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by

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Abstract

This paper examines the feasibility of collusion in capacity constrained duopoly supergames. In each period firms simultaneously set a price-quantity pair specifying the price for the period and the maximum quantity the firm is willing to sell at this price. Under price-quantity competition firms are able to ration their output below capacity. For a wide range of capacity pairs, the equilibrium path providing the smaller firm with its highest stationary perfect equilibrium payoff requires that it undercut its rival's price and ration demand. Furthermore, for some capacities and discount factors supporting security level punishments, price shading and rationing arise everywhere on the set of stationary perfect equilibrium paths yielding (constrained) Pareto optimal payoffs. That is, price shading may not only be consistent with successful collusion, it may be a requirement of successful collusion.

Keywords: Bertrand-Edgeworth, Supergame, Collusion, Capacity

JEL Classification: C73, D43, L13, L41

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1 Introduction

This paper examines the feasibility of collusion in price setting duopoly supergames with capacity constrained firms and constant unit costs of production up to capacity. In contrast to standard Bertrand-Edgeworth supergame models,[1] following Dixon (1992) we assume that in each period firms simultaneously set a price-quantity pair specifying the price for the period and the maximum quantity the firm is willing to sell at this price.[2]

In one-shot simultaneous move capacity constrained games, modifying Bertrand-Edgeworth (B-E) competition to allow for simultaneous price-quantity choice is innocuous. Simultaneous price-quantity games yield equilibria that are equivalent to the B-E equilibria: firms' equilibrium expected payoffs and distributions of prices and sales are the same under the two game forms (assuming an equal sharing rule when both firms price at unit cost and have capacities sufficiently large to generate the classical Bertrand equilibrium). One implication of this result is that the equilibrium behavior of real market variables, such as quantities produced and sold, prices, and profits in B-E and price-quantity games, is indistinguishable for any finite number of repetitions.

Price-quantity supergames are different.[3] In these games, the ability of firms to ration their output below capacity when they are low-priced helps relax incentive constraints in maintaining collusive behavior. Indeed, for a wide range of discount factors and asymmetric capacity pairs, the equilibrium path providing the smaller firm with its highest stationary perfect equilibrium payoff requires that it undercut its rival's price and ration consumers by setting a bound on its sales below both its demand and capacity. In fact, for some capacities and discount factors supporting security level perfect equilibrium punishments, asymmetric pricing and rationing arise everywhere on the set of stationary perfect equilibrium paths yielding constrained Pareto optimal payoffs.

The price-quantity approach to modeling collusion has important implications for business strategy and antitrust and regulatory policy. Historically, price shading by smaller firms has often been interpreted as a signal of the breakdown of attempts to collude (see, for instance, Stigler (1964)). Our results demonstrate that, contrary to conventional wisdom, asymmetric pricing and endogenous rationing of consumers by a small firm are behaviors consistent with, and sometimes required by, tacitly or overtly collusive behavior in an intuitive non-cooperative game theoretic model of

[1]See, for instance, Davidson and Deneckere (1984, 1990), Brock and Scheinkman (1985), Benoit and Krishna (1987, 1991), Lambson (1987, 1994, 1995), and Compte, Jenny and Rey (2002).

[2]This commitment to a quantity is made without incurring production costs. Costly production takes place "to order," only after the price- and ceiling-determined sales are realized.

[3]Capacity constrained price-quantity games differ from price setting games for other extensive forms as well. For instance, in the case where firms move sequentially, the two are not generally equivalent.

long run market interaction.

Rees (1993) applies Bertrand-Edgeworth supergames to analyze data from the Great Salt Duopoly in the UK in the 1980's. Rees claims that the data rule out one-shot Bertrand-Edgeworth equilibrium pricing behavior and concludes that the discount factors prevalent in the market were sufficient to support pure strategy collusive outcomes. However, he observes that in the Duopoly during this period the small firm's price was slightly below the large firm's and the small firm had a higher capacity utilization than the large firm.[4] We provide calculations of the extent of price dispersion in our model that are consistent with Rees's observations. Depending on the discount factor, on the perfect equilibrium path that maximizes the small firm's payoff, price dispersion can be as high as 16% of the price-cost margin. We also show that our model can generate patterns of capacity utilization on constrained Pareto optimal stationary perfect equilibrium paths that are consistent with Rees's observations and differ from those arising under Bertrand-Edgeworth competition. For example, for a sufficiently low discount factor, even if the large firm's capacity is just sufficient to serve the whole market at a price equal to marginal cost, in the price-quantity model the small firm may have a higher degree of capacity utilization than the large firm. This is impossible under Bertrand-Edgeworth competition.

In our model, rationing arises endogenously in a tacitly collusive pure strategy perfect equilibrium of an infinitely repeated game with complete information. This is in contrast to the previous literature on endogenous rationing which relies upon either incomplete information, as in Allen and Faulhaber (1991), DeGraba (1995) and Gilbert and Klemperer (2000), or some form of power to precommit in price, as in the sequential price setting models of Boyer and Moreaux (1988, 1989).

Our result is reminiscent of the logic of "judo economics" examined by Gelman and Salop (1983). In that paper, a unit cost advantaged incumbent is initially endowed with sufficient capacity to serve the entire market. A potential entrant moves first, deciding upon the amount of capacity to install and then its price, acting as a price leader. The incumbent then acts as a price follower. Although Gelman and Salop do not derive the entrant's optimal capacity choice explicitly, they do show that it is optimal for the entrant to choose a capacity and price that will deter the incumbent from undercutting or matching its price. The entrant remains small in order to avoid an aggressive price response from a more efficient incumbent.

Similar behavior also arises in the model of Allen, Deneckere, Faith and Kovenock (2000). They show that, for certain regions of cost (including regions where the second mover in capacity is more efficient), sequential capacity choice followed by simultaneous price setting leads to a unique perfect equilibrium in which the capacity follower sets a relatively small capacity and the equilibrium in the price setting stage is in non-degenerate mixed strategies. In the price setting subgame, the small firm

[4] In the Great Salt Duopoly, the small firm is British Salt and the large firm is ICI Weston Point.

has a lower expected price than the large firm.

Our paper obtains a "judo" outcome without an unrealistic exogenous sequencing of prices or use of non-degenerate mixed pricing strategies. We maintain a simultaneous move structure embedded in an infinitely repeated game and focus on pure strategy equilibria. A small firm reduces price and restricts its output below capacity, not as a result of preemption by a larger firm, but rather to avoid the defection of the larger firm from a collusive agreement and its resulting punishment. That is, judo behavior may be consistent with overtly or tacitly collusive behavior.

Our approach also addresses a potential weakness in the Bertrand-Edgeworth supergame literature. In that literature firms can collude in price but cannot collude (or even coordinate) in determining market shares. In much of the literature these shares are exogenously fixed. Lambson (1994) and Compte, Jenny and Rey (2002) relax this assumption by allowing firms to coordinate by endogenously sharing demand at identical prices, (but not at different prices). This has proven useful: when compared to exogenous sharing rules, endogenous sharing at identical prices facilitates both collusion along the initial path and the sustainability of symmetric price punishment paths, by allowing market shares to vary to balance the firms' incentives to cheat.

Even with endogenous sharing of demand at identical prices, there is a sense in which capacity is asked to do too much work in Bertrand-Edgeworth supergames. It serves to allocate market share in a collusive phase (at least at asymmetric prices) and determines the ability to punish following a deviation from that phase. Price-quantity games separate out these two functions by allowing quantity choice to determine the market sharing rule while leaving capacity as the measure of a firm's ability to punish. The qualitatively rich set of empirical implications of the price-quantity model relative to the B-E model serves as an illustration of the danger of asking a single strategic variable (in this case capacity) to do too much. This is in stark contrast to the spirit of the Kreps-Scheinkman (1983) analysis, which shows that a reduced form (quantity setting) may serve as an accurate proxy for a more complicated game with multiple strategic variables.

Another weakness of the Bertrand-Edgeworth approach is its reliance on suboptimal punishment paths to punish deviations from the collusive phase. For instance, in their analysis of the effect of mergers on the ability to collude, Compte, Jenny and Rey use endogenous sharing rules on symmetric price paths. However, imposing symmetry of punishment price paths is not without loss of generality and there is no reason to believe that such paths constitute optimal punishments. One might expect a restriction to symmetric equilibria to substantially reduce the ability to sustain collusion, especially in situations in which firms differ considerably in size. Indeed, one conclusion of Compte, Jenny and Rey's analysis is that asymmetry in firm size reduces the ability to collude. In this paper, we examine the class of simple penal codes having punishments with a 2-phase structure. Given such penal codes, security level punishments can be supported in a perfect equilibrium for a wider range of capacities and discount factors under price-quantity competition than under B-E competition. On the perfect equilibrium security level punishment paths we construct, firms set asymmetric prices in the first period of punishment. Moreover, when capacities are asymmetric, our 2-phase punishment paths punish deviations at least as severely, and sometimes strictly more severely, than the 2-phase punishment paths and the proportional penal codes of Lambson (1994), the symmetric paths of Compte, Jenny and Rey (2002) and the grim trigger strategy of reverting to the one-shot Nash equilibrium (Davidson and Deneckere (1984, 1990), Brock and Scheinkman (1985), Benoit and Krishna (1987, 1991)).

The formulation of our component game as one in which each firm sells at the price it announces is equivalent to a "pay-as-bid" auction, which is currently a popular tool in the analysis of electricity and other power markets. Our analysis therefore has direct and immediate implications for the nature and sustainability of collusion in "pay-as-bid" auctions in these markets. Another auction form common both in practice and the theoretical analysis of power markets is the uniform price auction. Although we do not analyze infinitely repeated uniform price auctions in this paper, the usefulness of the price-quantity approach extends to uniform price auctions as well.[5]

Section 2 introduces the cost and demand conditions employed in this paper and the basic simultaneous move price-quantity model that we have developed. An equivalence result is derived showing that for the relevant cost and demand structures the one shot equilibrium of the price-quantity game coincides with that of a corresponding B-E game. Section 3 introduces our price-quantity supergame and Section 4 characterizes the Pareto set within the set of stationary perfect equilibrium paths assuming security level punishments. In Section 5, we examine the class of simple penal codes having punishments with a 2-phase structure. We show that for a wide range of capacities, discount factors, and unit costs, relevant for the analysis in Section 4, perfect equilibrium 2-phase punishment paths exist and achieve the security level for both firms. Section 6 addresses the implications of our model for price dispersion and capacity utilization on the Pareto set of stationary perfect equilibrium paths. Section 7 concludes by showing that all of the behaviors described in this paper arise as subgame perfect equilibrium outcomes when there is endogenous choice of capacity.

[5]Preliminary analysis of repeated uniform price-quantity auctions, which perhaps better describe actual uniform price electricity auctions than, say, the B-E approach employed by Fabra (2002), Le Coq (2002) and Crampes and Creti (2002), indicates that such auctions may yield higher collusive equilibria than uniform price auctions in a B-E setting.

2 One-period simultaneous move game

2.1 The price-quantity game

Consider a market for a homogeneous good in which two firms, 1 and 2, face a demand curve:

$$D(p) = \begin{cases} M & \text{if } p \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The firms face capacity constraints k_i, $i = 1, 2$ with $k_1 \geq k_2$, and each incurs a constant unit cost of production c, $0 \leq c \leq 1$, up to its capacity. Thus, if firm i realizes sales s_i, its cost of production is $c_i(s_i) = cs_i$ for $s_i \leq k_i$. Output greater than k_i is infinitely costly. To simplify notation, we define a firm's effective capacity, $\hat{k}_i \equiv \min\{k_i, M\}$.

Assuming box demand allows us to abstract from issues related to the use of a particular rationing rule. For example, both the efficient and proportional rationing schemes define the same residual demand for firm i, $M - \hat{k}_j$.

In the component game, $G(k_1, k_2, c)$, firms simultaneously set price and quantity pairs (p_i, q_i) where q_i is interpreted as a credible pre-commitment not to produce an output greater than q_i. Firm i's strategy set is the set of the price-quantity pairs $S_i = \{(p_i, q_i) : p_i \leq 1 \text{ and } q_i \in [0, k_i]\}$.[6] Note that we do not *a priori* rule out prices below cost but, without loss of generality, assume that no firm will ever set a price beyond the choke price. In the continuation, let $\hat{q}_i = \min\{q_i, M\}$ be the effective maximum quantity ceiling of i. Let $\mathcal{M}i$ denote the set of mixed strategies (probability measures on the $\sigma-$algebra of Borel-Lebesgue measurable sets of S_i). For a given pair of pure strategies $(p_1, q_1), (p_2, q_2)$, firm i's sales are given by the function $s_i : S_i \times S_{-i} \to [0, k_i]$, $i = 1, 2$, where

$$s_i(p_1, q_1, p_2, q_2) = \begin{cases} \hat{q}_i & \text{if } p_i < p_j, p_i \leq 1, \\ \min\{\hat{q}_i, \frac{\hat{q}_i M}{\hat{q}_1 + \hat{q}_2}\} & \text{if } p_i = p_j \leq 1, \\ \min\{\hat{q}_i, M - \hat{q}_j\} & \text{if } 1 \geq p_i > p_j, \\ 0 & \text{otherwise.} \end{cases}$$

This assumes that in the event of a tie in prices, the firms sell their quantity ceiling, unless the sum of ceilings exceeds demand. In this case, demand is allocated in proportion to the the effective maximum quantity ceilings $\hat{q}_i$, $i = 1, 2$. Given this mapping from the vector of firms' price-quantity pairs to effective sales, firm i's profit is:

$$\pi_i(p_i, q_i, p_j, q_j) = (p_i - c)s_i(p_i, q_i, p_j, q_j).$$

This profit function may be extended in a natural way to an expected profit function on $\mathcal{M}_i \times \mathcal{M}_{-i}$. For any triple (k_1, k_2, c), the component price-quantity setting game

[6] Because of the strict capacity constraint, the restriction $q_i \in [0, k_i]$ also coincides with the no bankruptcy constraint of Dixon (1992).

is then a normal form game $\Gamma(k_1, k_2, c)$ with players $i = 1, 2$, strategy sets $\mathcal{M}_1$, $\mathcal{M}_2$, and expected payoff functions $\pi_i(\mu_1, \mu_2)$.

Finally, we define firm i's minmax payoff or security level in the price-quantity game to be:

$$\underline{\pi}_i = \min_{\mu_j} \max_{\mu_i} \pi_i(\mu_i, \mu_j). \tag{1}$$

It is simple to compute $\underline{\pi}_i$ by noting that for any given strategy by firm j, firm i can guarantee itself a payoff at least as great as $(1-c)(M-\hat{k}_j)$ by setting $(p_i, q_i) = (1, \hat{k}_i)$ with probability one. If firm j sets $p_j = c$ and $q_j = k_j$, the most firm i can obtain is $(1-c)(M-\hat{k}_j)$, which is the minmax profit $\underline{\pi}_i$.

2.2 The simultaneous move equilibrium

In this subsection, we derive an equivalence result showing that every equilibrium of the component price-quantity game coincides with an equilibrium of a corresponding one-shot B-E game and vice-versa. That is, firms' equilibrium expected profits and distributions of prices and sales are the same under the two game forms.

Ghemawat (1986) has shown that the B-E price-setting game, $G^E(k_1, k_2, c)$ has a unique equilibrium for any triple (k_1, k_2, c). Assuming without loss of generality that $k_1 \geq k_2$, if $k_2 \geq M$, the unique equilibrium in the game is $p_1 = p_2 = c$ and $\pi_1 = \pi_2 = 0$. By convention, demand is shared equally.[7] If $k_1 + k_2 > M > k_2$, the only equilibrium exhibits non-degenerate mixed strategies and firm i's expected profit is $\pi_i = \frac{\hat{k}_i}{\hat{k}_1}(1-c)(M-k_2)$. If $k_1 + k_2 \leq M$, the unique equilibrium is $p_1 = p_2 = 1$ and $\pi_i = (1-c)k_i$, $i = 1, 2$.

To demonstrate the equivalence of the B-E equilibrium and the set of equilibria in the simultaneous move price-quantity game $\Gamma(k_1, k_2, c)$, we first develop some notation. For any $\mu_i \in \mathcal{M}_i$, define μ_i^p to be the marginal distribution of firm i's price associated with the strategy μ_i. That is μ_i^p is the projection of μ_i onto the set of prices. Define l_i and u_i to be the greatest lower bound and the least upper bound of μ_i^p, $i = 1, 2$, respectively. Moreover, define $\gamma_i^\mu(p) = \{q \in [0, k_i] | (p, q) \in \text{supp}(\mu_i)\}$, where $\text{supp}(\mu_i)$ denotes the support of the probability measure μ_i.

Proposition 1 *For every (k_1, k_2, c), $G^E(k_1, k_2, c)$ and $\Gamma(k_1, k_2, c)$ have identical equilibrium distributions of profits, prices and sales.*

The proof of Proposition 1, which is relegated to the Appendix, is intuitive. The price and sales distribution of the unique equilibrium of the B-E game must clearly arise as one possible equilibrium outcome of the price-quantity game. If both firms offer their capacity for sale at any price they may set, then the B-E price distributions are

[7] Our analysis assumes that in the case in which $k_i \geq M$, $i = 1, 2$, firms share demand equally in the B-E equilibrium, $p_1 = p_2 = c$. If this sharing rule is not employed, only equilibrium prices and profits need coincide in this case.

best-responses to each other for these quantities. Since no single firm can increase its profit by offering a lower quantity, a strategy in the price-quantity game in which the marginal price distribution is the B-E price distribution and the capacity level is the only quantity in the support of the distribution is an equilibrium strategy for each firm.

Moreover, in any equilibrium of the price-quantity game, a firm would always lose sales with positive probability if its quantity at any price in its support is strictly lower than capacity unless (i) it has a capacity $k_i \geq M$ and $q_i \geq M$, or (ii) its price is undercut by the other firm with probability 1 and its quantity q_i is greater than or equal to residual demand $M - q_j$. Hence setting a quantity strictly below capacity cannot be part of a best response to the other firm's strategy unless one of these two cases holds or the firm's price is at or below unit cost.

Setting price below unit cost is clearly not equilibrium behavior in the price-quantity game. From dominance arguments similar to those used in the B-E game, unless capacities are in the classical Bertrand region ($k_i \geq M$, $i = 1, 2$), unit cost is also not an element of the equilibrium price distribution in the price-quantity game. An argument very similar to that for B-E competition shows that unit cost pricing is the unique equilibrium price pair in the price-quantity game when capacities are in the classical Bertrand region. This, however, clearly requires that $q_i \geq M$. Otherwise j would have incentive to raise price above unit cost, so there is no difference between the price-quantity equilibrium in this case and the B-E equilibrium.

Hence, the only potential source of differences in the equilibria of the B-E game and the price-quantity game appears in cases (i) and (ii) above. However, in neither of these cases do the relevant quantities set affect either the firms' sales or incentives to deviate. Since quantities set below capacity in a manner that affects sales cannot be part of an equilibrium strategy in the price-quantity game, and quantities set below capacity that do not affect sales also have no effect on the incentive to deviate, it follows that for every triple of capacities and marginal cost, every equilibrium of the price-quantity game generates the same price and sales distributions as the B-E game.

3 The price-quantity supergame

In this section, we examine the supergame $G^{\delta}(k_1, k_2, c)$ obtained by infinitely repeating the one shot game $G(k_1, k_2, c)$ and discounting payoffs with discount factor $\delta < 1$. In the supergame, a path τ is an infinite sequence of actions $\{(p_1^t, q_1^t, p_2^t, q_2^t)\}_{t=0}^{\infty}$. A pure strategy σ_i for firm i is a sequence of functions, $\{\sigma_i(t)\}_{t=0}^{\infty}$, such that for every t, $\sigma_i(t) : H_t \to S_i$. H_t is the set of possible histories $h_t = (p_1^0, q_1^0, p_2^0, q_2^0, ..., p_1^{t-1}, q_1^{t-1}, p_2^{t-1}, q_2^{t-1})$ up to time t. A strategy profile is a vector $\sigma = (\sigma_1, \sigma_2)$. Each strategy profile generates an infinite path $\tau(\sigma)$. Firm i's normalized discounted value from period s along a given path $\tau = \{(p_1^t, q_1^t, p_2^t, q_2^t)\}_{t=0}^{\infty}$ is given by:

$$V_i(\tau, s) = (1-\delta)\sum_{t=s}^{\infty} \delta^t \pi_i(p_1^t, q_1^t, p_2^t, q_2^t).$$

We refer to $V_i(\tau, t)$ for $t = 1, 2, 3\ldots$ as firm i's continuation value at t. We let $V_i(\tau) \equiv V_i(\tau, 0)$ denote the payoff associated with the entire path.

Following Abreu (1988), we define a simple strategy profile $\sigma(\tau_0, \tau_1, \tau_2)$, where τ_0 is the initial path, and τ_i is a punishment path started if player i unilaterally deviates from the ongoing prescribed path, $i = 1, 2$. A simple penal code is a vector of simple strategy profiles $(\sigma^1(\tau_1, \tau_1, \tau_2), \sigma^2(\tau_2, \tau_1, \tau_2))$. Letting $\tilde{V}_i(\sigma^i(\tau_i, \tau_1, \tau_2)) \equiv V_i(\tau_i)$, an optimal penal code is a vector of strategy profiles $(\underline{\sigma}^1, \underline{\sigma}^2)$ such that $\underline{\sigma}^i$ is a perfect equilibrium of the supergame and $\tilde{V}_i(\underline{\sigma}^i) = \min_\sigma\{\tilde{V}_i(\sigma) | \sigma \text{ is a perfect equilibrium}\}$. Note that an optimal penal code may not exist. If an optimal penal code exists, then we refer to $\tilde{V}_i(\underline{\sigma}^i)$ as the worst punishment value for firm i.

A security level punishment for firm i is a punishment path on which firm i obtains the discounted sum of its minmax profit, equal to $\underline{\pi}_i$ in normalized terms. A security level penal code is a penal code that contains security level punishment paths for both firms. Since the minmax payoff is the lowest payoff a firm can be held to in the supergame, if there exists a security level penal code then it is an optimal penal code.

4 Constrained Pareto optimal paths and collusion

In the following sections, we assume $k_1 \geq k_2$ and focus on the area in capacity space where $k_1 + k_2 > M$. The last restriction is without loss of generality since $k_1 + k_2 \leq M$ implies that the one-shot Nash equilibrium payoffs are Pareto optimal. If an optimal penal code exists and yields $\underline{V}_i$ to firm i, then $\tau = \{(p_1^t, q_1^t, p_2^t, q_2^t)\}_{t=0}^{\infty}$ is a perfect equilibrium if and only if it is sustainable by $(\underline{V}_1, \underline{V}_2)$ that is, for every i, j, $i \neq j$ and every t:

$$V_i(\tau, t) \geq (1-\delta)\pi_i^*(p_j^t, q_j^t) + \delta \underline{V}_i, \tag{2}$$

where $\pi_i^*(p_j^t, q_j^t)$ is firm i's optimal deviation profit when firm j charges p_j^t and offers q_j^t. We have:

$$\pi_i^*(p_j^t, q_j^t) = \sup_{(p_i^t, q_i^t)} \{\pi(p_i^t, p_j^t, q_i^t, q_j^t)\}.$$

In our model, firm i's optimal deviation profit is obtained by either slightly undercutting $p_j^t \leq 1$ and offering its whole capacity for sale, or by charging the monopoly price, 1, on its residual demand, $M - \hat{q}_j^t$.[8] Therefore, $\pi_i^*(p_j^t, q_j^t) = \max\{(1-c)(M -$

[8]If $p_j^t > 1$, firm i's optimal deviation is to offer its capacity for sale at a price of 1. It then obtains a payoff equivalent to what it would receive by slightly undercutting $p_j^t = 1$. If firm j

$\hat{q}_j^t), (p_j^t - c)\hat{k}_i\}$. Hence there exists a set A_i such that if $(p_j, q_j) \in A_i$ then firm i's optimal deviation is to slightly undercut and offer its capacity.

$$A_i = \left\{ (p_j, q_j) \in S_j | \hat{q}_j \geq M - \frac{(p_j - c)\hat{k}_i}{(1-c)} \right\}. \tag{3}$$

Let $\underline{p}_i$ be the unique price such that $(p, \hat{k}_j) \notin A_i$ for all $p < \underline{p}_i$. $\underline{p}_i$ is the highest price p such that if j offers its capacity for sale at p, firm i finds it most profitable to charge the monopoly price on its residual demand and offering any quantity between residual demand and its capacity.

From (2), it follows that along any stationary perfect equilibrium path (SPEP) $\{(p_1, q_1, p_2, q_2)\}$ supported by perfect equilibrium security level punishment paths, the following incentive constraints must be satisfied:[9]

$$(p_1 - c)s_1(p_1, q_1, p_2, q_2) \geq (1-\delta)\pi_1^*(p_2, q_2) + \delta(1-c)(M - \hat{k}_2), \tag{4}$$

$$(p_2 - c)s_2(p_1, q_1, p_2, q_2) \geq (1-\delta)\pi_2^*(p_1, q_1) + \delta(1-c)(M - \hat{k}_1). \tag{5}$$

(4) and (5) state that the payoff a firm obtains along a stationary path must be larger than the one-period deviation profit plus the punishment value.

We now state several definitions and an assumption that we will use throughout the rest of the paper.

Definition 1 (PO) *A SPEP τ^p is a Constrained Pareto Optimal path if there is no SPEP τ' such that $V_j(\tau') \geq V_j(\tau^p)$ for all j and $V_i(\tau') > V_i(\tau^p)$ for some i.*

Definition 2 (C) *A path τ^c is a collusive path if each firm receives a payoff on τ^c that is at least as great as the discounted sum of its one-shot Nash equilibrium payoff and at least one firm receives a strictly greater payoff.*

Definition 3 (SP) *A path $\tau^{sp} = \{(p_1^t, q_1^t, p_2^t, q_2^t)\}$ is a path exhibiting symmetric pricing if on τ^{sp}, $p_1^t = p_2^t$, $\forall t$.*

Assumption 1 (A1) *There exist pure strategy perfect equilibrium security level punishment paths for both firms.*

If a SPEP satisfies (C), then $s_2(p_1, q_1, p_2, q_2) \geq \alpha_2^N \equiv \frac{(M - \hat{k}_2)\hat{k}_2}{\hat{k}_1}$. If not, then for every price vector, firm 2 receives a payoff lower than the discounted sum of its one-shot Nash equilibrium payoff. Note that (PO) does not imply (C), nor does (C) imply (PO). For example, if the stationary path $\{(1, q_1, 1, M - q_1)\}$ is a SPEP, it satisfies

charges $p_j^t > 1$, then it obtains a zero payoff and firm i's deviation profit is maximized. Firm j could also secure a zero payoff by setting $q_j^t = 0$ and charging any price less than 1, thereby reducing firm i's deviation profit without affecting its payoff.

[9] When we focus on stationary paths, we drop the time superscript.

(PO), but it does not satisfy (C) if $q_1 > M - \alpha_2^N$. If the path $\{(1, M - \hat{k}_2, \tilde{p}, \hat{k}_2)\}$ is a SPEP, it satisfies (C) as long as $\tilde{p} > \underline{p}_2$, but in general, it does not satisfy (PO) if $\tilde{p} < 1$. In section 6, we show that A1 is satisfied for a wide range of parameter values by constructing perfect equilibrium punishment paths with a 2-phase structure along which both firms receive their security level.

Many papers note that in the classical B-E supergame, sustainable stationary collusive price paths involve symmetric pricing because symmetric pricing minimizes firms incentives to deviate along stationary paths (see Davidson and Deneckere, 1990, for example). Under B-E competition, with asymmetric pricing, the sales of the firm with the highest price are such that its discounted profit along any sustainable asymmetric stationary path is at or below the discounted sum of its one-shot Nash equilibrium profit. Sustainability requires that the lower price be such that the high-priced firm has no incentives to undercut, but this requires that the low-priced firm receives no more than the discounted sum of its Nash-equilibrium payoff. Therefore, there are no sustainable stationary paths with asymmetric pricing yielding collusive payoffs in the sense of (C). The ability to ration output at any price provides a profit sharing mechanism in price-quantity supergames that is not available in B-E supergames. The intuition behind Proposition 2 below is that it may be optimal for a firm to lower its price below that of its rival in order to lower the rival's deviation profit, and at the same time, ration its output so as to limit the decrease in the rival's sales. However, a firm that lowers its price and rations its output should sell a sufficiently large quantity that it does not find it profitable to raise price to slightly undercut the high-priced firm.

As a benchmark, we provide the following lemma, which states that if A1 holds and firms price symmetrically along a constrained Pareto optimal SPEP yielding collusive payoffs, then the path is sustainable if and only if $\delta \geq \max\left\{\frac{1}{2}, 1 - \frac{\hat{k}_2}{\hat{k}_1}\right\} \equiv \tilde{\delta}$. The lemma characterizes such paths. The proof of the lemma appears in the Appendix.

Lemma 1 *Suppose A1 holds. If there exists a SPEP $\tau^s = \{(p_1, q_1, p_2, q_2)\}$ satisfying (PO) and (SP) then $q_1 + q_2 \geq M$ and $p_i = 1$, $i = 1, 2$. Such a τ^s exists if and only if $\delta \geq \frac{1}{2}$. Moreover, τ^s also satisfies (C) if and only if $\delta \geq \tilde{\delta} \equiv \max\left\{\frac{1}{2}, 1 - \frac{\hat{k}_2}{\hat{k}_1}\right\}$.*

If $\delta \geq \frac{1}{2}$, there is a range of possible divisions of the market that can be supported as a SPEP satisfying (PO) and (SP). The quantities $\underline{\alpha}_i$ and $\overline{\alpha}_i$ defined below provide lower and upper bounds on firm i's sales on SPEP's satisfying (PO) and (SP). Define the quantities $\underline{\alpha}_i(\delta)$ and $\overline{\alpha}_i(\delta)$ as follows:

$$\underline{\alpha}_i(\delta) \equiv (1 - \delta)\hat{k}_i + \delta(M - \hat{k}_j),$$

$$\overline{\alpha}_i(\delta) \equiv (1 - \delta)(M - \hat{k}_j) + \delta\hat{k}_i.$$

for $i = 1, 2$ $j \neq i$. For $\delta \geq \frac{1}{2}$, $\underline{\alpha}_i(\delta) \geq \overline{\alpha}_i(\delta)$, $i = 1, 2$. If A1 is satisfied, $\underline{\alpha}_i(\delta)$ is the minimum quantity firm i can be allocated for its incentive constraint to be satisfied when both firms set a price equal to 1. Similarly, $\overline{\alpha}_i(\delta)$ is the maximum quantity firm 1 can be allocated for firm j's incentive constraint to be satisfied when both firms set a price equal to 1. It is clear that $\underline{\alpha}_i(\delta) = M - \overline{\alpha}_j(\delta)$, $i = 1, 2$, $i \neq j$.

To simplify notation, we also define the maximum incentive compatible quantity firm 2 can be allocated when firm 1 sets $p_1 = 1$ and firm 2 sets $p_2 < 1$:

$$\overline{\alpha}_2(p_2) = M - \frac{(1-\delta)(p_2 - c)\hat{k}_1 + \delta(1-c)(M - \hat{k}_2)}{1-c}. \tag{6}$$

The following proposition assumes pure strategy security level punishments exist and support a SPEP satisfying (PO), (C) and (SP) (from Lemma 1). The proposition shows that relaxing the assumption (SP) expands the set of collusive (C) constrained Pareto optimal (PO) SPEP's to include paths on which the small firm undercuts the large firm by setting a price lower than 1. In these equilibria, the small firm endogenously rations its output below its capacity.

Proposition 2 *Suppose A1 holds. For $\delta \geq \tilde{\delta}$, $\tau^s = \{(p_1^c, q_1^c, p_2^c, q_2^c)\}$ is a SPEP satisfying (PO) and (C) if and only if:*

$$\begin{aligned} p_1^c &= 1, \\ p_2^c \in P_2^c &\equiv [\min\{p_2^*, 1\}, 1], \\ q_2^c &= \overline{\alpha}_2(p_2^c) && \textit{for } p_2^c \in [p_2^*, 1), \\ q_1^c &\in [M - q_2^c, k_1] && \textit{for } p_2^c \in [p_2^*, 1), \\ (q_1^c, q_2^c) &\in Q^c && \textit{for } p_2^c = 1, \end{aligned}$$

where $p_2^* = (1-c)\left(\frac{M - \delta(M - k_2)}{2(1-\delta)k_1}\right) + c$ *and*

$$\begin{aligned} Q^c = \{(q_1, q_2) \in \ \times_{i=1,2}[0, k_i] \ &|q_1 + q_2 \geq M, s_i(1, q_1, 1, q_2) \geq \underline{\alpha}_i(\delta),\ \textit{i=1,2} \\ \textit{and} \quad &s_2(1, q_1, 1, q_2) \geq \alpha_2^N.\}. \end{aligned}$$

The proof of this proposition appears in the Appendix. Several implications are worth emphasizing. First, note that in any SPEP described by the proposition, each firm has sales constrained strictly below capacity. Furthermore, when δ is such that $\delta < \delta^r \equiv \frac{2\hat{k}_1 - M}{2\hat{k}_1 + \hat{k}_2 - M}$, the set P_2^c of SPEP prices for the small firm is a non-degenerate interval.

Under B-E competition, the set of stationary Pareto optimal paths consists of those paths along which both firms set $p_1 = p_2 = 1$. Since firms set the same price, assuming that side-payments are not feasible, they share collusive profits according to a given sharing rule. The intuition behind Proposition 2 is as follows. Since pricing symmetrically at the monopoly price yields the largest industry profit, if there exists a division of the market for which the monopoly price is sustainable, the resulting payoffs are constrained Pareto optimal. However, if the large firm has a binding incentive constraint at the monopoly price, it is necessary for the small firm to lower its price in order to relax this constraint. For payoffs to remain on the constrained Pareto frontier, the small firm's sales must increase more than proportionally (to the price cut) in order for its profit to increase. However, the small firm's sales must be strictly below its capacity to guarantee that the large firm's sales are sufficient for it to conform to the prescribed path. If the small firm's sales are equal to its capacity, incentive compatibility implies that the price it charges must be equal to $\underline{p}_1$, i.e., the Nash equilibrium payoffs are on the constrained Pareto frontier. But Nash equilibrium payoffs cannot be on the constrained Pareto frontier if $\delta \geq \tilde{\delta}$, since a SPEP exists which satisfies $p_1 = p_2 = 1$, (PO) and (C).

If $\delta \in [\tilde{\delta}, \delta^r)$, in the constrained Pareto optimal SPEP that provides the small firm its maximum profit, the small firm undercuts the large firm and rations its output below its capacity. The intuition behind the fact that only the small firm optimally lowers its price and rations its output is simple. The small firm has an incentive to set a price different from the large firm's price if it can increase its profit compared to what it obtains by charging the same price. The maximum profit the small firm can attain on a path satisfying (SP) is $V_2 = (1-c)\overline{\alpha}_2(\delta)$. If the small firm undercuts the large firm's price by an infinitesimal amount ϵ, but its sales do not change, its profit decreases by an amount equal to $\epsilon\overline{\alpha}_2(\delta)$. However, the large firm's incentive to deviate decreases as well. The amount by which the large firm's incentive constraint is relaxed is equal to $\epsilon(1-\delta)\hat{k}_1$. Therefore, firm 2's sales can increase by an amount equal to $\frac{\epsilon(1-\delta)\hat{k}_1}{1-c}$ while still satisfying firm 1's incentive constraint. If firm 2 adjusts the quantity it offers to sell exactly $\overline{\alpha}_2(\delta) + \frac{\epsilon(1-\delta)\hat{k}_1}{1-c}$ when it undercuts $p_1 = 1$ by ϵ, the first order effect on its profit is $\epsilon[(1-\delta)\hat{k}_1 - \overline{\alpha}_2(\delta)]$, which is strictly greater than zero if $\delta < \delta^r$. Applying the same reasoning to the large firm, we conclude that the large firm has an incentive to undercut $p_2 = 1$ and ration its output only if $(1-\delta)\hat{k}_2 > \overline{\alpha}_1(\delta)$. However, this inequality does not hold for any $\delta \geq \tilde{\delta}$.

We define the constrained Pareto optimal SPEP on which firm i's profit is maximized by:

$$\overline{\tau}_i(\delta) = \text{argmax}_\tau\{V_i(\tau)|\tau \text{ stationary}, (4), (5), \text{A1}, (PO) \text{ and } (C) \text{ hold}\}.$$

Proposition 2 ensures that $\tau_i(\delta)$ is well-defined for every $\delta \geq \tilde{\delta}$. Proposition 3 follows directly from Proposition 2.

Proposition 3 *Suppose A1 holds. If $\delta \in [\tilde{\delta}, \delta^r)$, then on $\overline{\tau}_2(\delta)$, the SPEP on which the small firm's profit is maximized, the small firm charges a price p_2^* lower than the monopoly price and rations its output below its capacity. The large firm charges the monopoly price (which equals the residual monopoly price) and serves residual* *demand. If* $\delta \geq \max\{\tilde{\delta}, \delta^r\}$*, then any SPEP satisfying (PO) also satisfies (SP).* $\tilde{\delta} < \delta^r$ *if and only if* $\hat{k}_1 > \frac{M+\hat{k}_2}{2}$.

Proof. Note that from Proposition 2, on every SPEP satisfying (PO) and (C), firm 1 sets a price equal to 1. From the definition of δ^r, it follows from Proposition 2 that if $\delta \geq \max\{\tilde{\delta}, \delta^r\}$ holds, $\min\{p_2^*, 1\} = 1$. Therefore $P_2^c = \{1\}$, hence, it follows from Proposition 2 that any SPEP satisfying (PO) and (C) also satisfies (SP). Assume for now that the interval $[\tilde{\delta}, \delta^r)$ is nonempty and that $\delta \in [\tilde{\delta}, \delta^r)$. It follows that $p_2^* < 1$. To prove Proposition 3, we show that if δ is in $[\tilde{\delta}, \delta^r)$, $\overline{\tau}_2(\delta) = \tau_2^*(\delta) \equiv \{(1, M - \overline{\alpha}_2(p_2^*)), p_2^*, \overline{\alpha}_2(p_2^*))\}$. To this effect, note that $M - \overline{\alpha}_2(p)$ is a strictly increasing function of p. Hence, $(1-c)(M - \overline{\alpha}_2(p)) > (1-c)(M - \overline{\alpha}_2(p_2^*))$ for all $p \in (p_2^*, 1]$. We now show that $\tau_2^*(\delta) = \overline{\tau}_2(\delta)$. Suppose to the contrary that $\overline{\tau}_2(\delta) = \hat{\tau} \equiv \{(1, q_1, p, q_2)\}$ where $p \in (p_2^*, 1]$. Then by definition of $\overline{\tau}_2(\delta)$, $V_2(\hat{\tau}) \geq V_2(\tau_2^*(\delta))$. But from Proposition 2, since $\hat{\tau}$ is a SPEP satisfying (PO) and (C), $V_1(\hat{\tau}) = (1-c)s(1, q_1, p, q_2) \geq (1-c)(M - \overline{\alpha}_2(p)) > (1-c)(M - \overline{\alpha}_2(p_2^*)) = V_1(\tau_2^*(\delta))$. Hence $V_2(\hat{\tau}) \geq V_2(\tau_2^*(\delta))$ cannot hold, otherwise $\tau_2^*(\delta)$ does not satisfy (PO), a contradiction to Proposition 2. Hence, it must be that $\tau_2^*(\delta) = \overline{\tau}(\delta)$. This implies that on the path $\overline{\tau}_2(\delta)$, firm 2 sets a price equal to p_2^*, which is strictly less than 1. From Proposition 2, it also follows that firm 1 sets a price equal to 1 and serves residual demand. Finally, using the definitions of δ^r and $\tilde{\delta}$, straightforward calculations yield $\tilde{\delta} < \delta^r$ if and only if $\hat{k}_1 > \frac{M+\hat{k}_2}{2}$. $\square$

One implication of Proposition 3 is that symmetric firms (indeed, firms with identical *effective* capacities) never charge different prices along constrained Pareto optimal SPEP's, since $\hat{k}_1 = \hat{k}_2$ implies $\hat{k}_1 \leq \frac{M+\hat{k}_2}{2}$.

Propositions 2 and 3 can be used to construct Figure 1. Figure 1 illustrates the constrained Pareto frontier of payoffs attainable along stationary perfect equilibrium paths for capacity pairs satisfying $\hat{k}_1 > \frac{M+\hat{k}_2}{2}$ and discount factor $\delta \in [\frac{1}{2}, \delta^r)$. Firm 1's normalized supergame payoff is indicated on the horizontal axis and firm 2's on the vertical axis. Firm 1's maximum payoff is attained on a path satisfying (SP) and is equal to $(1-c)\overline{\alpha}_1(\delta)$. Firm 2's payoff is then equal to $(1-c)\underline{\alpha}_2(\delta)$. The slope of the constrained Pareto frontier is -1 as (SP) implies that all payoffs on that portion of the frontier are obtained by transferring sales from one firm to the other at $p_1 = p_2 = 1$. The point $((1-c)\underline{\alpha}_1(\delta), (1-c)\overline{\alpha}_2(\delta))$ represents the maximum profit level for firm 2 which can be supported on a SPEP satisfying (SP).

As shown in Proposition 3, if $\hat{k}_1 > \frac{M+\hat{k}_2}{2}$ and $\delta \in [\frac{1}{2}, \delta^r)$, firm 2 can attain higher profit levels by undercutting firm 1. An implication is that the constrained Pareto frontier becomes non-linear.[10] Intuitively, on that portion of the frontier,

[10]Formally, an implication of Proposition 2 is that all (V_1, V_2) on the constrained Pareto frontier such that $V_1 < (1-c)\underline{\alpha}_1(\delta)$ must satisfy $V_1 = (1-c)(M - \overline{\alpha}_2(p_2))$ and $V_2 = (p_2 - c)\overline{\alpha}_2(p_2)$ for $p_2 \in P_2^c$. Using the expression for $\overline{\alpha}_2(p_2)$ given by (6) and solving for p_2 as a function of V_1

as V_1 decreases, all payoffs are obtained by transferring sales from the large firm to the small firm, but at a decreasing price for the small firm. Alternatively, note that firm 2's payoff on SPEP's satisfying (PO) in which firm 2 undercuts firm 1 is a non-linear function of firm 2's price. Moreover, it follows from Proposition 2 that for SPEP's satisfying (PO) and on which firm 2 undercuts firm 1, the price set by firm 2 determines each firm's payoff uniquely. Therefore any change in firm 2's price consistent with firms' payoffs remaining on the constrained Pareto frontier has a non-linear effect on firm 2's payoff, implying that the constrained Pareto frontier is non-linear.

If $\delta < \tilde{\delta}$, Lemma 1 implies that there is no sustainable stationary path satisfying both (PO) and (C) on which firms set the same price. However, if $\tilde{\delta} \leq \delta < \delta^r$, there exist constrained Pareto optimal SPEP yielding collusive payoffs on which the small firm sets a price less than 1, and the large firm sets its price equal to 1. Proposition 4 establishes that there is a range of discount factors below $\tilde{\delta}$ for which there exist SPEP's satisfying (PO) and (C) on which the small firm undercuts the large firm and rations its output below its capacity.

Proposition 4 *Suppose A1 holds. If $\hat{k}_1 > \frac{M+\hat{k}_2}{2}$, there exists a $\hat{\delta} < \tilde{\delta}$, such that for every $\delta \in (\hat{\delta}, \tilde{\delta})$, there exists a SPEP satisfying (PO) and (C). Furthermore on any such SPEP the small firm charges a price lower than the monopoly price and rations its output below its capacity. The large firm charges the monopoly price (which equals the residual monopoly price) and serves residual demand.*

Proof. From Lemma 1, the assumption $\delta < \tilde{\delta}$ implies that there exists no SPEP of the form $\{(1, q_1, 1, q_2)\}$ satisfying (PO). Moreover, since $\hat{k}_1 > \frac{M+\hat{k}_2}{2}$ by assumption, it follows from Proposition 3 that $\tilde{\delta} < \delta^r$. Thus, by definition of δ^r, if $\delta < \tilde{\delta}$, the set $[p_2^*, 1)$ is non-empty. We show that there exists $\hat{\delta} < \tilde{\delta}$ such that for all $\delta > \hat{\delta}$, there exists a SPEP satisfying (PO) and (C) satisfying the requirements of the proposition. If $\delta = \tilde{\delta}$, from Propositions 2 and 3, the path $\tau_2^*(\tilde{\delta}) = \{(1, M - \overline{\alpha}_2(p_2^*), p_2^*, \overline{\alpha}_2(p_2^*)\}$ satisfies (PO), (C) and maximizes firm 2's payoff. This implies that the following holds: $V_2(\tau_2^*(\tilde{\delta})) = p_2^*\overline{\alpha}_2(p_2^*) > (1-c)\overline{\alpha}_2(\tilde{\delta}) = (1-c)\max\{\alpha_2^N, \underline{\alpha}_2(\tilde{\delta})\}$, where the last equality follows from the definitions of $\tilde{\delta}$, $\overline{\alpha}_2(\delta)$, $\underline{\alpha}_2(\delta)$ and α_2^N. Thus by the definition of $\underline{\alpha}_2(\delta)$, if $\delta = \tilde{\delta}$, then on $\tau_2^*(\tilde{\delta})$, (5) is not binding, that is:

$$V_2(\tau_2^*(\tilde{\delta})) > (1-\tilde{\delta})(1-c)\hat{k}_2 + \tilde{\delta}(1-c)(M - \hat{k}_1).$$

Now, define the function $D(d) = V_2(\tau_2^*(d)) - (1-d)(1-c)\hat{k}_2 - d(1-c)(M - \hat{k}_1)$. Note that this function is strictly increasing in d because $V_2(\tau_2^*(d))$ is increasing in d and

yields $p_2 = \frac{V_1 - \delta(1-c)(M-\hat{k}_2)}{(1-\delta)\hat{k}_1} + c$. Substituting in V_2 yields $V_2 = \left(\frac{V_1 - \delta(1-c)(M-\hat{k}_2)}{(1-\delta)\hat{k}_1}\right)(M - \frac{V_1}{1-c})$. Differentiating and rearranging yields $\frac{\mathrm{d}V_2}{\mathrm{d}V_1} = \frac{(1-c)[M+\delta(M-\hat{k}_2)]-2V_1}{(1-c)(1-\delta)\hat{k}_1}$, which is less than zero, greater than -1, and decreasing in V_1, for all SPEP payoffs V_1 satisfying (PO) and (C).

$\hat{k}_1+\hat{k}_2>M$. Let $\hat{\delta}'$ be equal to $\min\{d\in\mathbb{R}|0<d<\tilde{\delta}$ and $D(d)=0\}$ if such number exists and to 0 otherwise. Then, if $\delta>\hat{\delta}'$, $D(\delta)\geq 0$. Therefore, by definition of $D(\delta)$, (5) holds on $\tau_2^*(\delta)$. Using the definition of $M-\overline{\alpha}_2(p_2)$, it is clear that (4) also holds on $\tau_2^*(\delta)$. It remains to show that $\tau_2^*(\delta)$ satisfies (C). A simple computation yields $p_2^*\overline{\alpha}_2(p_2^*)>\pi_2^N$ if and only if $\delta>\hat{\delta}''\equiv\frac{M-2\hat{k}_2}{M-\hat{k}_2}$. Straightforward computations yield $\hat{\delta}''<\tilde{\delta}$ for all (k_1,k_2) satisfying our assumptions. Letting $\hat{\delta}\equiv\max\{\hat{\delta}',\hat{\delta}''\}$, for $\delta\in(\hat{\delta},\tilde{\delta})$, $\tau_2^*(\delta)$ is a SPEP that satisfies (PO) and (C). □

Proposition 4 states that if $\hat{k}_1>\frac{M+\hat{k}_2}{2}$, then there exists a SPEP τ satisfying (PO) and (C) for δ in an interval below $\tilde{\delta}$. Such a SPEP cannot satisfy (SP) as we have shown in Lemma 1. The implication of Proposition 4 is that if the larger firm is sufficiently large, then asymmetric pricing and endogenous rationing by the small firm arise everywhere on the set of SPEP satisfying (PO) and (C). Note that such SPEP do not exist under B-E competition, since they rely on the firms' ability to ration output even when their prices differ.

5 Security level penal codes

It is well understood that in price-setting games with capacity constraints, including the price-quantity game analyzed in this paper, the grim trigger strategy of reverting to the one-shot Nash equilibrium (in mixed strategies) does not constitute an optimal punishment for the small firm for a wide range of discount factors and capacity pairs. We have shown that in the region of capacities where the classical B-E equilibrium is in non-degenerate mixed strategies, the small firm receives a payoff that strictly exceeds its minmax payoff in the equilibrium of the one-shot game. In the symmetric B-E supergame, Lambson (1987) shows that if there exists a worst pure strategy perfect equilibrium punishment, then it is a security level punishment for both firms. In the asymmetric B-E supergame, Lambson (1994) shows that if there exists a worst pure strategy perfect equilibrium punishment, then it is a security level punishment for the large firm.[11] However, it need not be the case for the small firm. Whenever they exist, perfect equilibrium strategies generating security level punishment paths for both firms constitute an optimal penal code, since no firm can be held down to a value strictly lower than its minmax payoff.

Abreu (1986) introduced the notion of a 2-phase punishment path. Following Abreu (1986), we call a 2-phase punishment path a path which is stationary after the first period. Despite the fact that these paths do not generally possess the optimality

[11]Lambson's analysis focuses, like ours, on pure strategies only. It is clear that if one allows for mixed strategies in the supergame, then repeating the one-shot Nash equilibrium forever achieves the worst punishment for the large firm.

properties in our game that they have in the Abreu analysis,[12] their application here is compelling. First, they are computationally tractable and include grim-trigger strategies as a special case. Second, they allow for more severe punishments than, and therefore improve upon, other punishments applied in the B-E literature (for a wide range of parameter values). We show that for a wide range of capacities, discount factors, and unit costs, perfect equilibrium 2-phase punishment paths exist and achieve the security level for both firms.

For $\delta \geq \frac{1}{2}$, optimal punishments within the class of 2-phase punishments have the following properties. If k_2 is sufficiently large, in the first period of the small firm's punishment path, the small firm offers and sells its whole capacity at a low price, possibly below unit cost. The large firm offers its capacity and sells an amount equal to its residual demand at the highest price that does not provide the small firm an incentive to deviate. The second phase consists of the constrained Pareto optimal SPEP (supported by security level punishments) along which the small firm obtains its lowest payoff. For smaller values of k_2, the firms' roles are reversed in the first period. The large firm offers and sells its whole capacity at the lowest price consistent with its incentive constraint being satisfied. The small firm offers its capacity, but sells to residual demand, at a price strictly above its rival's. From the second period on, firms revert to a stationary path satisfying (PO) which satisfies the following properties: the small firm obtains its security level on the entire path and all incentive constraints are satisfied in every period.

For $\delta \geq \frac{1}{2}$, the large firm's punishment path in an optimal 2-phase punishment takes two different forms. For a given discount factor, if the small firm's capacity is relatively low, in the first period, the large firm sets the highest price that does not provide the small firm an incentive to deviate, offers its capacity and sells to residual demand. In the first period, the small firm sets a lower price than the large firm, offers and sells its capacity. Firms then revert to a second phase in which the large firm may obtain more than its lowest constrained Pareto optimal SPEP payoff. If the small firm's capacity is sufficiently large, the form of the large firm's punishment path is similar that of the small firm's. In the first period, the small firm offers its capacity at the highest price that does not provide the large firm an incentive to deviate and sells to residual demand. The large firm sets a lower price, possibly below unit cost, offers and sells its capacity. The second phase consists of the constrained Pareto optimal SPEP on which the large firm obtains its lowest payoff.

We focus on stationary paths on which firms charge the same price for the second phase. We will see that for $\delta \geq \frac{1}{2}$, this is without loss of generality because the binding constraint is the large firm's constraint along the small firm's punishment path. Proposition 2 implies that for SPEP's supported by security level punishments,

[12] Later in this section, we construct a numerical example of a three-phase perfect equilibrium punishment path that supports a security level punishment for the small firm when the 2-phase path fails to achieve the security level for the small firm.

the large firm obtains its largest profit on SPEP's on which $p_1 = p_2 = 1$. Therefore, a second phase along which firms charge different prices would be more difficult to sustain.

Let $\delta \geq \frac{1}{2}$ and let $\tau^p \equiv \{(1, q_1, 1, q_2)\}$ be a constrained Pareto optimal SPEP supported by security level punishment paths. In the first phase of its punishment, let firm 2 charge a price p_2^s such that the following equality is satisfied:

$$(1-\delta)(p_2^s - c)\hat{k}_2 + \delta(1-c)s_2(1, q_2, 1, q_1) = \underline{\pi}_2. \tag{7}$$

Since τ^p is a SPEP, $s_2((1, q_2, 1, q_1)) > M - \hat{k}_1$. It follows from (7) that $p_2^s < \underline{p}_2$. Therefore if the punishing firm, 1, sets $p_1 = \underline{p}_2$, it sells to residual demand. Therefore, the only condition that needs to be satisfied for the 2-phase path to be sustainable is:

$$(1-\delta)(\underline{p}_2 - c)(M - \hat{k}_2) + \delta(1-c)s_1(1, q_2, 1, q_1) \geq \underline{\pi}_1. \tag{8}$$

Using this simple structure, we now show that the large firm is able to punish deviations by the small firm as harshly as possible for a large set of capacity pairs. In Proposition 5 below we characterize a sufficient condition under which the small firm may be held down to its security level in a 2-phase perfect equilibrium punishment. In our characterization, (8) is the binding constraint.

Note however that in the above formulation of the small firm's punishment, we do not *a priori* rule out negative or below cost pricing. While we believe that there may be conditions under which such pricing policies are relevant, before proceeding to Proposition 5, we first provide conditions on the parameters that insure that punishment prices do not become negative as δ approaches 1 (and therefore for all δ). To this effect, suppose that for $\delta \geq \frac{1}{2}$, $s_2(1, q_2, 1, q_1) = \underline{\alpha}_2(\delta)$ in (7). Solving for p_2^s, we obtain:

$$p_2^s = c + \frac{(1-c)[(M - \hat{k}_1) - \delta\underline{\alpha}_2(\delta)]}{(1-\delta)\hat{k}_2}.$$

Taking the derivative of p_2^s with respect to δ, it is straightforward to show that p_2^s is strictly decreasing in δ. Therefore:

$$\inf_{\delta}\{p_2^s\} = \lim_{\delta \uparrow 1} p_2^s = -(1-c)\left(\frac{\hat{k}_2 - 2(M - \hat{k}_1)}{\hat{k}_2}\right) + c.$$

The above limit is greater than or equal to zero if and only if:

$$c \geq \hat{c} \equiv \frac{1}{2} - \frac{M - \hat{k}_1}{2(\hat{k}_1 + \hat{k}_2 - M)} > 0.$$

We will maintain this assumption on unit cost in Proposition 5, although it can be removed if we allow for negative prices. The proof of the proposition appears in the Appendix.

Proposition 5 *Assume $\delta \geq \frac{1}{2}$, $k_1+k_2 > M$ and $c \geq \hat{c}$. Then if $\hat{k}_1 \geq \hat{k}_2 \geq \Psi(k_1, \delta) \equiv M - \min\left\{\hat{k}_1, \frac{\delta^2 M}{1-\delta(1-\delta)}\right\}$, there exists a pure strategy perfect equilibrium path τ_i^s along which firm i obtains its security level, $i = 1, 2$.*[13]

Our analysis of 2-phase punishment paths in price-quantity supergames generates security level punishments for a larger range of parameters than previous treatments of 2-phase punishment paths of B-E supergames. This arises for several reasons. First, when compared to 2-phase punishments in B-E supergames with a fixed sharing rule, the ability to ration output at any given price in price-quantity supergames allows for a higher payoff for the large firm along the constrained Pareto optimal SPEP constituting the second phase of the small firm's punishment. This, in turn, relaxes the large firm's incentive constraint in the first period of the small firm's punishment path.

Although previous B-E supergame analyses with endogenous sharing rules apply a second phase of punishment that coincides with that employed here, we also introduce two different types of first phase of punishment not previously analyzed in the literature. In our analysis, if the small firm's capacity is sufficiently large, the first phase of firm i's punishment requires that firm j set the lowest price greater than or equal to $\underline{p}_i$ that satisfies j's incentive constraint, and sell its effective capacity, $\hat{k}_j$. For smaller values of firm 2's capacity, in the first phase of firm 1's punishment, firm 1 sets a price equal to $\underline{p}_2$, firm 2 sets a price below firm 1's price and both firms offer their capacity. These first phases of punishment, which are also feasible in the B-E supergame, improve upon the 2-phase punishments proposed by Lambson (1994) and Compte, Jenny, and Rey (2002), which require that punishing firms set price equal to unit cost.[14]

In our analysis, the set of capacity pairs for which perfect equilibrium security level 2-phase punishment paths can be constructed for both firms becomes smaller as δ decreases. This can be easily seen by noting that $\Psi(k_1, \delta)$ is a non-decreasing function of δ. For $\delta < \frac{1}{2}$, Lemma 1 shows that the paths satisfying (SP) used in the second phase of the punishment paths constructed in Proposition 5 cannot be supported by security level punishment paths. However, from Proposition 4, for $\delta \in (\hat{\delta}, \frac{1}{2})$ and $\hat{k}_1 > \frac{M+\hat{k}_2}{2}$, constrained Pareto optimal SPEP's in which the small firm undercuts the large firm and rations its output can be supported by security level punishment paths. Moreover, in the limit as delta approaches $\frac{1}{2}$ from below, the large firm's payoff on its most favorable such path is arbitrarily close to the payoff it obtains on τ_1^p. Therefore, if additionally $\hat{k}_2 > \Psi(\hat{k}_1, \delta)$ holds, by using this path in the second phase of the small firm's punishment path, the large firm's incentive

[13]We may remove the assumption $c \geq \hat{c}$ if we allow for negative prices.

[14]In Lambson the punished firm sets a price lower than the punishing firm, while in Compte, Jenny and Rey all firms set a price equal to zero, which is both the unit cost and the lower bound of the one-shot strategy space.

constraint in the first phase of the small firm's punishment path is satisfied. Hence, there exists a region of capacity space for which 2-phase security level punishment paths can be constructed for both firms in a range of discount factors below $\frac{1}{2}$. That is, perfect equilibrium 2-phase security level punishment paths can be supported in price-quantity supergames for values of the discount factor for which they fail to exist in Bertrand-Edgeworth supergames.

In the next proposition, we show that the set of capacity pairs for which an optimal penal code is a security level penal code for both firms can be extended further. In Proposition 5 above, we showed that 2-phase punishment paths can be constructed if $\delta \geq \frac{1}{2}$ and $k_2 \geq \Psi(k_1, \delta)$. One characteristic of such punishments is that the large firm sets a higher price than the small firm in the first period and sells to residual demand. Below, we construct 2-phase punishment paths in the first period of which the small firm sets a higher price than the large firm. The large firm offers and sells its whole capacity at a price that keeps the small firm from deviating. The small firm offers its capacity but sells to residual demand only at a price that keeps the large firm from deviating. Then firms revert to a stationary path of the form $\{(1, q_1, 1, q_2)\}$, where $q_1 + q_2 = M$, in which the quantity ceilings are such that given the first period prices, the small firm's payoff on the entire path is equal to its security level and no firm has an incentive to deviate. A difficulty we did not encounter in Proposition 5 arises when constructing this type of path. When the large firm's capacity is close to M, a pair of first phase prices satisfying the incentive constraints and the requirement that the small firm obtains its security level does not exist. Such prices exist only if $k_1 < [1 - \delta(1 - \delta)]M$.

Note also that in Proposition 6, we assume $\hat{k}_2 < \Psi(k_1, \delta)$ so that the set of capacity pairs we characterize complements the set characterized in Proposition 5. This allows us to show straightforwardly and without additional assumption that all prices are non-negative on the paths we construct.

Proposition 6 *Assume $\delta \geq \frac{1}{2}$ and $k_1 + k_2 > M$. If (k_1, k_2) satisfies the following conditions:*

(i) $\frac{(1-\delta)M}{1-\delta(1-\delta)} \leq \hat{k}_1 < [1 - \delta(1 - \delta)]M$,

(ii) $\max\left\{\frac{(1-\delta)\hat{k}_1^2}{\hat{k}_1-(1-\delta)(M-\hat{k}_1)}, \frac{(1-\delta)\hat{k}_1}{1-\delta(1-\delta)}\right\} \leq \hat{k}_2 < \Psi(k_1, \delta)$,

there exists a pure strategy perfect equilibrium path τ_i^s along which firm i obtains its security level, $i = 1, 2$.

We have shown that punishment paths that become stationary after the first period are optimal for a large set of parameters. However, we have constructed examples in which the 2-phase paths above are not optimal. One example constructed includes a non-stationary constrained Pareto optimal second-phase that exhibits price shading and rationing. This suggests that paths with a simple structure in which the punished firm obtains a very low profit in the first period before reverting to a constrained Pareto optimal path exist and are optimal, at least for a range of parameters.[15] It also suggests that optimal punishment paths in price-quantity games that do not have a 2-phase structure are generally not replicated by B-E punishment paths. Characterizing such punishment paths entails characterizing the entire set of non-stationary constrained Pareto optimal perfect equilibrium paths, which remains a topic for future research.

6 Empirical implications

6.1 Price dispersion

Under price-quantity competition, the main departure from results obtained under B-E competition is the possibility of price dispersion in a pure strategy collusive equilibrium. In Proposition 2, we computed p_2^*, the minimum price set by the small firm on a constrained Pareto optimal SPEP. Since the large firm sets a price equal to 1, we define the maximum level price dispersion on a constrained Pareto optimal SPEP to be:

$$\Delta(\delta, k_1, k_2) = 1 - \min\{p_2^*, 1\}.$$

From Propositions 3 and 4, we have that $\Delta(\delta, k_1, k_2) > 0$ if and only if $\hat{k}_1 > \frac{M+\hat{k}_2}{2}$ and $\delta \in (\hat{\delta}, \delta^r)$. Moreover, using the expression for p_2^* in Proposition 2, it is clear that the maximum level of price dispersion increases as k_2 and δ become small and as $\hat{k}_1$ becomes large. Thus, price dispersion consistent with effective collusion is more likely to be observed in markets characterized by substantial size asymmetries. Table 1 displays values of p_2^* for various values of k_2 and δ.[16]

As an example of price shading arising on every constrained Pareto optimal SPEP, for $\delta = 0.49$ and $k_2 = 700$, p_2^c, the small firm's price in a constrained Pareto optimal SPEP must lie in the interval $[0.9025, 0.9266]$. The interval vanishes to a single point as k_2 increases.

Anecdotal evidence presented in Rees's (1993) analysis of the Great Salt Duopoly indicates that British Salt (BS), the smallest of the two firms, quoted prices per ton

[15] In this example, we set $M = 100$, $k_1 = 70$, $k_2 = \frac{200}{3}$, $c = 0.5$ and $\delta = \frac{1}{2}$. For these parameters, firm 1's constraint is binding in every period of firm 2's punishment path, τ_2^{sp}, defined in Proposition 5. It is possible to show that a non-stationary path with $(0.94, 1, 59.65, 40.35)$ as prices and quotas offered in the first period and the SPEP $\overline{\tau}_1(\frac{1}{2})$ starting in the second period exists and is a perfect equilibrium. Moreover, on this non-stationary path, firm 1's profit is higher than on $\overline{\tau}_1(\frac{1}{2})$, the stationary perfect equilibrium path on which firm 1's profit is maximized if $\delta = \frac{1}{2}$. Therefore using this path as a second phase of firm 2's punishment relaxes firm 1's constraint in the first period.

[16] The missing values correspond to values of the parameters for which we do not construct perfect equilibrium security level punishment paths for both firms.

(k_2, δ)	0.5	0.52	0.58	0.60	0.62	0.65
500	-	-	-	-	0.9447	0.9785
570	-	-	0.9361	0.9565	0.9790	1
670	0.9010	0.9177	0.9775	1	1	1
770	0.9310	0.9502	1	1	1	1

Table 1: p_2^* for various values of δ and k_2, setting $M = \hat{k}_1 = 1000$ and $c = 0.4$.

to its large customers below those quoted by the largest firm, ICI Weston Point (WP), by roughly 0.25%. Firms' capacities were such that WP could supply to the whole market at marginal cost ($k_1 \geq M$) except in 1980 and BS could supply to approximately two-thirds of the market ($k_2 \approx \frac{2}{3}M$). In our model, these capacities are such that price shading and endogenous rationing by the small firm arise on constrained Pareto Optimal SPEP's for sufficiently low discount factors (see Table 1). Moreover, despite the slight price difference, BS never operated at full capacity over the period of time examined by Rees, as is consistent with our endogenous rationing result.

6.2 Capacity utilization

Rees (1993) and Lambson and Richardson (1994) use firm-level data and a Bertrand-Edgeworth supergame approach to study the ability to collude in two different industries. They both provide data regarding the degree of capacity utilization during collusive periods. In Rees's study of the Great Salt Duopoly, the small firm used a significantly greater percentage of its capacity in collusive periods than did the large firm. In Lambson and Richardson's study of the US passenger car market, the large firm had a higher rate of capacity utilization than the small firm during the collusive periods.

For some values of the parameters, our model has implications regarding capacity utilization that differ from those obtained under standard B-E competition. For the B-E supergame, Lambson (1994) shows that if each firm's incentive constraint is binding along a collusive SPEP satisfying (SP) supported by optimal penal codes and the large firm's capacity is less than demand at the common price, the large firm must have a higher capacity utilization than the small firm.

In Lemma 3 in section 8.6 in the Appendix, we show that for $\delta \geq \tilde{\delta}$, if the large firm's capacity is not sufficient to serve the whole market, the large firm always has a higher capacity utilization than the small firm along the path that maximizes the large firm's payoff. However, the analogous statement for firm 2 is true only for a subset of discount factors. We show that if firm 1 is large relative to firm 2, but not necessarily larger than the market size, the small firm has a higher capacity utilization on the SPEP that maximizes its profit for values of the discount factor close to 1 or close to $\tilde{\delta}$. The intuition is outlined below and illustrated in Figure 2. Figure 2 shows the range of firm 1's sales on stationary constrained Pareto optimal SPEP also satisfying (C) when $\frac{M+k_2}{2} < k_1 \leq \min\{2k_2, M\}$ and A1 is satisfied. For such capacity pairs, $\tilde{\delta} = \frac{1}{2} < \delta^r$ holds. That is, there exists a range of discount factors for which constrained Pareto optimal SPEP include endogenous rationing. The figure is drawn showing firm 1's sales. The corresponding figure for firm 2 is obtained by noting that $s_2 = M - s_1$. For discount factors close to 1, the worst sustainable constrained Pareto optimal path for the large firm, $\{(1, \underline{\alpha}_1(\delta), 1, M - \underline{\alpha}_1(\delta))\}$ is such that it sells an amount close to residual demand after the small firm has sold its capacity. In this case, the small firm uses almost 100% of its capacity and has higher capacity utilization than the large firm. As the discount factor decreases, $\underline{\alpha}_1(\delta)$ increases so that eventually, the large firm has higher capacity utilization than the small firm. Since the conditions in Proposition 3 are satisfied, for discount factors in the interval $[\tilde{\delta}, \delta^r)$, on the constrained Pareto optimal SPEP that maximizes its payoff, the small firm undercuts the large firm. For such discount factors, the small firm's sales increase as the discount factor decreases. If firm 1 is sufficiently large, but not necessarily larger than M, there exists a level of δ below which the small firm has a higher capacity utilization than the large firm.

An implication of the above is that in price-quantity supergames, if the large firm is large enough, but not necessarily larger than the market size, at the critical value of the discount factor for which each firm's incentive constraint is binding, the small firm has a higher capacity utilization than the large firm. For that value of δ, on all constrained Pareto optimal SPEP's, the small firm sets a lower price than the large firm and rations its output below capacity. Thus, such paths are not feasible under B-E competition.

7 Conclusion

In contrast to the previous literature on collusion, this paper shows that price shading by a small firm may not only be consistent with successful collusion, it may be required. In the context of a capacity constrained price-quantity supergame, we show that collusion may be characterized by endogenous rationing and stable price dispersion.

Our analysis has been carried out taking capacities parametrically. A natural question arising in this context is whether the capacity pairs that are applied in our analysis arise endogenously in a game of simultaneous or sequential capacity choice followed by our capacity-contingent price-quantity supergame.

Consider first the dynamic game in which firms simultaneously build capacities. Following Benoit and Krishna's (1991) analysis of a dynamic Bertrand-Edgeworth duopoly, assume that firms do not randomize in the supergame except possibly to play the one-shot Nash equilibrium. Suppose also that the discount factor is in the range $\delta \geq \tilde{\delta}$. Then, for capacity pairs satisfying the conditions of Propositions 5 or 6, the worst subgame equilibrium for firm i is one in which it obtains its security level. On the other hand, since $\delta \geq \tilde{\delta}$, Proposition 2 implies that for such capacity pairs there exists a stationary perfect equilibrium path (SPEP) satisfying (PO) and (C). Furthermore, for capacity pairs that do *not* satisfy the conditions in either Proposition 5 or 6, the worst equilibrium yields a payoff no greater than the discounted sum of the one-shot Nash equilibrium payoff.

We claim that if $\delta \geq \tilde{\delta}$, (k_1^*, k_2^*) is a capacity pair that satisfies the conditions in either Proposition 5 or 6, and the cost of capacity is zero or is negligible, the following strategy forms a subgame perfect equilibrium of the dynamic game: "Set (k_1^*, k_2^*) in the first stage and play a SPEP that satisfies (PO) and (C) in the price-quantity supergame. If firm i deviates at the capacity stage to set $k_i^d \neq k_i^*$, revert to the path τ_i^s, on which firm i obtains its security level if such a path exists for capacities (k_i^d, k_{-i}^*). Otherwise, revert to the one-shot Nash equilibrium strategies forever." It is simple to check that all unilateral deviations in capacity from the above strategy yield a payoff in the overall game that is strictly less than the payoff from conforming to (k_1^*, k_2^*).[17]

Note also that the result holds whether the choice of capacity is made simultaneously or sequentially. It relies solely on firms' ability to punish deviations from the equilibrium capacity pair (k_1^*, k_2^*) by imposing a payoff for the deviating firm that is less than or equal to the discounted sum of the one-shot Nash equilibrium payoff evaluated at (k_1^*, k_2^*).

Proposition 7 *Assume that the unit cost of capacity is zero or is negligible. If $\delta \geq \tilde{\delta}$, every capacity pair for which there exist pure strategy perfect equilibrium security level punishment paths for both firms can be supported in a subgame perfect equilibrium of the dynamic game of either simultaneous or sequential capacity choice followed by the capacity constrained price-quantity supergame.*

The implication of this result is immediate. All of the behaviors described in this paper arise in some subgame perfect equilibrium when there is endogenous choice of capacity.

[17]First, k_i^d must be such that $k_i^d + k_{-i}^* \geq M$, otherwise, firm i could increase its payoff by increasing k_i^d slightly. Then, firm i's optimal deviation yields a payoff equal to either the security level, which is independent of firm i's capacity and is thus the same at (k_i^*, k_{-i}^*) and (k_i^d, k_{-i}^*), or the discounted sum of π_i^N evaluated at (k_i^d, k_{-i}^*), where $k_i^d < k_i^*$. Since the payoff from conforming is greater than or equal to both the security level and the discounted sum of π_i^N evaluated at (k_i^*, k_{-i}^*), it follows from the fact that the security level and π_i^N are non-decreasing in k_i that the payoff from conforming is greater than the payoff from deviating.

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8 Appendix

8.1 Proof of Proposition 1

Let (Φ_1^*, Φ_2^*) be a B-E equilibrium in $G^E(k_1, k_2, c)$. We will demonstrate that (μ_1^*, μ_2^*) satisfying $\mu_1^{*p} = \Phi_1^*$, $\mu_2^{*p} = \Phi_2^*$ and $\gamma_i^{\mu_i^*}(p) = k_i$, $\forall p \in \text{supp}(\mu_i^{*p})$, $i = 1, 2$, is an equilibrium of the price-quantity game.

Suppose μ_j^* satisfies $\mu_j^{*p} = \Phi_j^*$ and $\gamma_j^{\mu_j^*}(p) = k_j$ $\forall p \in \text{supp}(\mu_j^{*p})$. Suppose μ_i' is a best-response for firm i, $j \neq i$ in the game $\Gamma(k_1, k_2, c)$. Clearly, for any $p \geq c$, firm i's expected payoff setting $\gamma_i^{\mu_i'}(p) = k_i$ is at least as great as the expected payoff from setting $\gamma_i^{\mu_i'}(p) = q_i < k_i$. To see this, note that for any realization of p_j, $s_i(p_i, p_j, k_i, q_j) \geq s_i(p_i, p_j, q_i, q_j)$. Hence, for any $p_i \in \text{supp}(\mu_i')$, setting $\gamma_i^{\mu_i'}(p) = k_i$ cannot lower firm i's expected profit. However, since (Φ_1^*, Φ_2^*) is a B-E equilibrium, a strategy μ_i such that $\mu_i^p = \Phi_i^*$ and $\gamma_i^{\mu_i}(p) = k_i$ $\forall p \in \text{supp}(\mu_i^p)$ $i = 1, 2$, provides an expected payoff to firm i that is at least as great as any other distribution $\tilde{\mu}_i$ such that $\gamma_i^{\tilde{\mu}_i}(p) = k_i$ $\forall p \in \text{supp}(\tilde{\mu}_i)$. Hence, μ_i satisfying $\mu_i^p = \Phi_i^*$ and $\gamma_i^{\mu_i}(p) = k_i$ $\forall p \in \text{supp}(\mu_i)$ is a best-response to μ_j^* and the underlying distributions over sales and prices are identical to those of the B-E equilibrium (Φ_1^*, Φ_2^*).

Suppose (μ_1^*, μ_2^*) is an equilibrium of the price-quantity game $\Gamma(k_1, k_2, c)$. We will break down the analysis for different regions of the capacity space.

Suppose first that $k_1 + k_2 > M > k_1$. By examination, it is clear that for every p_i, p_j, q_j and $q_i < k_i$, $s_i(p_i, k_i, p_j, q_j) \geq s_i(p_i, q_i, p_j, q_j)$ with equality only if $p_i > p_j$ and $M - q_j \leq q_i$. Hence for any price p_i, firm i would strictly increase its expected profit π_i by increasing its quantity from q_i to k_i unless $p_i \leq c$, or $p_i > p_j$ almost everywhere with respect to μ_j and $M - q_j \leq q_i$ almost everywhere with respect to μ_j. A strict dominance argument rules out $p_i \leq c$ in the support of μ_i. If there exists a $\hat{p}_i \in \text{supp}(\mu_i^{*p})$, such that $\exists q_i \in \gamma_i^{\mu_i^*}(\hat{p}_i)$ with $q_i < k_i$ and satisfying $\hat{p}_i > p_j$, a.e. μ_j, and $M - q_j \leq q_i$, a.e. μ_j, then let $\tilde{\mu}_i$ be an element of $\mathcal{M}_i$ such that $\tilde{\mu}_i$ coincides with μ_i^* for all prices $p \neq \hat{p}_i$ and $\gamma_i^{\tilde{\mu}_i}(\hat{p}_i) = k_i$. $\tilde{\mu}_i$ achieves the same expected payoff and distribution of sales as μ_i^* against μ_j^*. Furthermore, μ_j^* remains a best response against $\tilde{\mu}_i$ since only the incentives to set prices $p_j \geq \hat{p}_i$ have been altered, and the payoff from these prices has decreased. Similarly, firm j's expected payoff and distribution of sales are unaltered against $\tilde{\mu}_i$. This demonstrates that $\tilde{\mu}_i$ is a best response to μ_j^* and μ_j^* is a best response to $\tilde{\mu}_i$. A similar argument demonstrates that if $\gamma_j^{\mu_j^*}(p) \neq k_j$, $\forall p \in \text{supp}(\mu_j^*)$, then we may construct a measure $\tilde{\mu}_j$ coinciding with μ_j^* for all prices except possibly some $\hat{p}_j$ defined analogously to $\hat{p}_i$. Such a $\tilde{\mu}_j$ remains a best response to $\tilde{\mu}_i$ and achieves the same expected payoff and distribution of sales as μ_j^*. Furthermore, $\tilde{\mu}_i$ remains a best response against $\tilde{\mu}_j$. Since $\gamma_i^{\tilde{\mu}_i}(p) = k_i$, $\forall p \in \text{supp}(\tilde{\mu}_i^p)$, $i = 1, 2$, and $(\tilde{\mu}_1, \tilde{\mu}_2)$ are best responses, $(\tilde{\mu}_1^p, \tilde{\mu}_2^p)$ must be the (unique) B-E equilibrium price distributions. Hence, (μ_1^*, μ_2^*) generate the same equilibrium payoffs, and price and sales distributions as the unique equilibrium of $G^E(k_1, k_2, c)$.

Arguments for the range of capacities $k_1 \geq M > k_2$ are similar to those above, except that quantities q_i such that $k_1 \geq q_1 \geq M$ are payoff and distributionaly equivalent to k_1.

The case $k_1 + k_2 \leq M$ is trivial since in the price-quantity game, $(p_i, q_i) = (1, k_i)$ is a strictly dominant strategy.

Suppose $k_1 \geq k_2 \geq M$. From the above arguments, if $q_i \in \gamma_i^{\mu_i^*}(p)$, then $q_i \geq M$ except possibly for $p \leq c$ or some $p = \hat{p}_i$ such that $\hat{p}$ is undercut with certainty by firm j. Suppose $c \in \text{supp}(\mu_j^{*p})$. Then $\pi_j^* = 0$, which implies that $\Pr_{\mu_i^*}(p_i \leq p) = 1$, $\forall p > c$. But this implies that $u_i = c$ and therefore $u_j \leq c$, since otherwise there would exist a $p > c$ such that $E_{\mu_j^*}\pi_i(p, q_i, p_j, q_j) > 0$. Hence $c \in \text{supp}(\mu_j^{*p})$ implies $u_i, u_j \leq c$. It is obvious that $l_i = l_j = c$ since, otherwise, one firm would either sell a positive amount at prices less than c or sell nothing at these prices ($q_i = 0$). Selling a positive amount at prices less than c is clearly not equilibrium behavior. Setting prices below c with positive probability but selling nothing would imply that the rival could sell to a positive residual demand at a price above c with positive probability, contradicting $u_i \leq c$, $i = 1, 2$. Hence $c \in \text{supp}(\mu_j^{*p})$ implies $u_1 = u_2 = l_1 = l_2 = c$, which yields the immediate conclusion that every $q_i \in \gamma_i^{\mu_i^*}(c)$ satisfies $q_i \geq M$, $i = 1, 2$, which is the B-E result.

Suppose $c \notin \text{supp}(\mu_j^{*p})$, $j = 1, 2$. By arguments similar to those above, we can rule out either firm pricing below c with positive probability. (Clearly setting (p_i, q_i) such that $p_i < c$ and $q_i > 0$ is not part of equilibrium behavior. If firm i sets prices below c with positive probability but sets $q_i = 0$, then firm j has positive residual demand with positive probability, and hence will set prices bounded above c. This would allow firm i to undercut j's support and earn a positive profit, again a contradiction.) Hence $l_i, l_j > c$. But by our previous argument, this implies that $q_i \in \gamma_i^{\mu_i^*}(p)$, which implies that $q_i \geq M$, except possibly for some $\hat{p}$ such that $\hat{p}$ is undercut with probability 1 by $j \neq i$. Suppose $u_i = u_j = u$. Clearly, both firms firms cannot place positive mass at u since this would imply that every $q_i \in \gamma_i^{\mu_i^*}(u)$ satisfies $q_i \geq M$, $i = 1, 2$, and there would be incentive to undercut u slightly. If only one firm has positive mass at u, that firm is undercut with certainty at u and faces zero residual demand there. It therefore earns zero equilibrium expected profit, which contradicts $l_i, l_j > 0$. Hence neither firm can place mass on $\{u\} \times [0, k_i]$. But this implies that $q_i \geq M, \forall q_i \in \gamma_i^{\mu_i^*}(p)$ and $\forall p \in \text{supp}(\mu_i^*)$ implying that $\lim_{p_j \uparrow u_j} \pi_j(p_j, \gamma_j^{\mu_j^*}(p_j), \mu_i^*) = 0$, again a contradiction to $l_j > 0$.

If $u_i > u_j$, $i \neq j$, it is clear that the above considerations imply that $\pi_i^* = 0$, again, a contradiction to $l_i > 0$. Hence we cannot have an equilibrium with $c \notin \text{supp}(\mu_j^*)$ for some j. $\square$

8.2 Proof of Lemma 1

We first demonstrate that if there exists a SPEP $\tau^s = \{(p_1, q_1, p_2, q_2)\}$ satisfying (PO) and (SP), then $q_1 + q_2 \geq M$ and $p_i = 1$, $i = 1, 2$. First, (SP) implies $p_1 = p_2 = \tilde{p}$ for some $\tilde{p}$. There are three possible cases: (i) $(\tilde{p}, q_j) \in A_i$ for $i = 1, 2$, (ii) $(\tilde{p}, q_j) \notin A_i$ for one and only one i, and (iii) $(\tilde{p}, q_j) \notin A_i$ for $i = 1, 2$.

In case (i), since A1 holds, the incentive constraints (4) and (5) may be written as follows:[18]

$$(\tilde{p} - c)[s_i(\tilde{p}, \tilde{p}, q_1, q_2) - (1 - \delta)\hat{k}_i] \geq \delta(1 - c)(M - \hat{k}_{-i}),\ i = 1, 2. \tag{9}$$

We first show that within the class of pricing policies satisfying (SP), the incentive constraints are never tightened by setting $p_1 = p_2 = 1$. From $\delta(1-c)(M - \hat{k}_j) \geq 0$, it follows that if $s_i - (1-\delta)\hat{k}_i < 0$, either (9) does not hold or $\tilde{p} < c$, both contradicting SPEP. If i is such that $s_i - (1 - \delta)k_i = 0$, (9) implies that $k_{-i} \geq M$. If $k_{-i} \geq M$ and $k_i < M$ then for $j \neq i$, $\delta(1-c)(M - k_{-j}) > 0$. But then, (9) implies $s_j - (1-\delta)k_j > 0$, so that the LHS of (9) is strictly increasing in $\tilde{p}$. Therefore setting $\tilde{p} = 1$ relaxes firm j's incentive constraint without affecting firm i's. If $k_{-i} \geq M$ $i = 1, 2$, and $s_i - (1 - \delta)k_i = 0$, then setting $\tilde{p} = 1$ does not affect the incentive constraint of any firm. Finally, $s_i - (1 - \delta)k_i > 0$ $i = 1, 2$ implies that the LHS of (9) is strictly increasing in $\tilde{p}$, so that setting $\tilde{p} = 1$ relaxes each firm's incentive constraint. Since setting $p_1 = p_2 = 1$ never tightens the incentive constraints, it is easy to see that in all cases setting $\tilde{p} < 1$ contradicts (PO).

In case (ii), let firm n be this firm for which $(\tilde{p}, q_{-n}) \notin A_n$. Since A1 holds, firm n's incentive constraint may be written as follows:

$$(\tilde{p} - c)s_n - (1 - \delta)(1 - c)(M - \hat{q}_{-n}) \leq \delta(1 - c)(M - \hat{k}_{-n}). \tag{10}$$

Using the same type of arguments as for case (i), $(\tilde{p} - c)s_n - (1 - \delta)(1 - c)(M - \hat{q}_{-n})$ contradicts the fact that τ^s is a SPEP. If $(\tilde{p} - c)s_n - (1 - \delta)(1 - c)(M - \hat{q}_{-n}) = 0$, then (10) implies $k_{-n} \geq M$. Since the LHS of (10) is strictly increasing in $\tilde{p}$, setting $\tilde{p} = 1$ relaxes (10). Arguments developed for case (i) show that setting $\tilde{p} = 1$ relaxes the other firm's incentive constraint, given by (9) for $i \neq n$, as well. Again, setting $\tilde{p} < 1$ would contradict (PO).

The proof for case (iii) follows directly from case (ii). This completes the proof that if τ^s is SPEP satisfying (PO) and (SP), then $p_1 = p_2 = 1$.

We now show that $p_1 = p_2 = \tilde{p} = 1$ on a SPEP τ^s implies that $(1, q_j) \in A_i$ $i = 1, 2$ $j \neq i$. Suppose $(1, q_j) \neq A_i$ for some i, j. Then it follows from (3), the definition of A_i, that $q_j < M - \hat{k}_i$, from which it follows that $\pi_j(1, 1, q_i, q_j) < \underline{\pi}_j$, a contradiction to SPEP.

[18] Where no confusion is possible, we drop the argument out of $s_i(\tilde{p}, \tilde{p}, q_1, q_2)$ for notational convenience.

Next we show that if τ^s satisfies (PO) and (SP), it must be the case that $q_1+q_2 \geq M$. Suppose $\{(1, q_1, 1, q_2)\}$ satisfies (PO) and $q_1 + q_2 < M$. Then firm i's profit is $\pi_i = (1-c)q_i$. However, setting $q_i' = M - q_j$, firm i obtains $\pi_i' = (1-c)(M-q_j) > (1-c)q_i$ and firm j's profit is unaffected, contradicting (PO).

Now we show that there exists a path $\tau^s = \{1, 1, q_1, q_2\}$ satisfying (PO) and (SP) if and only if $\delta \geq \frac{1}{2}$. There exists such a τ^s if and only if the following conditions are satisfied:

$$\begin{aligned}
(1-c)s_1 &\geq (1-\delta)(1-c)\hat{k}_1 + \delta(1-c)(M-\hat{k}_2), &(11)\\
(1-c)s_2 &\geq (1-\delta)(1-c)\hat{k}_2 + \delta(1-c)(M-\hat{k}_1), &(12)\\
s_1 &\geq M - \hat{k}_2, &(13)\\
s_2 &\geq M - \hat{k}_1. &(14)
\end{aligned}$$

From $q_1 + q_2 \geq M$, we have $s_2 = M - s_1$. Substituting for $M - s_1$ in (12) and rewriting (12) and (13) yields:

$$\delta \geq \delta'(s_1) \equiv \frac{\hat{k}_1 - s_1}{\hat{k}_1 + \hat{k}_2 - M}, \tag{15}$$

$$\delta \geq \delta''(s_1) \equiv \frac{\hat{k}_2 - M + s_1}{\hat{k}_1 + \hat{k}_2 - M}. \tag{16}$$

For a given quantity sold by firm 1, $\delta'(s_1)$ and $\delta''(s_1)$ define lower bounds on the discount factor for the existence of a SPEP τ^s satisfying (PO) and (SP). Therefore, if firm 1's sales are given by s_1, then τ^s is a SPEP satisfying (PO) and (SP) only if $\delta \geq \max\{\delta', \delta''\}$. Since δ' decreases linearly with s_1 and δ'' increases linearly with s_1, we can define s_1^* such that $\delta'(s_1^*) = \delta''(s_1^*)$. It follows that if $\delta < \delta'(s_1^*)$, there does not exist s_1 such that (11) and (12) are satisfied. Therefore, if $\delta < \delta'(s_1^*)$, there does not exist a SPEP satisfying (PO) and (SP). Solving for s_1^*, we obtain $s_1^* = \frac{1}{2}(\hat{k}_1 + M - \hat{k}_2)$. Substituting for s_1^* in δ', we obtain $\delta'(s_1^*) = \frac{1}{2}$. Consider the path $\tau^* = \{(1, 1, s_1^*, s_2^*)\}$. For every $\delta \geq \frac{1}{2}$, τ^* is a SPEP that satisfies (PO) and (SP). The proof that there exists a SPEP satisfying (PO) and (SP) if, and only if $\delta \geq \frac{1}{2}$ is now complete.

We now show that τ^s also satisfies (C) if, and only if $\delta \geq \tilde{\delta}$. We strengthen (14) to:

$$s_2 \geq \frac{(M - \hat{k}_2)\hat{k}_2}{\hat{k}_1}. \tag{17}$$

From (13) and (17), τ^s satisfies (C) only if additionally $M - \hat{k}_2 \leq s_1 \leq M - \left(\frac{M-\hat{k}_2}{\hat{k}_1}\right)\hat{k}_2$, with at least one strict inequality. It remains to establish that $\delta \geq \tilde{\delta}$ is necessary and sufficient for some τ^s satisfying (PO) and (SP) to also satisfy (C). We have $s_1^* > M - \hat{k}_2$, however, using (17), we obtain $s_1^* \leq M - \left(\frac{M-\hat{k}_2}{\hat{k}_1}\right)\hat{k}_2$ if, and only if $\hat{k}_2 \geq \frac{1}{2}\hat{k}_1$. So, $\hat{k}_2 < \frac{1}{2}\hat{k}_1$ implies that τ^* does not satisfy (C). In this case, $\delta'(M -$

$\left(\frac{M-\hat{k}_2}{\hat{k}_1}\right)\hat{k}_2) > \delta''(M-\left(\frac{M-\hat{k}_2}{\hat{k}_1}\right)\hat{k}_2)$. Simple algebra yields $\delta'(M-\left(\frac{M-\hat{k}_2}{\hat{k}_1}\right)\hat{k}_2) = 1-\frac{\hat{k}_2}{\hat{k}_1}$. For every $\delta \geq 1-\frac{\hat{k}_2}{\hat{k}_1}$, the path $\{(1, 1, M-\left(\frac{M-\hat{k}_2}{\hat{k}_1}\right)\hat{k}_2, \left(\frac{M-\hat{k}_2}{\hat{k}_1}\right)\hat{k}_2)\}$ is a SPEP that satisfies (PO), (C) and (SP). □

8.3 Proof of Proposition 2

Assuming A1, $\tau^s = \{(p_1^c, p_2^c, q_1^c, q_2^c)\}$ is a SPEP satisfying (PO) and (C) if and only if $(p_1^c, p_2^c, q_1^c, q_2^c)$ solves the following problem:

$$\max_{\{p_1,p_2,q_1,q_2\}} (p_2-c)s_2(p_1,q_1,p_2,q_2) \tag{18}$$

$$\text{subject to } (p_1-c)s_1(p_1,q_1,p_2,q_2) \geq \overline{\pi}, \tag{19}$$

$$s_2(p_1,q_1,p_2,q_2) \geq \alpha_2^N, \text{ (4) and (5),}$$

where $(1-c)(M-\alpha_2^N) \geq \overline{\pi} \geq (1-c)(M-\hat{k}_2)$. First, we show that if $\overline{\pi}$ is such that when firms' sales are given by $s_1 = \frac{\overline{\pi}}{1-c}$ and $s_2 = M - \frac{\overline{\pi}}{1-c}$ and firms set $p_1 = p_2 = 1$ neither (4) nor (5) are binding, the vector $(1, q_1, 1, q_2)$ such that $s_1(1, q_1, 1, q_2) = \frac{\overline{\pi}}{1-c}$ and $s_2(1, q_1, 1, q_2) = M - \frac{\overline{\pi}}{1-c}$ solves (18). Note that given any $p_1, p_2 > c$, it is optimal to set $q_1 + q_2 \geq M$, since if $q_2 < M - q_1$, the objective can be increased without affecting firm 1's sales and profit and neither firm's incentive constraints. Therefore, (PO) implies $s_2(p_1, q_1, p_2, q_2) = M - s_1(p_1, q_1, p_2, q_2)$. We solve for s_1 from (19) satisfied with equality and substitute for $M-s_1$ in the objective function. Maximizing with respect to p_1 and p_2 yields the unique solution $p_1 = p_2 = 1$. Therefore, using (19), we obtain $s_1(1, q_1, 1, q_2) = \frac{\overline{\pi}}{1-c}$, which implies that for such $\overline{\pi}$, $(1, q_1, 1, q_2)$ solves (18) if and only if $q_1 + q_2 \geq M$, $s_1(1, q_1, 1, q_2) = \frac{\overline{\pi}}{1-c}$ and $s_2(1, q_1, 1, q_2) = M - \frac{\overline{\pi}}{1-c}$.

Second, we examine the case in which $\overline{\pi} \leq \overline{\pi}'$, where $\overline{\pi}'$ is such that if firms set $p_1 = p_2 = 1$ and sell $s_1 = \frac{\overline{\pi}'}{1-c}$ and $s_2 = M - \frac{\overline{\pi}'}{1-c}$, then (4) is binding. By definition of $\underline{\alpha}_1(\delta)$, $\overline{\pi}' \equiv (1-c)\underline{\alpha}_1(\delta)$. Furthermore for $\overline{\pi} < \overline{\pi}'$, (4) does not hold on any path on which firms set $p_1 = p_2 = 1$ and sell $s_1 = \frac{\overline{\pi}}{1-c}$ and $s_2 = M - \frac{\overline{\pi}}{1-c}$. This implies that if $\overline{\pi} < \overline{\pi}'$, (4) must be binding at a solution to (18). An argument similar to that used above establishes that for all $\overline{\pi} \leq \overline{\pi}'$, (PO) implies $q_1 + q_2 \geq M$, in which case $s_1 + s_2 = M$. For given prices and quantities, both firms' sales are then uniquely defined by (4) satisfied with equality and $s_1 + s_2 = M$. Suppose (p_1', q_1', p_2', q_2') solves (18) and $(p_2', q_2') \in A_1$ (we show below that the latter does hold). Solving for s_1 from (4), setting $s_2 = M - s_1$ and substituting for s_2 in the objective of (18), it must be the case that (p_1', q_1', p_2', q_2') solves the following problem:

$$\max_{\{p_1,p_2\}} \quad (p_2-c)\left(M - \frac{(1-\delta)(p_2-c)\hat{k}_1 + \delta(1-c)(M-\hat{k}_2)}{(p_1-c)}\right) \tag{20}$$

$$\text{subject to} \quad (1-\delta)(p_2-c)\hat{k}_1 + \delta(1-c)(M-\hat{k}_2) \geq \overline{\pi} \text{ and (5).}$$

Note that the objective function is strictly increasing in p_1. We now show that $p_1' = 1$ must hold. Suppose $p_1' < 1$, then, since the objective function is strictly increasing in p_1, it must the case that (5) is binding at p_1', that is:

$$(p_2' - c)s_2 = (1-\delta)\pi_2^*(p_1', q_1') + \delta(M - \hat{k}_1). \tag{21}$$

However, we have shown above that $\{(1, M - \overline{\alpha}_2(\delta), 1, \overline{\alpha}_2(\delta))\}$ is a SPEP on which $\pi_1(1, M - \overline{\alpha}_2(\delta), 1, \overline{\alpha}_2(\delta)) \geq \overline{\pi}$ and:

$$\pi_2(1, M - \overline{\alpha}_2(\delta), 1, \overline{\alpha}_2(\delta)) \geq (1-\delta)(1-c)k_2 + \delta(1-c)(M - \hat{k}_1) > (p_2' - c)s_2,$$

where the last inequality follows from (21). But this contradicts the fact that (p_1', q_1', p_2', q_2') solves (18). Hence, $p_1' = 1$, implying that $s_2(p_1', q_1', p_2', q_2') = \overline{\alpha}_2(p_2')$ and $\pi_1(p_1', q_1', p_2', q_2') = (1-c)(M - \alpha_2(p_2'))$, by definition of $\overline{\alpha}_2(p_2)$. Therefore, the only constraint left is $(1-\delta)(p_2' - c)\hat{k}_1 + \delta(1-c)(M - \hat{k}_2) \geq \overline{\pi}$. If $\overline{\pi}$ is such that the latter constraint is binding, p_2' is uniquely defined by $(1-c)(M - \overline{\alpha}_2(p_2')) = \overline{\pi}$. From the latter inequality, it is easy to see that p_2' decreases as $\overline{\pi}$ decreases since $M - \overline{\alpha}_2(p_2)$ is increasing in p_2. We now determine the lowest such p_2' firm 2 ever sets on a SPEP satisfying (PO). This price, which we denote by p_2^* is the solution to the first order condition to (20) assuming (5) and the remaining constraint hold:

$$(1-c)M - 2(1-\delta)(p_2^* - c)\hat{k}_1 - \delta(1-c)(M - \hat{k}_2) = 0. \tag{22}$$

which yields $p_2^* = \frac{(1-c)(M - \delta(M - \hat{k}_2))}{2(1-\delta)\hat{k}_1} + c$.[19] If there is no interior solution for which $p_2^* \leq 1$, then the solution is a corner solution at $p_2^* = 1$. We have thus shown that if $\delta \geq \tilde{\delta}$, there exists p_2^* such that for all $p_2^c \in [\min\{p_2^*, 1\}, 1]$, there exists a path $\tau^c = \{(1, M - \overline{\alpha}_2(p_2^c), p_2^c, \overline{\alpha}_2(p_2^c))\}$ that satisfies (PO) and (C).

It remains to show that $(p_2^c, \overline{\alpha}_2(p_2^c)) \in A_1, \forall p_2^c \in P_2^c$. If $\min\{p_2^*, 1\} = 1$, this is obvious. Suppose $\min\{p_2^*, 1\} = p_2^* < 1$. Consider the path $\{(1, M - \overline{\alpha}_2(p_2^c)), p_2^c, \overline{\alpha}_2(p_2^c))\}$, $p_2^c \in P_2^c$. Suppose, to obtain a contradiction that $(p_2^c, \overline{\alpha}_2(p_2^c)) \notin A_1$. By definition of $\overline{\alpha}_2(p_2^c)$:

$$(1-c)(M - \overline{\alpha}_2(p_2^c)) = (1-\delta)(p_2^c - c)k_1 + \delta(1-c)(M - \hat{k}_2). \tag{23}$$

By the definition of A_1, if $(p_2^c, \overline{\alpha}_2(p_2^c)) \notin A_1$:

$$(1-c)(M - \overline{\alpha}_2(p_2^c)) > (p_2^c - c)\hat{k}_1.$$

Thus, from (23) and the fact that $\delta \in (0,1)$, it follows that:

$$(p_2^c - c)\hat{k}_1 < (1-c)(M - \hat{k}_2).$$

Hence, using the definition of $\underline{p}_1$, $p_2^c < \underline{p}_1 < 1$ must hold. Thus $\underline{p}_1 \in P_2^c$. We show that this is is impossible. From $p_2^* \leq p_2^c < 1$ and the fact that for $[p_2^*, 1]$, $p_2\overline{\alpha}_2(p_2)$

[19] The second order condition is $-2(1-\delta)\hat{k}_1 < 0$.

increases as p_2 decreases, we have $(p_2^* - c)\overline{\alpha}_2(p_2^*) \geq (p_2 - c)\overline{\alpha}_2(p_2) > (1-c)\overline{\alpha}_2(\delta) \geq \pi_2^N = (\underline{p}_1 - c)\hat{k}_2$ for every $p_2 \in P_2^c$. Thus, at every $p_2 \in [p_2^*, 1)$, firm 2 obtains a payoff strictly greater than the maximum it can obtain by setting $\underline{p}_1$. Firm 1 obtains a payoff $(1-c)(M - \overline{\alpha}_2(p_2))$ that is no less than $(1-c)(M - \hat{k}_2)$, the maximum payoff it can obtain when firm 2 sells its capacity at $\underline{p}_1$. Hence, by (PO), $\underline{p}_1 \notin P_2^c$, a contradiction. Therefore, if $p_2^c \in P_2^c$, $(p_2^c, \overline{\alpha}_2(p_2^c)) \in A_1$.

Therefore, for $\delta \geq \tilde{\delta}$, we have established that for all values of firm 1's payoffs in an interval, $\overline{\pi} \in [(1-c)(M - \overline{\alpha}_2(\min\{p_2^*, 1\}), (1-c)(M - \max\{\alpha_2^N, \underline{\alpha}_2(\delta)\}]$, there exists a unique p_2^c in an interval $P_2^c \equiv [\min\{p_2^*, 1\}, 1]$, such that $(1, q_1^c, p_2^c, q_2^c)$ solves (18) if and only if $q_1^c + q_2^c \geq M$, and sales satisfy $s_1(1, p_2^c, q_1^c, q_2^c) = M - \overline{\alpha}_2(p_2^c)$ and $s_2(1, p_2^c, q_1^c, q_2^c) = \overline{\alpha}_2(p_2^c)$ for $p_2^c \in [p_2^*, 1)$, and $s_2(1, p_2^c, q_1^c, q_2^c) \geq \max\{\alpha_2^N, \underline{\alpha}_2(\delta)\}$ and $s_1(1, p_2^c, q_1^c, q_2^c) \geq \underline{\alpha}_1(\delta)$ for $p_2^c = 1$. Therefore, there exists a unique SPEP that achieves payoffs $(\overline{\pi}, V_2)$, where V_2 is the value of the objective function at a solution to (18) given $\pi \in [(1-c)(M - \overline{\alpha}_2(\min\{p_2^*, 1\}), (1-c)(M - \max\{\alpha_2^N, \underline{\alpha}_2(\delta)\}]$. Hence, all such SPEP satisfy (PO). Moreover, note that firm 2's minimum payoff on such SPEP is $(1-c)\alpha_2^N = \pi_2^N$ and firm 1's payoff is strictly greater than π_1^N. Thus all such SPEP satisfy (C) as well.

To complete the proof of the proposition, we must show that firm 1 cannot obtain a payoff greater than $(1-c)\overline{\alpha}_1(\delta)$ on a SPEP that satisfies (PO) and (C). To this effect, suppose π'' is such that (5) is binding at $(1, q_1', 1, q_2')$ when $s_1(1, q_1', 1, q_2') = \frac{\pi''}{1-c}$ and $s_2(1, q_1', 1, q_2') = M - \frac{\pi''}{1-c}$. This clearly implies $\pi'' \equiv (1-c)\overline{\alpha}_1(\delta)$, by definition of $\overline{\alpha}_1(\delta)$. Thus it follows that if $\overline{\pi} > \pi''$, (5) does not hold on any path $\{(1, q_1', 1, q_2')\}$ for which $s_1(1, q_1, 1, q_2) = \frac{\overline{\pi}}{1-c}$ and $s_2(1, q_1, 1, q_2) = M - \frac{\overline{\pi}}{1-c}$. Therefore, for all $\overline{\pi} > \pi''$, in any solution to (18), it must be the case that $p_1 < 1$. Indeed, let $\overline{\pi} > \overline{\pi}''$ and suppose $p_1 = 1$, then $s_2 = M - \frac{\overline{\alpha}_1(\delta)}{p_1 - c}$, and $p_1 = 1$ imply that $(p_2 - c)s_2$ is maximized at $p_2 = 1$, where $s_2 = \underline{\alpha}_2(\delta)$, contradicting the fact that (19) holds. We now show that $\pi'' = (1-c)\overline{\alpha}_1(\delta)$ is the maximum profit attainable by firm 1 on a SPEP satisfying (PO). If this is not the case, then there exists a SPEP $\{(p_1, q_1, p_2, q_2)\}$ satisfying (PO) for which $p_1 < 1$ and $(p_1 - c)s_1 > \pi'' = (1-\delta)\overline{\alpha}_1(\delta) = \delta(1-c)\hat{k}_1 + (1-\delta)(1-c)(M - \hat{k}_2)$. Suppose such a SPEP exists. It is clear that on a SPEP, if firms set prices $p_1 > c$ and $p_2 > c$, the maximum amount of sales s_1 firm 1 can obtain is $M - s_2$, where s_2 is the solution in s_2 to firm 2's incentive compatibility constraint (5) satisfied with equality when firm 2's optimal deviation is to undercut firm 1's price:

$$s_2 = \frac{1}{(p_2 - c)}[(1-\delta)(p_1 - c)\hat{k}_2 + \delta(1-c)(M - \hat{k}_1)].$$

Thus, s_1 is strictly increasing in p_2. Therefore, to maximize firm 1's sales, set $p_2 = 1$. Now, maximizing $(p_1 - c)s_1$ with respect to p_1 yields the first order condition, satisfied at an interior solution $p_1^* < 1$:

$$(1-c)M - 2(1-\delta)(p_1^* - c)\hat{k}_2 - \delta(1-c)(M - \hat{k}_1) = 0 \tag{24}$$

But from $\delta \geq \tilde{\delta}$, it follows that the LHS of (24) is always strictly positive, so that the solution is rather a corner solution, $p_1^* = 1$, thus yielding a contradiction. But then, $(p_1^* - c)s_1 = (1-c)\overline{\alpha}_1(\delta)$. Since firm 1 cannot obtain a greater profit on a path on which both $p_1 < 1$ and $p_2 < 1$, we have shown that firm 1's profit on a SPEP that satisfies (PO) and (C) is less than or equal to $(1-c)\overline{\alpha}_1(\delta)$, which is what we had to prove. □

8.4 Proof of Proposition 5

Consider the case $\hat{k}_1 = \hat{k}_2 = M$. By Proposition 1, $\Gamma(k_1, k_2, c)$ and $G^E(k_1, k_2, c)$ have identical distributions of prices and sales and it is straightforward to show that setting $p_1 = p_2 = c$ and $q_1 = q_2 = M$ is a pure strategy equilibrium in the one-shot game providing each firm its security level, $\pi_i(c, c, M, M) = 0$, $i = 1, 2$. Repeating this one-shot equilibrium provides the desired path.

Suppose $k_2 < M$. We begin by showing that if there exists a pure strategy two-phase perfect equilibrium punishment path that forces the large firm down to its security level, then there also exists a pure strategy two-phase perfect equilibrium security level punishment for the small firm. Then we show that such a two-phase perfect equilibrium security level punishment indeed exists for the large firm.

Consider the following punishment path τ_2^s for the small firm. In the first period, firm 1 sets a price equal to $\underline{p}_2$ and a quantity ceiling equal to $\hat{k}_1$. Firm 2 sets a price p_2^s satisfying

$$(1-\delta)(p_2^s - c)\hat{k}_2 + \delta(1-c)\underline{\alpha}_2(\delta) = \underline{\pi}_2 \tag{25}$$

and $0 \leq p_2^s < \underline{p}_2$, and a quantity ceiling equal to $\hat{k}_2$. We show below that such a price p_2^s exists. After the first period and assuming no deviation, firms revert to the stationary path $\{(1, 1, \overline{\alpha}_1(\delta), \underline{\alpha}_2(\delta))\}$ from period 2 on. Firm 2's deviations are punished by restarting τ_2^s. We assume that there exists a perfect equilibrium security level punishment path for firm 1. We show that under the stated assumptions, the path τ_2^s is a security level perfect equilibrium punishment path for the small firm.

We first show that there exists a unique price p_2^s satisfying (25) and $0 \leq p_2^s < \underline{p}_2$. Since $\hat{k}_2 > 0$, the price p_2^s defined by (25) is unique. Furthermore, since $\underline{\alpha}_2(\delta) > M - \hat{k}_1$ and $(\underline{p}_2 - c)\hat{k}_2 \equiv (1-c)(M - \hat{k}_1) = \underline{\pi}_2$, $p_2^s < \underline{p}_2$. Finally, $p_2^s \geq 0$ follows from the assumption $c \geq \hat{c}$.

Next, we show that firms' incentive constraints are satisfied in the first period of τ_2^s. First, note that firm 2's incentive constraint in the first period is given by (25) and is therefore satisfied by definition of p_2^s. Second, since we assumed that there exists a security level perfect equilibrium punishment path for firm 1, its incentive constraint in the first period of τ_2^s is given by:

$$(1-\delta)(\underline{p}_2 - c)(M - \hat{k}_2) + \delta(1-c)\overline{\alpha}_1(\delta) \geq \underline{\pi}_1. \tag{26}$$

Solving for $\hat{k}_2$ in (26), we obtain that (26) is satisfied if and only if:

$$\hat{k}_2 \geq M - \min\left\{\hat{k}_1, \frac{\delta^2 M}{1-\delta(1-\delta)}\right\}. \tag{27}$$

We now show that on τ_2^s, both firms' incentive constraints are satisfied from period 2 on. Since firm 2's deviations are punished by restarting τ_2^s, using (25), firm 2's incentive constraint in any period of the stationary path $\{(1, 1, \overline{\alpha}_1(\delta), \underline{\alpha}_2(\delta))\}$ is:

$$(1-c)\underline{\alpha}_2(\delta) \geq (1-\delta)(1-c)\hat{k}_2 + \delta\underline{\pi}_2.$$

Since $\underline{\pi}_2 = (1-c)(M - \hat{k}_1)$, for $\delta \geq \frac{1}{2}$, the above incentive constraint is satisfied by definition of $\underline{\alpha}_2(\delta)$.

Since firm 1 is punished down to its security level if it deviates, firm 1's incentive constraint in any period of the stationary path $\{(1, 1, \overline{\alpha}_1(\delta), \underline{\alpha}_2(\delta))\}$ is:

$$(1-c)\overline{\alpha}_1(\delta) \geq (1-\delta)(1-c)\hat{k}_1 + \delta\underline{\pi}_1.$$

Since $\underline{\pi}_1 = (1-c)(M - \hat{k}_2)$, for $\delta \geq \frac{1}{2}$, the above incentive constraint is satisfied by definition of $\overline{\alpha}_1(\delta)$.

Thus we have established that under the stated assumptions, all incentive constraints are satisfied on the path τ_2^s. Thus τ_2^s is a 2-phase perfect equilibrium security level punishment path for the small firm.

We now turn to the large firm's punishment. Consider the following path, $\tau_1^{s\prime}$, constructed in a manner similar to firm 2's punishment above. In the first period, firm 2 sets a price equal to $\underline{p}_1$ and a quantity ceiling equal to $\hat{k}_2$. Firm 1 sets p_1^s satisfying

$$(1-\delta)(p_1^s - c)\hat{k}_1 + \delta(1-c)\underline{\alpha}_1(\delta) = \underline{\pi}_1, \tag{28}$$

and $0 \leq p_1^s \leq \underline{p}_2$, and a quantity ceiling equal to $\hat{k}_1$. We show below that such a price p_1^s exists. From the second period on, firms revert to the stationary path $\{(1, 1, \underline{\alpha}_1(\delta), \overline{\alpha}_2(\delta))\}$. We assume that firm 1's deviations are punished by restarting $\tau_1^{s\prime}$ in the first period and that firm 2's deviations are punished by reverting to τ_2^s.

We show below that if $\hat{k}_2 \geq \max\left\{\frac{\hat{k}_1}{1+\delta}, \Psi(k_1, \delta)\right\}$, then $\tau_1^{s\prime}$ is a 2-phase perfect equilibrium security level punishment path for firm 1.

We begin by deriving a condition under which there exists a unique price p_1^s satisfying (28) and $0 \leq p_1^s < \underline{p}_2$. First, since $\hat{k}_1 > 0$, if such a price p_1^s exists, it is unique. Second, it is simple to check that if $c \geq \hat{c}$, p_1^s solving (28) satisfies $p_1^s \geq 0$. Finally, straightforward computations yield that p_1^s solving (28) also satisfies $p_1^s \leq \underline{p}_2$ if and only if $\hat{k}_2 \geq \hat{k}_1/(1+\delta)$, with equality in one of these inequalities if and only if there is equality in the other.

We now show that firm 2's incentive constraint is satisfied in the first period of $\tau_1^{s\prime}$. Since firm 2's deviations are punished by reverting to τ_2^s, firm 2's incentive constraint in the first period of $\tau_1^{s\prime}$ is:

$$(1-\delta)(\underline{p}_1 - c)(M - \hat{k}_1) + \delta(1-c)\overline{\alpha}_2(\delta) \geq \underline{\pi}_2, \tag{29}$$

which, after solving for $\hat{k}_1$, is equivalent to:

$$\hat{k}_1 \geq M - \min\left\{\hat{k}_2, \frac{\delta^2 M}{1-\delta(1-\delta)}\right\}. \tag{30}$$

If the minimum in the bracketed expression in (30) is obtained at $\hat{k}_2$, the inequality holds from the assumption $k_1 + k_2 > M$. If the minimum in the bracketed expression in (30) is not obtained at $\hat{k}_2$, the inequality holds since by assumption $\hat{k}_1 \geq \hat{k}_2 \geq \Psi(k_1, \delta)$. Furthermore, firm 1's constraint in the first period is given by (28) and is therefore satisfied by definition of p_1^s.

Finally, we show that on $\tau_1^{s\prime}$, both firms' incentive constraints are satisfied from period 2 on. Since firm 2's deviations are punished by restarting τ_2^s, firm 2's incentive constraint in any period of the stationary path $\{(1, 1, \underline{\alpha}_1(\delta), \overline{\alpha}_2(\delta))\}$ is:

$$(1-c)\overline{\alpha}_2(\delta) \geq (1-\delta)(1-c)\hat{k}_2 + \delta\underline{\pi}_2.$$

Since $\underline{\pi}_2 = (1-c)(M - \hat{k}_1)$, for $\delta \geq \frac{1}{2}$, the above incentive constraint is satisfied by definition of $\overline{\alpha}_2(\delta)$.

Since firm 1's deviations are punished by restarting $\tau_1^{s\prime}$ in the first period, firm 1's incentive constraint in any period of the stationary path $\{(1, 1, \underline{\alpha}_1(\delta), \overline{\alpha}_2(\delta))\}$ is:

$$(1-c)\underline{\alpha}_1(\delta) \geq (1-\delta)(1-c)\hat{k}_1 + \delta\underline{\pi}_1.$$

Since $\underline{\pi}_1 = (1-c)(M - \hat{k}_2)$, for $\delta \geq \frac{1}{2}$, the above incentive constraint is satisfied by definition of $\overline{\alpha}_1(\delta)$.

Thus if $\hat{k}_2 \geq \max\left\{\frac{\hat{k}_1}{1+\delta}, \Psi(k_1, \delta)\right\}$, we have constructed a perfect equilibrium punishment path on which the large firm obtains its security level.

To complete the proof that under the assumptions of the proposition, there exists a 2-phase perfect equilibrium punishment that drives the large firm down to its security level, suppose $\hat{k}_2 \in \left[\Psi(k_1, \delta), \frac{\hat{k}_1}{1+\delta}\right]$.

We will show that the following path $\tau_1^{s\prime\prime}$ is a 2-phase perfect equilibrium security level punishment path for the large firm. In the first period, firm 1 sets a price equal to $\underline{p}_2$ and a quantity ceiling equal to $\hat{k}_1$ (but sells to residual demand only). From the second period on, firm 1 sets a price equal to 1 and a quantity ceiling equal to $q_1^s \in [\underline{\alpha}_1(\delta), \overline{\alpha}_1(\delta)]$, where q_1^s is defined below. In the first period, firm 2 sets a price p_2^p satisfying

$$(1-\delta)(p_2^p - c)\hat{k}_2 + \delta(1-c)(M - q_1^s) \geq \underline{\pi}_2, \tag{31}$$

and $0 \leq p_2^p < \underline{p}_2$, and a quantity ceiling equal to $\hat{k}_2$. From the second period on, firm 2 sets a price equal to 1 and a quantity ceiling equal to $M - q_1^s$. Firm 1's deviations are punished by restarting $\tau_1^{s\prime\prime}$ in the first period and firm 2's deviations are punished by reverting to τ_2^s.

To demonstrate that the path $\tau_1^{s\prime\prime}$ satisfies the required conditions, we first show that there exists a quantity ceiling q_1^s such that if on $\tau_1^{s\prime\prime}$, firm 1 sells to residual demand at $\underline{p}_2$ in the first period, and from the second period on, firms revert to the stationary path $\{(1, 1, q_1^s, M - q_1^s)\}$, firm 1 obtains the payoff $\underline{\pi}_1$. From the argument following (28), note first that if $\hat{k}_2 = \frac{\hat{k}_1}{1+\delta}$,

$$(1-\delta)(\underline{p}_2 - c)\hat{k}_1 + \delta(1-c)\underline{\alpha}_1 = \underline{\pi}_1.$$

It follows that if $\hat{k}_2 \leq \frac{\hat{k}_1}{1+\delta}$:

$$(1-\delta)(\underline{p}_2 - c)(M - \hat{k}_2) + \delta(1-c)\underline{\alpha}_1 \leq \underline{\pi}_1. \tag{32}$$

with equality if and only if $\hat{k}_2 = \frac{\hat{k}_1}{1+\delta}$ and $\underline{p}_2 = c$. Moreover, by definition of $\Psi(k_1, \delta)$, we have

$$(1-\delta)(\underline{p}_2 - c)(M - \hat{k}_2) + \delta(1-c)\overline{\alpha}_1 \geq \underline{\pi}_1,$$

Thus it follows from the strict monotonicity and continuity of the left-hand side of (32) in firm 1's sales in the second phase, that there exists $q_1^s \in [\underline{\alpha}_1(\delta), \overline{\alpha}_1(\delta)]$ satisfying

$$(1-\delta)(\underline{p}_2 - c)(M - \hat{k}_2) + \delta(1-c)q_1^s = \underline{\pi}_1.$$

This is the required quantity ceiling.

Now we show that there exists p_2^p satisfying (31) and $0 \leq p_2^p \leq \underline{p}_2$. Let p_2^r be the unique solution to

$$(1-\delta)(p_2 - c)\hat{k}_2 + \delta(1-c)(M - q_1^s) = \underline{\pi}_2,$$

By definition p_2^r satisfies (31) and since $q_1^s < \hat{k}_1$ and $(\underline{p}_2 - c)\hat{k}_2 \equiv (1-c)(M - \hat{k}_1) = \underline{\pi}_2$, we have $p_2^r < \underline{p}_2$. Thus letting $p_2^p = p_2^r$, p_2^p satisfies (31) and $p_2^p < \underline{p}_2$. If p_2^p so defined satisfies $p_2^p \geq 0$, we are finished. If not, then $p_2^r < 0$. It then follows from the definition of p_2^r and the fact that $\hat{k}_2$ is strictly greater than zero that:

$$(1-\delta)(0 - c)\hat{k}_2 + \delta(1-c)(M - q_1^s) > \underline{\pi}_2.$$

Hence, it follows from $0 < \hat{c} \leq c \leq \underline{p}_2$, that setting $p_2^p = 0$, p_2^p satisfies (31) and $0 \leq p_2^p < \underline{p}_2$. Therefore we have shown that there exists a price p_2^p satisfying (31) and $0 \leq p_2^p < \underline{p}_2$.

Finally, we show that on $\tau_1^{s\prime\prime}$, both firms' incentive constraints are satisfied from period 2 on. Since firm 2's deviations are punished by reverting to τ_2^s, firm 2's incentive constraint in any period of the stationary path $\{(1, 1, q_1^s, M - q_1^s)\}$ is:

$$(1-c)(M - q_1^s) \geq (1-\delta)(1-c)\hat{k}_2 + \delta(1-c)(M - \hat{k}_1).$$

To show that the above constraint is satisfied, note that since $q_1^s \leq \overline{\alpha}_1(\delta)$, $M - q_1^s \geq M - \overline{\alpha}_1(\delta) = \underline{\alpha}_2(\delta)$. Thus

$$(1-c)(M-q_1^s) \geq (1-c)\underline{\alpha}_2(\delta) = (1-\delta)(1-c)\hat{k}_2 + \delta(1-c)(M-\hat{k}_1),$$

where the last inequality follows from the definition of $\underline{\alpha}_2(\delta)$. Therefore, firm 2's incentive constraint is satisfied.

Since firm 1's deviations are punished by restarting $\tau_1^{s\prime\prime}$ in the first period, firm 1's incentive constraint in any period of the stationary path $\{(1, 1, q_1^s, M - q_1^s)\}$ is:

$$(1-c)q_1^s \geq (1-\delta)(1-c)\hat{k}_1 + \delta(1-c)(M-\hat{k}_2).$$

To show that the above constraint is satisfied, note that since $q_1^s \geq \underline{\alpha}_1(\delta)$:

$$(1-c)q_1^s \geq (1-c)\underline{\alpha}_1(\delta) = (1-\delta)(1-c)\hat{k}_1 + \delta(1-c)(M-\hat{k}_2),$$

where the last inequality follows from the definition of $\underline{\alpha}_1(\delta)$. Therefore, firm 1's incentive constraint is satisfied.

Hence, we have shown that all incentive constraints hold by construction in every period of the path $\tau_1^{s\prime\prime}$. Therefore $\tau_1^{s\prime\prime}$ is a pure strategy 2-phase perfect equilibrium security level punishment path for firm 1.

Letting $\tau_1^s = \tau_1^{s\prime}$ if $\hat{k}_2 \geq \max\left\{\frac{\hat{k}_1}{1+\delta}, \Psi(k_1,\delta)\right\}$ and $\tau_1^s = \tau_1^{s\prime\prime}$ if $\hat{k}_2 \in \left[\Psi(k_1,\delta), \frac{\hat{k}_1}{1+\delta}\right]$, we have shown that under the stated assumptions, there exists a pure strategy 2-phase perfect equilibrium security level punishment path τ_1^s for the large firm.

Thus, under the assumptions of the proposition, there exist pure strategy perfect equilibrium punishment paths for both firms. □

8.5 Proof of Proposition 6

The proof is by construction. We construct a single perfect equilibrium path τ^s on which both firms obtain their security level. We construct τ^s in the following way: in the first period, firm 1 sets a price p_1^s satisfying $0 \leq p_1^s \leq \underline{p}_2$ and offers its capacity. From the second period on, firm 1 sets its price equal to 1 and offers q_1^s, where q_1^s satisfies:

$$(1-\delta)(p_1^s - c)\hat{k}_1 + \delta(1-c)q_1^s = (1-c)(M-k_2) = \underline{\pi}_1. \tag{33}$$

and $q_1^s \in [\underline{\alpha}_1(\delta), \overline{\alpha}_1(\delta)]$. We show below that such p_1^s and q_1^s exist.

In the first period of τ^s, firm 2 sets a price p_2^s, and offers its capacity. From the second period on, firm 2 sets its price equal to 1 and offers $q_2^s = M - q_1^s$. Suppose $k_1 < M$ and let p_2^s be given by

$$(1-\delta)(p_2^s - c)(M-\hat{k}_1) + \delta(1-c)q_2^s = (1-c)(M-\hat{k}_1) = \underline{\pi}_2, \tag{34}$$

and $p_1^s < p_2^s \leq \underline{p}_1$. We show below that such p_2^s exists. Assume that unilateral deviations from τ^s are punished by restarting τ^s in the first period. We now show that τ^s is a perfect equilibrium path.

First, define p_1^u to be the unique solution in p_1 to:

$$(1-\delta)(p_1 - c)\hat{k}_1 + \delta(1-c)\overline{\alpha}_1(\delta) = \underline{\pi}_1.$$

The assumption $k_2 < \Psi(k_1, \delta)$, (26) and (27) imply that $p_1^u \geq 0$.

Second, let $k_1 < M$, and define p_2^u to be the unique solution in p_2 to:

$$(1-\delta)(p_2 - c)(M - \hat{k}_1) + \delta(1-c)(M - \overline{\alpha}_1(\delta)) = \underline{\pi}_2. \tag{35}$$

Substituting for the value of $\overline{\alpha}_1(\delta)$ in both equations and solving for p_1^u and p_2^u, we obtain after some computations:

$$p_1^u < p_2^u \iff k_1 < [1 - \delta(1-\delta)]M < M.$$

Next we show that $p_1^u \leq \underline{p}_2$ and $p_2^u \leq \underline{p}_1$. Straightforward computations yield:

$$p_1^u \leq \underline{p}_2 \iff k_2 \geq \frac{(1-\delta)M}{1-\delta(1-\delta)},$$

and

$$p_2^u \leq \underline{p}_1 \iff M > k_1 \geq \frac{M}{1+\delta}. \tag{36}$$

Thus, for $[1-\delta(1-\delta)]M > k_1 \geq M/(1+\delta)$ and $\Psi(k_1, \delta) > k_2 \geq (1-\delta)M/[1-\delta(1-\delta)]$, let $p_1^s = p_1^u$, $p_2^s = p_2^u$ and $q_1^s = \overline{\alpha}_1(\delta)$. From the properties of p_1^s and p_2^s, we obtain that each firm's incentive constraint is satisfied in period 1. Furthermore, using arguments similar to those used for the second phase of paths described in Proposition 5, $\delta \geq \frac{1}{2}$ and the definition of $\overline{\alpha}_1(\delta)$ imply that both firms' incentive constraints are satisfied from period 2 on. Thus for the set of capacity pairs described above, τ^s is a security level perfect equilibrium path for both firms.

To complete the proof of the proposition, suppose $k_1 < M/(1+\delta)$ in addition to the assumptions of the proposition. To construct τ^s for such capacity pairs, first let $\hat{q}_2$ be the solution in q_2 to the following equation:

$$(1-\delta)(\underline{p}_1 - c)(M - k_1) + \delta(1-c)q_2 = \underline{\pi}_2.$$

Since $k_1 < M/(1+\delta)$ by assumption, it follows from (35) and (36) that

$$(1-\delta)(\underline{p}_1 - c)(M - k_1) + \delta(1-c)(M - \overline{\alpha}_1(\delta)) < \underline{\pi}_2,$$

and from $k_1 \geq (1-\delta)M/[1-\delta(1-\delta)]$, (29) and (30), it follows that:

$$(1-\delta)(\underline{p}_1 - c)(M - k_1) + \delta(1-c)\overline{\alpha}_2(\delta) \geq \underline{\pi}_2.$$

Since $M - \overline{\alpha}_1(\delta) = \underline{\alpha}_2(\delta)$, we have that $\hat{q}_2 \in [\underline{\alpha}_2(\delta), \overline{\alpha}_2(\delta)]$. Solving for $\hat{q}_2$ explicitly, we obtain

$$\hat{q}_2 = (M - k_1)\left[\frac{k_1 - (1-\delta)(M - k_2)}{\delta k_1}\right].$$

Moreover, tedious but straightforward computations show that the following holds

$$(1-\delta)(\underline{p}_2 - c)k_1 + \delta(1-c)(M - \hat{q}_2) \geq \underline{\pi}_1$$

if and only if

$$k_2 \geq \frac{(1-\delta)\hat{k}_1^2}{\hat{k}_1 - (1-\delta)(M - \hat{k}_1)}.$$

Hence, from the assumptions in the statement of Proposition 6 and the above argument, if p_1' satisfies

$$(1-\delta)(p_1' - c)k_1 + \delta(1-c)(M - \hat{q}_2) = \underline{\pi}_1,$$

then p_1' is such that $0 \leq p_1^{'} \leq \underline{p}_2$. Therefore, if $\Psi(k_1, \delta) > k_2 \geq [(1-\delta)\hat{k}_1^2]/[\hat{k}_1 - (1-\delta)(M - \hat{k}_1)]$ and $M/(1+\delta) > k_1 \geq (1-\delta)M/[(1-\delta(1-\delta)]$ hold, letting $p_1^s = p_1'$, $p_2^s = \underline{p}_1$ and $q_1^s = M - \hat{q}_2$, the triple (p_1^s, p_2^s, q_1^s) satisfies all the required properties. For both firms, incentive constraints are satisfied in period 1. Moreover, from $\delta \geq \frac{1}{2}$ and $q_1^s \in [\underline{\alpha}_1(\delta), \overline{\alpha}_1(\delta)]$, we obtain that incentive constraints are also satisfied from period 2 on.

Therefore, letting $\tau_i^s = \tau^s$ for $i = 1, 2$, we have shown that under the assumptions of the proposition, there exists a perfect equilibrium security level punishment path τ_i^s for $i = 1, 2$. $\square$

8.6 Capacity Utilization

In this section of the Appendix, we characterize the pattern of capacity utilization, $U_i \equiv \frac{s_i}{k_i}$, $i = 1, 2$, on constrained Pareto optimal collusive SPEP's.

Lemma 2 *Suppose A1 holds. Let τ^s be a SPEP satisfying (PO). For every δ such that $\delta \geq \frac{1}{2}$, the small firm has a higher capacity utilization than the large firm on τ^s if and only if $s_2 \geq \frac{k_2 M}{k_1 + k_2}$.*

Proof. From Proposition 2, we have that if τ^s satisfies (PO), then on τ^s, $s_1 + s_2 = M$. Therefore, $U_1 \leq U_2$ if and only if $\frac{M - s_2}{k_1} \leq \frac{s_2}{k_2}$. Rearranging this inequality yields $s_2 \geq \frac{k_2 M}{k_1 + k_2}$, which proves the lemma. $\square$

Lemma 3 *Suppose A1 holds. For every δ such that $\delta \geq \tilde{\delta}$,*

(i) On the SPEP $\overline{\tau}_1(\delta)$, $U_1 < U_2$ if and only if one of the following conditions holds:
$k_1 \geq k_2 \geq M$ *and* $\delta < \frac{k_1}{k_1+k_2}$,
$k_2 < M$ *and* $k_1 > M + \frac{k_2^2}{M-k_2}$,
$k_2 \in [\frac{1}{2}M, M)$, $k_1 \in [2M - k_2, M + \frac{k_2^2}{M-k_2}]$ *and* $\delta < 1 - \frac{M}{k_1+k_2}$,

(ii) On the SPEP $\overline{\tau}_2(\delta)$, $U_2 < U_1$ if and only if one of the following conditions holds:
$k_2 < M$, $k_1 < M - \frac{(M-k_2)^2}{M+k_2}$ *and* $\delta < \frac{\hat{k}_1}{k_1+k_2}$,
$k_2 < M$, $k_1 \geq M - \frac{(M-k_2)^2}{M+k_2}$ *and* $\delta \in (\delta^c, \frac{\hat{k}_1}{k_1+k_2})$,

where $\delta^c \equiv \left(\frac{k_1-k_2}{k_1+k_2}\right)\left(\frac{M}{M-k_2}\right)$.

Proof. From Lemma 1 and Proposition 2, we have $\overline{\tau}_1(\delta) = \{(1, 1, \overline{\alpha}_1(\delta), \underline{\alpha}_2(\delta))\}$ if $\tilde{\delta} = \frac{1}{2}$ and $\frac{1}{2} \leq \delta \leq \frac{k_2}{\hat{k}_1}$, $\overline{\tau}_1(\delta) = \{(1, 1, M - \alpha_2^N, \alpha_2^N)\}$ if $\delta \geq \frac{k_2}{\hat{k}_1}$. If $\tilde{\delta} = 1 - \frac{k_2}{\hat{k}_1}$, $\overline{\tau}_1(\delta) = \{(1, 1, M - \alpha_2^N, \alpha_2^N)\}$ for all $\delta \geq \tilde{\delta}$.

To prove the first statement in (i), note that $\hat{k}_1 = \hat{k}_2 = M$ implies $\tilde{\delta} = \frac{1}{2}$ and $\frac{k_2}{\hat{k}_1} = 1$. Therefore $\tau_1(\delta) = \{(1, 1, \overline{\alpha}_1(\delta), \underline{\alpha}_2(\delta)\}$ and $s_2 = \underline{\alpha}_2(\delta)$. Using the definition of $\underline{\alpha}_2(\delta)$ yields $\underline{\alpha}_2(\delta) = (1-\delta)M = s_2 > \frac{k_2 M}{k_1+k_2}$ if and only if $\delta < \frac{k_1}{k_1+k_2} (> \frac{1}{2})$. The first part of (i) then follows from Lemma 2.

We now prove the second statement in (i). First note that since $\overline{\tau}_1(\delta)$ satisfies (C), for every $\delta \geq \tilde{\delta}$, on $\overline{\tau}_1(\delta)$, $s_2 \geq \alpha_2^N$. Straightforward computations yield

$$\alpha_2^N > \frac{k_2 M}{k_1 + k_2} \iff k_1 > M + \frac{k_2^2}{M - k_2} > M.$$

Thus if $k_1 > M + \frac{k_2^2}{M-k_2}$, the fact that $s_2 \geq \alpha_2^N$, the above equation and Lemma 2 imply that on $\overline{\tau}_1(\delta)$, $U_1 < U_2$. This proves the second statement.

To prove the third statement, assume $M \leq k_1 \leq M + \frac{k_2^2}{M-k_2}$. From the second statement, we obtain that if $\delta \geq \frac{k_2}{\hat{k}_1} = \frac{k_2}{M}$, then $U_1 \geq U_2$ since in this case, $s_2 = \alpha_2^N$. Therefore, if $U_1 < U_2$ occurs on $\overline{\tau}_1(\delta)$ for the capacity pairs described in the third statement, it must be the case that $s_2 = \underline{\alpha}_2(\delta)$. If $k_2 < \frac{1}{2}M$, then $\tilde{\delta} = 1 - \frac{k_2}{\hat{k}_1}$, thus, on $\overline{\tau}_1(\delta)$, for every $\delta \geq \tilde{\delta}$, $s_2 = \alpha_2^N$. It then follows immediately that $U_1 \geq U_2$. To complete the proof of the statement, assume $k_2 \geq \frac{1}{2}M$. Straightforward computations yield

$$\underline{\alpha}_2(\delta) > \frac{k_2 M}{k_1 + k_2} \iff \delta < \frac{k_2(k_1 + k_2 - M)}{(\hat{k}_1 + k_2 - M)(k_1 + k_2)}. \tag{37}$$

Substituting for $\hat{k}_1 = M$ in (37) and using Lemma 2, we obtain that $U_1 < U_2$ on $\overline{\tau}_1(\delta)$ if and only if $\tilde{\delta} \leq \delta < 1 - \frac{M}{k_1+k_2}$ holds. We now find conditions under which $1 - \frac{M}{k_1+k_2} > \tilde{\delta}$. Since $k_2 \geq \frac{1}{2}M$, $\tilde{\delta} = \frac{1}{2}$. We have $1 - \frac{M}{k_1+k_2} > \frac{1}{2} = \tilde{\delta}$ if and only if $k_1 > 2M - k_2 > M$. Thus, on $\overline{\tau}_1(\delta)$, if $M > k_2 \geq \frac{1}{2}M$, $U_1 < U_2$ if and only if $k_1 > 2M - k_2$ and $\tilde{\delta} \leq \delta < 1 - \frac{M}{k_1+k_2}$ hold. This completes the proof of the third statement of (i).

We now prove (ii). To this effect, we consider two cases depending on the value of δ: $\delta \geq \delta^r$ and $\tilde{\delta} \leq \delta < \delta^r$. First, if $\delta \geq \delta^r$, it follows from Propositions 2 and 3 that $\overline{\tau}_2(\delta) = \{(1, 1, \underline{\alpha}_1(\delta), \overline{\alpha}_2(\delta))\}$. Using the definition of $\overline{\alpha}_2(\delta)$, straightforward computations yield:

$$\overline{\alpha}_2(\delta) < \frac{k_2 M}{k_1 + k_2} \iff \delta < \frac{\hat{k}_1(k_1 + k_2) - k_1 M}{(\hat{k}_1 + k_2 - M)(k_1 + k_2)}. \tag{38}$$

The second inequality in (38) reduces to $\delta < \frac{M}{k_1+k_2} = \frac{\hat{k}_1}{k_1+k_2}$ if $\hat{k}_1 = M$ and $\delta < \frac{k_1}{k_1+k_2} = \frac{\hat{k}_1}{k_1+k_2}$ if $\hat{k}_1 < M$. Using Proposition 3, if $\hat{k}_1 \leq \frac{M+\hat{k}_2}{2}$, then $\delta \geq \tilde{\delta}$ implies $\delta \geq \delta^r$. Straightforward computations yield $\frac{k_1}{k_1+k_2} (> \tilde{\delta})$ if and only if $k_1 > M - k_2$, which holds by assumption. Therefore, it follows from the above argument that if (a) $\hat{k}_1 \leq \frac{M+\hat{k}_2}{2}$, then on $\tau_2(\delta)$, $U_2 < U_1$ if and only if $\delta < \frac{k_1}{k_1+k_2} (> \tilde{\delta})$ holds.

Now assume $\hat{k}_1 > \frac{M+\hat{k}_2}{2}$. From Proposition 3, $\delta^r > \tilde{\delta}$ holds. If $\delta \geq \delta^r$, from Propositions 2 and 3, $\overline{\tau}_2(\delta) = \{(1, 1, \underline{\alpha}_1(\delta), \overline{\alpha}_2(\delta))\}$. Hence, arguments developed above for the case $\hat{k}_1 \leq \frac{M+\hat{k}_2}{2}$ apply. Furthermore, straightforward computations yield that $\delta^r < \frac{k_1}{k_1+k_2}$ if and only if $k_1 > M - k_2$, which holds by assumption. It follows that if (b) $\hat{k}_1 > \frac{M+\hat{k}_2}{2}$ and $\delta \geq \delta^r$, then on $\overline{\tau}_2(\delta)$, $U_2 < U_1$ if and only if $\delta < \frac{k_1}{k_1+k_2}$ holds.

Finally, consider the case $\tilde{\delta} \leq \delta < \delta^r$. From Propositions 2 and 3, $\overline{\tau}_2(\delta) = \{(1, p_2^*, q_1^c, \overline{\alpha_2}(p_2^*))\}$. Therefore on $\overline{\tau}_2(\delta)$, $s_2 = \overline{\alpha}_2(p_2^*)$ and $s_1 = M - \overline{\alpha}_2(p_2^*)$. Using the definitions for $\overline{\alpha}_2(p_2^*)$ and p_2^* yields $s_2 = \frac{M-\delta(M-k_2)}{2}$. We have:

$$\frac{M - \delta(M - k_2)}{2} < \frac{k_2 M}{k_1 + k_2} \iff \delta > \left(\frac{k_1 - k_2}{k_1 + k_2}\right)\left(\frac{M}{M - k_2}\right) = \delta^c. \tag{39}$$

If $\delta^c < \tilde{\delta}$, we are finished. We derive conditions under which $\delta^c \geq \tilde{\delta}$ holds. First, straightforward computations show that $1 - \frac{k_2}{\hat{k}_1} \leq \delta^c < \delta^r$ for every capacity pair. Thus, if $\tilde{\delta} = 1 - \frac{k_2}{\hat{k}_1} > \frac{1}{2}$, $\delta^c \geq \tilde{\delta}$ holds. Second, suppose $\tilde{\delta} = \frac{1}{2} \geq 1 - \frac{k_2}{\hat{k}_1}$ holds. Note that since $\hat{k}_1 > \frac{M}{2}$, this implies $k_2 > \frac{1}{3}M$. Moreover, $\delta^c \geq \frac{1}{2}$ if and only if:

$$k_1 \geq \max\{M - \frac{(M - k_2)^2}{M + k_2}, M - k_2\}.$$

Straightforward computations yield $\max\{M - \frac{(M-k_2)^2}{M+k_2}, M - k_2\} = M - \frac{(M-k_2)^2}{M+k_2}$ if and only if $k_2 \geq \frac{1}{3}M$. Therefore, we have shown that if (c) $k_1 > \frac{M+\hat{k}_2}{2}$ and $\tilde{\delta} \leq \delta < \delta^r$, then on $\overline{\tau}_2(\delta)$, $U_2 < U_1$ if and only if either $k_1 < M - \frac{(M-k_2)^2}{M+k_2}$ or $k_1 \geq M - \frac{(M-k_2)^2}{M+k_2}$ and $\delta \in (\delta^c, \delta^r)$ hold. Finally, it is simple to show that for $k_1 > M - k_2$, which holds by assumption, $M - \frac{(M-k_2)^2}{M+k_2} > \frac{M+k_2}{2}$.

Therefore, if $k_1 < M - \frac{(M-k_2)^2}{M+k_2}$, it follows from (a), (b) and (c) that on $\overline{\tau}_2(\delta)$, $U_2 < U_1$ if and only if $\delta < \frac{k_1}{k_1+k_2}$. Finally, from (b) and (c), it follows that on $\overline{\tau}_2(\delta)$, if $k_1 \geq M - \frac{(M-k_2)^2}{M+k_2}$, $U_2 < U_1$ if and only if $\delta \in (\delta^c, \frac{k_1}{k_1+k_2})$. This completes the proof of (ii). $\square$

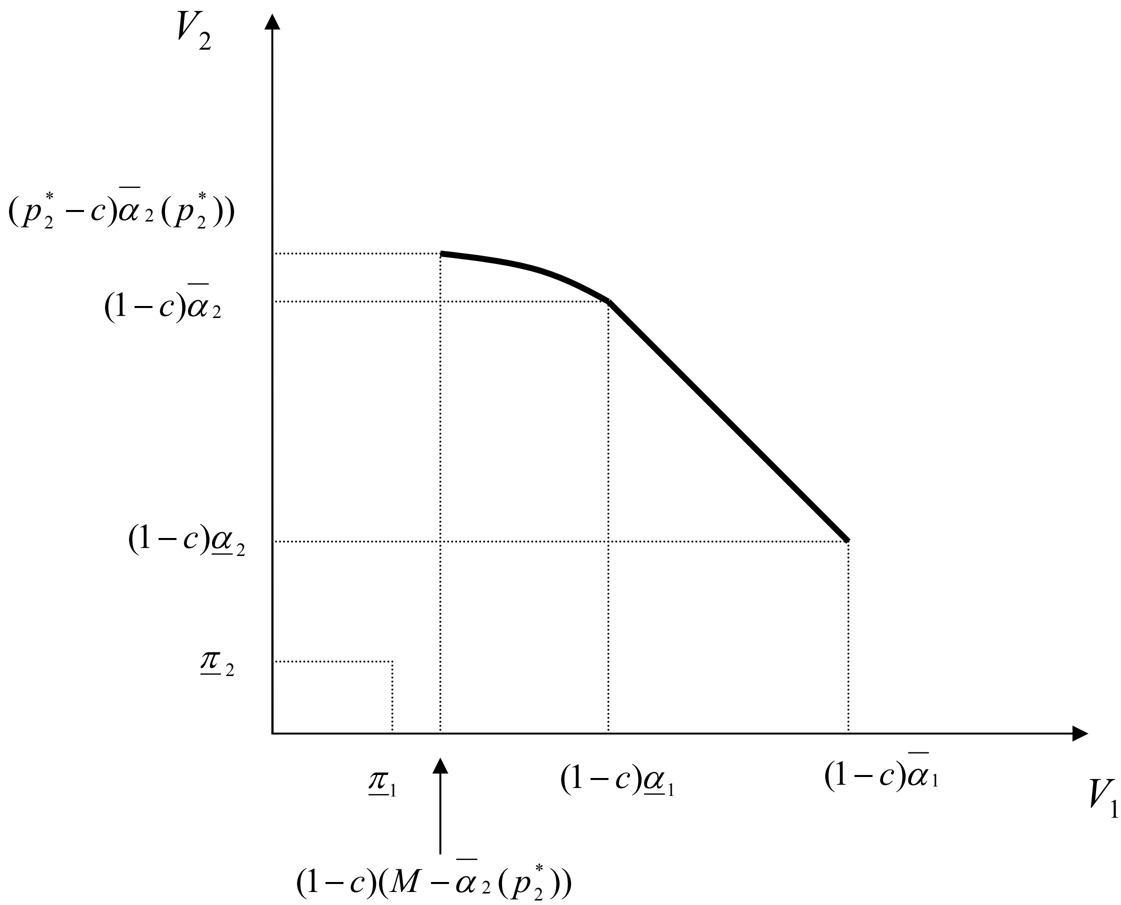

Figure 1: Frontier of payoffs attainable on constrained Pareto optimal SPEP for some $\delta \in [\tilde{\delta}, \delta^r)$ and $\frac{M+\hat{k}_2}{2} < \hat{k}_1 \leq 2\hat{k}_2$.

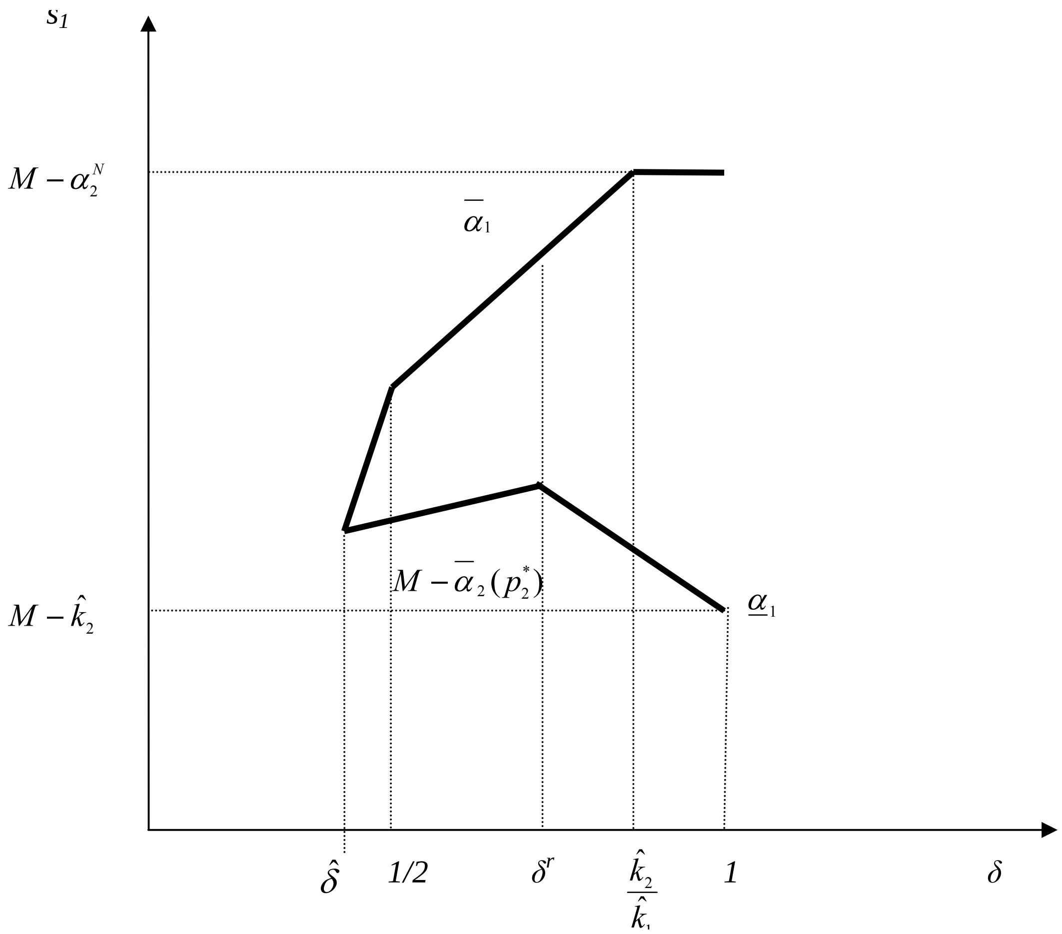

Figure 2: Firm 1's range of sales on constrained Pareto optimal SPEP for different values of δ and $\frac{M+\hat{k}_2}{2} < \hat{k}_1 \leq 2\hat{k}_2$.