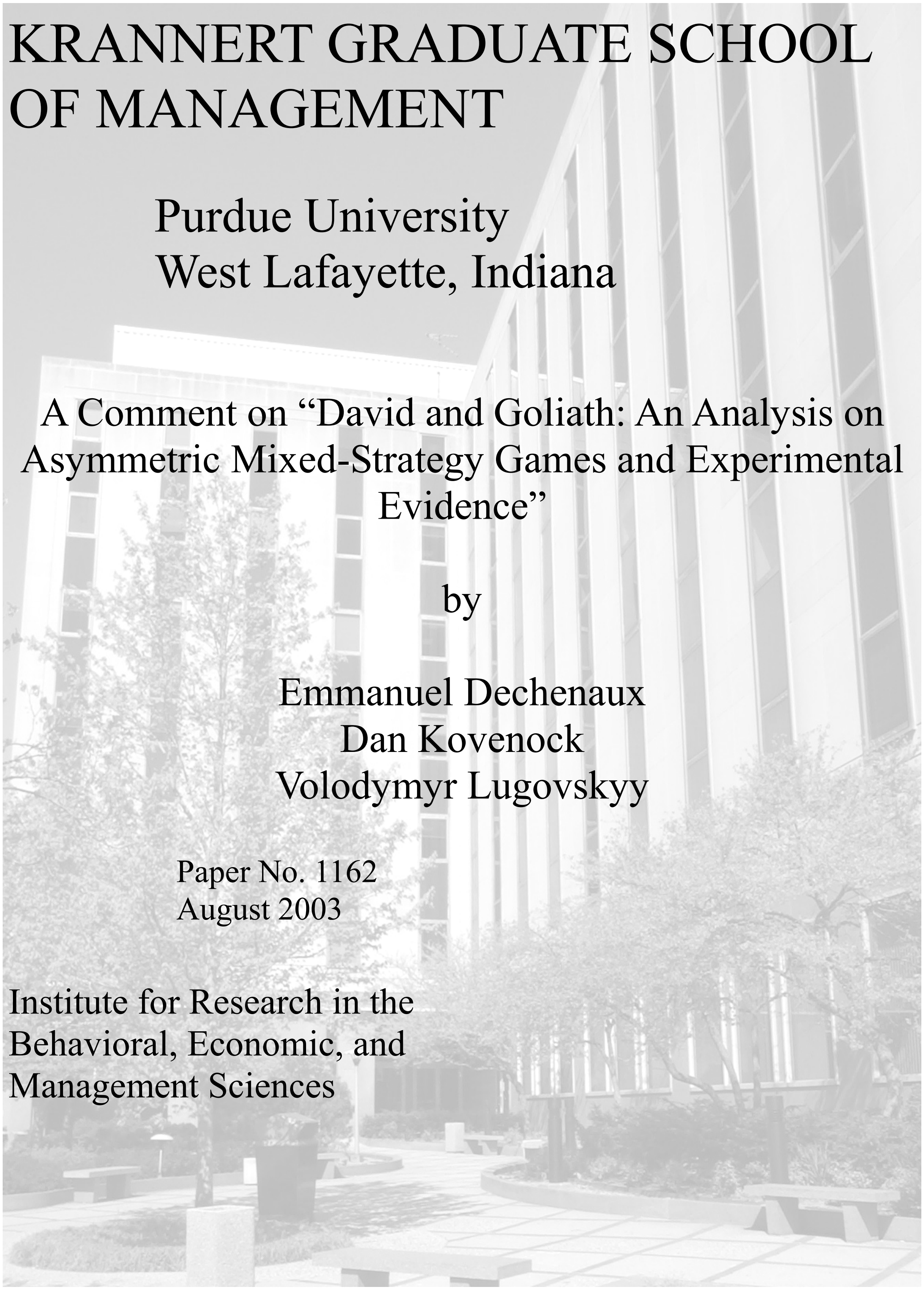

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A Comment on "David and Goliath: An Analysis on Asymmetric Mixed-Strategy Games and Experimental Evidence"

by

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A Comment on "David vs. Goliath: An Analysis of Asymmetric Mixed-Strategy Games and Experimental Evidence"

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Abstract

In this note, we characterize the full set of equilibria of the 2-firm patent race analyzed by Amaldoss and Jain (Management Science, 48(8), August 2002, pp. 972-991). Contrary to Amaldoss and Jain's (2002) claim, we show that the equilibrium is not always unique and that the set of equilibria is non-robust to changes in the (discrete) set of available strategies. In some equilibria, the qualitative results are the reverse of those in the only equilibrium Amaldoss and Jain identify. Our findings have important implications for the analysis of the data from Amaldoss and Jain's experiments, as well as other experiments appearing in the literature.

Keywords: All-Pay Auction, Contests, Experimental Economics, Competitive Strategy, R&D.

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1 Introduction

In a recent paper appearing in this journal, Amaldoss and Jain (2002) develop a simple model of a contest. Two firms compete to win a patent, the "prize", by simultaneously investing in R&D. The firm that invests the higher amount wins the prize and receives a payoff equal to its value of the prize minus its investment. The firm that invests the lower amount loses an amount equal to its investment. Both firms face a symmetric financial constraint which prevents them from investing above a certain amount. This cap on investment is lower than the value of the prize for either firm.

Although the authors focus on a particular application, the game analyzed is a type of all-pay auction with (identical) bid caps. The all-pay auction has been applied in the literature on rent seeking and it is often noted that it can be used as a reduced form to model R&D races.[1] In fact, a very similar all-pay auction has been analyzed by Che and Gale (1998) to model lobbying with expenditure caps.[2]

There are two major differences between the Amaldoss-Jain (henceforth A-J) and the Che-Gale (henceforth C-G) treatments. One is that C-G assume that in the event of a tie in expenditure the prize is either split or allocated with a fair randomizing device (henceforth "partial dissipation"), whereas A-J's main theoretical treatment assumes that in the event of a tie in expenditure the value of the prize is completely dissipated and no one receives the prize (henceforth "full dissipation").[3] A second major difference lies in the fact that A-J assume a discrete (pure) strategy space, whereas the strategy space in C-G is continuous.[4]

In this note we demonstrate that several of the results obtained by A-J are erroneous, including Proposition 1, the one and only proposition in the article. Contrary to the claim of A-J's Proposition 1, we show that under full dissipation the equilibrium is not always unique. We provide an exhaustive characterization of the set of equilibria and show that

[1] See for instance Dasgupta (1986), Rosen (1988), and Baye, Kovenock and de Vries (1996). Baye and Hoppe (2003) demonstrate that many other models of patent races appearing in the literature can be formulated as Tullock rent seeking games, of which the all-pay auction is a special case.

[2] Che and Gale (1996a, 1996b) extend the analysis of the all-pay auction with financially constrained bidders and a continuous strategy space to the case of incomplete information and $N \geq 2$ bidders.

[3] A-J also note that other tie breaking rules can be used and analyze an example that uses the prize splitting rule employed by C-G. We comment further on this case below.

[4] For a treatment of the all-pay auction with a discrete strategy space in the absence of bid caps see Baye, Kovenock and de Vries (1994).

there is more than one equilibrium under fairly general conditions. We suspect that this failure to recognize the existence of other equilibria stems from an erroneous claim used in the proof of Proposition 1, namely that both firms always earn an expected profit of zero in a non-degenerate mixed-strategy equilibrium. We also examine some of the implications of our analysis for the conclusions of the experimental investigation undertaken by A-J. Specifically, we show that the error of imposing a zero profit constraint carries over to the partial dissipation experiment analyzed by A-J. The vectors of probabilities presented as equilibrium strategies by A-J on page 984 of their article do not constitute a Nash equilibrium of the game. In equilibrium, under "partial dissipation", the firm with the lower value does not necessarily invest more aggressively than the firm with the higher value as the authors claim on page 977 (even with the additional assumptions mentioned on that page).

Below, we begin by introducing some notation. Then, we demonstrate that Amaldoss and Jain's major theoretical claims are erroneous and correct them. We end by discussing the empirical implications of our results.

2 The model

Suppose there are two firms indexed by j, $j \in \{H, L\}$. Firm j's pure strategy space is given by $X_j = \{0, c/k, 2c/k, 3c/k, ..., c\}$ with generic element x_j. Thus firms have identical strategy spaces. Let r_j denote firm j's value for the prize. Following A-J, let $r_H > r_L > c$. Firm j's payoff from playing x_j when the other firm plays x_{-j} is denoted by $\Pi_j(x_j, x_{-j})$. Let t denote the fraction of the prize each firm obtains when both firms invest the same amount. We have:

$$\Pi_j(x_j, x_{-j}) = \begin{cases} r_j - x_j & \text{if } x_j > x_{-j}, \\ tr_j - x & \text{if } x_j = x_{-j} = x, \\ -x_j & \text{if } x_j < x_{-j}. \end{cases}$$

In the mixed extension of the game, let $p_j(x)$ denote a probability distribution over the elements of X_j. $\Pi_j(p_j, p_{-j})$ is then firm j's expected payoff.

3 Equilibria under the “full dissipation” assumption

We show that equilibria that are qualitatively distinct from the equilibrium suggested by A-J exist under fairly general conditions in the game they examine. This directly contradicts their Proposition 1 and has important implications for empirical analyses of this type of all-pay auction.

We begin by restating A-J's Proposition 1 and its assumptions as they appear in the paper. Recall that the result below assumes full dissipation ($t = 0$) so that firm j's payoff is given by:

$$\Pi_j(x_j, x_{-j}) = \begin{cases} r_j - x_j & \text{if } x_j > x_{-j}, \\ -x_j & \text{otherwise.} \end{cases}$$

Proposition 1 (Amaldoss and Jain) *If $k > 1$, the unique equilibrium of this discrete game is for firm H to invest ic/k discrete units of capital (where $i = 1, ..., k$) with probability:*

$$p_H\left(i\frac{c}{k}\right) = \begin{cases} \frac{c}{k}\frac{1}{r_L} & \textit{if } i = 0, 1, ..., k-1, \\ 1 - \frac{c}{r_L} & \textit{if } i = k. \end{cases} \tag{1}$$

Similarly, for firm L we have:

$$p_L\left(i\frac{c}{k}\right) = \begin{cases} \frac{c}{k}\frac{1}{r_H} & \textit{if } i = 0, 1, ..., k-1, \\ 1 - \frac{c}{r_H} & \textit{if } i = k. \end{cases} \tag{2}$$

The corresponding c.d.f. for firms H and L converges to (3) and (4), respectively, as $k \to \infty$ where:

$$F_H(x) = \begin{cases} \frac{x}{r_L} & 0 \leq x < c \\ 1 & x \geq c. \end{cases} \tag{3}$$

$$F_L(x) = \begin{cases} \frac{x}{r_H} & 0 \leq x < c \\ 1 & x \geq c. \end{cases} \tag{4}$$

Claim 1 below provides a complete characterization of the set of equilibria of the discrete “full dissipation” game with discrete strategy space. It shows that equilibria that do not satisfy (1) and (2) exist fairly generally. These equilibria have an alternating structure, where one firm's support contains the even points of the strategy space and the other firm's contains the odd points. In these alternating equilibria, one firm earns a strictly positive expected profit, and the other firm earns zero.

Claim 1 *Assume $k > 1$. If k is odd, there exist three Nash equilibria. One equilibrium is characterized by (1) and (2) and satisfies $\Pi_H^* = \Pi_L^* = 0$. The two other equilibria are given by (p_j^*, p_{-j}^*), $j \in \{H, L\}$ where:*

$$p_j^*\left(i\frac{c}{k}\right) = \begin{cases} \frac{c}{k}\frac{2}{r_{-j}} & \text{if } i = 1,3..., k-2 \\ 1 - \left(\frac{k-1}{k}\right)\left(\frac{c}{r_{-j}}\right) & \text{if } i = k \\ 0 & \text{if } i = 0, 2, ..., k-1 \end{cases} \tag{5}$$

and

$$p_{-j}^*\left(i\frac{c}{k}\right) = \begin{cases} 1 - \left(\frac{k-1}{k}\right)\left(\frac{c}{r_j}\right) & \text{if } i = 0 \\ \frac{c}{k}\frac{2}{r_j} & \text{if } i = 2, 4, ..., k-1 \\ 0 & \text{if } i = 1, 3, ..., k \end{cases} \tag{6}$$

In such equilibria, expected payoffs are equal to $\Pi_j^* = r_j - c > 0$ and $\Pi_{-j}^* = 0$, $j \in \{H, L\}$. If k is even, there exists a unique equilibrium characterized by (1) and (2).

The proof of claim 1 appears in the Appendix. The reason why A-J claim uniqueness of equilibrium in their Proposition 1 whereas our Claim 1 demonstrates nonuniqueness is that the proof of A-J's Proposition 1, provided in part A of their Technical Appendix, contains an error.[5] The statement directly preceding Equation (A12) in A-J's Lemma A4 is not correct. Showing that the system (A12) has a unique solution is not sufficient to prove uniqueness of a Nash equilibrium in mixed strategies. The system (A12) is written with an equality in every row, while it should be written with an inequality ($\leq$) in every row, since when a pure strategy is played with probability zero it may indeed yield an expected payoff that is strictly below the specified equilibrium level. This is exactly the reasoning used in the proof of Claim 1 in this note, which follows the general programming framework for solving for the Nash equilibria of all-pay auctions with discrete strategy spaces developed in Baye, Kovenock, and De Vries (1994).[6]

[5] The Technical Appendices can be found on the INFORMS website at http://mansci.pubs.informs.org/ecompanion.html.

[6] The second part of A-J's Proposition 1 concerns the continuous strategy space equilibrium that is the limit of the discrete equilibria that they identify as the mesh of the discrete grid of feasible expenditures

4 Empirical implications

Based on Claim 1, we now reconsider Results 2 and 3 appearing on Page 975 of Amaldoss and Jain (2002). Result 2 states that "[o]n average, firm L invests more than firm H." Result 3 states that "[f]irm L is more likely to win the patent." These two results are not true in the equilibrium where the high-value firm earns a strictly positive expected profit. Therefore, the "counterintuitive" result that the low-value firm invests more aggressively in the patent race than the high-value firm is specific to a particular class of equilibria in which the high-value firm earns zero expected profit.

Claim 2 *There exists an equilibrium in which Results 2 and 3 of A-J are reversed.*

Suppose that $k > 1$ and k is an odd number:

(i) In the equilibrium in which $\Pi_H^* > 0$, firm L invests <u>less</u> than firm H on average and is <u>less</u> likely to win the patent than firm H.

(ii) In all equilibria in which $\Pi_H^* = 0$, firm L invests more than firm H on average and is more likely to win the patent than firm H.

Suppose that $k > 1$ and k is an even number:

(iii) In the unique equilibrium, firm L invests more than firm H on average and is more likely to win the patent than firm H.

The proof of Claim 2 appears in the Appendix. In Amaldoss and Jain (2002), Proposition 1 is used as a theoretical prediction against which data from experiments are evaluated. The strategy space chosen by the authors for the experiments is such that $k = 2$. In this particular case, we showed in Claim 1 that the equilibrium is unique and coincides with the one described in Proposition 1 of A-J. However, we question whether an all-pay auction with only three possible levels of investment is apt for modeling a patent race. More specifically, for the parameters chosen for the sessions of the experiments with full dissipation, the equilibrium predicts that firms' mixed strategies yield a tie in investment,

goes to zero. Corresponding to the equilibria that we identify are continuous strategy space equilibria that have the property that one firm places a mass point at zero and the other firm a mass point at c. A correct analysis of the continuous case under "partial dissipation" may be found in Che and Gale (1998).

and thus no firm wins the patent race, in roughly 43% of the cases. Hence, the full dissipation assumption provides a player with a strong incentive to avoid investing the same amount as the other player. We argue that such incentives do not seem to reflect the reality of a patent race.[7] In support of the qualitative nature of their findings with a three point strategy space, the authors note (A-J, p. 978, footnote 7) that related work by Rapoport and Amaldoss (2000) shows that the results are robust to increases in the set of feasible strategies. Indeed, Rapoport and Amaldoss (2000) examine a game with six feasible strategies, a space that, contrary to the claims of the authors, yields multiple Nash equilibria. However, in the Rapoport and Amaldoss (2000) experiments, players place on average higher probability at a 0 investment than is predicted by the only equilibrium they identify (which corresponds to that in Proposition 1 of A-J with $r_H = r_L$). In a separate note (Dechenaux, Kovenock and Lugovskyy 2003), we examine how the experimental findings of Rapoport and Amaldoss (2000) appear consistent with at least a subset of players sometimes playing asymmetric equilibria of the type derived in this note.

Above we argued that it is the failure to allow for the possibility that a single player need not play all feasible pure strategies with positive probability in equilibrium that causes A-J to arrive at the false conclusion (A-J, p. 976) that "if a firm gets zero from not investing, then its expected payoffs from investing a positive amount must also be zero." This leads A-J to erroneously conclude that equilibrium profits must be zero for both firms. Indeed, the error of assuming that both firms must earn zero profit in equilibrium carries over to A-J's treatment of the partial dissipation case, where even a zero expenditure does not generally lead to a zero expected profit. The authors examine the behavioral implications of the partial dissipation case where $c = 2, r_L = 2.2, r_H = 2.9$ and $t = \frac{1}{2}$. A-J claim that "the equilibrium solutions for this case are (A-J, p. 984):"

$$\begin{aligned} p_H &= (p_H(0), p_H(1), p_H(2)) = (0.0909, 0.7272, 0.1818)\,, \\ p_L &= (p_L(0), p_L(1), p_L(2)) = (0.3103, 0.0689, 0.6207)\,. \end{aligned}$$

We claim that the statement above is erroneous, as under the assumptions made by the authors, (p_H, p_L) does not constitute an equilibrium of the game. In a Nash equilibrium, if

[7] The literature on contest approaches to patent races also generally assumes partial dissipation in the event of a tie and not full dissipation. See Dasgupta (1986).

x and z are in the support of firm j's mixed strategy:

$$\Pi_j(x, p_{-j}) = \Pi_j(z, p_{-j}).$$

But then, using the probabilities suggested by the authors:

$$\begin{aligned} \Pi_L(0, p_H) &= 0.0909 \times 0.5 \times 2.2 = 0.09999 > \\ \Pi_L(1, p_H) &= 0.0909 \times 2.2 + 0.7272 \times 0.5 \times 2.2 - 1 = -0.0001. \end{aligned}$$

On page C1 of A-J's Technical Appendix, the statement preceding equations (C1), (C2) and (C3) is incorrect. If both firms play 0 with positive probability, they obtain a strictly positive expected profit at 0. Thus, the system of equations that follows the statement does not characterize an equilibrium.

We now compute the correct equilibrium probabilities assuming that every pure strategy is in the support of each firm for a general k and $t = \frac{1}{2}$. Recall that a firm's payoff under partial dissipation is given by:

$$\Pi_j(x_j, x_{-j}) = \begin{cases} r_j - x_j & \text{if } x_j > x_{-j}, \\ \frac{1}{2}r_j - x & \text{if } x_j = x_{-j} = x, \\ -x_j & \text{if } x_j < x_{-j}. \end{cases}$$

A-J focus on a class of equilibria in which each firm's support contains all pure strategies. Knowing that in equilibrium at each point of its support a firm earns the same expected payoff, Π_j^*, we can characterize this type of equilibrium by the following system of equations for $j \in \{L, H\}$:

$$\begin{cases} \Pi_j^* = \frac{1}{2}p_{-j}(0)r_j \\ \Pi_j^* = [p_{-j}(0) + \frac{1}{2}p_{-j}(\frac{c}{k})]r_j - \frac{c}{k} \\ \dots \\ \Pi_j^* = \sum_{i=0}^{k-1} p_{-j}(i\frac{c}{k})r_j + \frac{1}{2}p_{-j}(c)r_j - c. \end{cases} \tag{7}$$

In addition, we know that all probabilities are nonnegative and the probabilities sum to one for each firm $j \in \{L, H\}$:

$$\sum_{i=0}^{k} p_j(i\frac{c}{k}) = 1. \tag{8}$$

Now from (7) we can express all probabilities for firm j in terms of its probability of bidding zero, $p_j(0)$:

$$p_j(i\frac{c}{k}) = \frac{\frac{c}{k}(1 + (-1)^{i+1})}{r_{-j}} + (-1)^i p_j(0). \tag{9}$$

By plugging this result into (8) we obtain:

$$\sum_{i=0}^{k} p_j(i\frac{c}{k}) = \frac{c}{r_{-j}} + \frac{c}{k}\frac{1-(-1)^k}{2r_{-j}} + \frac{1+(-1)^k}{2}p_j(0) = 1. \tag{10}$$

For k odd, $c\left(1+\frac{1}{k}\right) = r_{-j}$ should hold. But this contradicts the fact that $r_H > r_L$ and thus an equilibrium with the specified properties does not exist for k odd. For k even, $p_j(0) = 1 - \frac{c}{r_{-j}}$ so for each firm:

$$p_j\left(i\frac{c}{k}\right) = \begin{cases} 1-\frac{c}{r_{-j}} & \text{for } i \text{ even,} \\ \frac{c}{r_{-j}}\left(\frac{k+2}{k}\right) - 1 & \text{for } i \text{ odd.} \end{cases} \tag{11}$$

Using (11) to calculate the equilibrium solution yields:

$$\begin{aligned} p_H &= (p_H(0), p_H(1), p_H(2)) = (0.0909, 0.8181, 0.0909)\,, \\ p_L &= (p_L(0), p_L(1), p_L(2)) = (0.3103, 0.3793, 0.3103)\,. \end{aligned}$$

On page 977 A-J claim that the qualitative implications of their theoretical results derived for $t = 0$ hold for every $t \in [0, \frac{1}{2}]$ given that, in equilibrium, both firms randomize over all pure strategies. One of these implications is that firm L invests more on average than firm H. Using (11) we can calculate expected investment for each firm j, $j \in \{H, L\}$:

$$\begin{aligned} E[x_j] &= \sum_{i=0}^{k} p_j(i\frac{c}{k})\left(i\frac{c}{k}\right) \\ &= [\frac{c}{r_{-j}}\left(\frac{k+2}{k}\right) - 1]\frac{c}{k}(1+3+\ldots+k-1) \\ &\quad + \left(1-\frac{c}{r_{-j}}\right)\frac{c}{k}(2+4+\ldots+k) \\ &= \frac{c}{2}. \end{aligned} \tag{12}$$

As we can see, in equilibrium both firms invest an equal amount on average, which contradicts the claim made by A-J.

5 Conclusion

In this note, we have applied methods developed in Baye, Kovenock, and de Vries (1994) to illustrate some of the pitfalls of characterizing the complete set of Nash equilibria in contests with a discrete strategy space. We have shown that Amaldoss and Jain (2002) have erroneously characterized the set of Nash equilibria for discrete strategy spaces under full dissipation (A-J, Proposition 1) and have misspecified the equilibrium strategies in their experimental test of the partial dissipation case.

The experiments carried out by A-J for the full dissipation case employ a game with a strategy space with three feasible strategies and a unique Nash equilibrium. A-J justify the choice of such a restricted strategy set by appealing to its simplicity and to the robustness of the set of equilibria to increases in the number of possible strategies. However, our theoretical results show that the set of equilibria is not robust to increases in the number of possible strategies, invalidating A-J's justification. In a companion piece (Dechenaux, Kovenock, and Lugovskyy 2003), we examine how the experimental findings of Rapoport and Amaldoss (2000) appear consistent with at least a subset of players sometimes playing asymmetric equilibria of the type derived in this note.

Since experimental testing requires discrete strategy spaces, careful attention should be paid to completely characterizing the equilibria of the game to be tested. It is somewhat troubling that the set of equilibria in the all-pay auction with common caps on expenditures is non-robust to the cardinality of the strategy space. Indeed, it is hard to understand how experimental subjects could uncover (unstable) mixed strategy equilibria when it is difficult for trained researchers in the management sciences to do so.

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7 Appendix

Proof of Claim 1:

We prove Claim 1 through a series of Lemmata. It follows from Nash (1950) that an equilibrium exists. Throughout the proof, let $S \equiv \{0, \frac{c}{k}, 2\frac{c}{k}, \ldots, c\}$ and let (p_H^*, p_L^*) be an equilibrium of the game. For a given equilibrium (p_H^*, p_L^*), let $p_j^*(x)$ be the associated probability that firm j plays x, S_j the associated support of firm j's distribution, and Π_j^* its expected profit in that equilibrium, $j \in \{H, L\}$.

Lemma 1 *In any equilibrium (p_H^*, p_L^*), $S_H \cup S_L = S$. Consequently, $\Pi_j^* = 0$ for at least one j, $j \in \{H, L\}$.*

Proof. First, we show that (i) if a point $v\frac{c}{k}$, $v > 0$, is in the support of at least one firm, then all points $n\frac{c}{k}$ where $0 \leq n < v$ must be in the support of at least one firm. This implies that (ii) 0 is in the support of at least one firm. Then, we use (ii) to show that (iii), c is in the support of at least one firm. Combining (i) and (iii) completes the proof of the lemma.

We first show (i). Let v be a strictly positive integer with $v\frac{c}{k} \in S_j$ for some $j \in \{H, L\}$. Suppose contrary to our claim that there exists an $n < v$ such that $n\frac{c}{k} \notin S_H \cup S_L$. This implies that there exists some $u \leq v$ such that $u\frac{c}{k} \in S_l$ for some $l \in \{H, L\}$ and $(u-1)\frac{c}{k} \notin S_H \cup S_L$.

Then, firm l's expected payoff of playing $u\frac{c}{k}$ is given by:

$$\Pi_l(u\frac{c}{k}) = \sum_{i<u} p_{-l}^*(i\frac{c}{k})r_l - u\frac{c}{k},$$

and firm l's expected profit of playing $(u-1)\frac{c}{k}$ is given by:

$$\Pi_l((u-1)\frac{c}{k}) = \sum_{i<u-1} p_{-l}^*(i\frac{c}{k})r_l - (u-1)\frac{c}{k}.$$

But since $\sum_{i<(u-1)} p_{-l}^*(n\frac{c}{k}) = \sum_{i<u} p_{-l}^*(n\frac{c}{k})$, we have $\Pi_l(u\frac{c}{k}) < \Pi_l((u-1)\frac{c}{k})$. Therefore, $u\frac{c}{k}$ cannot be in the support of firm l's equilibrium distribution, a contradiction. Thus, we have established (i).

We now turn to (ii). First note that firm j's expected profit of playing 0 is, regardless of its opponent's strategy:

$$\Pi_j(0) = 0.$$

Suppose to the contrary that $0 \notin S_L \cup S_H$. Let $\underline{i} \equiv \min_i \{i | i\frac{c}{k} \in S_H \cup S_L\}$. Suppose without loss of generality that $\underline{i}\frac{c}{k} \in S_{l'}$ for some $l' \in \{H, L\}$. It follows from $\underline{i} > 0$, that:

$$\Pi_{l'}(\underline{i}\frac{c}{k}) = \sum_{n<\underline{i}} p^*_{-l'}(n\frac{c}{k})r_{l'} - \underline{i}\frac{c}{k} = 0 - \underline{i}\frac{c}{k} < 0,$$

since $p^*_{-l'}(n\frac{c}{k}) = 0$, $\forall n < \underline{i}$, a contradiction to the fact that $\underline{i}\frac{c}{k} \in S_{l'}$. Thus $0 \in S_L \cup S_H$.

To complete the proof of the lemma, we show (iii). Suppose to the contrary that $c \notin S_H \cup S_L$. Then, any firm j's expected profit from playing c is equal to:

$$\Pi_j(c) = \sum_{n<k} p^*_{-j}(n\frac{c}{k})r_j - c = r_j - c > 0, \text{ for } j \in \{H, L\},$$

whereas its expected profit from playing 0 is:

$$\Pi_j(0) = 0.$$

But, by (ii) above, 0 must be in the support of at least one firm j, a contradiction. Hence, we have established (iii).

Combining (i) and (iii), we have shown that $S_H \cup S_L = S$. Consequently, $\Pi^*_j = \Pi_j(0) = 0$ for at least one j, which proves the lemma. Q.E.D.

Lemma 2 *In any equilibrium* (p^*_H, p^*_L), *if* $p^*_H(i\frac{c}{k}) > 0$ *and* $p^*_L(i\frac{c}{k}) > 0$, *then* $p^*_H(n\frac{c}{k}) > 0$ *and* $p^*_L(n\frac{c}{k}) > 0$, *for every* n *such that* $0 \leq n < i$.

Proof. Suppose to the contrary that there exists an investment $v\frac{c}{k}$ such that $p^*_j(v\frac{c}{k}) > 0$, $j \in \{H, L\}$ and $n < v$ such that $p^*_l((v-1)\frac{c}{k}) = 0$ for some $l \in \{H, L\}$. This implies that there exists u, $u \leq v$ such that $u\frac{c}{k} \in S_H \cap S_L$ and such that $(u-1)\frac{c}{k} \notin S_l$ for some $l \in \{H, L\}$. From Lemma 1 it follows that since $p^*_l((u-1)\frac{c}{k}) = 0$, then $p^*_{-l}((u-1)\frac{c}{k}) > 0$. Furthermore, firm $-l$'s expected profit at $u\frac{c}{k}$ is equal to:

$$\Pi_{-l}(u\frac{c}{k}) = \sum_{n<u} p^*_l(n\frac{c}{k})r_{-l} - u\frac{c}{k},$$

while its expected profit at $(u-1)\frac{c}{k}$ is:

$$\Pi_{-l}((u-1)\frac{c}{k}) = \sum_{n<u-1} p^*_l(n\frac{c}{k})r_{-l} - (u-1)\frac{c}{k}.$$

Since $p^*_l((u-1)\frac{c}{k}) = 0$, $\sum_{n<(u-1)} p^*_l(n\frac{c}{k}) = \sum_{n<u} p^*_l(n\frac{c}{k})$. It then follows that $\Pi_{-l}((u-1)\frac{c}{k}) > \Pi_{-l}(u\frac{c}{k})$, contradicting $u\frac{c}{k} \in S_{-l}$. Q.E.D.

Lemma 3 *In any equilibrium* (p_H^*, p_L^*), *if* $\Pi_l^* > 0$ *for some* $l \in \{H, L\}$, *then:*

(i) $S_H \cap S_L = \emptyset$,

(ii) Firm l *randomizes over all pure strategies* $i\frac{c}{k}$ *for which* i *is odd and firm* $-l$ *randomizes over all pure strategies* $i\frac{c}{k}$ *for which* i *is even,*

(iii) $\Pi_{-l}^* = 0$.

Proof. First $\Pi_l^* > 0$ implies $0 \notin S_l$. Thus, from Lemma 1, $0 \in S_{-l}$. It follows that $\frac{c}{k} \notin S_{-l}$ since firm $-l$ can increase its expected profit by moving mass from $\frac{c}{k}$ to 0. Then, from Lemma 1, $\frac{c}{k} \in S_l$. Suppose $2\frac{c}{k} \in S_l$. Then whether $2\frac{c}{k}$ is in S_{-l} or not, firm l can increase its expected payoff by moving all mass from $2\frac{c}{k}$ to $\frac{c}{k}$. Thus $2\frac{c}{k} \notin S_l$, from which it follows that $2\frac{c}{k} \in S_{-l}$. Suppose now that $3\frac{c}{k} \in S_{-l}$. Then, whether $3\frac{c}{k}$ is in S_l or not, firm $-l$ can increase its expected payoff by moving all mass from $3\frac{c}{k}$ to $2\frac{c}{k}$ (recall that firm l puts no mass at $2\frac{c}{k}$). Since k is finite, it is straightforward to see that the same argument applies recursively to every $i \leq k$. That is, suppose $i\frac{c}{k} \in S_m$ and $i\frac{c}{k} \notin S_{-m}$, then $(i+1)\frac{c}{k} \notin S_m$ and $(i+1)\frac{c}{k} \in S_{-m}$, $\forall i \in \{0, 1, \ldots, k\}$ and $m \in \{H, L\}$.

(iii) follows immediately from the fact that $0 \in S_{-l}$. Q.E.D.

Lemma 4 *In any equilibrium* (p_H^*, p_L^*), *if* $\Pi_H^* = \Pi_L^* = 0$, *then* $S_H = S_L = S$.

Proof. Let $\bar{i}_j \equiv \max_i\{i | i\frac{c}{k} \in S_j\}$ and $\bar{i} \equiv \max\{\bar{i}_H, \bar{i}_L\}$. We claim that $\bar{i}_H = \bar{i}_L = \bar{i} = k$. Suppose $\bar{i}_l > \bar{i}_{-l}$. Then it is clear that $\Pi_l(\bar{i}_l) > 0$, a contradiction to $\Pi_l^* = 0$. Thus $\bar{i}_H = \bar{i}_L = \bar{i}$. Applying Lemma 1, $\bar{i} = k$. The claim then follows from Lemma 2. Q.E.D.

Lemma 5 *There exists an equilibrium* (p_H^*, p_L^*) *with payoffs* $\Pi_H^* = \Pi_L^* = 0$. *In this equilibrium* $S_L = S_H = S$ *and:*

$$p_H\left(i\frac{c}{k}\right) = \begin{cases} \frac{c}{k}\frac{1}{r_L} & \text{if } i = 0, 1, ..., k-1, \\ 1 - \frac{c}{r_L} & \text{if } i = k. \end{cases}$$

$$p_L\left(i\frac{c}{k}\right) = \begin{cases} \frac{c}{k}\frac{1}{r_H} & \text{if } i = 0, 1, ..., k-1, \\ 1 - \frac{c}{r_H} & \text{if } i = k. \end{cases}$$

Proof. If (p_H^*, p_L^*) is an equilibrium with payoffs $\Pi_H^* = \Pi_L^* = 0$, then from Lemma 4, it follows immediately that $S_L = S_H = S$. We establish existence by construction. Since $S_L = S_H = S$ and both firms obtain an expected profit of 0, if the following pair of matrix equations is satisfied (with equality) for $j = H, L$, we have constructed such an equilibrium:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ r_j & 0 & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ r_j & r_j & ... & r_j & r_j & 0 \end{bmatrix} \begin{bmatrix} p_{-j}^*(0) \\ p_{-j}^*(\frac{c}{k})) \\ . \\ p_{-j}^*(c) \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{c}{k} \\ . \\ c \end{bmatrix} \tag{13}$$

For a given $j \in \{H, L\}$, (13) is a system of $k+1$ equations and $k+1$ unknowns. If the leftmost matrix is non-singular, then (13) has a unique solution. The determinant of the leftmost matrix is equal to $(-1)^{k+2}(r_j)^k \neq 0$. Therefore (13) has a unique solution.

It is now straightforward to compute the solution to (13). We have

$$p_{-j}^*(0)r_j - \frac{c}{k} = \Pi_j^* = 0,$$

which yields $p_{-j}^*(0) = \frac{c}{r_j k} > 0$. The remaining probabilities can be solved by repeated substitution of $\frac{c}{r_j k}$ in place of $p_{-j}^*(0)$ in (13), which yields $p_{-j}^*(i\frac{c}{k}) = \frac{c}{k}\frac{1}{r_j}$ for $i \in \{0, 1, ..., k-1\}$ and $p_{-j}^*(c) = 1 - \frac{c}{r_j}$. Since $r_j > c > i\frac{c}{k}$ by assumption, the solutions to (13) are probabilities. Uniqueness of an equilibrium satisfying the conditions in the lemma follows immediately from Lemma 4 and the fact that such an equilibrium must satisfy the system (13), for $j \in \{H, L\}$. Q.E.D.

Lemma 6 *An equilibrium with $\Pi_l^* > 0$ for some l, $l \in \{H, L\}$, exists if and only if k is odd. If k is odd, then there exist exactly two such equilibria (p_l^*, p_{-l}^*), $l = H, L$, given by:*

$$p_l^*\left(i\frac{c}{k}\right) = \begin{cases} \frac{c}{k}\frac{2}{r_{-l}} & \text{if } i = 1,3..., k-2 \\ 1 - \left(\frac{k-1}{k}\right)\left(\frac{c}{r_{-l}}\right) & \text{if } i = k \\ 0 & \text{if } i = 0, 2, ..., k-1 \end{cases}$$

and

$$p_{-l}^*\left(i\frac{c}{k}\right) = \begin{cases} 1 - \left(\frac{k-1}{k}\right)\left(\frac{c}{r_l}\right) & \text{if } i = 0 \\ \frac{c}{k}\frac{2}{r_l} & \text{if } i = 2, 4, ..., k-1 \\ 0 & \text{if } i = 1, 3, ..., k \end{cases}$$

Proof. We prove existence by construction. Let l be the firm obtaining $\Pi_l^* > 0$ and let the other firm be $-l$. From Lemma 3, the equilibrium must be of the alternating form, and firm l must play pure strategies $i\frac{c}{k}$ for which i is an odd number.

Suppose k is even. It follows that c is in S_{-l} but not in S_l. Therefore $\Pi_{-l}^* = r_{-l} - c > 0$. But this and $\Pi_l^* > 0$ contradict Lemma 3. Therefore no equilibria with $\Pi_l^* > 0$ exist when k is even, which proves the "only if" part of the statement.

Suppose k is odd. Then by Lemma 3, $S_l = \{1, 3\frac{c}{k}, ..., c\}$ and $S_{-l} = \{0, 2\frac{c}{k}, ..., (k-1)\frac{c}{k}\}$. It follows that $\Pi_l^* = r_l - c$. From Lemma 3, for firm $-l$, the strictly positive $p_{-l}^*(x)$'s must be the solution to the following system of $\frac{k+1}{2}$ equations:

$$\begin{cases} \Pi_l^* = p_{-l}^*(0) r_l - \frac{c}{k} \\ \Pi_l^* = [p_{-l}^*(0) + p_{-l}^*(2\frac{c}{k})] r_l - 3\frac{c}{k} \\ \dots \\ \Pi_l^* = [p_{-l}^*(0) + p_{-l}^*(2\frac{c}{k}) + ... + p_{-l}^*((k-1)\frac{c}{k})] r_l - k\frac{c}{k} \end{cases}$$

This system of equations can be written in matrix form:

$$\begin{bmatrix} r_l & 0 & 0 & ... & 0 \\ r_l & r_l & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ r_l & r_l & r_l & ... & r_l \end{bmatrix} \begin{bmatrix} p_{-l}^*(0) \\ p_{-l}^*(2\frac{c}{k}) \\ ... \\ p_{-l}^*((k-1)\frac{c}{k}) \end{bmatrix} = \begin{bmatrix} \Pi_l^* + \frac{c}{k} \\ \Pi_l^* + \frac{3c}{k} \\ ... \\ \Pi_l^* + k\frac{c}{k} \end{bmatrix}. \tag{14}$$

Note that the number of equations coincides with the number of unknowns. The determinant of the first matrix equals $(r_l)^{(k+1)/2} > 0$. This is sufficient to prove uniqueness of the solution. Moreover, in equilibrium $\Pi_l^* = r_l - c = p_{-l}^*(0) r_l - \frac{c}{k}$. Solving the system by repeated substitution yields $p_{-l}^*(0) = 1 - \left(\frac{k-1}{k}\right)\left(\frac{c}{r_l}\right) > 0$ and $p_{-l}^*(i\frac{c}{k}) = \frac{2c}{r_l k} > 0$ for $i \in \{2, \dots, (k-1)\frac{c}{k}\}$. We check that the probabilities sum to one:

$$p_{-l}^*(0) + \sum_{i=2}^{k-1} p_{-l}^*(i\frac{c}{k}) = 1 - \frac{(k-1)\,c}{r_l k} + \left(\frac{k-1}{2}\right)\frac{2c}{r_l k} = 1,$$

so they are indeed probabilities.

We now turn to firm l's strategy. Note that firm $-l$ obtains an expected payoff of 0 in equilibrium, $\Pi_{-l}^* = 0$. From Lemma 3, the strictly positive $p_l^*(x)$'s must solve the following system of $\frac{k-1}{2}$ equations:

$$\begin{cases} \Pi_{-l}^{*} = p_l^{*}(\frac{c}{k})r_{-l} - 2\frac{c}{k} \\ \Pi_{-l}^{*} = [p_l^{*}(\frac{c}{k}) + p_l^{*}(3\frac{c}{k})]r_{-l} - 4\frac{c}{k} \\ \dots \\ \Pi_{-l}^{*} = [p_l^{*}(\frac{c}{k}) + ... + p_l^{*}((k-2)\frac{c}{k})]r_{-l} - (k-1)\frac{c}{k} \end{cases}$$

This system of equations can be written in matrix form:

$$\begin{bmatrix} r_{-l} & 0 & 0 & ... & 0 \\ r_{-l} & r_{-l} & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ r_{-l} & r_{-l} & r_{-l} & ... & r_{-l} \end{bmatrix} \begin{bmatrix} p_l^{*}(\frac{c}{k}) \\ p_l^{*}(3\frac{c}{k}) \\ ... \\ p_l^{*}((k-2)\frac{c}{k}) \end{bmatrix} = \begin{bmatrix} 2\frac{c}{k} \\ 4\frac{c}{k} \\ ... \\ (k-1)\,\frac{c}{k} \end{bmatrix}. \tag{15}$$

Note that the number of equations coincides with the number of unknowns. The determinant of the first matrix equals $(r_{-l})^{(k-1)/2} > 0$. This is sufficient to prove uniqueness of the solution. Solving the system by repeated substitution yields $p_l^{*}(i\frac{c}{k}) = \frac{2c}{r_{-l}k} > 0$, where $\frac{2c}{r_{-l}k} < 1$, for all $i \in \{1, 3, ..., k-2\}$. Using the fact that probabilities must sum to 1 to solve for $p_l^{*}(c)$, we obtain $p_l^{*}(c) = 1 - \left(\frac{k-1}{k}\right)\left(\frac{c}{r_{-l}}\right) > 0$.

Uniqueness of an equilibrium (p_j^{*}, p_h^{*}) satisfying $\Pi_j^{*} > 0$ and $\Pi_h^{*} = 0$ follows from Lemma 3 and the fact that such an equilibrium must satisfy the systems (14) and (15). Thus using Lemma 3, if k is odd, there exists exactly one equilibrium in which $\Pi_H^{*} > 0$ and exactly one equilibrium in which $\Pi_L^{*} > 0$. Q.E.D.

To complete the proof of the claim, it suffices to note that Lemma 1 implies that there are no equilibria in which both firms earn a strictly positive expected profit. Therefore, all equilibria of the game are characterized by Lemma 5 and Lemma 6. Q.E.D.

Proof of Claim 2:

The proof of (iii) can be found on Pages 975-976 of A-J, below Results 2 and 3. Uniqueness of the equilibrium in this case follows from Claim 1.

Now consider the equilibria in which $\Pi_j^{*} > 0$ and $\Pi_{-j}^{*} = 0$. First, from Claim 1, such equilibria exist if and only if, k is odd. Second, using (5) and (6), we compute each firm's expected investment:

$$E[x_j] = \left[\frac{2}{r_{-j}}\frac{c}{k}(1+3+\ldots+k-2) + \left(1 - \frac{k-1}{k}\frac{c}{r_{-j}}\right)k\right]\frac{c}{k}, \tag{16}$$

and

$$E[x_{-j}] = \frac{2}{r_j}\left(\frac{c}{k}\right)^2(2+4+\ldots+k-1). \tag{17}$$

Using the fact that:

$$1+3+5+\ldots+k = \frac{(k+1)^2}{4},$$

we rewrite (16):

$$E[x_j] = \left[\frac{2}{r_{-j}}\left(\frac{c}{k}\right)\frac{(k+1)^2}{4} + \left(1 - \frac{k-1}{k}\frac{c}{r_{-j}}\right)k\right]\frac{c}{k}.$$

Straightforward computations yield:

$$E[x_j] = \left[\frac{c + k^2(2r_{-j} - c)}{2kr_{-j}}\right]\frac{c}{k}.$$

Now using the fact that:

$$2+4+6+\ldots+k-1 = \frac{k^2-1}{4},$$

we rewrite (17):

$$E[x_{-j}] = \left[\frac{c(k^2-1)}{2kr_j}\right]\frac{c}{k}.$$

Using the expressions for $E[x_j]$ and $E[x_j]$, we obtain:

$$\begin{aligned}
E[x_j] - E[x_{-j}] &= \left[\frac{c + k^2(2r_{-j} - c)}{2kr_{-j}} - \frac{c(k^2-1)}{2kr_j}\right]\frac{c}{k} \\
&= \left[\frac{2k^2 r_j r_{-j} + cr_j(1-k^2) + cr_{-j}(1-k^2)}{2kr_j r_{-j}}\right]\frac{c}{k} \\
&> \left[\frac{k^2 2(r_j r_{-j} - c\max_l\{r_l\})}{2kr_j r_{-j}}\right]\frac{c}{k} \\
&> 0.
\end{aligned}$$

Therefore in the equilibrium in which $\Pi_j^* > 0$, firm j invests more than firm $-j$ on average, $j \in \{H, L\}$.

Now we compute the probability of winning for each firm, again using (5) and (6):

$$\Pr[j \text{ wins}] = \sum_{z=0}^{(k-3)/2} \frac{2}{r_{-j}}\frac{c}{k}\left[1 - \left(\frac{k-1}{k}\right)\left(\frac{c}{r_j}\right) + z\frac{2}{r_j}\frac{c}{k}\right] + 1 - \left(\frac{k-1}{k}\right)\left(\frac{c}{r_j}\right),$$

and,

$$\Pr[-j \text{ wins}] = \sum_{z=1}^{(k-1)/2} \frac{2}{r_j}\frac{c}{k}\left[z\frac{2}{r_{-j}}\frac{c}{k}\right].$$

Straightforward computations yield:

$$\Pr[j \text{ wins}] - \Pr[-j \text{ wins}] = 1 - \left(\frac{c}{r_j}\right)\left(\frac{c}{r_{-j}}\right)\left(\frac{k^2-1}{k^2}\right) > 0,$$

Therefore in the equilibrium in which $\Pi_j^* > 0$, firm j is more likely to win the patent than firm $-j$, $j \in \{H, L\}$. The argument for the equilibrium in which $\Pi_H^* = \Pi_L^* = 0$ in (ii) is similar to the one used to prove (iii). This completes the proof of Claim 2. Q.E.D.