\to\infty} E_{n_1,n_2,\theta_1,\theta_2}\left[\, f(X_1,X_2)\,\right] \;=\; f(\theta_1,\theta_2)\,. \qquad \square$$

We name the expression $E_{n_1,n_2,\theta_1,\theta_2}[f(X_1,X_2)]$ the *bivariate functional form generator*. This result can easily be generalized to the d-variable case and the *d-dimensional functional form generator* can be expressed as follows

$$\begin{aligned} & E_{n_1,\ldots,n_d,\theta_1,\ldots,\theta_d}\left[\, f(X_1,\ldots,X_d)\,\right] \\ = & \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1,\ldots,x_d)\, G_{n_1,\ldots,n_d,\theta_1,\ldots,\theta_d}(x_1,\ldots,x_d)\, dx_1\,\ldots\, dx_d\,. \end{aligned}$$

We get one example of Theorem 1 by taking X_1, X_2 to be independent Binomial random variables, i.e., $X_i \sim \mathrm{Bin}(n_i, x_i)$, $x_i \in$[0,1]. Then the resulting functional form is the sequence of two-variable Bernstein polynomials [see Kingsley (1951)] and it has the following form:

$$\begin{aligned} B^f_{n_1,n_2}(x_1,x_2) &= \sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2} f\left(\frac{v_1}{n_1},\frac{v_2}{n_2}\right)\binom{n_1}{v_1}\binom{n_2}{v_2} \\ &\qquad {x_1}^{v_1}{x_2}^{v_2}(1-x_1)^{n_1-v_1}(1-x_2)^{n_2-v_2} \\ &= \sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2} f\left(\frac{v_1}{n_1},\frac{v_2}{n_2}\right) P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2)\,, \end{aligned}$$

where

$$P_{v,n}(x) = \binom{n}{v} x^v(1-x)^{n-v},$$

$$(x_1, x_2) \in [0,1] \times [0,1] \,, \text{ and } n_1,\, n_2 \in I\!N \,.$$

Corollary: $\lim_{n_1 \to \infty, n_2 \to \infty} B^f_{n_1,n_2}(x_1, x_2) = f(x_1, x_2)$. *If f is continuous on the closed unit square, then the convergence is uniform.*

Using the d-dimensional functional form generator, the d-variable Bernstein polynomials can be expressed as follows:

$$B^f_{n_1,\ldots,n_d}(x_1, \ldots, x_d) = \sum_{v_1=0}^{n_1} \cdots \sum_{v_d=0}^{n_d} f\left(\frac{v_1}{n_1}, \cdots, \frac{v_d}{n_d}\right) \binom{n_1}{v_1} \cdots \binom{n_d}{v_d} P_{v_1,n_1}(x_1) \cdots P_{v_d,n_d}(x_d) \,,$$

$$(x_1, \ldots, x_d) \in [\,0\,,\,1\,] \times \cdots \times [\,0\,,\,1\,] \,, \text{ and } n_1, \ldots, n_d \in I\!N \,.$$

In the rest of this paper we shall mainly concentrate on the bivariate Bernstein polynomials.

3 Rates of Convergence of Bernstein Polynomials

In this section, we consider the rate of convergence of the Bernstein sequence $\{B^f_{n_1,n_2}\}$ to its determining function f. The degree of approximation of a function $f(x_1, x_2)$, $x_1, x_2 \in [a, b] \times [a, b]$, by polynomials may be expressed in terms of its *modulus of continuity*, denoted as $\omega(\delta) = \omega^f(\delta)$.

Lemma 1: *Let $f(x_1, x_2)$ be a continuous function on the closed unit square $S:\, 0 \le x_1 \le 1,\, 0 \le x_2 \le 1$. Let*

$$\omega(\delta) = sup\Big\{ |\, f(\tilde{Q}_1) - f(\tilde{Q}_2)|\, :\, \tilde{Q}_1, \tilde{Q}_2 \in S \text{ such that } \rho(\tilde{Q}_1, \tilde{Q}_2) < \delta \Big\} \,,$$

and let $\delta > 0$, Q_1 and Q_2 be two arbitrary points inside S. Suppose that $\lambda =$

$\lambda(Q_1, Q_2;\ \delta) = \lfloor \frac{\rho(Q_1,Q_2)}{\delta} \rfloor$ *is the integer part of* $\frac{\rho(Q_1,Q_2)}{\delta}$. *Then*

$$|f(Q_1) - f(Q_2)| \leq (\lambda + 1)\omega(\delta)\,.$$

Proof: If $\rho(Q_1, Q_2) < \delta$, then $\lambda = \lfloor \frac{\rho(Q_1,Q_2)}{\delta} \rfloor = 0$. It follows immediately that $|f(Q_1) - f(Q_2)| \leq \omega(\delta)$. Suppose that $\rho(Q_1, Q_2) \geq \delta$. Then $\lambda \geq 1$. Now let

$$\bar{Q}_j = Q_1 + \frac{j}{\lambda + 1}(Q_2 - Q_1), \quad j = 0, 1, ..., \lambda + 1.$$

Note that $\bar{Q}_0 = Q_1$ and $\bar{Q}_{\lambda+1} = Q_2$, and $\rho(\bar{Q}_j, \bar{Q}_{j+1}) < \delta$, which implies that $|f(\bar{Q}_j) - f(\bar{Q}_{j+1})| \leq \omega(\delta),\ j = 0, 1, ..., \lambda$. Therefore

$$\begin{aligned} |f(Q_1) - f(Q_2)| &\leq \sum_{j=0}^{\lambda} |f(\bar{Q}_j) - f(\bar{Q}_{j+1})| \\ &\leq (\lambda + 1)\omega(\delta)\,, \end{aligned}$$

as required. □

Theorem 2: *If* f *is continuous in* S *and* $\omega(\delta)$ *is the modulus of continuity of* $f(x_1, x_2)$*, then for* $n_1 = n_2 = n$ *we have*

$$\Big|\, f(x_1, x_2) \ - \ B^f_{n_1,n_2}(x_1, x_2) \,\Big| \ \leq \ \frac{5}{4}\,\omega(\delta)\,,$$

where

$$\delta = \left(\frac{2}{n}\right)^{\frac{1}{2}}.$$

[See Lorentz (1953) for the one-variable version.]

Proof:

$$\begin{aligned} &\Big|\, f(x_1, x_2) \ - \ B^f_{n_1,n_2}(x_1, x_2) \,\Big| \\ \leq \ &\omega(\delta)\Bigg\{ \sum_{v_1=0}^{n_1} \sum_{v_2=0}^{n_2} \left[1 + \lambda\left((x_1, x_2), \left(\frac{v_1}{n_1}, \frac{v_2}{n_2}\right);\ \delta \right) \right] P_{v_1,n_1}(x_1) P_{v_2,n_2}(x_2) \Bigg\} \end{aligned}$$

$$
\begin{aligned}
&\le \quad \omega(\delta)\left\{1+\frac{1}{\delta}\sum_{v_1,v_2:\,\lambda\ge 1}\rho\left((x_1,x_2),\left(\frac{v_1}{n_1},\frac{v_2}{n_2}\right)\right)P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2)\right\}\\
&\le \quad \omega(\delta)\left\{1+\frac{1}{\delta^2}\sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2}\rho^2\left((x_1,x_2),\left(\frac{v_1}{n_1},\frac{v_2}{n_2}\right)\right)P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2)\right\}\\
&\le \quad \omega(\delta)\left\{1+\frac{1}{4\delta^2}\left(\frac{n_1+n_2}{n_1n_2}\right)\right\}.
\end{aligned}
$$

Now choose

$$
n = n_1 = n_2 \quad \text{and} \quad \delta = \left(\frac{2}{n}\right)^{\frac{1}{2}}.
$$

Then we have

$$
\left| f(x_1,x_2) \; - \; B^f_{n_1,n_2}(x_1,x_2)\right| \;\le\; \frac{5}{4}\,\omega(\delta)\,. \qquad \square
$$

If f is a non-negative, monotone non-decreasing function in S, then $\omega(\delta)$ will be the variation of the function in the first δ-radius from the origin.

Remark: Let f_1 and f_2 be the first derivatives of f with respect to x_1 and x_2, and suppose that they are continuous over S. Let

$$
\omega_1(\delta) = sup\left\{| f_1(\tilde{Q}_1) - f_1(\tilde{Q}_2)| : \tilde{Q}_1,\tilde{Q}_2 \in S \text{ such that } \rho(\tilde{Q}_1,\tilde{Q}_2) < \delta\right\},
$$

$$
\omega_2(\delta) = sup\left\{| f_2(\tilde{Q}_1) - f_2(\tilde{Q}_2)| : \tilde{Q}_1,\tilde{Q}_2 \in S \text{ such that } \rho(\tilde{Q}_1,\tilde{Q}_2) < \delta\right\},
$$

and let $\delta > 0$, Q_1, Q_2 be two arbitrary points in S. Then we get results similar to the result in Lemma 1:

$$
|f_1(Q_1) - f_1(Q_2)| \le (\lambda+1)\omega_1(\delta)\,,
$$

$$
|f_2(Q_1) - f_2(Q_2)| \le (\lambda+1)\omega_2(\delta)\,,
$$

where $\lambda = \lambda(Q_1,Q_2;\,\delta) = \lfloor \frac{\rho(Q_1,Q_2)}{\delta} \rfloor$.

Now suppose that the first derivatives of f with respect to x_1 and x_2 are both defined and continuous on S. Do we get any improvement on the rate of convergence? This question will be answered by the next theorem.

Theorem 3: *Let f be a continuous function with continuous first partial derivatives f_1 and f_2 on S. Let $\delta > 0$. If $\omega_1(\delta)$ and $\omega_2(\delta)$ are the moduli of continuity of f_1 and f_2 respectively, then for $n_1{=}n_2{=}n$ we have*

$$\left| f(x_1, x_2) \; - \; B^f_{n,n}(x_1, x_2) \right| \; \leq \; \frac{3}{2\sqrt{n}} \, \tilde{\omega}(\delta) \; ,$$

where

$$\delta \; = \; \frac{1}{\sqrt{n}} \; ,$$

$$\tilde{\omega}(\delta) \; = \; max \left\{ \, \omega_1(\delta) \, , \, \omega_2(\delta) \, \right\} .$$

[The one-variable result can be found in Lorentz (1953).]

Proof: Note that

$$\begin{aligned} & \left| f(x_1, x_2) \; - \; B^f_{n_1,n_2}(x_1, x_2) \right| \\ = \; & \Bigg| \sum_{v_1=0}^{n_1} \sum_{v_2=0}^{n_2} \left[f(x_1, x_2) - f\left(\frac{v_1}{n_1}, x_2\right) + f\left(\frac{v_1}{n_1}, x_2\right) - f\left(\frac{v_1}{n_1}, \frac{v_2}{n_2}\right) \right] \\ & P_{v_1,n_1}(x_1) P_{v_2,n_2}(x_2) \Bigg| \; . \end{aligned}$$

By mean value theorem, the last expression is equal to

$$\begin{aligned} & \Bigg| \sum_{v_1=0}^{n_1} \sum_{v_2=0}^{n_2} \left\{ \left(x_1 - \frac{v_1}{n_1}\right) f_1(x_1, x_2) + \left(x_1 - \frac{v_1}{n_1}\right) \left[f_1(\xi, x_2) - f_1(x_1, x_2) \right] \right. \\ + \; & \left. \left(x_2 - \frac{v_2}{n_2}\right) f_2\left(\frac{v_1}{n_1}, x_2\right) + \left(x_2 - \frac{v_2}{n_2}\right) \left[f_2\left(\frac{v_1}{n_1}, \eta\right) - f_2\left(\frac{v_1}{n_1}, x_2\right) \right] \right\} \\ & P_{v_1,n_1}(x_1) P_{v_2,n_2}(x_2) \Bigg| \; , \end{aligned}$$

where ξ is between x_1 and $\frac{v_1}{n_1}$, and η is between x_2 and $\frac{v_2}{n_2}$. Note that the last expression does not exceed

$$\left| \sum_{v_1=0}^{n_1} \sum_{v_2=0}^{n_2} \left(x_1 - \frac{v_1}{n_1}\right) f_1(x_1, x_2) P_{v_1,n_1}(x_1) P_{v_2,n_2}(x_2) \right|$$

$$
\begin{aligned}
&+ \left|\sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2}\Big(x_2-\frac{v_2}{n_2}\Big)f_2\Big(\frac{v_1}{n_1},x_2\Big)P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2)\right| \\
&+ \omega_1(\delta)\Bigg\{\sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2}\Big|x_1-\frac{v_1}{n_1}\Big|\Big[\lambda_1\Big((x_1,x_2),\Big(\frac{v_1}{n_1},x_2\Big);\ \delta\Big)+1\Big] \\
&\quad P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2)\Bigg\} \\
&+ \omega_2(\delta)\Bigg\{\sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2}\Big|x_2-\frac{v_2}{n_2}\Big|\Big[\lambda_2\Big(\Big(\frac{v_1}{n_1},x_2\Big),\Big(\frac{v_1}{n_1},\frac{v_2}{n_2}\Big);\ \delta\Big)+1\Big] \\
&\quad P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2)\Bigg\},
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_1\Big((x_1,x_2),\Big(\frac{v_1}{n_1},x_2\Big);\ \delta\Big) &= \left\lfloor\frac{\rho\Big((x_1,x_2),\big(\frac{v_1}{n_1},x_2\big)\Big)}{\delta}\right\rfloor, \\
\lambda_2\Big(\Big(\frac{v_1}{n_1},x_2\Big),\Big(\frac{v_1}{n_1},\frac{v_2}{n_2}\Big);\ \delta\Big) &= \left\lfloor\frac{\rho\Big(\big(\frac{v_1}{n_1},x_2\big),\big(\frac{v_1}{n_1},\frac{v_2}{n_2}\big)\Big)}{\delta}\right\rfloor.
\end{aligned}
$$

Using the fact [Lorentz (1953), p.14] that

$$
T_{n_1,1}(x_1) \;=\; \sum_{v_1=0}^{n_1}(v_1-n_1x_1)P_{v_1,n_1}(x_1) \;=\; 0\,,
$$

$$
T_{n_2,1}(x_2) \;=\; \sum_{v_2=0}^{n_2}(v_2-n_2x_2)P_{v_2,n_2}(x_2) \;=\; 0\,,
$$

we obtain that

$$
\begin{aligned}
&\Big|\,f(x_1,x_2)\;-\;B^f_{n_1,n_2}(x_1,x_2)\,\Big| \\
\le\;& \omega_1(\delta)\Bigg\{\sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2}\Big|x_1-\frac{v_1}{n_1}\Big|P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2) \\
+\;& \sum_{v_1,v_2:\,\lambda_1\ge 1}\Big|x_1-\frac{v_1}{n_1}\Big|\lambda_1\Big((x_1,x_2),\Big(\frac{v_1}{n_1},x_2\Big);\ \delta\Big)P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2)\Bigg\} \\
+\;& \omega_2(\delta)\Bigg\{\sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2}\Big|x_2-\frac{v_2}{n_2}\Big|P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2) \\
+\;& \sum_{v_1,v_2:\,\lambda_2\ge 1}\Big|x_2-\frac{v_2}{n_2}\Big|\lambda_2\Big(\Big(\frac{v_1}{n_1},x_2\Big),\Big(\frac{v_1}{n_1},\frac{v_2}{n_2}\Big);\ \delta\Big)P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2)\Bigg\}\,.
\end{aligned}
$$

Since

$$\lambda_1\left((x_1, x_2), \left(\frac{v_1}{n_1}, x_2\right);\ \delta\right) \ \le\ \frac{1}{\delta}\left|x_1 - \frac{v_1}{n_1}\right| ,$$
$$\lambda_2\left(\left(\frac{v_1}{n_1}, x_2\right), \left(\frac{v_1}{n_1}, \frac{v_2}{n_2}\right);\ \delta\right) \ \le\ \frac{1}{\delta}\left|x_2 - \frac{v_2}{n_2}\right| ,$$

we have

$$\begin{aligned}
&\left| f(x_1, x_2) \ -\ B^f_{n_1,n_2}(x_1, x_2) \right| \\
\le\ & \omega_1(\delta)\left\{ \sum_{v_1=0}^{n_1} \left[\left|x_1 - \frac{v_1}{n_1}\right| + \frac{1}{\delta}\left(x_1 - \frac{v_1}{n_1}\right)^2 \right] P_{v_1,n_1}(x_1) \right\} \\
+\ & \omega_2(\delta)\left\{ \sum_{v_2=0}^{n_2} \left[\left|x_2 - \frac{v_2}{n_2}\right| + \frac{1}{\delta}\left(x_2 - \frac{v_2}{n_2}\right)^2 \right] P_{v_2,n_2}(x_2) \right\} \\
\le\ & \omega_1(\delta)\left\{ \frac{1}{2\sqrt{n_1}} + \frac{1}{4n_1\delta} \right\} + \omega_2(\delta)\left\{ \frac{1}{2\sqrt{n_2}} + \frac{1}{4n_2\delta} \right\} .
\end{aligned}$$

Now choose

$$n \ =\ n_1 \ =\ n_2 \,,$$
$$\delta \ =\ \frac{1}{\sqrt{n}} \,,$$
$$\tilde{\omega}(\delta) \ =\ max\left\{ \omega_1(\delta)\,, \omega_2(\delta) \right\} .$$

Hence we have

$$\left| f(x_1, x_2) \ -\ B^f_{n,n}(x_1, x_2) \right| \ \le\ \frac{3}{2\sqrt{n}}\,\tilde{\omega}(\delta) \,. \qquad \square$$

Note that for large $n_1,\ n_2$, when f has continuous first partial derivatives on $[0,1] \times [0,1]$, we get an improvement on the rate of convergence over that given in Theorem 2.

If our true functions f have continuous second partial derivatives on S, then we have the following way of expressing the rate of convergence of the Bernstein sequence of f at each point in the domain.

Theorem 4: *Let f be bounded and be twice continuously differentiable on S. Then*

$$\lim_{n\to\infty}\left\{n\Big[B^f_{n,n}(x_1,x_2)-f(x_1,x_2)\Big]\right\}=\frac{1}{2}\Big[x_1(1-x_1)f_{11}+x_2(1-x_2)f_{22}\Big]\,,$$

and the convergence is uniform on S.

[See Lorentz (1953) for the univariate result.]

Proof: Since all the partial derivatives of f exist and they are continuous, using a Taylor series expansion we have

$$\begin{aligned} & f(x_1+h,x_2+k) \\ = \;& f(x_1,x_2)+hf_1(x_1,x_2)+kf_2(x_1,x_2)+\frac{1}{2}\left[h^2f_{11}(x_1+\alpha_1h,x_2+\alpha_2k)\right. \\ + \;& \left.2hkf_{12}(x_1+\alpha_1h,x_2+\alpha_2k)+k^2f_{22}(x_1+\alpha_1h,x_2+\alpha_2k)\right], \end{aligned}$$

where

$$0<\alpha_1,\alpha_2<1\,.$$

Let

$$h=\left(\frac{v_1}{n_1}-x_1\right)\qquad\text{and}\qquad k=\left(\frac{v_2}{n_2}-x_2\right).$$

Then multiply the expression $f(x_1+h,x_2+k)$ by

$$P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2)=\binom{n_1}{v_1}\binom{n_2}{v_2}{x_1}^{v_1}{x_2}^{v_2}(1-x_1)^{n_1-v_1}(1-x_2)^{n_2-v_2}\,,$$

and sum up over v_1 and v_2, so we have

$$\begin{aligned} & B^f_{n_1,n_2}(x_1,x_2) \\ = \;& \sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2}f\left(\frac{v_1}{n_1},\frac{v_2}{n_2}\right)P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2) \\ = \;& f(x_1,x_2)+\sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2}\Big[hf_1+kf_2\Big]P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2) \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\Bigg\{\sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2}\Big[h^2 f_{11}(x_1+\alpha_1 h, x_2+\alpha_2 k) + 2hk f_{12}(x_1+\alpha_1 h, x_2+\alpha_2 k) \\
&+ \quad k^2 f_{22}(x_1+\alpha_1 h, x_2+\alpha_2 k)\Big] P_{v_1,n_1}(x_1) P_{v_2,n_2}(x_2)\Bigg\} \\
&= \quad f(x_1,x_2) + \frac{1}{2}\Bigg\{\sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2}\Big[h^2 f_{11}(x_1+\alpha_1 h, x_2+\alpha_2 k) \\
&+ \quad 2hk f_{12}(x_1+\alpha_1 h, x_2+\alpha_2 k) + k^2 f_{22}(x_1+\alpha_1 h, x_2+\alpha_2 k)\Big] \\
&\quad\quad P_{v_1,n_1}(x_1) P_{v_2,n_2}(x_2)\Bigg\}.
\end{aligned}
$$

Let

$$
\begin{aligned}
\eta_1(h,k) &= \frac{1}{2} f_{11}(x_1+\alpha_1 h, x_2+\alpha_2 k) - \frac{1}{2} f_{11}\,, \\
\eta_2(h,k) &= f_{12}(x_1+\alpha_1 h, x_2+\alpha_2 k) - f_{12}\,, \\
\eta_3(h,k) &= \frac{1}{2} f_{22}(x_1+\alpha_1 h, x_2+\alpha_2 k) - \frac{1}{2} f_{22}\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\quad B^f_{n_1,n_2}(x_1,x_2) \\
&= \quad f(x_1,x_2) + \sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2}\Bigg\{h^2\Big[\eta_1(h,k) + \frac{1}{2} f_{11}\Big] + hk\Big[\eta_2(h,k) + f_{12}\Big] \\
&+ \quad k^2\Big[\eta_3(h,k) + \frac{1}{2} f_{22}\Big]\Bigg\} P_{v_1,n_1}(x_1) P_{v_2,n_2}(x_2) \\
&= \quad f(x_1,x_2) + \frac{x_1(1-x_1)}{2n_1} f_{11} + \frac{x_2(1-x_2)}{2n_2} f_{22} \\
&+ \quad \sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2}\Big\{h^2\eta_1(h,k) + hk\eta_2(h,k) + k^2\eta_3(h,k)\Big\} P_{v_1,n_1}(x_1) P_{v_2,n_2}(x_2)\,.
\end{aligned}
$$

Note that $\eta_i(h,k)$'s are bounded, that is, $|\eta_i(h,k)| < H$, $i = 1,2,3$. Given $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $|h| < \delta$, $|k| < \delta$, we have $|\eta_i(h,k)| < \epsilon$, $i = 1,2,3$. It follows that

$$
\sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2}\Big\{h^2\eta_1(h,k) + hk\eta_2(h,k) + k^2\eta_3(h,k)\Big\} P_{v_1,n_1}(x_1) P_{v_2,n_2}(x_2)
$$

$$\le \left\{\sum_{|h|\le\delta}\sum_{|k|\le\delta} + \sum_{|h|\le\delta}\sum_{|k|>\delta} + \sum_{|h|>\delta}\sum_{|k|\le\delta} + \sum_{|h|>\delta}\sum_{|k|>\delta}\right\}$$

$$\left\{h^2|\eta_1(h,k)| + |h||k||\eta_2(h,k)| + k^2|\eta_3(h,k)|\right\}P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2)$$

$$\le \epsilon\sum_{|h|\le\delta}\sum_{|k|\le\delta}\left\{h^2 + |h||k| + k^2\right\}P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2)$$

$$+ H\left\{\sum_{|h|\le\delta}\sum_{|k|>\delta} + \sum_{|h|>\delta}\sum_{|k|\le\delta} + \sum_{|h|>\delta}\sum_{|k|>\delta}\right\}$$

$$\left\{h^2 + |h||k| + k^2\right\}P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2)\,.$$

Observe that $|h| \le 1$ and $|k| \le 1$. So the last expression does not exceed

$$\epsilon\sum_{|h|\le\delta}\sum_{|k|\le\delta}\left\{2h^2 + 2k^2\right\}P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2)$$

$$+ 3H\left\{\sum_{|h|\le\delta}\sum_{|k|>\delta} + \sum_{|h|>\delta}\sum_{|k|\le\delta} + \sum_{|h|>\delta}\sum_{|k|>\delta}\right\}P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2)$$

$$\le \epsilon\left\{\frac{2x_1(1-x_1)}{n_1} + \frac{2x_2(1-x_2)}{n_2}\right\}$$

$$+ 3H\left\{\sum_{|h|\le\delta}\sum_{|k|>\delta} + \sum_{|h|>\delta}\sum_{|k|\le\delta} + \sum_{|h|>\delta}\sum_{|k|>\delta}\right\}P_{v_1,n_1}(x_1)P_{v_2,n_2}(x_2)\,.$$

Note that $x_1(1-x_1) \le 1$ and $x_2(1-x_2) \le 1$. Use the result in 1.5(7) (with $s = 2$) of Lorentz (1953), the last expression is no larger than

$$2\epsilon\left(\frac{1}{n_1} + \frac{1}{n_2}\right) + 3H\left(\frac{C_1}{n_2^2} + \frac{2C_2}{n_1^2}\right),$$

where C_1 and C_2 are positive constants depending on δ. Now take $n_1 = n_2 = n$ and $C = 3H(C_1 + 2C_2)$. We obtain

$$\left|n\left[B^f_{n,n}(x_1,x_2) - f(x_1,x_2)\right] - \frac{1}{2}\left[x_1(1-x_1)f_{11} + x_2(1-x_2)f_{22}\right]\right| \le 4\epsilon + \frac{C}{n}\,.$$

Hence $\forall\ \epsilon > 0$

$$\limsup_{n\to\infty}\left|n\left[B^f_{n,n}(x_1,x_2) - f(x_1,x_2)\right] - \frac{1}{2}\left[x_1(1-x_1)f_{11} + x_2(1-x_2)f_{22}\right]\right| \le 4\epsilon\,.$$

So

$$\limsup_{n\to\infty} \left| n\Big[B^f_{n,n}(x_1,x_2) - f(x_1,x_2)\Big] - \frac{1}{2}\Big[x_1(1-x_1)f_{11} + x_2(1-x_2)f_{22}\Big]\right| = 0\,.$$

Therefore

$$\lim_{n\to\infty} \left\{ n\Big[B^f_{n,n}(x_1,x_2) - f(x_1,x_2)\Big]\right\} = \frac{1}{2}\Big[x_1(1-x_1)f_{11} + x_2(1-x_2)f_{22}\Big]\,. \qquad \Box$$

4 Flexibility and Other Properties of Bernstein Polynomials

We look at some properties of Bernstein polynomials in this section. One elegant property, which is based on a theorem in Butzer (1953), is described in the next theorem.

Theorem 5: [Butzer (1953)] *If f is continuous on the closed unit square with continuous m^{th} partial derivatives on the open unit square: $0 < x_1 < 1,\ 0 < x_2 < 1$, then we have the following result:*

$$\lim_{n_1\to\infty,\, n_2\to\infty} \frac{\partial^m B^f_{n_1,n_2}}{\partial x_1^q \partial x_2^{m-q}}(x_1,x_2) \;=\; \frac{\partial^m f}{\partial x_1^q \partial x_2^{m-q}}(x_1,x_2), \qquad m = 0, 1, \cdots,$$

provided that, for any two finite positive numbers r and t,

$$0 < r \le \frac{n_1+1}{n_2+1} \le t < \infty\,.$$

If the partial derivatives of f are continuous on the closed unit square, then the convergence is uniform without restriction on n_1 and n_2.[2]

[2] This result is based on a theorem in Butzer (1953). Butzer's theorem is more general than this, allowing f to be merely bounded on $[0,1]\times[0,1]$, and asserting convergence at any point (x_1,x_2) for which the total differential of f exists.

Proof: See Butzer (1953) and Kingsley (1951).

The above results show that, given the restriction on the growth of n_i, the sequences of derivative functions of the bivariate Bernstein polynomials converge at least pointwise to those of their respective true functions as $n_1, n_2 \to \infty$. This means that two-dimensional Bernstein polynomials are shape-preserving functions for large n_1 and n_2, and hence global flexibility is guaranteed. If the true function has continuous derivatives, then we have uniform convergence and *uniform flexibility.*

Corollary: *Let $H(f)$ denote the Hessian of f. Then*

$$det\Big[H(B^f_{n_1,n_2}(x_1,x_2))\Big] \quad \longrightarrow \quad det\Big[H(f(x_1,x_2))\Big] \quad as\ n_1\,, n_2\ \to\ \infty\,,$$

wherever all second partial derivatives are defined.

Remark: If $f(x_1,x_2) \in C^2$ and if $\det[H(f(x_1,x_2))] > 0$, $\forall\ (x_1,x_2) \in [0,1]\times[0,1]$, as in the case when f is strictly concave, then there exists an N such that for $n_1,\ n_2 > N$, we have $\det[H(B^f_{n_1,n_2}(x_1,x_2))] > 0,\ \forall\ x_1,\ x_2$.

The consequence of this remark will be exploited in sections 5 and 6 when we discuss concavity violation of Bernstein polynomials in the case of two variables.

Next we show that two-variable Bernstein polynomials $B^f_{n_1,n_2}$ for biaffine functions agree with their defining functions. This is the content of Theorem 6.

Theorem 6: $B^g_{n_1,n_2} = g$, $\forall\ n_1, n_2$ *iff* g *is of the form* $g(x_1,x_2) = a + bx_1 + cx_2 + dx_1x_2$, *where* a, b, c, d *are arbitrary constants, that is,* g *is biaffine (*g *is affine in each individual variable).*

Proof: If $g(x_1, x_2) = a + bx_1 + cx_2 + dx_1x_2$, then

$$\begin{aligned} B^g_{n_1,n_2}(x_1, x_2) &= E[g(X_1, X_2)] \\ &= a + bE[X_1] + cE[X_2] + dE[X_1X_2] \\ &= a + bx_1 + cx_2 + dx_1x_2 \,. \end{aligned}$$

On the other hand, if f agrees with all its Bernstein polynomials, that is, $f(x_1, x_2) = B^f_{n_1,n_2}(x_1, x_2)$, $\forall\ n_1, n_2$, then f is a polynomial of degree ≤ 2 [since degree $B^f_{n_1,n_2}(x_1, x_2) \leq n_1 + n_2$]. So if $n_1 + n_2 = 2$, then $n_1 = n_2 = 1$. This implies that f is biaffine. □

Remark: In general, in the multivariate case, the only functions that agree with their Bernstein polynomials $\forall\ n_1, \ldots, n_n$ are just the multiaffine functions.

In the case that f is concave, we have a strong result. This result depends on an important property of the Bernstein polynomials that tells us that when f is concave, in either the one-variable or two-variable case, the Bernstein polynomials approach the limit f in a very orderly way. We call this result the Stacking Theorem. The one-variable case [where $f(x)$ is a convex function] is due to Bonnie Averbach and can be found in Schoenberg (1959). Here we state and prove the two-variable case.

Theorem 7: (Stacking Theorem) *Suppose f is a bivariate concave function. Then B_{n_1,n_2} are bimonotone in n_1 and n_2, that is*

$$B^f_{n_1+1,n_2}(x_1, x_2) \geq B^f_{n_1,n_2}(x_1, x_2) \quad \text{and} \quad B^f_{n_1,n_2+1}(x_1, x_2) \geq B^f_{n_1,n_2}(x_1, x_2)$$

$$\forall\ (x_1, x_2) \in [0, 1] \times [0, 1] \ \text{ and } \ \forall\ n_1, n_2.$$

Strict inequalities hold if f is a strictly concave function.

Proof:

$$
\begin{aligned}
& B^f_{n_1+1,n_2}(x_1,x_2) \;-\; B^f_{n_1,n_2}(x_1,x_2) \\
= & \sum_{v_2=0}^{n_2}\Bigg\{\sum_{v_1=0}^{n_1+1} f\Big(\frac{v_1}{n_1+1},\frac{v_2}{n_2}\Big)\binom{n_1+1}{v_1} x_1^{v_1}(1-x_1)^{n_1+1-v_1} \\
- & \sum_{v_1=0}^{n_1} f\Big(\frac{v_1}{n_1},\frac{v_2}{n_2}\Big)\binom{n_1}{v_1} x_1^{v_1}(1-x_1)^{n_1-v_1}\Bigg\} P_{v_2,n_2}(x_2)\,.
\end{aligned}
$$

Now the expression above is divided by $(1-x_1)^{n_1+1}$ with $x_1 \neq 1$, and we obtain

$$
\begin{aligned}
& \frac{B^f_{n_1+1,n_2}(x_1,x_2) \;-\; B^f_{n_1,n_2}(x_1,x_2)}{(1-x_1)^{n_1+1}} \\
= & \sum_{v_2=0}^{n_2}\Bigg\{\sum_{v_1=0}^{n_1+1} f\Big(\frac{v_1}{n_1+1},\frac{v_2}{n_2}\Big)\binom{n_1+1}{v_1}\Big(\frac{x_1}{1-x_1}\Big)^{v_1} \\
- & \sum_{v_1=0}^{n_1} f\Big(\frac{v_1}{n_1},\frac{v_2}{n_2}\Big)\binom{n_1}{v_1}\Big(\frac{x_1}{1-x_1}\Big)^{v_1}\frac{1}{1-x_1}\Bigg\} P_{v_2,n_2}(x_2)\,.
\end{aligned}
$$

Let

$$
\begin{aligned}
Z &= \frac{x_1}{1-x_1}\,, \\
\tilde f\Big(\frac{v_1}{n_1}\Big) &= f\Big(\frac{v_1}{n_1},\frac{v_2}{n_2}\Big)\,, \\
\tilde f\Big(\frac{v_1}{n_1+1}\Big) &= f\Big(\frac{v_1}{n_1+1},\frac{v_2}{n_2}\Big)\,.
\end{aligned}
$$

Then we have the following

$$
\begin{aligned}
& \frac{B^f_{n_1+1,n_2}(x_1,x_2) \;-\; B^f_{n_1,n_2}(x_1,x_2)}{(1-x_1)^{n_1+1}} \\
= & \sum_{v_2=0}^{n_2}\Bigg\{\sum_{v_1=0}^{n_1+1} \tilde f\Big(\frac{v_1}{n_1+1}\Big)\binom{n_1+1}{v_1} Z^{v_1} \\
- & (Z+1)\sum_{v_1=0}^{n_1} \tilde f\Big(\frac{v_1}{n_1}\Big)\binom{n_1}{v_1} Z^{v_1}\Bigg\} P_{v_2}(x_2) \\
= & \sum_{v_2=0}^{n_2}\Bigg\{\sum_{v_1=0}^{n_1+1} \tilde f\Big(\frac{v_1}{n_1+1}\Big)\binom{n_1+1}{v_1} Z^{v_1} - \sum_{v_1=0}^{n_1} \tilde f\Big(\frac{v_1}{n_1}\Big)\binom{n_1}{v_1} Z^{v_1+1} \\
- & \sum_{v_1=0}^{n_1} \tilde f\Big(\frac{v_1}{n_1}\Big)\binom{n_1}{v_1} Z^{v_1}\Bigg\} P_{v_2}(x_2)
\end{aligned}
$$

$$= \sum_{v_2=0}^{n_2} \left\{ \sum_{v_1=1}^{n_1} \left[\tilde{f}\left(\frac{v_1}{n_1+1}\right) \binom{n_1+1}{v_1} - \tilde{f}\left(\frac{v_1-1}{n_1}\right) \binom{n_1}{v_1-1} \right.\right.$$
$$\left.\left. - \ \tilde{f}\left(\frac{v_1}{n_1}\right) \binom{n_1}{v_1} \right] Z^{v_1} \right\} P_{v_2}(x_2) \,.$$

Note that

$$B^f_{n_1+1,n_2}(x_1,x_2) \ - \ B^f_{n_1,n_2}(x_1,x_2) \ \geq \ 0$$

if

$$\tilde{f}\left(\frac{v_1}{n_1+1}\right) \binom{n_1+1}{v_1} - \tilde{f}\left(\frac{v_1-1}{n_1}\right) \binom{n_1}{v_1-1} - \tilde{f}\left(\frac{v_1}{n_1}\right) \binom{n_1}{v_1} \ \geq \ 0 \,.$$

Let

$$\alpha(v_1,n_1) = \tilde{f}\left(\frac{v_1}{n_1+1}\right) \binom{n_1+1}{v_1} - \tilde{f}\left(\frac{v_1-1}{n_1}\right) \binom{n_1}{v_1-1} - \tilde{f}\left(\frac{v_1}{n_1}\right) \binom{n_1}{v_1} \,.$$

Observe that

$$\frac{v_1-1}{n_1} \ < \ \frac{v_1}{n_1+1} \ < \ \frac{v_1}{n_1} \,.$$

Since $\tilde{f}$ is concave (a plane section of a concave surface is concave), we have the following

$$\frac{\tilde{f}\left(\frac{v_1}{n_1+1}\right) - \tilde{f}\left(\frac{v_1-1}{n_1}\right)}{\frac{v_1}{n_1+1} - \frac{v_1-1}{n_1}} \ \geq \ \frac{\tilde{f}\left(\frac{v_1}{n_1}\right) - \tilde{f}\left(\frac{v_1-1}{n_1}\right)}{\frac{v_1}{n_1} - \frac{v_1-1}{n_1}} \,.$$

This comes to

$$\frac{1}{n_1(n_1+1)} \left[(n_1+1)\tilde{f}\left(\frac{v_1}{n_1+1}\right) - v_1 \tilde{f}\left(\frac{v_1-1}{n_1}\right) \right.$$
$$\left. -(n_1-v_1+1)\tilde{f}\left(\frac{v_1}{n_1}\right) \right] \geq 0 \,.$$

Hence

$$\alpha_(v_1,n_1)$$
$$= \ n_1 \binom{n_1+1}{v_1} \frac{1}{n_1(n_1+1)}$$
$$\left[(n_1+1)\tilde{f}\left(\frac{v_1}{n_1+1}\right) - v_1 \tilde{f}\left(\frac{v_1-1}{n_1}\right) - (n_1-v_1+1)\tilde{f}\left(\frac{v_1}{n_1}\right) \right] \geq 0 \,.$$

Thus

$$B^f_{n_1+1,n_2}(x_1,x_2) \;\geq\; B^f_{n_1,n_2}(x_1,x_2)\,. \tag{$*$}$$

Similarly, we can obtain

$$B^f_{n_1,n_2+1}(x_1,x_2) \;\geq\; B^f_{n_1,n_2}(x_1,x_2)\,. \tag{$**$}$$

By continuity, equations $(*)$ and $(**)$ also hold for $x_1 = 1$. Thus, $B^f_{n_1,n_2}(x_1,x_2)$ is bimonotone in n_1 and n_2, $\forall\ (x_1,x_2) \in [0,1]\times[0,1]$. □

Theorem 8: *Let f be a concave function on $[0,1]\times[0,1]$. Assume that there exists a point $(z_1,z_2) \in (0,1)\times(0,1)$ and a pair of integers n_1, $n_2 \geq 1$ such that $f(z_1,z_2) = B^f_{n_1,n_2}(z_1,z_2)$. Then f is an affine function.*

Proof: By concavity, there exists a "tangent plane" $t(x_1,x_2) = a + bx_1 + cx_2$ such that $f(z_1,z_2) = t(z_1,z_2)$ and $t \geq f$. Then

$$\begin{aligned}
f(z_1,z_2) &= \sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2} f\Big(\frac{v_1}{n_1},\frac{v_2}{n_2}\Big) P_{v_1,n_1}(z_1)P_{v_2,n_2}(z_2) \\
&\leq \sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2} t\Big(\frac{v_1}{n_1},\frac{v_2}{n_2}\Big) P_{v_1,n_1}(z_1)P_{v_2,n_2}(z_2) \\
&= t(z_1,z_2) \quad \text{(Theorem 6)} \\
&= f(z_1,z_2)\,.
\end{aligned}$$

So the above inequality is an equality. Since $f \leq t$ and $P_{v_i,n_i}(z_i) > 0$ for every $v_i = 0,1,\cdots,n_i$, $i = 1,2$, we deduce that $f\big(\frac{v_1}{n_1},\frac{v_2}{n_2}\big) = t\big(\frac{v_1}{n_1},\frac{v_2}{n_2}\big)$ for every v_1 and v_2. Thus $f-t$ is a non-positive concave function that equals 0 at the points $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. Hence $f-t$ is identically 0 on $[0,1]\times[0,1]$. Therefore f is identically equal to the tangent plane t, and this proves the desired result. □

5 Regularity of Bernstein Polynomials

In this section, we examine the regularity conditions, namely continuity, non-negativity, monotonicity, concavity and homogeneity of degree one of Bernstein polynomials. Clearly, Bernstein polynomials preserve continuity, non-negativity, but not homogeneity of their determining functions. As for monotonicity and concavity conditions, they are inherited by the unvariate Bernstein polynomials [see Lorentz (1953)], but only monotonity and concavity in each individual variable are preserved by the bivariate Bernstein polynomial [see Chak (2001)]. Concavity of the function is, however, not necessarily preserved because, for arbitrary n_1 and n_2 the determinant of the Hessian of $B^f_{n_1,n_2}(x_1,x_2)$ can be positive, negative or zero, where the determinant of the Hessian of $B^f_{n_1,n_2}(x_1,x_2)$ is given by:

$$\begin{aligned}
& n_1n_2(n_1-1)(n_2-1)\left\{\sum_{v_1=0}^{n_1-2}\sum_{v_2=0}^{n_2}\left[f\left(\frac{v_1+2}{n_1},\frac{v_2}{n_2}\right)\right.\right. \\
- \; & \left.\left. 2f\left(\frac{v_1+1}{n_1},\frac{v_2}{n_2}\right)+f\left(\frac{v_1}{n_1},\frac{v_2}{n_2,n_2}\right)\right]P_{v_1,n_1-2}(x_1)P_{v_2}(x_2)\right\} \\
& \left\{\sum_{v_1=0}^{n_1}\sum_{v_2=0}^{n_2-2}\left[f\left(\frac{v_1}{n_1},\frac{v_2+2}{n_2}\right)-2f\left(\frac{v_1}{n_1},\frac{v_2+1}{n_2}\right)\right.\right. \\
+ \; & \left.\left. f\left(\frac{v_1}{n_1},\frac{v_2}{n_2}\right)\right]P_{v_1,n_1}(x_1)P_{v_2,n_2-2}(x_2)\right\} \\
- \; & n_1^2n_2^2\left\{\sum_{v_1=0}^{n_1-1}\sum_{v_2=0}^{n_2-1}\left[f\left(\frac{v_1+1}{n_1},\frac{v_2+1}{n_2}\right)\right.\right. \\
- \; & \left. f\left(\frac{v_1+1}{n_1},\frac{v_2}{n_2}\right)-f\left(\frac{v_1}{n_1},\frac{v_2+1}{n_2}\right)+f\left(\frac{v_1}{n_1},\frac{v_2}{n_2}\right)\right] \\
& \left. P_{v_1,n_1-1}(x_1)P_{v_2,n_2-1}(x_2)\right\}^2.
\end{aligned}$$

For functions $f(x_1, x_2)$ which are additively separable,[3] the cross-partial derivatives of the $B^f_{n_1,n_2}(x_1, x_2)$ will vanish;[4] the determinants of the Hessians of these Bernstein polynomials are always non-negative, and thus $B^f_{n_1,n_2}(x_1, x_2)$ is concave, $\forall\, n_1, n_2$. On the other hand, for functions $f(x_1, x_2)$ which are not additively separable, the determinants of the Hessians of $B^f_{n_1,n_2}(x_1, x_2)$ are not guaranteed to be non-negative, and hence concavity may be violated. Fortunately, this violation shrinks as n_1, n_2 get large. In fact, as we remarked in section 4, if $f(x_1, x_2) \in C^2$ and if the determinant of the Hessian of $f(x_1, x_2)$ is greater than zero, $\forall (x_1, x_2) \in [0, 1]\times[0, 1]$, then there exists an N such that for $n_1, n_2 > N$, the determinant of the Hessian of $B^f_{n_1,n_2}(x_1, x_2)$ will be non-negative, $\forall\ x_1,\ x_2$, and so the concavity violation of $B^f_{n_1,n_2}(x_1, x_2)$ disappears after a finite number of indices. In this case, $B^f_{n_1,n_2}(x_1, x_2)$ is concave, $\forall\, n_1, n_2 > N$. However, if the determinant of the Hessian of $f(x_1, x_2)$ is zero, for example, a concave Cobb-Douglas function $[f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}]$ which is homogeneous of degree one, our result is that, for large n_1, n_2, the concavity violation of $B^f_{n_1,n_2}(x_1, x_2)$ will be small; that is, as $n_1, n_2 \to \infty$, the determinant of the Hessian of $B^f_{n_1,n_2}(x_1, x_2)$ will go to zero, which implies that $B^f_{n_1,n_2}(x_1, x_2)$ is "asymptotically" concave. This problem will be discussed in more detail in the next section.

6 Functions Whose Hessians Have Zero Determinants and Their Bernstein Polynomials

In this section, we are going to investigate the properties of functions with their Hessians having zero determinants in order to understand better the concavity

[3] A function is additively separable if its cross-partial derivatives are zero.

[4] See the proof of Lemma 2 in Butzer (1953).

violation of the Bernstein polynomials. It is an easy consequence of Euler's theorem that all the functions which are homogeneous of degree one have the determinants of their Hessians equal zero.

Theorem 9: *If $f(x_1, x_2)$ is a homogeneous-of-degree-one function, then*

$$det[H(f)] = det \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = 0 \ ,$$

$\forall\ x_1,\ x_2$ *wherever the second partial derivatives are defined.*

Proof: See Chak (2001). □

The implication of this theorem is that if $f(x_1, x_2)$ is a homogeneous function of degree one, then the graph of $f(x_1, x_2)$ is a ruled surface (since $det[H(f)]$=0), in fact, a "cone" with apex at the origin (we assume that $f(0,0) = 0$), that is, a surface generated by a line fixed at one point at the origin and moving along a curve. We can illustrate this with the Cobb-Douglas function: $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, $0 \leq \alpha \leq 1$ and $0 \leq x_1, x_2 \leq 1$. Consider the surface $\tilde{S} :\ Z = f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ (this surface is contained in a unit cubic box). Intersecting $\tilde{S}$ with a plane $\Pi_\lambda :\ x_2 = \lambda x_1,\ \lambda \in [0, \infty)$, we have a curve which can be described as follows:

$$C_{\lambda,f} = \tilde{S} \cap \Pi_\lambda = \Big\{(x_1, x_2, Z)\ :\ Z = \lambda^{1-\alpha} x_1,\ x_2 = \lambda x_1,\ \lambda \in [0, \infty)\Big\}\ .$$

This intersection is in fact a straight line through the origin. Thus $\tilde{S} = \bigcup_{\lambda=0}^{\infty} C_{\lambda,f}$, and $(0,0,0) \in C_{\lambda,f}$; that is, the surface is the union of a family of straight lines which intersect at the origin. This fact helps us understand how $B^f_{n_1,n_2}(x_1, x_2)$ can violate the concavity property. With $B^f_{n_1,n_2}(0,0) = f(0,0)$ and $B^f_{n_1,n_2}(1,1) = f(1,1)$, if f is a homogeneous-of-degree-one function, then for any finite n_1, n_2 we must have some points (x_1, x_2) along the line segment joining $(0,0)$ and $(1,1)$ where the Hessians of $B^f_{n_1,n_2}(x_1, x_2)$ have negative determinants (except when f is a plane), as all the Bernstein surfaces have to lie below or are equal to f (because of the Stacking Theorem). However, due to the uniform convergence of the sequences of Bernstein polynomials and their derivatives to a determining function and its derivatives, the concavity violation of the Bernstein polynomials must diminish as n_1, n_2 get large.

7 Graphic and Numerical Results

In this section, we consider some examples of graphs and tables of Bernstein polynomials as well as their first and second derivatives to view the quality of the Bernstein approximation.

For the one-variable case, we choose two strictly concave functions: $f(x) = x^{\frac{1}{2}}$ and $f(x) = x - \frac{1}{2}x^2$, that is, a one-variable Cobb-Douglas function and a one-variable quadratic function. The graphs of these two functions as well as of their Bernstein polynomials are shown in Figures 1 and 4. To show the rates of convergence of these Bernstein polynomials to their true functions, we choose n=2, 5, 10, 20. From these two figures, we can see that the Bernstein polynomials are shape-preserving functions and approximate their true functions quite well. When $n = 20$, the Bernstein polynomials for the univariate Cobb-Douglas and the univariate quadratic functions are very close to their the true functions. By examining Figures 1 and 4 closely, we find that the sequence of Bernstein polynomials for the univariate quadratic function converges to its true function at a faster rate than the one for the univariate Cobb-Douglas function. Since both functions are concave, for a given $\delta > 0$, the maximum variations of the univariate Cobb-Douglas and the univariate quadratic functions, $\omega^c(\delta)$ and $\omega^q(\delta)$, can be found in the δ-interval from the origin. Also note that for the points within the δ-interval from the origin, the univariate Cobb-Douglas function has much steeper slopes than those of the univariate quadratic function. In fact, none of the derivatives of the univariate Cobb-Douglas function beyond the 0^{th} is defined at the origin. When we choose $\delta = x$, $\omega^c(\delta) = \delta^{\frac{1}{2}} > \omega^q(\delta) = \delta - \frac{1}{2}\delta^2$, $\forall\ \delta \in (0, 1]$. Thus by Theorem 1.6.1 of Lorentz (1953)[5] the sequence of Bernstein polynomials for the univariate Cobb-Douglas function has a slower rate of convergence.

Figures $2, 3, 5$ and 6 show how well the first and second derivatives of the univariate Cobb-Douglas and the univariate quadratic functions are approximated by the derivatives of their Bernstein polynomials. Due to the fact that the first and second derivatives of the univariate Cobb-Douglas function are not defined at $x = 0$, we choose the interval $[0.01,\ 1]$ for x; while the interval of x for the derivatives of the univariate quadratic function remains [0,1]. As expected, in these two cases, when n gets larger, the first and second derivatives of the Bernstein polynomials approach to those of their true functions.

For the two-variable case, we also examine the two functions: $f(x_1, x_2) = (x_1 x_2)^{\frac{1}{2}}$ (a regular function) and $f(x_1, x_2) = 2x_1 - x_1^2 + x_2 - \frac{1}{2}x_2^2$ (a non-negative, strictly monotone increasing and strictly concave function), that is, a two-variable Cobb-Douglas function and a two-variable quadratic function. Figures $8, 9, 10$ and 11 show the Bernstein polynomials for the bivariate Cobb-Douglas function with $n_1{=}n_2{=}20, 10, 5, 2$ respectively. Note that all these graphs illustrate the concavity violation around the origin, but the region of violation shrinks towards the origin as n_1, n_2 get large. On the other hand, all the graphs of the Bernstein polynomials

[5] Theorem 1.6.1 is the one-variable version of our Theorem 2.

for the bivariate quadratic function shown in Figures $13, 14, 15$ and 16 exhibit no concavity violation. It is due to the fact that the bivariate quadratic function is an additively separable function. All the graphic results for two variables are in agreement with the theory that we develop in sections 5 and 6: concavity violation does not occur in the Bernstein polynomials for concave, additively separable functions, but occurs in the Bernstein polynomials for concave, non-separable functions whose Hessians have zero determinants, where the region of violation diminishes as n_1, n_2 become large.

Regarding the rates of convergence for the two-variable case, our graphs in Figures 17 and 18, once again, show that the sequence of Bernstein polynomials of the bivariate quadratic function converges to its true function faster than the sequence of Bernstein polynomials of the bivariate Cobb-Douglas function to its true function. This is due to the same reason as in the one-variable case. Because of the difficulty of three-dimensional visualization, we use tables instead of graphs to compare the rates of convergence of the first and second derivatives of the Bernstein polynomials for these two functions. We choose three points: (0.1,0.2), (0.3,0.5) and (0.7,0.9). In Tables $1, 2$ and 3, where the true function is the bivariate Cobb-Douglas function, we can see that the concavity violation ($\det[H(B^f_{n_1,n_2}(x_1, x_2))] < 0$) occurs at the point (0.1,0.2) when n_1=n_2=$2, 5, 10$, and the point (0.3,0.5) when n_1=n_2=$2, 5$, but that there is no concavity violation at the point (0.7,0.9). As n_1=n_2=20, all the derivatives are quite close to those of the true function, except the second derivative with respect to x_1 at the point (0.1,0.2). This happens because (0.1,0.2) is quite close to the origin where concavity violation still takes place when n_1=n_2=20.

On the other hand, in Tables 4, 5 and 6, where the true functions is the bivariate quadratic function, there are no concavity violations and the results are exceptionally good: all the derivatives of the Bernstein polynomials at the three points are very close to those of the true function for n_1, $n_2 \geq 10$, especially the cross-partial derivatives.

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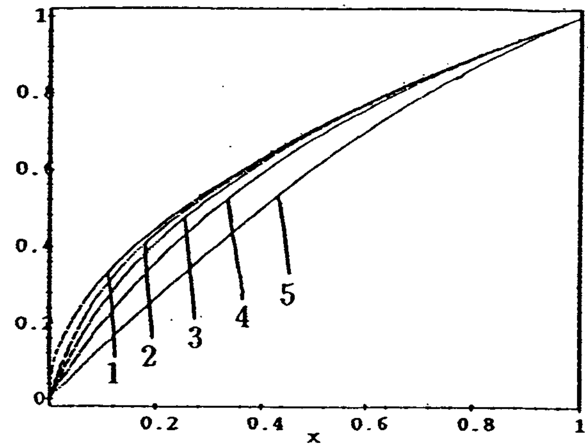
True Function: $f(x) = x^{\frac{1}{2}}$

Figure 1



$1 \cdots f$

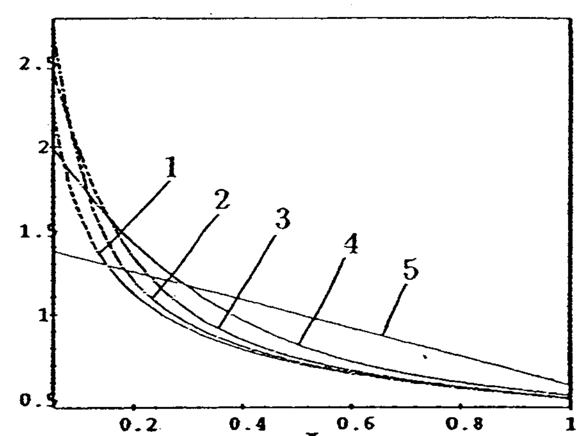
$2 \cdots B^f_{20}$

$3 \cdots B^f_{10}$

$4 \cdots B^f_5$

$5 \cdots B^f_2$

Figure 2



$1 \cdots \frac{df}{dx}$

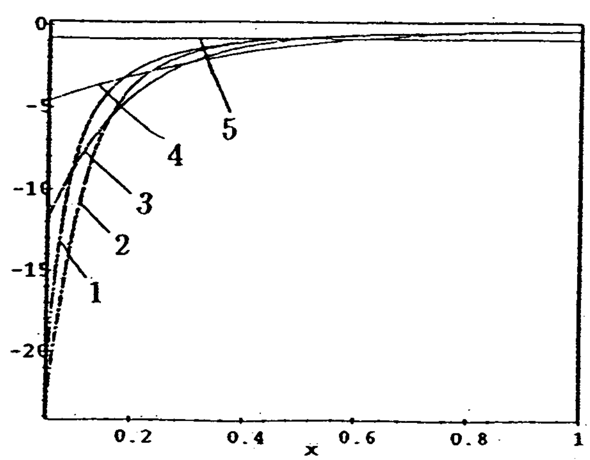
$2 \cdots \frac{dB^f_{20}}{dx}$

$3 \cdots \frac{dB^f_{10}}{dx}$

$4 \cdots \frac{dB^f_5}{dx}$

$5 \cdots \frac{dB^f_2}{dx}$

Figure 3



$1 \cdots \frac{d^2 f}{dx^2}$

$2 \cdots \frac{d^2 B^f_{20}}{dx^2}$

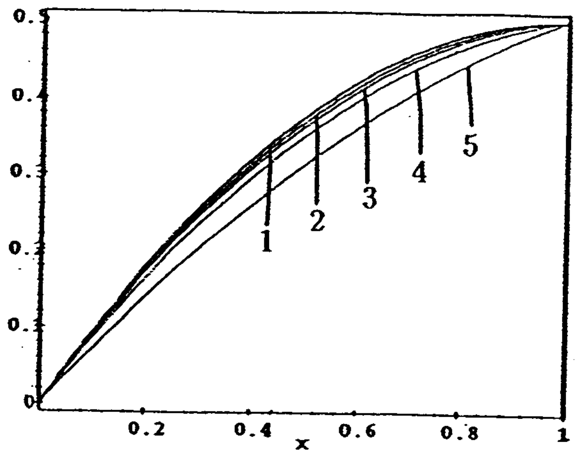
$3 \cdots \frac{d^2 B^f_{10}}{dx^2}$

$4 \cdots \frac{d^2 B^f_5}{dx^2}$

$5 \cdots \frac{d^2 B^f_2}{dx^2}$

True Function: $f(x) = x - \frac{1}{2}x^2$

Figure 4



$1 \cdots f$

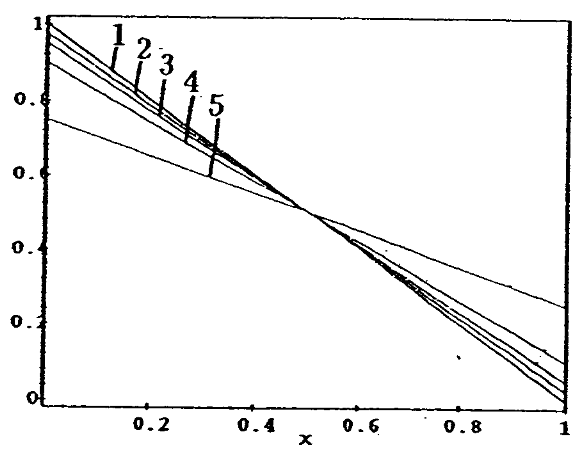
$2 \cdots B_{20}^f$

$3 \cdots B_{10}^f$

$4 \cdots B_5^f$

$5 \cdots B_2^f$

Figure 5



$1 \cdots \dfrac{df}{dx}$

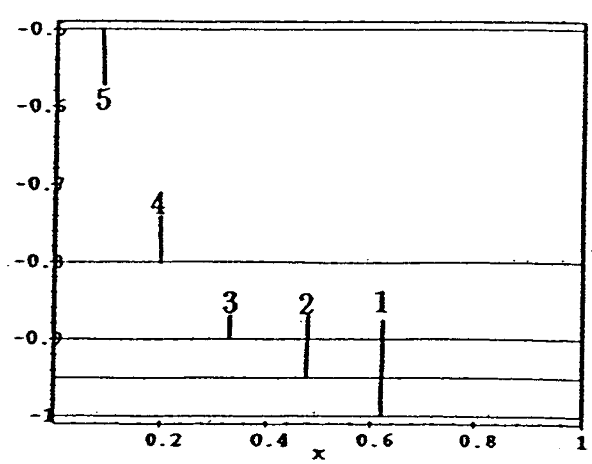
$2 \cdots \dfrac{dB_{20}^f}{dx}$

$3 \cdots \dfrac{dB_{10}^f}{dx}$

$4 \cdots \dfrac{dB_5^f}{dx}$

$5 \cdots \dfrac{dB_2^f}{dx}$

Figure 6



$1 \cdots \dfrac{d^2 f}{dx^2}$

$2 \cdots \dfrac{d^2 B_{20}^f}{dx^2}$

$3 \cdots \dfrac{d^2 B_{10}^f}{dx^2}$

$4 \cdots \dfrac{d^2 B_5^f}{dx^2}$

$5 \cdots \dfrac{d^2 B_2^f}{dx^2}$

True Function: $f(x_1, x_2) = (x_1 x_2)^{\frac{1}{2}}$

Figure 7 f

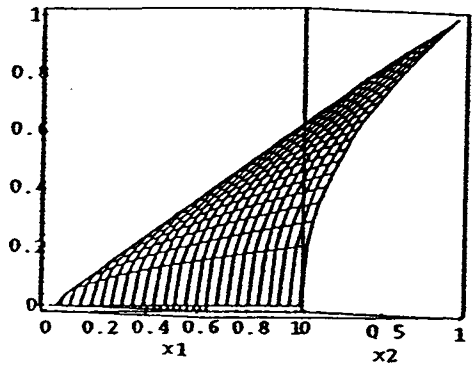


Figure 8 $B^f_{20,20}$

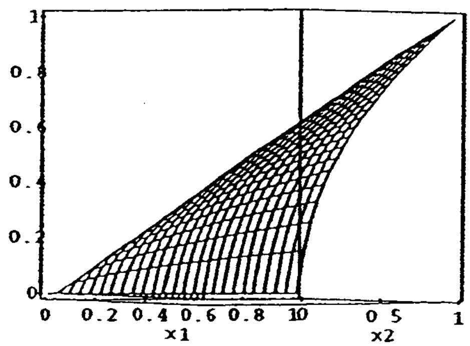


Figure 9 $B^f_{10,10}$

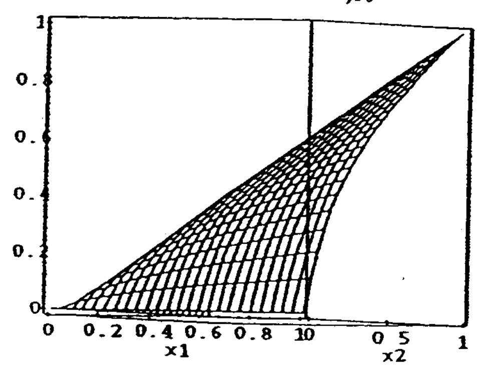


Figure 10 $B^f_{5,5}$

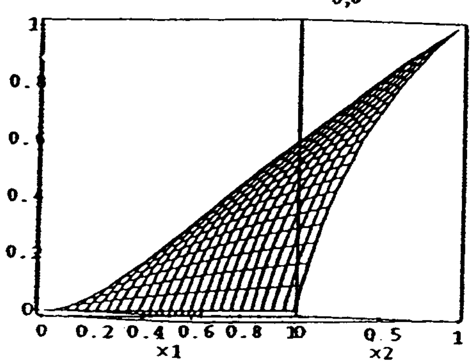
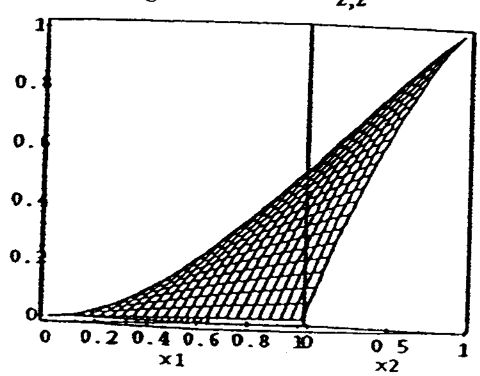


Figure 11 $B^f_{2,2}$



True Function: $f(x_1, x_2) = 2x_1 - x_1^2 + x_2 - \frac{1}{2}x_2^2$

Figure 12 f

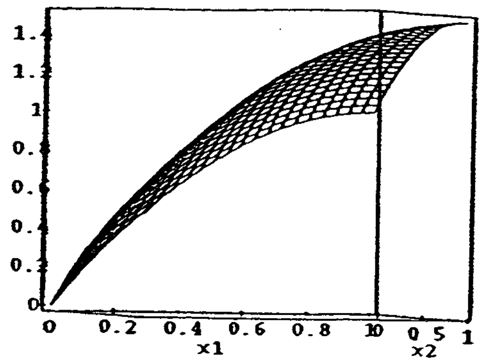


Figure 13 $B^f_{20,20}$

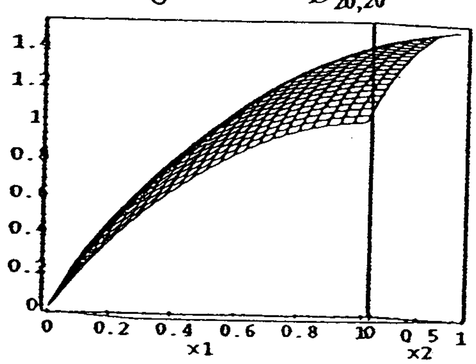


Figure 14 $B^f_{10,10}$

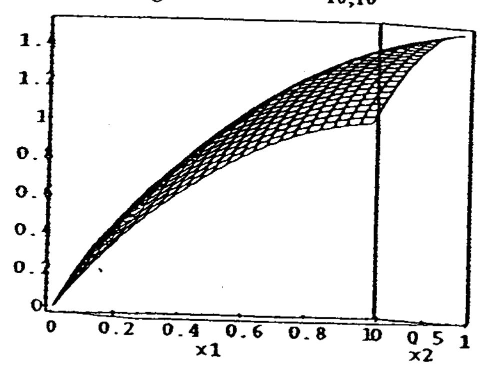


Figure 15 $B^f_{5,5}$

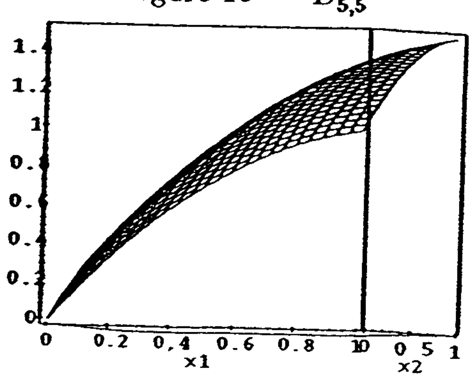
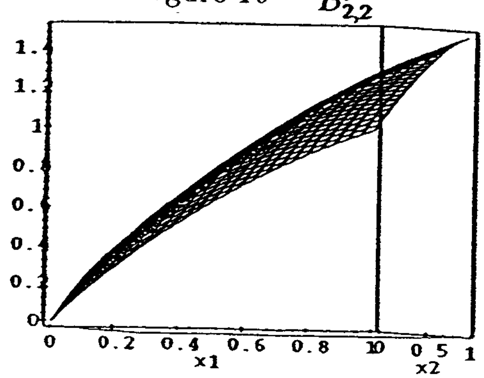
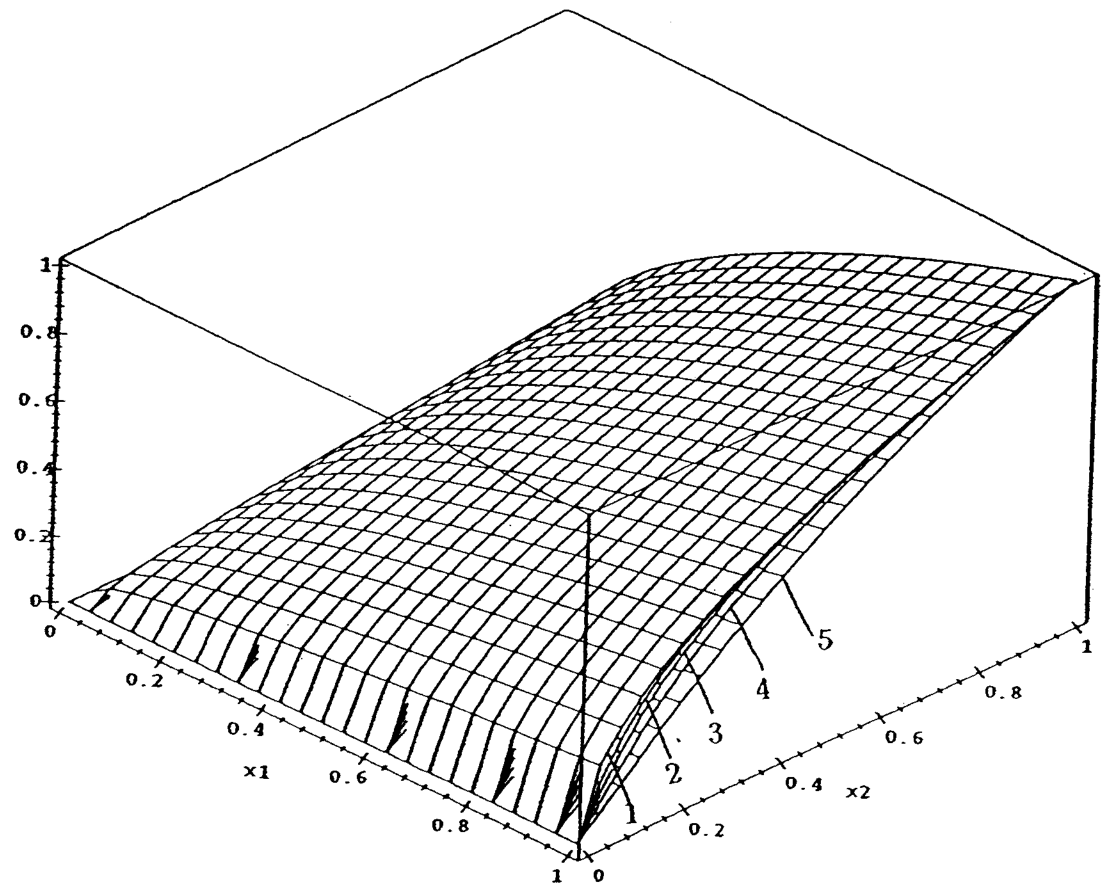


Figure 16 $B^f_{2,2}$



True Function: $f(x_1, x_2) = (x_1 x_2)^{\frac{1}{2}}$

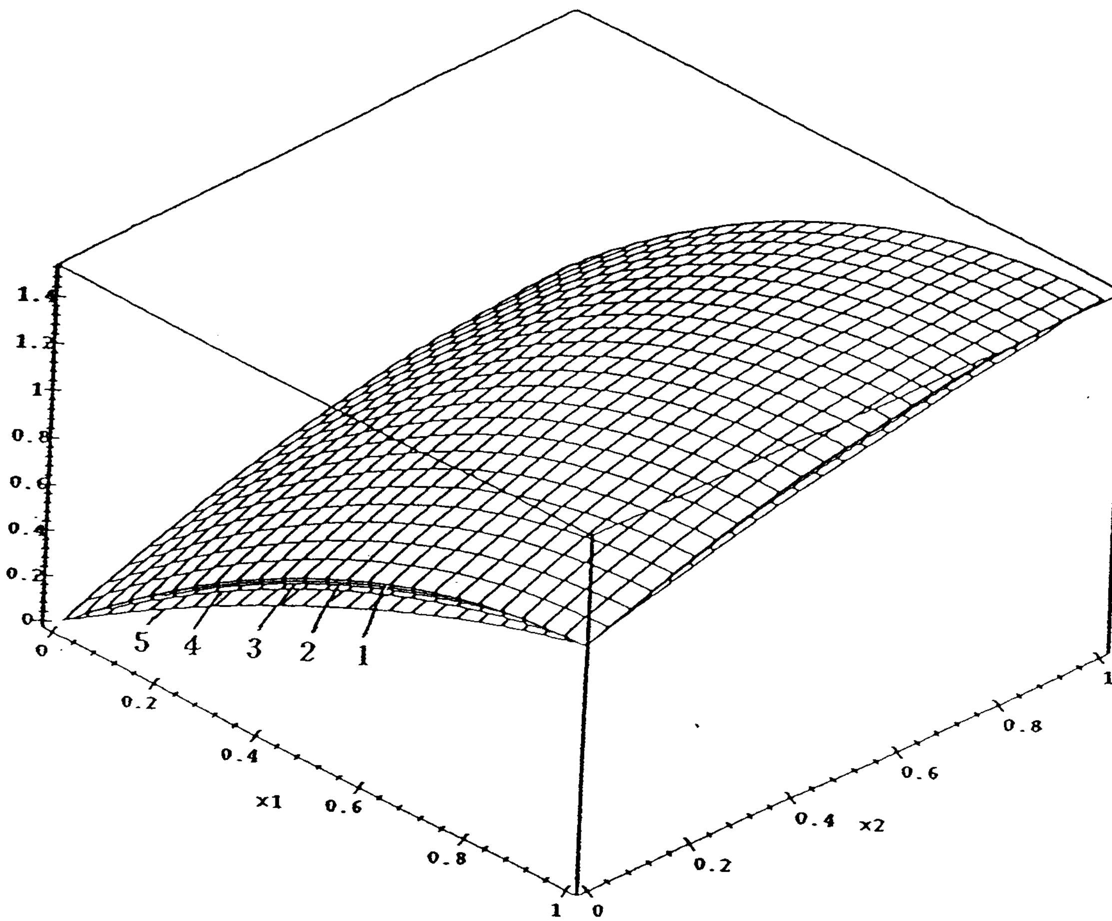
Figure 17



$1 \cdots f$
$2 \cdots B^f_{20,20}$
$3 \cdots B^f_{10,10}$
$4 \cdots B^f_{5,5}$
$5 \cdots B^f_{2,2}$

True Function: $f(x_1, x_2) = 2x_1 - x_1^2 + x_2 - \frac{1}{2}x_2^2$

Figure 18



1 --- f

2 --- $B^f_{20,20}$

3 --- $B^f_{10,10}$

4 --- $B^f_{5,5}$

5 --- $B^f_{2,2}$

True Function: $f(x_1, x_2) = (x_1 x_2)^{\frac{1}{2}}$

Table 1: Point $(0.1, 0.2)$

	function	$\frac{\partial}{\partial x_1}$	$\frac{\partial}{\partial x_2}$	$\frac{\partial^2}{\partial x_1 \partial x_2}$	$\frac{\partial^2}{\partial x_1^2}$	$\frac{\partial^2}{\partial x_2^2}$	det$[H]$
f	0.141	0.707	0.354	1.768	-3.536	-0.884	0.000
$B^f_{20,20}$	0.124	0.811	0.343	2.236	-5.097	-1.057	0.388
$B^f_{10,10}$	0.102	0.799	0.332	2.603	-3.470	-1.070	-3.063
$B^f_{5,5}$	0.072	0.636	0.283	2.520	-1.448	-0.610	-5.467
$B^f_{2,2}$	0.037	0.355	0.171	1.662	-0.221	-0.114	-2.737

Table 2: Point $(0.3, 0.5)$

	function	$\frac{\partial}{\partial x_1}$	$\frac{\partial}{\partial x_2}$	$\frac{\partial^2}{\partial x_1 \partial x_2}$	$\frac{\partial^2}{\partial x_1^2}$	$\frac{\partial^2}{\partial x_2^2}$	det$[H]$
f	0.387	0.645	0.387	0.645	-1.076	-0.387	0.000
$B^f_{20,20}$	0.379	0.664	0.388	0.682	-1.213	-0.402	0.023
$B^f_{10,10}$	0.366	0.710	0.389	0.756	-1.582	-0.442	0.128
$B^f_{5,5}$	0.331	0.787	0.399	0.948	-1.544	-0.575	-0.011
$B^f_{2,2}$	0.224	0.704	0.386	1.166	-0.500	-0.321	-1.199

Table 3: Point $(0.7, 0.9)$

	function	$\frac{\partial}{\partial x_1}$	$\frac{\partial}{\partial x_2}$	$\frac{\partial^2}{\partial x_1 \partial x_2}$	$\frac{\partial^2}{\partial x_1^2}$	$\frac{\partial^2}{\partial x_2^2}$	det$[H]$
f	0.794	0.567	0.441	0.315	-0.405	-0.245	0.000
$B^f_{20,20}$	0.791	0.576	0.446	0.324	-0.420	-0.251	0.001
$B^f_{10,10}$	0.788	0.585	0.451	0.335	-0.441	-0.259	0.002
$B^f_{5,5}$	0.781	0.614	0.462	0.363	-0.566	-0.285	0.030
$B^f_{2,2}$	0.738	0.782	0.526	0.558	-0.776	-0.652	0.195

True Function: $f(x_1, x_2) = 2x_1 - x_1^2 + x_2 - \frac{1}{2}x_2^2$

Table 4: Point $(0.1, 0.2)$

	function	$\frac{\partial}{\partial x_1}$	$\frac{\partial}{\partial x_2}$	$\frac{\partial^2}{\partial x_1 \partial x_2}$	$\frac{\partial^2}{\partial x_1^2}$	$\frac{\partial^2}{\partial x_2^2}$	det$[H]$
f	0.370	1.800	0.800	0.000	-2.000	-1.000	2.000
$B_{20,20}^f$	0.362	1.760	0.785	0.000	-1.900	-0.950	1.805
$B_{10,10}^f$	0.353	1.720	0.770	0.000	-1.800	-0.900	1.620
$B_{5,5}^f$	0.336	1.640	0.740	0.000	-1.600	-0.800	1.280
$B_{2,2}^f$	0.285	1.400	0.650	0.000	-1.000	-0.5000	0.500

Table 5: Point $(0.3, 0.5)$

	function	$\frac{\partial}{\partial x_1}$	$\frac{\partial}{\partial x_2}$	$\frac{\partial^2}{\partial x_1 \partial x_2}$	$\frac{\partial^2}{\partial x_1^2}$	$\frac{\partial^2}{\partial x_2^2}$	det$[H]$
f	0.885	1.400	0.500	0.000	-2.000	-1.000	2.000
$B_{20,20}^f$	0.869	1.380	0.500	0.000	-1.900	-0.950	1.805
$B_{10,10}^f$	0.852	1.360	0.500	0.000	-1.800	-0.900	1.620
$B_{5,5}^f$	0.818	1.320	0.500	0.000	-1.600	-0.800	1.280
$B_{2,2}^f$	0.718	1.200	0.500	0.000	-1.000	-0.500	0.500

Table 6: Point $(0.7, 0.9)$

	function	$\frac{\partial}{\partial x_1}$	$\frac{\partial}{\partial x_2}$	$\frac{\partial^2}{\partial x_1 \partial x_2}$	$\frac{\partial^2}{\partial x_1^2}$	$\frac{\partial^2}{\partial x_2^2}$	det$[H]$
f	1.405	0.600	0.100	0.000	-2.000	-1.000	2.000
$B_{20,20}^f$	1.392	0.620	0.120	0.000	-1.900	-0.950	1.805
$B_{10,10}^f$	1.380	0.640	0.140	0.000	-1.800	-0.900	1.620
$B_{5,5}^f$	1.354	0.680	0.180	0.000	-1.600	-0.800	1.280
$B_{2,2}^f$	1.278	0.800	0.300	0.000	-1.000	-0.500	0.500