A STRONG NO SHOW PARADOX IS A COMMON FLAW IN CONDORCET VOTING CORRESPONDENCES

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ABSTRACT

The No Show Paradox (there is a voter who would rather not vote) is known to affect every Condorcet voting function. This paper analyses a strong version of this paradox (there is a voter whose favorite candidate loses the election if she votes honestly, but gets elected if she abstains) in the context of Condorcet voting correspondences. All Condorcet correspondences satisfying some weak domination properties are shown to be affected by this strong form of the paradox. On the other hand, with the exception of the Simpson-Cramer Minmax, all the Condorcet correspondences that (to the best of our knowledge) are proposed in the literature suffer this paradox.

1. INTRODUCTION

In the theory of voting, the prospects of finding a best voting method have been disappointing, due to the negative results obtained through the systematic axiomatic analysis employed during the last half of this century, including the Arrow Impossibility Theorem and the Gibbard-Satterthwaite result, with its subsequent developments and refinements. We know now that no voting method simultaneously fulfills some minimal properties that apparently are required by any reasonable method, that is to say, no method is free from paradoxes (failures to satisfy some intuitively compelling properties).

However, it still makes sense to analyze and compare methods in order to select a reasonable one for a given setting. In this task of confronting methods and choosing the right one, perhaps the two main families are the Condorcet and the Positional family.

The interest and relevance of Condorcet voting methods stem from their fidelity to the democratic principle which asserts that if there exists a candidate that is favored by a majority of voters (in a face to face comparison) over any other, this candidate should be the only one chosen. This is called the Condorcet property. For the definition and analysis of the best known Condorcet methods, see Fishburn (1977), Tideman (1987), Laffond *et al* (1995), and Peris and Subiza** (1999).

On the other hand, the Positional or Scoring methods, and in particular the Borda method, aggregate the preferences of voters through a scoring technique which in some way extracts a measure of the intensity of these preferences. These methods have a normatively appealing consistency property (if two electorates are combined, the global result is coherent with the partial results). For the definition and analysis of the Positional methods, see Young (1974, 1975) and Saari (1990).

Young and Levenglick (1978) have established the incompatibility of the Condorcet and consistency properties. A similar, but independent, property called Participation (none of the voters is disillusioned by submitting his true ballot) has also been shown in Moulin (1988) to be incompatible with the Condorcet property. Hence, every Condorcet method suffers what has been termed as the No Show

Paradox.

This paper, which follows Moulin (1988) and extends some results from Pérez (1995), explores the incidence in Condorcet voting correspondences of a strong form of the paradox (from now on called Strong No Show Paradox, or SNSP for short) in which there is a voter V_1 whose favorite candidate loses the election if V_1 votes honestly, but gets elected if V_1 abstains.

Although not all Condorcet methods suffer from the Strong No Show Paradox, the Simpson-Cramer Minmax method is, as far as I know, the only exception among those proposed in the literature.

Section 2 presents the basic terminology and some known results. Section 3 defines the SNSP, and identify some weak properties that imply the paradox. Section 4 analyses, with the help of these properties (whenever possible), which known correspondences abide by the paradox, and section 5 concludes the paper with some additional remarks.

2. TERMINOLOGY AND SOME KNOWN RESULTS.

The terminology of Fishburn (1977) and Laffond *et al* (1995) will be used whenever possible, with few modifications.

Let $X = \{x_1,x_2,...,x_n\}$ be a finite set with two or more candidates. Preferences of any voter are supposed to take the form of a complete ranking, that is to say, a linear (strict and complete) order l over X. We say $l = xyzt...$ to denote the preference order in which x is the most preferred candidate, y is the second one, and so on, and $a\ l\ b$ means that a is preferred to y in l.

Given the set of candidates $X = \{x_1,x_2,...,x_n\}$, and any finite set $V = \{1,2,...,m\}$ with one or more voters, we call a **Situation** any pair (X,p), where p is a preference profile over X from V, that is to say, a m-tuple of orders over X, each one meaning the preferences of a voter over X.

We call **Voting Correspondence** (from now on **VC**) any function f which maps any situation (X,p) to a non-empty subset of X, $f(X,p)$. The elements of $f(X,p)$ are the chosen candidates (the winners) over X from the preference profile p. Given any two VCs f and g, we say f is a **refinement** of g if and only if $f(X,p) \subseteq g(X,p)$ for every situation (X,p).

Since we will consider only anonymous VCs (all voters are equally considered), a preference profile over X from V can also be described by specifying how many of the m voters from V sustain any of the $n!$ linear orders on X.

Given any X, any two disjoint sets of voters $V_1=\{1,2,...,m_1\}$ and $V_2=\{m_1+1, m_1+2,...,m_1+m_2\}$, and any two preference profiles p_1 and p_2 over X from, respectively, V_1 and V_2, we can merge these two profiles in order to obtain a new profile over X, but now originated from $V_1 \cup V_2$. This new profile will be called p_1+p_2.

Let (X,p) be any situation with n candidates and m voters:

Given any two different candidates x, y from X, $p(x,y)$ means the number of voters in p which prefer x to y. Because ties are not allowed in any voter's ballot, $p(x,y)+p(y,x)=m$. The square nxn matrix M_p, whose entries are $p(x,y)$, will be called the **Paired Comparison Matrix** for (X,p). In this matrix, for any given candidate there is a row (say the *i-th* row) and a column (the *i-th* column). For every candidate x, the sum of the off-diagonal row entries in M_p is called the **Borda Score** of x.

Candidate x is said to **beat** y, denoted by xW_py, if and only if $p(x,y)>p(y,x)$. If we use $\geq$ instead of $>$, we have the relation x **beats or ties** y, which is denoted by xU_py. Candidate x is said to **beat indirectly** (**beat or tie indirectly**), denoted by xWW_py (xUU_py) if and only if there are k candidates $x_1, x_2, ..., x_k$ in X such that $x= x_1 R\ x_2\ R ... R\ x_k = y$, being $R=W_p$ ($R=U_p$). If x beats any other, then x is called the **Condorcet** candidate.

The situation in which the set of candidates is $Z \subseteq X$, and the profile is the restriction to Z of profile p, is called $(Z,p/Z)$. The comparison matrices of this new situation are $M_{p/Z}$ and $T_{p/Z}$.

For every order l: $x_1x_2...x_n$ of the candidates, the sum $\sum_{i<j} p(x_i, x_j)$ will be called the **Kemeny Score** of l. Given an order l : $x_1x_2...x_n$ of the candidates, we say $\boldsymbol{x}$

attains *y* through *l* in (X,p) if and only if there is a sequence of distinct candidates $a_1, a_2, ..., a_j$, with $x=a_1$ and $y=a_j$, such that $a_i \; l \; a_{i+1}$ and $p(a_i , a_{i+1}) \geq p(a_j , a_1)$ for i=1,...,j-1. The order l is called a **stack** of (X,p) if and only if $x_i \; l \; x_j$ implies that x_i attains x_j through l in (X,p).

Let us call **Bipartisan Plurality Game** of (X, p) the two-player symmetric constant-sum game in which X is the set of pure strategies, and the payoffs for the profile of strategies (x,y) are:

If $x \neq y$, $\quad u_1(x,y) = p(x,y)$ and $u_2(x,y)=p(y,x)$

If $x=y$, $\quad u_1(x,y) = u_2(x,y) = m/2$

In order to completely relate the terms $p(x,y)$ and the payoffs of this game, we will suppose that all the entries $p(x,x)$ of the main diagonal will be set as equal to $m/2$. This technical convention does not significantly affect any previous concept or result, and facilitates some definitions and computations. Thus, M_p is the payoff matrix of the row player.

Any change in M_p by which a unit is added to the off-diagonal entry $p(x,y)$ and subtracted from $p(y,x)$, is called an **elemental interchange in M_p**. Any change in (X,p), by which two consecutive candidates x and y in a voter's order interchange their position in that voter's order, is called an **elemental interchange in (X,p)**.

Following Fishburn (1977), we will distinguish three types of VCs. The correspondence f is said a **C1**, **C2** or **C3** Correspondence if, respectively:

C1: For every situation (X,p), the set of winners $f(X,p)$ depends only on the W_p relation.

C2: f is not a C1-Correspondence and for every situation (X,p), $f(X,p)$ depends only on the Paired Comparison Matrix M_p.

C3: f is neither a C1-Correspondence nor a C2-Correspondence.

Definition 1: *A VC f is called **Condorcet** if and only if for every situation* (X,p), $(xW_p y \quad \forall y \in X \setminus \{x\})$ *implies* $f(X,p)=\{x\}$.

That is to say, if there is a Condorcet candidate, it will be the only winner.

Definition 2: *A VC f satisfies the **Consistency** property if and only if for any two* *situations* (X,p_1) *and* (X,p_2), $f(X,p_1) \cap f(X,p_2) \neq \emptyset$ *implies* $f(X,p_1+p_2)=f(X,p_1) \cap f(X,p_2)$.

In other words, if some candidates are chosen for profile p_1 and profile p_2, they, and only they, are chosen when the two profiles are merged. This property characterizes, along with Anonymity and Neutrality, the positional choice correspondences, whose best known examples are the **Plurality rule** (the winners are those who are the most preferred candidates by a highest number of voters) and the **Borda rule** (the winners are those who obtain the highest Borda Score). See Young (1975), and see also Young (1974) for a characterization of the Borda rule where the Consistency property plays a fundamental role.

The following property, from Moulin (1988), is defined in the context of Voting Functions (VCs which, for any situation, chose only one candidate), which he calls Voting Rules.

Definition 3: *A Voting Function f satisfies the* ***Participation*** *property if and only if for any given pair of situations* (X,p) *and* (X,v), *where profile v has only one voter,* $(f(X,p) = \{x\}$ *and x is preferred to y in v*$)$ *implies* $f(X,p+v) \neq \{y\}$.

That is to say, if x is the winner for a situation and a new voter who prefers x to y is added, candidate y will not become the winner. From the point of view of the new voter, he would do better if he abstained, because submitting his ballot would result in the election of a less preferred candidate. If we apply Moulin's terminology, failing to satisfy Participation means that the **No Show paradox** sets in.

2.1 Some known results.

The incompatibility of the above two properties with the Condorcet property is shown in propositions 1 and 2 below, established respectively in Young and Levenglick (1978) and Moulin (1988).

Proposition 1: *No Condorcet VC satisfies the Consistency property*.

Proposition 2: *No Condorcet Voting Function satisfies the Participation property*.

Consistency and Participation are not logically related. Moreover, Moulin (1988) proved that Participation does not imply nor is implied by the Reinforcement property, which is a natural translation of Consistency to the Voting Functions framework.

The following definition is a natural translation of the Participation property to the Voting Correspondences framework.

Definition 4: *A VC f satisfies the* ***VC-Participation*** *property if and only if for any given pair of situations* (X,p) *and* (X,v)*, where profile v has only one voter,*

If $x \in f(X,p)$ *and x is preferred to y in v, then* ($y \in f(X,p+v)$ *implies* $x \in f(X,p+v)$).

In other words, if candidate *x* is chosen for a situation and a new voter is added who strictly prefers *x* to *y*, candidate *y* will not be chosen if she is not accompanied by candidate *x*.

An easy adaptation of the proof (ii) in Moulin's statement (1988, p. 57-59), allows to establish proposition 3 as below. See Pérez (1995).

Proposition 3: *No Condorcet VC satisfies the VC-Participation property*.

3. THE STRONG NO SHOW PARADOX

The following property can be easily shown to be a weakening of both Consistency and VC-Participation, and it may be seen as the minimum to require to any VC, concerning the coherence in the set of winning candidates when new voters are added.

Definition 5: *A VC f satisfies the* ***Positive Involvement*** *property if and only if for any given pair of situations* (X,p) *and* (X,v), *where profile v has only one voter,*
If $x \in f(X,p)$ *and x is preferred to any y in v, then* $x \in f(X,p+v)$.

In other words, if candidate *x* is chosen, *x* will remain chosen when a new voter is added who prefers *x* to any other candidate. Saari (1994) defines (in a slightly different way) Positive Involvement and shows that Correspondences defined by sequential pairwise comparisons according to a specified agenda fail to satisfy it.

The failure by an VC *f* to satisfy this property means that *f* suffers an acute form of Moulin's No Show Paradox, which we will call **Strong No Show Paradox** (**SNSP**).

Some Condorcet VCs that satisfy Positive Involvement **do exist**, as will be shown in proposition 6 below.

3.1 Impossibility result for some families of Condorcet VCs.

The following domination properties will be needed to identify families of Condorcet VCs that fail to satisfy Positive Involvement.

Definition 6: *Given a situation* (X,p) *and two candidates* x *and* s, *we say* s ***is C1-dominated by*** x *if and only if the two following conditions hold:*

a) $p(x,s) > p(s,x)$

b) For any $z \in X \backslash \{x,s\}$, *if* $p(s,z) \geq p(z,s)$ *then* $p(x,z) > p(z,x)$.

In other words, x beats s and x beats any candidate z beaten by, or tied with, s.

Note: If we consider only the information conveyed by the W_p relation (thus focusing on the underlying Tournament structure of the situation), we can say:

1) The C1-domination relation coincides with the covering relation defined in the context of strict tournaments, in which W_p is a complete relation ($\forall x,y$, if $x \neq y$ then xW_py or yW_px). See Fishburn (1977) and Laffond *et al* (1995).

2) The C1-domination relation is stronger than any of the covering relations defined in the context of weak tournaments, in which W_p is the asymmetric part of a complete relation. Thus, if s is C1-dominated by x then s is covered by x. See Peris and Subiza (1999).

Definition 7: *Given a situation* (X,p) *and two candidates* x *and* s, *we say that* s ***is C2-dominated by*** x *if and only if the two following conditions hold*:

a) $p(x,s) > p(s,x)$

b) *For any* $z \in X \backslash \{x,s\}$, $p(x,z) \geq p(s,z)$.

In other words, x beats s, and x performs equal or better than s in the matrix M_p, in her confrontation with any other candidate. Both C1 and C2 domination concepts are generalizations of the Pareto Domination relation.

Definition 8: *Given a situation* (X,p) *and two candidates* x *and* s, *we say that* s ***is C2-quasidominated in differences by*** x *if and only if the three following conditions hold*:

a) $p(x,s) > p(s,x)$.

b) $p(x,z) \geq p(s,z)$ *for any* $z \in X\text{-}\{x,s\}$ *except perhaps for a unique* z.

c) *If* $p(x,z) < p(s,z)$, *then* $p(x,s) - p(s,s) > p(s,z) - p(x,z)$.

In other words, x beats s, x performs better than s in her confrontation with any other candidate, except perhaps with only one, say the z candidate, and the difference in favor of x in her confrontation with s (as expressed by the difference $p(x,s)$-$p(s,s)$) more than compensates for the difference in favor of s when both are confronted with z. Candidate z can be called the *weak point* of x with respect to s.

Definition 9: *Let* (X,p) *be any situation. Given three different candidates* x, y *and* s, *we say that* s **is C2-dominated by the pair** $\{x,y\}$ *if and only if the two following conditions hold*:

a) *Both* x *and* y *C2-quasidominate* s *in differences*.

b) *If* $w\in\{x,y\}$ *and* $p(w,z) < p(s,z)$, *then* $p(y,z)-p(s,z) \geq p(s,z)-p(x,z)$.

That is to say, besides the fact that both x and y C2-quasidominate s, the performance of any of them at the weak point of the other is enough to compensate the poor performance of the other at its own weak point. This compensation causes that, in every column of M_p, the entry corresponding to candidate s is equal or lower than the average of the entries corresponding to candidates x and y.

Although stronger, this definition is formulated in the spirit of the concept of weak domination of a pure strategy by a mixed strategy in finite strategic-form games. In fact, in the Bipartisan Plurality Game associated to a situation (X,p), defined in Laffond *et al* (1993, 1994), if a candidate s is C2-Dominated by the pair $\{x,y\}$, then the pure strategy s is weakly dominated by the mixed strategy $0.5x+0.5y$.

Every concept of domination among candidates tell us that, from the perspective of this concept, a dominated candidate, being surpassed by other(s), does not perform sufficiently well in the preferences of voters and, therefore, does not deserve to win the election. So, for every concept of domination, it is relevant to pose the question of which VCs respect that concept, by not electing as a winner the dominated candidate or, at least, by not electing it as the unique winner. The following definition, being of a general character, is applicable to the four concepts of domination as defined above.

Definition 10: *A VC f* ***Weakly Respects the Q-Domination*** *if and only if for any given situation* (X,p), (s *is Q-Dominated implies that* $f(X,p) \neq \{s\}$).

(*We say f* ***Respect the Q-Domination*** *if the consequent of the implication is* $s \notin f(X,p)$)

There is an obvious relation between the just defined properties of a VC and its refinements. Indeed, for any domination concept, if f is a refinement of g, the two following statements hold: a) If g respects the Domination so does f.

b) If f weakly respects the Domination so does g.

The following, the main proposition of the paper, establishes a logical incompatibility between Positive Involvement and some of the above defined domination concepts.

Proposition 4: *No Condorcet VC that weakly respects the C1-Domination or weakly respects the C2-Domination by a pair, satisfies the Positive Involvement property*.

Proof: The following lemma, whose proof is an easy adaptation of the proof of an analogous result in Moulin (1988a, p. 57) will be needed.

Lemma 1: *Given any Condorcet VC f satisfying Positive Involvement, any situation* (X,p) *and any two candidates x and z*, $p(x,z) < Min_{y\in X}\, p(z,y)$ *implies* $x \notin f(X,p)$

Proof of the lemma: Let m be the number of voters in profile p, and suppose $p(x,z) < \text{Min}_{y \in X}\, p(z,y)$. Iteratively adding to p a number $h = p(z,x) - \text{Min}_{y \in X}\, p(z,y)$ of new voters, all with identical preference order xz..., the minimal entry of the z row in the new profile p' is $p'(z,x) = p(z,x)$. On the other hand, $p'(x,z) < p'(z,x)$, because $p'(x,z) = p(x,z) + h = p(x,z) + p(z,x) - \text{Min}_{y \in X}\, p(z,y) = p(z,x) + [p(x,z) - \text{Min}_{y \in X}\, p(z,y)] < p(z,x) = p'(z,x)$. Hence, the minimal entry on the z row in profile p', $p'(z,x)$, is higher than $(m+h)/2$, making z a Condorcet candidate in the new situation and, because of the supposed Condorcet property, the only candidate chosen.

Therefore, candidate x can not be chosen for (X,p), because in that case, as the new h voters are added, it would necessarily happen at some point that x will not be chosen, thus contradicting the Positive Involvement property.

To complete the proof of proposition 4, let $X = \{x,y,z,u,t\}$ and p the following profile: [*yxtuz* (11 voters), *uzytx* (10 voters), *xztyu* (10 voters), *uztyx* (2 voters), *utzyx* (2 voters), *zyxtu* (2 voters), *tzyxu* (1 voter), *xytuz* (1 voter)]. The paired comparison matrix is:

		x	y	z	u	t
	x	19.5	11	22	25	24
M_p:	y	28	19.5	12	25	24
	z	17	27	19.5	13	24
	u	14	14	26	19.5	14
	t	15	15	15	25	19.5

Suppose f is Condorcet and satisfies Positive Involvement. From Lemma 1 applied to the pairs (x,y), (y,z), (z,u) and (u,t), candidates x, y, z, and u are not chosen for (X,p), and thus candidate t is the only winner.

However, both x and y C1-Dominate t (y is the weak point of x and z is the weak point of y), and the pair $\{x,y\}$ C2-Dominates t (the difference in favor of t at the weak point of x is $p(t,y)$-$p(x,y)$=4, while $p(x,t)$-$p(t,t)$=4.5 and $p(y,y)$-$p(t,y)$=4.5; in a similar way, the difference in favor of t at the weak point of y is $p(t,z)$-$p(y,z)$=3, while $p(y,t)$-$p(t,t)$=4.5 and $p(x,z)$-$p(t,z)$=7). Therefore, f fails to weakly respect the C1-Domination and also fails to weakly respect the C2-Domination by a pair, hence concluding the proof.

Thus, proposition 4 shows that all sensible Condorcet C1-Correspondences (those that weakly respect the C1-Domination) and the wide family of C2-Correspondences that weakly respect the C2-Domination by a pair, suffer the Strong No Show Paradox.

The following proposition identifies a family of C2-Correspondences respecting the C2-Domination by a pair, and to which Proposition 4 is consequently applicable.

Given any situation (X,p), let $w=\{w_k\}_{k\in\{1,2,...,n-1\}}$ be a nonnegative real vector with *n-1* components, such that $w_1=1$. For every order l: $x_1x_2...x_n$ of candidates, let us suppose that the rows and columns of M_p are ordered according to l. The sum $\Sigma_{i<j}w_ip_{ij}$, where $p_{ij}=p(x_i, x_j)$. will be called the ***w*-Generalized Kemeny Score** *of l*, abbreviated $K(l,w)$. We say that f is a ***w*-Generalized Kemeny Correspondence** if and only if, for every situation (X,p), the winners are those candidates who are at the top of an order which has a maximal *w*-Generalized Kemeny Score.

It is easy to see that the Borda VC and the Kemeny VC can be selected as particular cases of this definition. Borda is selected if $w_k=0$ when $k>1$, while Kemeny is selected if $w_k=1$ for every k.

Proposition 5: *Every w-Generalized Kemeny Correspondence f respects the C2-Domination by a pair.*

Proof: Let (X,p) be a situation in which t is C2-Quasidominated by x, $w=\{w_k\}_{k\in\{1,2,...,n-1\}}$ be the weights vector of f, and l: $x_1...x_r...x_n$ be an order of X where $x_1=t$ and $x_r=x$. Let x_s be the weak point of x with respect to t. We will prove that interchanging in l the first candidate t with candidate x, the resulting l_{xt} order has a *w*-Generalized Kemeny Score higher than that of l.

The sums defining the scores of l and l_{xt} differ only in two rows, the first row and the r-row, so that $K(l_{xt}, w) - K(l, w) =$

$= (\Sigma_{r\neq j} w_1 p_{rj} - \Sigma_{1<j} w_1 p_{1j}) + (\Sigma_{r<j} w_r p_{1j} - \Sigma_{r<j} w_r p_{rj}) =$

$=(w_1 p_{r1} + \Sigma_{1<j<r} w_1 p_{rj} + \Sigma_{r<j} w_1 p_{rj} - w_1 p_{1r} - \Sigma_{1<j<r} w_1 p_{1j} - \Sigma_{j<r} w_1 p_{1j}) + (\Sigma_{r<j} w_r p_{1j} - \Sigma_{r<j} w_r p_{rj}) =$

$= w_1 (p_{r1} - p_{1r}) + w_1 \Sigma_{1<j<r} (p_{rj}-p_{1j}) + w_1 \Sigma_{r<j} (p_{rj} - p_{1j}) + w_r \Sigma_{r<j} (p_{1j} - p_{rj}) =$

$=w_1 (p_{r1} - p_{1r}) + w_1 \Sigma_{1<j<r} (p_{rj} - p_{1j}) + (w_1 - w_r)\Sigma_{r<j} (p_{rj} - p_{1j})$.

As $x=x_r$ quasi dominates $t=x_1$ in differences, and $w_1>w_1-w_r$:

If $s<r$, $K(l_{xt}, w) - K(l, w) \geq w_1(p_{r1} - p_{1r}) + w_1(p_{rs} - p_{1s}) > 0$

If $s>r$, $K(l_{xt}, w) - K(l, w) \geq w_1 (p_{r1} - p_{1r}) + (w_1 - w_r) (p_{rs} - p_{1s}) > 0$

Therefore, in any case, the order l_{xt} has a *w*-Generalized Kemeny Score higher than that of *l*, which excludes *t* from being a winner, thus completing the proof.

The situation described in the proof of proposition 4, in which there is no Condorcet Candidate, along with proposition 5, allows us to conclude that no Condorcet VC which coincides with a *w*-Generalized Kemeny Correspondence when there is no Condorcet candidate, satisfies the Positive Involvement property.

4. INCIDENCE OF THE PARADOX IN KNOWN CONDORCET CORRESPONDENCES.

4.1 C1-Correspondences.

As shown by proposition 4, all reasonable Condorcet C1-Correspondences (that is, those weakly respecting the C1-Domination) suffer the Strong No Show Paradox. In order to see that no Condorcet C1-Correspondence proposed in the literature (as far as I know) is free from the paradox, we only need to analyze in detail the **Top Cycle** (f_{TC}) and the **Uncovered Set** (f_{US}) correspondences:

$$f_{TC}(X,p) \equiv \{x \in X: \textit{there is no } y \in X \textit{ such that } yWW_p x \textit{ and } (\textit{not } xWW_p y)\}$$

$$f_{US}(X,p) \equiv \{x \in X: \textit{there is no } y \in X \textit{ such that } (xW_p z \quad \textit{implies} \quad yW_p z)\}$$

If *s* is C1-dominated by *x*, then *s* is beaten by *x*, so that *s* can not belong to f_{TC} (X,p) if *x* does not. Therefore, f_{TC} weakly respects the C1-Domination and, by proposition 4, suffers SNSP.

On the other hand, if *s* is C1-dominated by *x*, then *s* is covered by *x*, so that *s* can not belong to f_{US} (X,p). Therefore, f_{US} respects the C1-Domination and, by proposition 4, suffers SNSP. Observe that this is a valid argument for any definition of the covering relation, thus it applies to the correspondences known as **Fishburn**'s function and **Miller's Uncovered set** correspondence, and also to those Uncovered Set correspondences defined in Peris and Subiza (1999) for weak tournaments.

Furthermore, all the others neutral C1-correspondences proposed in the literature are, to our knowledge, refinements of the Uncovered set correspondence, hence they all respect the C1-Domination. This is the case of the following correspondences, and also of their counterparts in weak tournaments: **Copeland**, **Slater**, **Kendall-Wei**, **Dutta's Minimal Covering**, **Banks**, **Laffond's Bipartisan Tournament Set**, and **Schwartz's Tournament Equilibrium**. See Fishburn (1977), Moulin (1986), Dutta (1988), Laffond *et al* (1993, 1995), Levin and Nalebuff (1995) and Peris and Subiza (1999).

4.2 C2-Correspondences.

Proposition 4 shows that all Condorcet C2-Correspondences satisfying a very weak compensation property (that is, those weakly respecting the C2-Domination by a pair) suffer the Strong No Show Paradox.

On the other hand, all *w*-Generalized Kemeny correspondences are, by proposition 5, shown to respect the C2-Domination by a pair. Let us begin analyzing the **Black** and **Kemeny** correspondences (f_{BLACK} and f_{KEM}). See Fishburn (1977), Young and Levenglick (1978), and Young (1995).

$$f_{BLACK}(X,p) \equiv \begin{cases} \{c\} & \textit{if a Condorcet candidate c exists,} \\ \{x \in X\text{: } \textit{x has a maximal Borda Score}\}, & \textit{in any other case.} \end{cases}$$

$$f_{KEM}(X,p) \equiv \{x \in X\text{: } \textit{x is at the top of an order l with maximal Kemeny Score}\}$$

The first correspondence applies the Borda algorithm when no Condorcet Candidate exists, while the second applies always the Kemeny algorithm. Therefore, both are *w*-Generalized Kemeny correspondences (w_1=1 and w_k=0 when k>1, for the case of Black, and w_k=1 for every k, for the case of Kemeny). Therefore, both suffer SNSP.

Now we will show that **Nanson**'s and **Simplified Dogdson**'s correspondences (f_{NAN} and $f_{S.DOG}$) respect the C2-Domination by a pair, because no quasidominated in differences candidate is a winner. See Fishburn (1977) and

Young (1995).

$f_{NAN}(X,p) \equiv lim_{j\to\infty} X_j$, *where* $X_1 = X$ *and*

$X_{j+1}=$ X_j , *if all candidates in* X_j *have the same Borda Score on* $(X_j\ ,\ p/X_j)$.

$X_j \setminus \{x \in X_j$: *The Borda Score of* x *on* $(X_j,\ p/X_j)$ *is minimal*$\}$, *in any other case.*

(Observe that the algorithm operates in an iterative fashion, eliminating all candidates with a worst Borda Score in the actual situation, except when all candidates have the same Borda Score)

$f_{S.\ DOG}(X,p) \equiv$ $\{x \in X$: *x needs a minimal number of elemental interchanges in*

M_p *to become a Condorcet Candidate*$\}$

If t is quasidominated by x in differences, the Borda score of t is lower than that of x at any step of the elimination process. Hence, candidate t (which will be eliminated before x) is not a winner in $f_{NAN}(X,p)$. On the other hand, the number of elemental interchanges in M_p needed by t to become a Condorcet Candidate is obviously higher than that needed by x. This implies that t cannot be a winner in $f_{S.DOG}(X,p)$. Therefore, both f_{NAN} and $f_{S.DOG}$ respect the C2-Domination by a pair, and suffer SNSP.

Let us analyze now the **Laffond's Bipartisan Plurality Set** correspondence (f_{BPS}). Defined in Laffond *et al* (1994), it respects the C2-Domination by a pair. Let (X,p) be any situation in which $p(x,y)\neq p(y,x)$ for every $x\neq y$.

$f_{BPS}(X,p) \equiv$ $\{x \in X$: *x belongs to the support of the unique symmetric Nash Equilibrium of the Bipartisan Plurality Game of* $(X,p)\}$

Let (X,p) be any situation in which, as in that of the proof of Proposition 4, $p(u,v)\neq p(v,u)$ for every two different candidates u and v. Laffond *et al* (1994) have shown that the Bipartisan Plurality Game of this situation has a unique Nash Equilibrium, and that this equilibrium is symmetric. Let us suppose now that candidate t is C2-Dominated by the pair $\{x,y\}$. If a player of the game plays strategy t with a strictly positive probability ϕ, the best response of the other player can not include t in its support (Indeed, he would obtain a strictly higher payoff by transferring the probability ϕ from t to $0.5x+0.5y$). Therefore, t cannot be in the support of the unique symmetric Nash Equilibrium of the game, and thus t is not a winner in $f_{BPS}(X,p)$. Hence, f_{BPS} respects the C2-Domination by a pair in this type of situations and suffers SNSP.

Let us show now that the **Tideman's Ranked Pairs** correspondence (f_{RP}), defined and studied in Tideman (1987) and Zavist and Tideman (1989), suffers the SNSP despite the fact that it does not weakly respect the C2-Domination by a pair nor the C1-Domination.

$f_{RP}(X,p) \equiv \{x \in X$: *x is the top candidate of a stack l*$\}$

In the situation of the proof of proposition 4, candidate t is the only winner because *tuzyx* is the unique stack. Thus, f_{RP} does not weakly respect the C2-Domination by a pair nor the C1-Domination. Nevertheless, let $X = \{x,y,z,u\}$ and p the following 11-voters profile: [*uzyx* (3 voters), *xzyu* (3 voters), *yuxz* (3 voters), *zyxu* (1 voter), *xyuz* (1 voter)]. Let p' be the profile $p+v$ where v is the one-voter profile with preferences *xyuz*. The comparison matrices are:

M_p	x	y	z	u
x	5.5	4	7	5
y	7	5.5	4	8
z	4	7	5.5	4
u	6	3	7	5.5

$M_{p'}$	x	y	z	u
x	6	5	8	6
y	7	6	5	9
z	4	7	6	4
u	6	3	8	6

We will prove that x is elected in (X,p) but not in (X,p'). Let us first see that the order *l: xzyu* is a stack of (X,p). Candidate x attains z through l in (X,p) because $p(x,z)>p(z,x)$, attains y because $p(x,z) \geq p(y,x)$ and $p(z,y) \geq p(y,x)$, and attains u

because $p(x,z) \geq p(u,x)$, $p(z,y) \geq p(u,x)$ and $p(y,u) \geq p(u,x)$. Candidate z attains y because $p(z,y)>p(y,z)$, and attains u because $p(z,y) \geq p(u,z)$ and $p(y,u) \geq p(u,z)$. Candidate y attains u because $p(y,u) > p(u,y)$. Therefore, candidate x is a winner.

Let us now see that no order with x at the top can be a stack of (X,p'). For the order l: $xz_1z_2z_3$ to be a stack, it is necessary that any candidate in l beats or ties in (X,p') to his immediate successor. The only orders with x at the top and satisfying this necessary condition are l_1:$xzyu$ and l_2:$xuzy$. However, l_1:$xzyu$ is not a stack because z fails to attain u through l_1 in (X,p') and l_2:$xuzy$ is not a stack because u fails to attain y through l_2 in (X,p'), thus x is not a winner in (X,p'). Therefore, f_{RP} fails to satisfy Positive Involvement and suffers SNSP.

The last C2-correspondence to be analyzed, the **Simpson-Cramer Minmax** correspondence (f_{MINMAX}), is (as far as I know) the only one known Condorcet correspondence not affected by the paradox. See Fishburn (1977) and Young (1995).

$f_{MINMAX}(X,p)\equiv\{x\in X$: *The minimal off-diagonal term of row x in M_p is maximal*$\}$

Let us show that f_{MINMAX} satisfies the Positive Involvement property. Let (X,p) be any situation, z_1 a winner candidate for (X,p), and (X,v) be any one-voter's situation with preferences l: $z_1...z_n$. Call p' the profile $p+v$. The matrix $M_{p'}$ of the new situation is:

$$p'(x,y) = \begin{cases} p(x,y) + 1 & \text{if } x\neq y \text{ and } xly. \\ p(x,y) + ½ & \text{if } x=y \\ p(x,y) & \text{in any other case} \end{cases}$$

As supposed, the z_1 row has a maximal minimal off-diagonal entry in M_p. Let $p(z_1, z_j)$ a minimal off-diagonal entry of the z_1 row in M_p. Since $p'(z_1, z)= p(z_1, z)+1$ for every $z\neq z_1$, while $p'(x, y)\leq p(x, y)+1$ for every x,y, it is obvious that $p'(z_1, z_j)$ is also a minimal off-diagonal entry of the z_1 row in $M_{p'}$, and that the z_1 row has a maximal minimal off-diagonal entry in M_p. Therefore, z_1 is a winner in the new situation.

4.3 C3-Correspondences.

The statement of proposition 4 is not applicable to these correspondences. However, some of the arguments used in this proposition may be applicable, and in fact are. Let us analyze the two C3-correspondences proposed in the literature, the **Dogdson** and the **Young** correspondences (f_{DOG} and f_{YOUNG}). See Fishburn (1977). We will see that both suffer the SNSP.

f_{DOG} (X,p) ≡ {$x \in X$: *The number of elemental interchanges in (X,p) needed by x to become a Condorcet Candidate is minimal*}

f_{YOUNG}(X,p) ≡ {$x \in X$: *The number of excluded voters in (X,p) needed by x to become a Condorcet Candidate is minimal*}

Note: Fishburn (1977) provides slightly different definitions of Dogdson and Young correspondences. Firstly, they are based on the concept of QuasiCondorcet Candidates (those that beat or tie every other) and secondly, he introduces a limit process in order to avoid that these correspondences fail satisfying homogeneity. The results obtained in this paper are not affected by these modifications.

In the situation (X,p) described in the proof of proposition 4, since candidate t needs more than 12 elemental interchanges in (X,p) to become a Condorcet Candidate, while y needs only 8 (obtained by switching y and z in 8 voters with preferences *uzytx*), t is not chosen in f_{DOG}(X,p). On the other hand, and for the same situation, since candidate u needs only the removal of 12 voters (the ten voters with preferences *xztyu* and the two with preferences *zyxtu*) to become a Condorcet Candidate, while t needs more than 12 voters removed, t is not chosen in f_{YOUNG}(X,p). Therefore, both f_{DOG} and f_{YOUNG} fail to satisfy Positive Involvement and suffer SNSP.

5. FINAL REMARKS.

Remark 1: A practical question, which has not been dealt with here, refers to the number of candidates and voters that are necessary to invoke the paradox. Although a situation with 5 candidates and 39 voters was needed in the proof of proposition 4, usually a simpler situation (typically of 4 candidates and a number of voters between 15 and 30) is enough to build a counterexample of the Positive Involvement property for any given method.

Remark 2 (a stronger and a weaker version of the paradox): We can define an even stronger no show paradox, called $SNSP^+$, in the following way:

Definition 11: *A VC f is said to satisfy the **Weak Positive Involvement** property if and only if for any situation* (X,p), *there is a winner x such that*:

If (X,v) *is a one-voter situation with favorite candidate x, then* $x \in f(X,p+v)$.

In other words, at least one winner *x* will remain a winner when a new voter, who prefers *x* to any other candidate, is added. An VC *f* that fails to satisfy this property is said to suffer $SNSP^+$.

The proof of proposition 4 remains obviously valid for the following alternative statement: "*No Condorcet VC that respects the C1-Domination or respects the C2-Domination by a pair, satisfies the Weak Positive Involvement property*". Among the VCs studied in this paper which suffer SNSP, all suffer this new paradox, except the Top Cycle VC and (perhaps) the Tideman's Ranked Pairs VC. The reason is that in the situation (X,p) described in the proof of proposition 4, none of them chooses candidate *t* as a winner.

On the other hand, if we allow ties in the voter's preferences, a paradox weaker than SNSP, and affecting every Condorcet VC, can be defined in the following way:

Definition 12: *A VC f satisfies the **Positive Involvement with ties allowed** property if and only if for any given pair of situations* (X,p) *and* (X,v), *where profile v has only one voter,*

If $x \in f(X,p)$ *and x is preferred or tied to any y in v, then* $x \in f(X,p+\mathrm{v})$.

In other words, if candidate x is a winner, x will remain a winner when a new voter is added for whom no candidate is strictly preferred to x.

To establish the following proposition, let us make some necessary, but natural modifications in the definition of M_p:

$p(x,y) = \Sigma_{i=1,\dots,n}\ (1/i)p_i(x,y)$, where $p_1(x,y)$ is the number of voters who strictly prefer x over y and $p_j(x,y)$, when $j>1$, is the number of voters who have x sharing with y a j-candidates tie. If there are no ties, $p(x,y)$ has the usual meaning.

Proposition 6: *No Condorcet VC f satisfies the Positive Involvement with ties allowed property*.

Proof: Let f be a Condorcet VC, $X=\{x,y,z\}$ and p be the following classical symmetric profile $p=[xyz$ (1 voter), yzx (1 voter), zxy (1 voter)]. Let us suppose that, without any loss of generality, x is a winner. Then, if we add to p two new voters with preferences $x \sim z > y$, candidate z becomes a Condorcet candidate and, as a consequence, the only winner. Thus x becomes a loser when the first voter is added or when the second voter is added. Therefore, f fails to satisfy the Positive Involvement (with ties allowed) property.

REFERENCES

Dutta B (1988): Covering sets and a New Condorcet Choice Correspondence. J Econ
Theory 44: 63-80

Fishburn PC (1977): Condorcet Social Choice Functions. SIAM J Appl Math, 33: 469-489.

Laffond G, Laslier JF, Le Breton M (1993): The Bipartisan Set of a Tournament Game. Games Econ Behav 5: 182-201

Laffond G, Laslier JF, Le Breton M (1994): Social-Choice Mediators. American Economic Review Papers and Proceedings 84: 448-453

Laffond G, Laslier JF, Le Breton M (1995): Condorcet choice correspondences: A set-theoretical comparison. Mathematical Social Sciences 30: 23-35

Levin J, Nalebuff B (1995): An Introduction to Vote-Counting Schemes. J Econ Perspectives 9: 3-26

Moulin H (1986): Choosing from a Tournament. Soc Choice Welfare 3: 271-291

Moulin H (1988): Condorcet's Principle Implies the No Show Paradox. J Econ Theory 45: 53-64

Pérez J (1995): Incidence of no-show paradoxes in Condorcet choice functions. Investigaciones Económicas, XIX (1), 139-154

Peris JE, Subiza B (1999): Condorcet choice correspondences for a weak tournament. Soc Choice Welfare 16: 217-231

Saari DG (1990): The Borda dictionary Soc Choice Welfare 7: 279-319

Saari DG (1994): Basic Geometry of Voting. Springer-Verlag. New York.

Tideman TN (1987): Independence of clones as a criterion for voting rules. Soc Choice Welfare 4: 185-206

Young HP (1974): An axiomatization of Borda's rule, J Econ Theory 9: 43-52

Young HP (1975): Social Choice Scoring Functions. SIAM J Appl Math, 28: 824-838

Young HP, Levenglick A (1978): A consistent extension of Condorcet election principle. SIAM J Appl Math 35: 285-300

Young HP (1995): Optimal voting rules, J Econ Perspectives 9: 51-64.

Zavist TM, Tideman TN (1989): Complete Independence of Clones in the Ranked Pairs Rule. Soc Choice Welfare 6: 167-173