

# KRANNERT GRADUATE SCHOOL OF MANAGEMENT

Purdue University  
West Lafayette, Indiana

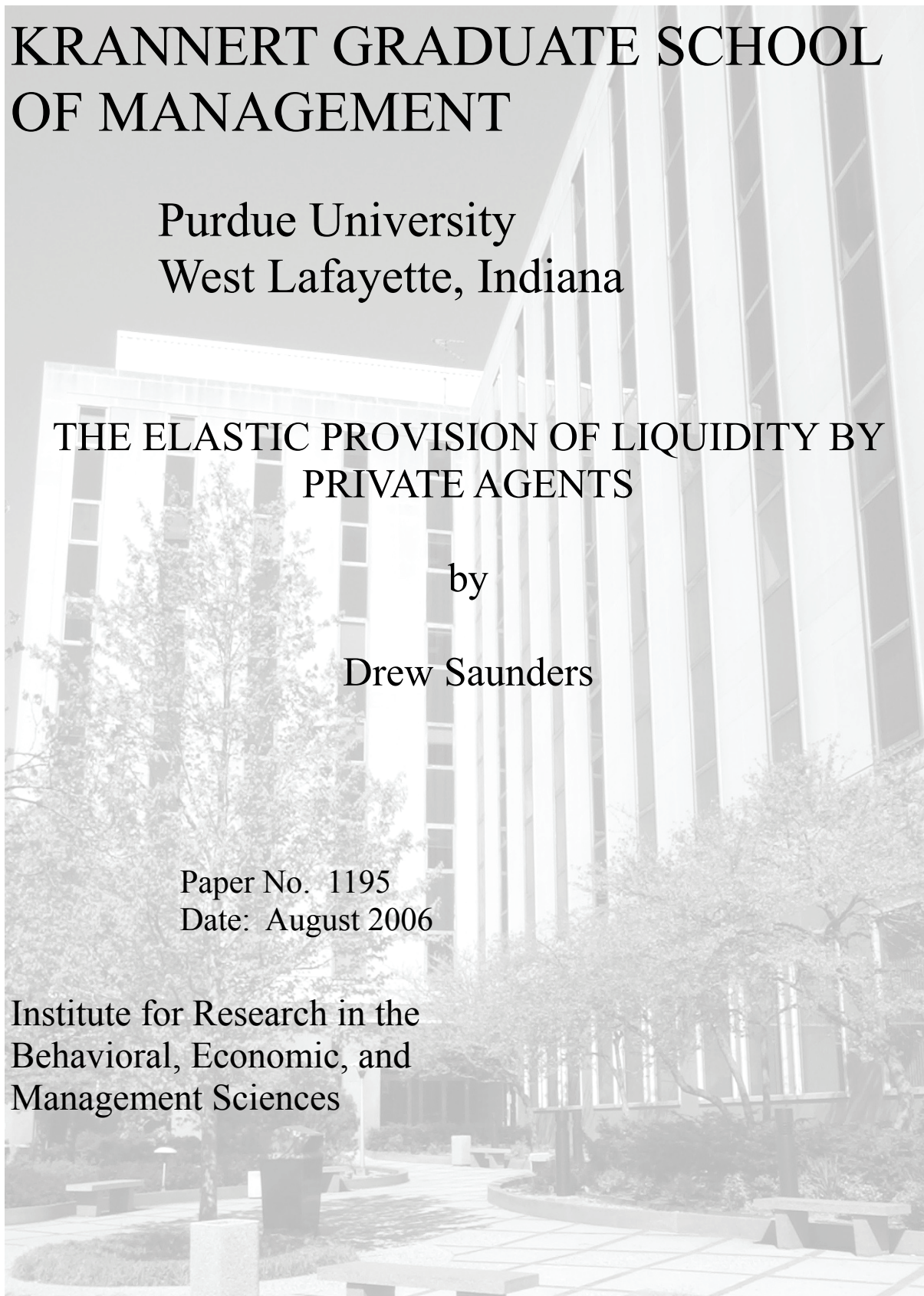
## THE ELASTIC PROVISION OF LIQUIDITY BY PRIVATE AGENTS

by

Drew Saunders

Paper No. 1195  
Date: August 2006

Institute for Research in the  
Behavioral, Economic, and  
Management Sciences



# The Elastic Provision of Liquidity by Private Agents

Drew Saunders<sup>\*,†</sup>

Purdue University

August 2006

## Abstract

I study a model of entrepreneurial investment in which investment projects are heterogeneous with respect to their exposure to an aggregate liquidity shock. A firm that is affected by the shock will mitigate its exposure by purchasing claims issued by a firm that is not. Liabilities of the unaffected firm may earn a liquidity premium due to their fungibility; and, because they are backed by productive investment, their supply is elastic to the demand. This segmentation implies that an aggregate liquidity shock has different consequences across sectors. The unaffected firm plays a role like that of a bank by supplying liquidity to other firms; this mechanism recalls the “real bills” doctrine of classical monetary theory.

*JEL Classification:* E44, E51, E22.

*Keywords:* Liquidity, money supply elasticity.

---

\*I have benefited enormously from the help and support of Andres Almazan, Dean Corbae, Scott Freeman, and Bruce Smith. Thanks are due also to seminar participants at Purdue University, the University of Texas Departments of Economics and Finance, and the 2003 Midwest Macroeconomics Conference. All errors are my own. This essay was previously circulated under the title “Endogenous Liquidity Provision”.

†Please address correspondence to the author at saunders@mgmt.purdue.edu.

# 1 Introduction

The fact that corporations maintain substantial holdings of liquid securities may be explained by the existence of a wedge between the cost of internal funds and the cost of external funds, and the need of these firms to be responsive to investment opportunities or cost shocks.<sup>1</sup> The theoretical foundations for a connection between liquidity and the efficiency of firms' investment policies was established in Jensen and Meckling (1976), Myers (1977), and Myers and Majluf (1984); and some consequences for the macroeconomy have been explored by Holmstrom and Tirole (1998, 2001) and Kiyotaki and Moore (1997, 2005). The latter groups of authors investigate the effects of aggregate tightness in the market for liquid claims on investment by firms that require liquidity to invest efficiently. These models abstract from consideration of an effect on the investment of firms positioned to *issue* such claims. In this paper, I construct a general equilibrium model of liquidity constrained investment incorporating a non-trivial endogenous supply of liquid liabilities by private agents. The theory suggests an important link between the properties of available investment projects and the distribution of those projects, and the demand and supply in the market for liquidity. In particular, I show how the liquidity supply function inherits the elasticity properties of the investment of a certain class of producers. The model also admits a surprising explanation for liquidation of investment projects in an environment of liquidity crisis under perfect information.

In Holmström and Tirole (1998) and Kiyotaki and Moore (2005), liquidity is defined as the means by which wealth can be stored intertemporally and accessed readily. In each of these papers, two factors induce an entrepreneur to hold this kind of security. First, moral hazard at the time of production (in the future) generates the wedge between the private rate of return to investment and the rate of return demanded by the market. Second, the entrepreneur's project faces the possibility of a liquidity shock (a cost shock or an random investment opportunity) at a time when its cash flow is low. Kiyotaki and Moore (2005) show that a

---

<sup>1</sup>See Opler et al. (1999) for empirical evidence on firms' holdings of liquid securities supporting this view.

fungible productive asset in fixed supply can earn an extra-fundamental “liquidity premium”, a price higher than that consistent with its marginal product, and circulate like money. In their model, land serves a collateral function in addition to its role as an input to production. Here the scarcity of land is an exogenous property, and its price is determined endogenously. In Holmström and Tirole (1998), the liquidity need may be met by a government asset that is supplied perfectly elastically at a price fixed exogenously. They show that the asset may be demanded even if the price is higher than the fundamental one.

I adopt salient features of the model of Holmström and Tirole (1998) into my model, but I depart from them by requiring that liquidity be supplied endogenously; there is no government and no land in my model. Instead, a second producer (whom I call the “banker”) is introduced who is capable of issuing securities, backed by his project dividends, that can be used by the entrepreneur to mitigate his liquidity needs. Notice that the liquid securities are backed by investment, and to the extent that the banker’s investment depends on the market rate of interest, his issue of liquid liabilities will do so as well. This phenomenon induces an upward-sloping liquidity supply curve, which contrasts the vertical one in Kiyotaki and Moore (2005) and the horizontal one in Holmström and Tirole (1998).

Somewhat surprisingly, it is possible for liquidation of the banker’s assets to occur in equilibrium. This is because I assume that the rate of return wedge described above affects also the banker, except when the banker terminates his project early. That is, the moral hazard problem associated with management of the banker’s project may be circumvented by liquidation. Therefore, even though the net return on liquidated investment may be low, it may be that the banker can promise to pay more to outside stakeholders when his project is liquidated than when it is allowed to mature. This property engenders an interesting gambling behavior wherein the banker terminates his investment in a state in which the liquidity need of the entrepreneur is particularly acute.

With respect to the interpretation of the transactions role of the liabilities of firms, my model is related also to that of Gorton and Pennacchi (1990). They show that riskless

private liabilities offer transactions services to Diamond-Dybvig-style liquidity traders at an informational disadvantage with respect to the quality of other securities in the market. The instruments used in these transactions are interpreted alternatively as corporate debt or bank deposits. Because the scale of investment projects of firms that issue liquid claims is fixed exogenously, however, Gorton and Pennacchi (1990) are unable to investigate the elasticity of the supply of liquidity in their model.

The position that the supply and availability of private liabilities affects the allocation of real resources can be supported empirically. First, it is known that broad measures of money incorporating interest-bearing private marketable liabilities are more highly correlated with output than the monetary base or M1.<sup>2</sup> This fact led Friedman and Schwartz (1963) to focus on M2 in their famous study of the connexion between money and the real economy. Secondly, Friedman and Kuttner (1993) have found that the six-month commercial paper rate is more informative with respect to movements of output than the rate of interest on the three-month T-bill, and other researchers have verified that this finding is quite robust. At a minimum, this evidence suggests that the liabilities of the government do not determine the financial environment independent of the positions and capabilities of other participants.

The mechanism for liquidity provision in my paper in the model brings to mind the “real bills doctrine” that private financial instruments backed by appropriate assets should be allowed to supplement other media of exchange.<sup>3</sup> This doctrine holds that creation and circulation of notes backed by the proceeds of commerce impart a beneficial elasticity to the supply of liquidity. The present model contributes to the understanding of the elasticity of the transactions medium by showing how, when contracting frictions exist, a “shortage” of liabilities suitable for circulation may arise. A scarcity of the transactions medium cannot exist in the monetary models of Sargent and Wallace (1982) and Champ, Smith and Williamson (1996), for example, because these models abstract from commitment frictions that might disqualify certain assets from serving as security for a note.

---

<sup>2</sup>See, for example, Cooley and Hansen (1995).

<sup>3</sup>See Mints (1945) for a discussion of the real bills doctrine in the history of banking theory.

The rest of the paper is structured as follows. In the next section, I introduce the formal model and assumptions. In the third section, I conduct the formal analysis of the model. The fourth section discusses my findings. The last section summarizes and concludes.

## 2 Environment

### 2.1 Time, Preferences, and Endowments

The economy is inhabited by three agents, whom I label as the banker, the entrepreneur, and the worker. I index these agents by  $i \in \{b, e, w\}$ . There are three dates  $t \in \{0, 1, 2\}$ .

There is a single good in each period in the economy. The good is useful for consumption at any date, and the dates 0 and 1 goods are useful also for investment in production projects as discussed below. The date  $t$  good perishes if it is not consumed or invested by the end of that date. At date 0, the banker and the entrepreneur have endowments  $\omega^b$  and  $\omega^e$ , respectively. The worker has no endowment of the good at date 0, and no agent is endowed with goods at dates 1 or 2.

The worker has an unlimited quantity of labor at dates 0 and 1 that may be converted one-for-one into the contemporaneous good. Producing goods in this manner has a disutility for the worker equivalent to one unit of consumption. The worker's labor is inalienable, so that a promise from the worker to provide labor in the future can be reneged with impunity.<sup>4</sup>

The banker and the entrepreneur each have a production project that can be used to produce date 2 goods subject to a pattern of investment of goods at date 0 and 1. Projects are discussed in detail in the next subsection.

All agents are risk neutral. The banker and the entrepreneur evaluate outcomes according to the sum of non-negative consumption at the three dates. The worker evaluates outcomes according to the sum of consumption at each date minus labor expended. At each date, agents act in order to maximize the expectation of their payoff at the current and future

---

<sup>4</sup>I provide a precise mathematical representation for this concept below.

date(s). Agents do not discount the future.

## 2.2 Production Projects

Production in the economy is affected by the realization of a random variable that is observed at date 1. The random variable takes on the value  $H$  with probability  $\phi$ , and  $L$  with probability  $1 - \phi$ . For convenience of notation, I will sometimes write  $s \in S = \{H, L\}$  for a generic outcome, and I write  $\phi_s$  for the probability that the outcome is  $s$ . I will refer to a pairing of the date and the realization as the state; that is, state  $1s$  indicates date 1 when the realized outcome is  $s$ . Abusing the notation, I may refer to the state at date 0 as “state 0”.

I first describe the project of the entrepreneur. At date 0, the entrepreneur chooses investment level  $I^e \in \mathbb{R}_+$ . At date 1, the project may need additional investment of goods in order to continue. Precisely, if the outcome from the random variable at date 1 is  $H$ , then the project requires additional investment of  $(1 - \lambda_H^e)I^e\rho$ , where  $\lambda_s^e \in [0, 1]$  is the fraction of his project that is discontinued in state  $1s$ , and  $\rho > 0$ .<sup>5</sup> For simplicity, I assume that no additional investment is required in state  $L$ . Discontinuance or “liquidation” of the entrepreneur’s project yields no residual, so that the fraction that is liquidated is simply lost. At the beginning of date 2, the entrepreneur has an opportunity to abscond with a booty of  $(1 - \lambda_s^e)I^e\gamma^e$  from the project and consume it, in which case he leaves the remnants of the project valueless. If he chooses rather to allow the project to mature, then it yields an amount  $(1 - \lambda_s^e)I^eR^e$  to be divided according to agreements reached by the entrepreneur with other agents. Note that the other agents have no recourse against the entrepreneur when he absconds, but claims against the yield of a project are enforceable once the entrepreneur has chosen to allow it to mature.

The project available to the banker is similar to that of the entrepreneur, but never requires additional investment at date 1. At date 0, the banker chooses an investment level

---

<sup>5</sup>In order to establish a more generalized and convenient notation, I will sometimes write  $\rho_L := 0$  and  $\rho_H := \rho$ .

$I^b \in \mathbb{R}_+$ . I allow for liquidation of a fraction  $\lambda_s^b \in [0, 1]$  of the banker's project in state  $1s$ . Different from the entrepreneur's project, there is a positive yield  $\lambda_s^b I^b L$  available from the liquidated portion of the banker's project in state  $1s$ , where  $L < 1$ . Like the entrepreneur, the banker can abscond with some amount  $(1 - \lambda_s^b) I^b \gamma^b$  at the beginning of date 2, scuttling the remainder of the project without recourse by other agents. If he chooses instead to allow the project to mature, then the project yields a dividend  $(1 - \lambda_s^b) I^b R^b$  to be divided according to any agreement arranged at previous dates.

**Assumption 1.** I assume (i) that the mature yield of each project is greater than the private value obtained by the entrepreneur or the banker from absconding,  $R^j > \gamma^j$  for  $j \in \{e, b\}$ ; and (ii) that the net surplus available from investment in each project is positive in expectation,

$$R^e - \phi\rho - 1 > 0 \text{ and } R^b - 1 > 0.$$

## 2.3 Market Institutions

I assume that, at each date  $t \in \{0, 1\}$ , a competitive market exists for state-contingent claims to goods at date  $t + 1$ . I write  $q_{1s}$  for the date 0 price of claims to goods in state  $1s$ , and I write  $q_{2s}$  for the state  $1s$  price of claims to goods in state  $2s$ . I denote the consumption of agent  $i$  in state  $\sigma$  by  $c_\sigma^i$ , and I denote the net claims to state  $ts$  consumption held by agent  $i$  at the end of date  $t - 1$  by  $B_{ts}^i$ . Since the worker's utility depends only on his consumption net of his labor supply in each period, I will describe his actions only up to this net supply of labor at dates 0 and 1 denoted  $x_0 := n_0 - c_0^w$  and  $x_{1s} := n_{1s} - c_{1s}^w$ , respectively, where  $n_\sigma$  his gross amount of labor supplied in state  $\sigma$ .

### 2.3.1 The Worker's Problem

The worker takes present and future claims prices as given, and chooses net claims holdings, and net labor supply to maximize his expected lifetime consumption. The worker's choices are subject to a sequence of budget constraints, as well as participation constraints



reflecting the inalienability of his labor. Formulated mathematically, the worker chooses  $(x, c_2^w, B^w)$  to maximize

$$-x_0 + \sum_{s \in S} \phi_s (-x_{1s} + c_{2s}^w). \quad (1)$$

subject to the budget constraints

$$-x_0 + \sum_{s \in S} q_{1s} B_{1s}^w \leq 0 \quad (2)$$

$$-x_{1s} + q_{2s} B_{2s} \leq B_{1s}^w \text{ for each } s \in S \quad (3)$$

$$c_{2s}^w \leq B_{2s}^w \text{ for each } s \in S, \quad (4)$$

and the participation constraints

$$-x_{1s} + c_{2s}^w \geq 0 \text{ for each } s \in S. \quad (5)$$

The constraints (2) and (3) require that the net purchases of the worker in the dates 0 and 1 financial markets are no greater than his wealth in the appropriate state. The wealth of the worker at date 0 is zero, and he affords financial asset purchases only by working more than he consumes. Similarly, his wealth in state 1s is given by his accumulation of assets from the previous period  $B_{1s}^w$ . The date 2 budget constraints (4) show that the worker consumes no more in state 2s than afforded by his accumulation of claims  $B_{2s}^w$ . The last set of constraints (5) reflect the fact that the worker is free to renege on any agreement to provide labor that does not benefit him ex post; this is the mathematical manifestation of inalienable labor.

### 2.3.2 The Entrepreneur's Problem

The entrepreneur takes claims prices as given and chooses net claims holdings, non-negative consumption and investment, and a project liquidation rule  $(c^e, I^e, \lambda^e, B^e)$  to max-

imize his expected lifetime payoff,

$$c_0^e + \sum_s \phi_s (c_{1s}^e + c_{2s}^e). \quad (6)$$

The entrepreneur faces the budget constraints

$$c_0^e + I^e + \sum_{s \in S} q_{1s} B_{1s}^e \leq \omega^e \quad (7)$$

$$c_{1s}^e + I^e (1 - \lambda_s^e) \rho_s + q_{2s} B_{2s}^e \leq B_{1s}^e \text{ for each } s \in S \quad (8)$$

$$c_{2s}^e \leq B_{2s}^e + I^e (1 - \lambda_s^e) R^e \text{ for each } s \in S. \quad (9)$$

The date 0 budget constraint (7) says that the entrepreneur can apply no more than his endowment to date 0 consumption, investment, and purchase of financial claims. The date 1 budget constraints (8) say that accumulated claims must be used to fund consumption, additional investment, and the portfolio to be held at date 2. Finally, (9) says that date 2 consumption must be funded out of accumulated claims and project dividends.

The last budget constraint is valid under the assumption that the entrepreneur does not abscond with the project dividend. Since the entrepreneur can always achieve consumption of  $(1 - \lambda_s^e) I^e \gamma^e$  at date 2 by his choice to abscond, an additional constraint applies to the choice problem. To see this, suppose that the entrepreneur holds a negative claims position  $B_{2s}^e < -I^e (1 - \lambda_s^e) (R^e - \gamma^e) < 0$  for some state at date 2. Then (9) implies that  $c_{2s}^e < (1 - \lambda_s^e) I^e \gamma^e$ , and the entrepreneur can achieve a higher payoff by absconding. Under perfect information, no creditor would buy a quantity of claims from the entrepreneur that would induce him to abscond at date 2. Equivalently, it must be the optimal policy of the entrepreneur satisfies the *incentive compatibility constraints*

$$c_{2s}^e \geq (1 - \lambda_s^e) I^e \gamma^e \text{ for each } s \in S. \quad (10)$$

The problem of the entrepreneur may now be stated as that of maximizing (6) subject

to (7)-(10).

The quantity  $\tilde{R}^e := R^e - \gamma^e$ , which is positive by Assumption 1, plays an important role in the analysis to follow. The moral hazard problem described above creates a wedge between the internal rate of return available to the entrepreneur through investment, and the share that can be pledged to outsiders. As a result,  $\tilde{R}^e$  is the *marketable share* of the entrepreneur's project, the maximal amount that can credibly be pledged to outside stakeholders per unit of investment.

### 2.3.3 The Banker's Problem

The problem of the banker is very similar to that of the entrepreneur. The banker takes prices as given and chooses net claims holdings, non-negative consumption and investment, and a project liquidation rule  $(c^b, I^b, \lambda^b, B^b)$  to maximize the objective function

$$c_0^b + \sum_{s \in S} \phi_s (c_{1s}^b + c_{2s}^b) \quad (11)$$

subject to the budget constraints

$$c_0^b + I^b + \sum_{s \in S} q_{1s} B_{1s}^b \leq \omega^b \quad (12)$$

$$c_{1s}^b + q_{2s} B_{2s}^b \leq B_{1s}^b + I^b \lambda_s^b L \text{ for each } s \in S \quad (13)$$

$$c_{2s}^b \leq B_{2s}^b + I^b (1 - \lambda_s^b) R^b \text{ for each } s \in S, \quad (14)$$

and the incentive compatibility constraints

$$c_{2s}^b \geq I^b (1 - \lambda_s^b) \gamma^b \text{ for each } s \in S. \quad (15)$$

The constraints (12)-(14) are to be interpreted analogously to (7)-(9) for the entrepreneur, with the principle exception that the banker has no need of additional funding at date 1, and instead may obtain goods at that date by liquidating a portion of his project. The incentive

compatibility constraint (15) is derived and interpreted analogously to (10).

The *marketable share* of the banker's project,  $\tilde{R}^b := R^b - \gamma^b$ , plays a role analogous to that of the entrepreneur's project as discussed in the previous subsection. The following assumption implies that, for sufficiently large investment  $I^e$  (respectively,  $I^b$ ), the entrepreneur (banker) will be unable to credibly promise to repay  $I^e - \omega^e$  ( $I^b - \omega^b$ ) to outside creditors, so that the entrepreneur (banker) will be unable to finance an arbitrarily high amount of investment.

**Assumption 2.** The marketable share of the entrepreneur satisfies

$$(1 - \phi) \lambda_L^e \tilde{R}^e + \phi \lambda_H^e (\tilde{R}^e - \rho) < 1$$

for all liquidation rules  $\lambda^e \in [0, 1]$ ; and the marketable share of the banker is less than one,  $\tilde{R}^b < 1$ .

### 2.3.4 Definition of Equilibrium

An *equilibrium* is an allocation  $\{c, n, I, \lambda, B, q\}$  of consumption, labor, investment, liquidation policies, net assets holdings, and asset prices such that (i) each agent's problem is solved; and (ii) the markets for contingent claims to consumption clear at dates 0 and 1, i.e., for each  $t \in \{1, 2\}$  and  $s \in \{H, L\}$ ,  $\sum_i B_{ts}^i = 0$ .<sup>6</sup>

It is easy to see that each agent's marginal rate of substitution of consumption at date 0 for consumption in state 1s is  $\phi_s$ , and that of consumption in state 1s for consumption in state 2s is 1. Therefore, I will refer to the price system defined by  $q_{1s} = \phi_s$  and  $q_{2s} = 1$  for each  $s$  as the *fundamental* one. As will be seen in the next section, this price system need not support an equilibrium.

The following assumption implies that the liquidity shock to the entrepreneur's project

---

<sup>6</sup>The focus of the analysis to follow will be on the issue and purchase of claims by agents in the model. Thus, I have defined an equilibrium in terms of these markets, rather than those for goods. As usual, imposition of goods market clearing conditions in the definition of equilibrium instead of those for the claims markets would lead to an equivalent concept.

may induce a liquidity problem in the sense that the entrepreneur will need to hoard assets in order to continue his project in the bad state.

**Assumption 3.** The marketable share of the entrepreneur is less than the additional investment required in state  $1H$ ,  $\tilde{R}^e - \rho < 0$ .

## 3 Analysis

### 3.1 The Role of the Worker

Inspecting the worker's problem, it is immediate that a finite solution exists only if  $q_{1s} \geq \phi_s$  and  $q_{2s} \geq 1$ . These properties are a manifestation of the worker's infinite capacity to purchase assets that yield a positive return. For example, if it were true that  $q_{1s} < \phi_s$ , then the worker could profit by converting an arbitrarily large amount of date 0 labor into date 0 goods, and trading the date 0 goods for claims to state  $1s$  consumption. Since none of his constraints are violated by making  $x_0$  arbitrarily positive while setting  $x_{1s} = -x_0/q_{1s}$ , and since the net contribution of the scheme to his lifetime expected payoff is  $\phi_{1s}/q_{1s} - 1 > 0$  per unit of net labor expended, the worker's problem has no solution and there can be no equilibrium.

On the other hand, there is no symmetric argument available that shows that  $q_{1s}$  cannot be higher than  $\phi_s$  in equilibrium. To see why, use the first-order condition for  $c_{2s}^w$  to derive the complementary slackness condition  $q_{2s}c_{2s}^w = c_{2s}^w$ , and note that this implies (from (3) and (4)) that

$$B_{1s}^w = -x_{1s} + c_{2s}^w.$$

Now the participation constraints (5) imply that  $B_{1s}^w \geq 0$  for each  $s$ . Thus, whereas the worker can *buy* assets to take advantage of a *low* price of date 1 consumption, his inability to commit to supply future labor prohibits his *issuing* assets to take advantage of *high* prices. The following proposition summarizes and extends these points. All of the proofs in

the paper are relegated to the Appendix.

**Proposition 1** *If  $q$  is an equilibrium price system, then  $q_{1s} \geq \phi_s$  and  $q_{2s} \geq 1$  for each  $s$ . Furthermore, there is an optimal policy for the worker with claims holdings  $B^w$  if and only if the following hold for each  $s$ :*

- (i)  $B_{1s}^w \geq 0$  with equality if  $q_{1s} > \phi_s$ ; and
- (ii)  $B_{2s}^w \geq 0$  with equality if  $q_{2s} > 1$ .

### 3.2 A Simple Special Case

In this subsection, I analyze the special case of the model in which the probability that the additional investment by the entrepreneur will be necessary is one; that is  $\phi = 1$ . This case focuses attention immediately on the date 0 market of state  $1H$  claims, which will remain central to the analysis of the model more generally. In this setting, no agent will be induced to liquidate a project, since (under perfect foresight) he could always do better by simply reducing the scale of his investment ex ante. I take advantage of these simplifications by imposing that  $\lambda_H^b = \lambda_H^e = 0$  a priori, and by ignoring reference to the outcome of the random process where there can be no confusion. For example, I will write  $q_1$  rather than  $q_{1H}$ , and I will write  $c_1^b$  instead of  $c_{1H}^b$ . I do this in the present subsection only.

From the analysis of the previous subsection, it can be seen that the market return on two-period saving is less than the return available to the banker on funds invested internally. That is,  $R \leq 1 < R^b$ , where  $R := (q_1 q_2)^{-1}$  is the two-period interest rate, and the second inequality follows from Assumption 1. In this environment, the banker will find it advantageous to borrow as much as possible against project proceeds and apply all of his marketable lifetime wealth toward investment.<sup>7</sup> This means that his incentive compatibility constraint will bind, and the budget constraints can be solved for the optimal investment of the banker,

$$I^b = \frac{\omega^b}{1 - q_1 q_2 \tilde{R}^b}.$$

---

<sup>7</sup>These results are stated formally and proved for the general case in Lemma 1 below.

To focus on the pattern of claims holdings adopted by the banker under equilibrium prices, use the budget constraints once more to derive that<sup>8</sup>

$$B_1^b = -q_2 I^b \tilde{R}^b = -\frac{q_2 \omega^b \tilde{R}^b}{1 - q_1 q_2 \tilde{R}^b} \quad (16)$$

and

$$B_2^b = -I^b \tilde{R}^b = -\frac{\omega^b \tilde{R}^b}{1 - q_1 q_2 \tilde{R}^b}. \quad (17)$$

Interpretation is facilitated by defining the expression  $\tilde{R}^b \left(1 - \tilde{R}^b/R\right)^{-1}$  as the *leverage ratio* of the banker. These equations show that, as the market interest rate, the rate demanded by lenders, is reduced, the banker will issue more and more liabilities, the amount tending to infinity as the market rate approaches the marketable share of the banker. Thus, any finite demand for liquid liabilities can be met by the supply of the banker for some  $R > \tilde{R}^b$ .

In contrast to the role of the banker in the model, the entrepreneur's need of liquidity at date 2 implies that he will be a net buyer of claims at date 0. Since, as shown above, the price of these claims may be higher than the fundamental price, the need of liquidity may impinge upon the profitability of the entrepreneur's project in a non-fundamental way.

If the fundamental prices hold, then the entrepreneur will desire to invest as much in his project as possible. This case leads him to behave as the banker does, consuming nothing at dates 0 and 1, and taking the maximal amount of leverage against project proceeds by setting date 2 consumption so that his incentive compatibility constraint binds. On the other hand, if the price of future claims rises above fundamentals, then the entrepreneur will desire to transfer as much of his future consumption to date 0 as possible for any given level of investment. The principal difference between the two cases is that it may be optimal for the entrepreneur to consume his endowment (rather than invest) at date 0 if the price of liquidity is sufficiently high.

---

<sup>8</sup>The hypothesis of "equilibrium prices" implies that the denominator in the expressions on the right-hand sides of (16) and (17) must be positive. To see this, notice that the banker could finance unlimited consumption if  $1 - q_1 q_2 \tilde{R}^b$  were negative.

These arguments imply that  $c_1^e = 0$  and  $c_2^e = I^e \gamma^e$ , and now the dates 1 and 2 budget constraints can be used to write optimal claims holdings of the entrepreneur as

$$B_1^e = I^e \left( \rho - q_2 \tilde{R}^e \right) \quad (18)$$

and

$$B_2^e = -I^e \tilde{R}^e. \quad (19)$$

Now from (17) and (19) it can be seen that  $q_2 = 1$  in equilibrium; for if not, then Proposition 1 implies that  $B_2^w = 0$ , and  $\sum_i B_2^i < 0$  contradicts the market clearing condition. In meeting the liquidity need, the entrepreneur sells at date 1 the securities he purchases at date 0 for capital goods. Thereafter, the worker holds all of the outstanding liabilities of the other two agents, a position he will only accept if the price of claims is consistent with fundamentals.

On the other hand, because the entrepreneur is a net buyer of claims to date 1 goods (that is,  $B_1^e \geq 0$ ), the market for such claims may not clear at the fundamental price. This can be seen more easily by looking at a reduced form of the entrepreneur's problem. Solving the entrepreneur's budget constraints for date 0 consumption by eliminating his claims holdings and substituting the result into his objective function, the problem becomes that of choosing  $I^e$  to maximize

$$I^e \left[ \gamma^e - q_1 \left( \rho - \tilde{R}^e \right) - 1 \right] + \omega^e$$

subject to

$$I^e \leq \frac{\omega^e}{1 + q_1 \left( \rho - \tilde{R}^e \right)},$$

where the constraint inheres from the non-negativity of date 0 consumption. If the expression in square brackets in the objective function is positive, then each unit of investment increases the payoff of the entrepreneur, and he will invest the maximal amount allowed by



the constraint. But this expression is non-positive for  $q_1 \geq \hat{q}$ , where

$$\hat{q} := \frac{\gamma^e - 1}{\rho - \tilde{R}^e} > 1,$$

and the inequality follows from Assumption 1 after some algebra. The next proposition is the important corollary of this discussion.

**Proposition 2** *Suppose that additional investment by the entrepreneur is required with probability one. At equilibrium prices  $q$ , there is an optimal policy for the entrepreneur with claims holdings  $B^e$  if and only if*

$$B_1^e = \frac{\omega^e (\rho - \tilde{R}^e)}{1 + q_1 (\rho - \tilde{R}^e)} \chi$$

and

$$B_2^e = -\frac{\omega^e \tilde{R}^e}{1 + q_1 (\rho - \tilde{R}^e)} \chi$$

for some  $\chi \in \Xi(q_1)$ , where

$$\Xi(q_1) := \begin{cases} 1, & \text{for } q_1 < \hat{q} \\ [0, 1], & \text{for } q_1 = \hat{q} \\ 0, & \text{for } q_1 > \hat{q}. \end{cases}$$

It is intuitive to think of the amount of date 1 claims *issued* by the banker at date 0 (that is, the quantity  $-B_1^b$ ) as the *supply of liquidity*. From (16) and the fact that  $q_2 = 1$  in equilibrium, the supply of liquidity is given by

$$\frac{\omega^b \tilde{R}^b}{1 - q_1 \tilde{R}^b}.$$

The family of broken curves in Figure 1 represent the supply of liquidity for different values of the banker's date 0 endowment  $\omega^b$ . The *demand for liquidity* is the entrepreneur's holding

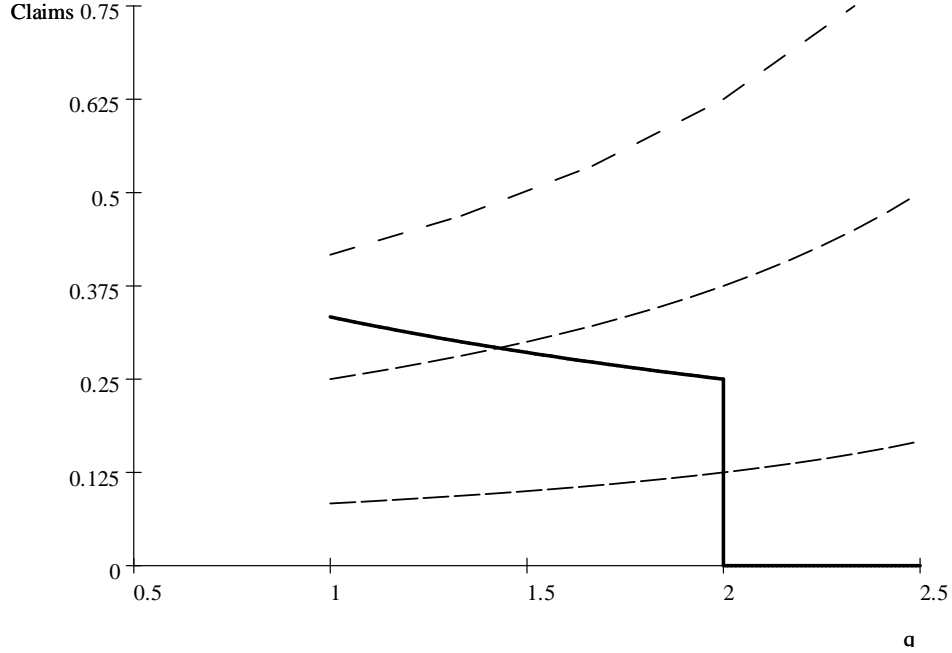


Figure 1: “Supply” and “Demand” of Liquidity for the Parameterization of Example 1

of date 1 claims given in Proposition 3; this is shown by the solid curve in the figure.<sup>9</sup>

The supply and demand of liquidity represent a convenient device for characterizing the equilibrium price of liquidity  $q_1$  as follows. First, if the curves do not intersect for some  $q_1 \geq \phi (= 1)$ , then the supply of liquidity by the banker at the fundamental price exceeds the demand of the entrepreneur; this is the case for the highest broken curve representing the largest value of  $\omega^b$  in the figure. From Proposition 1, it is clear that the worker will be able and willing to purchase excess liquid claims supplied by the banker at the fundamental price  $q_1 = 1$ ; apparently, this is the equilibrium price in this case.

Next, if the curves intersect for some price between the fundamental price and  $\hat{q}$ , as they do for the intermediate of the supply curves in the figure, then the equilibrium exhibits a liquidity price premium on liquid claims and no liquidation is required by the entrepreneur. The price premium induces the banker to borrow more than he would under fundamental prices, essentially subsidizing his investment. At the same time, the entrepreneur reduces

<sup>9</sup>The parameters used in the plot are those of the Example 1 below.

investment relative to what he would choose otherwise, because the added price of providing for the continuation of his project impinges on his ability to raise capital and invest. In this case, the price premium dissuades the worker from purchasing any of these claims, and all of the claims issued by the banker are purchased by the entrepreneur.

The third possibility is that the curves intersect at the price  $\hat{q}$ , as they do for the lowest supply curve in the figure. In this case, the entrepreneur is indifferent with respect to investment, and in equilibrium, he invests exactly the amount that can be continued at date 1 using the claims issued by the banker at this price. The investment of the entrepreneur is  $\omega^e \chi / \left[ 1 + \hat{q} (\rho - \tilde{R}^e) \right]$ , where  $\chi$  is given implicitly by

$$\frac{\omega^b \tilde{R}^b}{1 - \hat{q} \tilde{R}^b} = \frac{\omega^e (\rho - \tilde{R}^e)}{1 + \hat{q} (\rho - \tilde{R}^e)} \chi,$$

the equation of the liquidity supplied by the banker to the demand of the entrepreneur.

**Example 1.** The parameters used in the plot are  $R^b = \frac{5}{4}$ ,  $\gamma^b = 1$ ,  $R^e = 3$ ,  $\gamma^e = 2$ , and  $\rho = \frac{3}{2}$ . One can calculate that  $\tilde{R}^b = \frac{1}{4}$ ,  $\tilde{R}^e = 1$ , and  $\hat{q} = 2$ . The endowment of the entrepreneur is  $\omega^e = 1$ , and the endowments of the banker for the three curves are  $\omega^b = \frac{5}{4}, \frac{3}{4}, \frac{1}{4}$ . ■

### 3.3 More General Cases

In this subsection, we explore the new possibilities that exist when the liquidity shock is a random event. For simplicity, I will restrict the analysis by assuming that the probability of the liquidity shock is low enough that the entrepreneur's project may be profitably undertaken even when it will be fully liquidated in the bad state. Formally, I assume in the remainder of the text that

$$(1 - \phi) R^e - 1 > 0. \tag{20}$$

This assumption is not necessary, but it simplifies the exposition without significant conceptual loss.<sup>10</sup>

With respect to investment at date 0, the motivation of the banker is little changed in the general environment from that of the special case considered above. The restrictions derived on equilibrium prices in subsection 3.1 imply that the private rate of return available from investing will always exceed the market rate of interest, and the banker behaves optimally by maximizing the size of his investment. The following lemma contains the formal statement of this fact.

**Lemma 1** *At equilibrium prices, the banker will choose to consume nothing at dates 0 and 1, and he will choose date 2 consumption so that his incentive compatibility constraint is binding in each state; that is,  $c_0^b = 0$ , and  $c_{1s}^b = 0$  and  $c_{2s}^b = (1 - \lambda_s^b) I^b \gamma^b$  for each  $s$ .*

Under the assumption (20), we have eliminated the possibility that the entrepreneur may be indifferent about investment in equilibrium. The next lemma shows that the entrepreneur invests all he can in the present case.

**Lemma 2** *Suppose that (20) holds. At equilibrium prices, the entrepreneur will choose to consume nothing at dates 0 and 1, and he will choose date 2 consumption so that his incentive compatibility constraint is binding in each state; that is,  $c_0^e = 0$ , and  $c_{1s}^e = 0$  and  $c_{2s}^e = (1 - \lambda_s^e) I^e \gamma^e$  for each  $s$ .*

At an optimum under equilibrium prices  $q$ , Lemma 1 and the binding budget constraints of the banker's problem can be seen to yield

$$I^b = \frac{\omega^b}{1 - \sum_{s \in S} q_{1s} \left[ \lambda_s^b L + q_{2s} (1 - \lambda_s^b) \tilde{R}^b \right]}. \quad (21)$$

---

<sup>10</sup>From the previous subsection, one can get the flavor of the case excluded here. When  $\phi$  is high (i.e., when (20) is violated) and as the price of liquidity rises, the entrepreneur will prefer to reduce his a priori investment rather than liquidate his project after it is begun. In contrast, as will be seen, when (20) holds, he will invest as much as possible and liquidate if necessary. In terms of the market for liquidity, the two modes of behavior are qualitatively similar, since the liquid claims desired by the entrepreneur will be proportional to the product of his investment and the fraction of the project that he will continue,  $I^e (1 - \lambda_H^e)$ .

It is easy to see that the denominator of this expression must be positive at equilibrium prices for all feasible liquidation rules of the banker. If this were not the case, inspection of the problem shows that a feasible policy exists that gives the banker any arbitrarily large payoff. Intuitively, the liquid claims issued by the banker will be in proportion to his payoff, so that any finite demand for these claims can be met by the supply of the banker at a price lower than that affording him infinite payoff.

Now using Lemma 1 and eliminating investment using (21), some algebra shows that the optimal liquidation rule for the banker maximizes

$$\gamma^b \left\{ \frac{\omega^b \sum_s \phi_s (1 - \lambda_s^b)}{1 - \sum_s q_{1s} [\lambda_s^b L + q_{2s} (1 - \lambda_s^b) \tilde{R}^b]} \right\}. \quad (22)$$

This objective function has a simple interpretation. The factor  $\gamma^b$  is the banker's date 2 consumption per unit of residual (i.e., un-liquidated) investment at that date. The expression in braces is the expectation over states of the residual investment, which can usefully be dissected further as follows. The factor

$$\kappa^b := \left\{ 1 - \sum_s q_{1s} [\lambda_s^b L + q_{2s} (1 - \lambda_s^b) \tilde{R}^b] \right\}^{-1}$$

is the maximal ratio of investment to the banker's initial endowment that is afforded by the incentive compatibility constraints, which is the leverage ratio of the banker. Therefore  $\omega^b \kappa^b$  is the investment of the banker. The last factor,  $\sum_s \phi_s (1 - \lambda_s^b)$ , is the unconditional expectation of the fraction of the banker's project that will not be liquidated before date 2 under the chosen policy. Obviously, the banker gets no utility from liquidating a portion of his project.

Under condition (20), the same logic can be applied to the case of the entrepreneur to

show that his investment is<sup>11</sup>

$$I^e = \frac{\omega^e}{1 - \sum_s q_{1s} (1 - \lambda_s^e) (q_{2s} \tilde{R}^e - \rho_s)}, \quad (23)$$

and his liquidation decision boils down to choosing  $\lambda^e \in [0, 1]^2$  to maximize

$$\gamma^e \left\{ \frac{\omega^e \sum_s \phi_s (1 - \lambda_s^e)}{1 - \sum_s q_{1s} (1 - \lambda_s^e) (q_{2s} \tilde{R}^e - \rho_s)} \right\}.$$

The latter admits an interpretation similar to that for (22).

Before completing the characterization of optimal policies, it is useful to make the following observation.

**Corollary 1** *At equilibrium prices, it must be that  $q_{1L} = 1 - \phi$  and  $q_{2s} = 1$  for each  $s$ .*

The proof in the Appendix proceeds by showing that the extra-fundamental prices in any market except that for state  $1H$  claims must either violate market clearing or the result of Proposition 1. In particular, in each market except that for claims in state  $1H$ , both the banker and the entrepreneur will be net issuers (negative claims holders). Therefore, the relevant market clearing conditions imply that the claims holdings of the worker must be positive for these markets, and then Proposition 1 shows that the fundamental prices must prevail.

It is now possible to characterize the equilibrium with reference only to the market for liquidity (that is, state  $1H$  claims) by pinning down the price of liquidity  $q_{1H}$ . For the banker, one has the following.

**Lemma 3** *Suppose that  $q$  is an equilibrium price system.*

1. *If  $\tilde{R}^b \geq L$  then the banker will never liquidate his project in any state in equilibrium.*

---

<sup>11</sup>Analogous to the case of the banker, in order for the entrepreneur's problem to have a solution, the denominator of the expression on the RHS must be positive for all  $\lambda^e \in [0, 1]^2$ ; therefore, equilibrium prices must have this property.

2. If  $\tilde{R}^b < L$ , then, in equilibrium, the banker will never liquidate his project in the good state, but liquidation will occur in the bad state if the price of state 1H claims is high enough. More precisely, the optimal liquidation policy for the banker has  $\lambda_L^b = 0$  and  $\lambda_H^b \in \Lambda^b(q_{1H})$ , where

$$\Lambda^b(q_{1H}) := \begin{cases} 0, & \text{if } q_{1H} < q^b \\ [0, 1], & \text{if } q_{1H} = q^b \\ 1, & \text{if } q_{1H} > q^b, \end{cases}$$

and

$$q^b := \frac{1 - (1 - \phi) \tilde{R}^b}{L - (1 - \phi) \tilde{R}^b} \phi.$$

It may be surprising that the banker would ever liquidate his project in this environment, since he can always choose his investment and structure his claims portfolio so that it is not necessary to do so. The key obviously lies in the condition that the marketable share of the banker's project must be less than its liquidation value for liquidation to be optimal. In this case, the banker can raise more outside funds by promising to liquidate. Therefore, though the banker gets no payoff in the event, liquidation in the bad state allows him to increase his investment. If the price of liquidity is high enough, the increased investment afforded may sufficiently increase his payoff in the good state to compensate (in expectation) for the sacrifice of any payoff in the bad state. This result will be discussed at greater length in the next section.

For the entrepreneur, the optimal liquidation policy abides the following.

**Lemma 4** *Assume that condition (20) holds. At equilibrium prices, the entrepreneur will never liquidate his project in the good state, and there is a cutoff level  $\bar{q} > \phi$  of the price of liquidity such that liquidation is optimal in the bad state if and only if  $q_{1H} \geq \bar{q}$ . More*

precisely, optimality has  $\lambda_L^e = 0$  and  $\lambda_H^e \in \Lambda^e(q_{1H})$ , where

$$\Lambda^e(q_{1H}) := \begin{cases} 0, & \text{if } q_{1H} < \bar{q} \\ [0, 1], & \text{if } q_{1H} = \bar{q} \\ 1, & \text{if } q_{1H} > \bar{q}, \end{cases}$$

and

$$\bar{q} := \frac{1 - (1 - \phi) \tilde{R}^e}{(1 - \phi) (\rho - \tilde{R}^e)} \phi.$$

Writing the investment of the banker under equilibrium prices  $q$  as

$$I^b = \mathcal{I}^b(q_{1H}, \lambda^b) := \frac{\omega^b}{1 - (1 - \phi) \tilde{R}^b - q_{1H} [\lambda_H^b L + (1 - \lambda_H^b) \tilde{R}^b]},$$

the banker's claims holding can be summarized as follows.

**Proposition 3** *Suppose that  $q$  is an equilibrium price system.*

1. *The banker will be a net issuer of claims in each state; that is, his holdings of claims will be non-positive.*
2. *If the liquidation value of the banker's project is no greater than its marketable share, then his holdings of claims will be strictly negative and equal in each state; precisely, if  $L \leq \tilde{R}^b$ , then*

$$B_{ts}^b = -\mathcal{I}^b(q_{1H}, 0) \tilde{R}^b = -\frac{\omega^b \tilde{R}^b}{1 - (1 - \phi + q_{1H}) \tilde{R}^b}$$

for  $t \in \{0, 1\}$  and  $s \in \{H, L\}$ .

3. *If the liquidation value of the banker's project is greater than its marketable share, then (i) his holdings of claims will be negative in state 1s for each  $s$ , and in state 2L; but (ii) his state 2H holdings may reflect liquidation of a portion of his project and retirement of liabilities. More precisely, there is an optimal policy for the banker with*



claims holdings  $B^b$  if and only if

$$\begin{aligned} B_{1L}^b &= B_{2L}^b = -\mathcal{I}^b(q_{1H}, \lambda_H^b) \tilde{R}^b \\ B_{1H}^b &= -\mathcal{I}^b(q_{1H}, \lambda_H^b) \left[ \lambda_H^b L + (1 - \lambda_H^b) \tilde{R}^b \right] \\ B_{2H}^b &= -\mathcal{I}^b(q_{1H}, \lambda_H^b) (1 - \lambda_H^b) \tilde{R}^b, \end{aligned}$$

for some  $\lambda_H^b \in \Lambda^b(q_{1H})$ .

The following proposition characterizes the set of state  $1H$  claims holdings that can be optimal for the entrepreneur at equilibrium prices. This knowledge is all one needs to know about the behavior of the entrepreneur in order to characterize the equilibrium prices. The optimal state  $1H$  claims holdings of the entrepreneur define his demand for liquidity.

**Proposition 4** *Suppose that (20) holds. At equilibrium prices, there is an optimal policy for the entrepreneur with claims holdings  $B^e$  only if*

$$B_{1H}^e = \beta^e(q_{1H}, \lambda_H^e) := \frac{\omega^e (1 - \lambda_H^e) (\rho - \tilde{R}^e)}{1 - (1 - \phi) \tilde{R}^e + q_{1H} (1 - \lambda_H^e) (\rho - \tilde{R}^e)}$$

for some  $\lambda_H^e \in \Lambda^e(q_{1H})$ .

Now the price of liquidity, the only element of the equilibrium price system that has not been pinned down, may be characterized according to the algorithm described in the following proposition. In stating the result, I write  $\beta^b(q_{1H}, \lambda_H^b)$  for the banker's issue of claims when the equilibrium price is  $q_{1H}$  and his optimal liquidation decision specifies that he liquidate the share  $\lambda_H^b$  of his project in the bad state. Then by Proposition 3,

$$\beta^b(q_{1H}, \lambda_H^e) := \frac{\omega^b \left[ \lambda_H^b L + (1 - \lambda_H^b) \tilde{R}^b \right]}{1 - (1 - \phi) \tilde{R}^b + q_{1H} \left[ \lambda_H^b L + (1 - \lambda_H^b) \tilde{R}^b \right]}.$$

**Proposition 5** *Suppose that (20) holds.*

1. *If the banker's issue of liquid claims is at least as great as the entrepreneur's demand for them at the fundamental price, then this price supports an equilibrium without liquidation by the entrepreneur. More precisely, if  $\beta^b(\phi, 0) + \beta^e(\phi, 0) \leq 0$ , then in equilibrium  $q_{1H} = \phi$ ; there is no liquidation; and agents' state 1H claims are given by  $B_{1H}^b = \beta^b(\phi, 0)$ ,  $B_{1H}^e = \beta^e(\phi, 0)$ , and  $B_{1H}^w = -B_{1H}^b - B_{1H}^e$ .*
2. *If the banker's issue of liquid claims is less than the entrepreneur's demand for them at the fundamental price, and the marketable share of the banker's project is at least as great as its liquidation value, then the equilibrium price of liquidity will be greater than the fundamental price, and the entrepreneur may be required to liquidate a portion of his project in the bad state. Precisely, if  $\beta^b(\phi, 0) + \beta^e(\phi, 0) > 0$  and  $\tilde{R}^b \geq L$ , then  $q_{1H}$  uniquely satisfies  $\beta^b(q_{1H}, 0) + \beta^e(q_{1H}, \lambda_H^e) = 0$  with  $\lambda_H^e \in \Lambda^e(q_{1H})$ .*
3. *If the banker's issue of liquid claims is less than the entrepreneur's demand for them at the fundamental price, and the marketable share of the bankers project is less than its liquidation value, then the equilibrium price of liquidity will be greater than the fundamental price, and one or both of the producers may choose to liquidate a portion of his project in the bad state. Precisely, if  $\beta^b(\phi, 0) + \beta^e(\phi, 0) > 0$  and  $\tilde{R}^b < L$ , then  $q_{1H}$  uniquely satisfies  $\beta^b(q_{1H}, \lambda_H^b) + \beta^e(q_{1H}, \lambda_H^e) = 0$  with  $\lambda_H^b \in \Lambda^b(q_{1H})$  and  $\lambda_H^e \in \Lambda^e(q_{1H})$ .*

Two additional examples are presented in the next section.

## 4 Discussion

### 4.1 Liquidity Provision and Output Across Sectors

The liquidity value of collateral securities has been investigated in varied environments by Holmström and Tirole (1998,2001) and Kiyotaki and Moore (1998, 2005). Holmström and Tirole (1998), whose modeling devices I have essentially incorporated here, investigate (inter

alia) the utility of a government bond to ameliorate the liquidity problem. In their model, the supply of the liquid security is exogenous and perfectly elastic. In Kiyotaki and Moore (1998, 2005), collateral is in fixed (perfectly inelastic) supply. Thus, each of these papers abstracts from the topic of primary interest here, the elasticity of the supply of liquidity.

While each of these papers examines how liquidity problems affect firms that experience them, the important innovation of the present model is a theory of how liquidity may be provided by the issue of liquid securities by firms that do not. One implication is that, as long as some firms are unaffected, and as long as those firms are capable of issuing fungible securities, liquidity problems can have qualitatively different effects in different sectors. The following example illuminates the possibility that some sector may be benefited by such episodes.

**Example 2.** Let  $R^b = \frac{12}{7}$ ,  $R^e = 2$ ,  $\gamma^b = \gamma^e = \frac{8}{7}$ ,  $\rho = \frac{8}{7}$ ,  $\phi = \frac{1}{4}$ ,  $\omega^b = \frac{2}{7}$ ,  $\omega^e = \frac{5}{7}$ , and suppose that  $L \leq \frac{4}{7}$ . Thus,  $\tilde{R}^b = \frac{4}{7}$ ,  $\tilde{R}^e = \frac{6}{7}$ , and  $\rho - \tilde{R}^e = \frac{2}{7}$ . Note that (20) holds (the left-hand side equals  $\frac{1}{2}$ ), and one can calculate that  $\bar{q} = \frac{5}{12}$ .

At the fundamental prices, (21) and Lemma 3 (part 1) show that the banker would invest

$$\frac{2/7}{1 - 4/7} = \frac{2}{3}$$

in his project. From Proposition 3 (part 2), it can be seen that his investment would be financed by his issue of  $\frac{2}{3} \cdot \frac{4}{7} = 0.381$  claims for each state. At these prices, (23) and Lemma 4 show that the entrepreneur would choose to invest

$$\frac{5/7}{1 - \frac{3}{4} \left(\frac{6}{7}\right) + \frac{1}{4} \left(\frac{2}{7}\right)} = \frac{5}{3}$$

in his project. To finance his investment, Proposition 4 shows that he would like to buy the net amount  $\frac{5}{3} \cdot \frac{2}{7} = 0.476$  claims for the bad state.

In the present example, the amount of claims desired by the entrepreneur exceeds the

amount that would be issued by the banker at the fundamental prices. Since the worker is precluded from issuing claims, the market equilibrium will therefore reflect a premium price on liquid claims. Precisely, Proposition 5 (part 2) shows that the market will clear at the price  $q_{1H}$  satisfying

$$\begin{aligned} -\beta^b(q_{1H}, 0) &= \frac{\frac{2}{7} \cdot \frac{4}{7}}{1 - \frac{3}{4} \left(\frac{4}{7}\right) - q_{1H} \left(\frac{4}{7}\right)} = \frac{8}{28(1 - q_{1H})} \\ = \beta^e(q_{1H}, \lambda_H^e) &= \frac{\frac{5}{7} (1 - \lambda_H^e) \left(\frac{2}{7}\right)}{1 - \frac{3}{4} \left(\frac{6}{7}\right) + q_{1H} (1 - \lambda_H^e) \left(\frac{2}{7}\right)} = \frac{20(1 - \lambda_H^e)}{35 + 28q_{1H}(1 - \lambda_H^e)} \end{aligned}$$

for some  $\lambda_H^e$  such that

$$\lambda_H^e \in \begin{cases} 0, & \text{if } q_{1H} \in \left[\frac{1}{4}, \frac{5}{12}\right] \\ [0, 1], & \text{if } q_{1H} = \frac{5}{12} \\ 1, & \text{if } q_{1H} > \frac{5}{12}. \end{cases}$$

The solution has  $\lambda_H^e = 0$  and  $q_{1H} = \frac{5}{14}$ . The supply and demand curves for this market are depicted in Figure 2, where the broken curve is  $\beta^b(\cdot, 0)$ , and the solid curve is  $\beta^e(\cdot, \lambda_H^e)$  for  $\lambda_H^e \in \Lambda^e(q_{1H})$ .

In the equilibrium, the entrepreneur's investment can be seen to be  $\frac{14}{9}$ , less than he would choose in an environment with surplus liquidity. On the other hand, the liquidity price-premium represents an implicit subsidy for the investment of the banker. The latter invests more  $\left(\frac{7}{9}\right)$  and issues more claims  $\left(\frac{4}{9}\right)$  to goods in each state. Thus, an interesting feature of the model is illuminated: that the liquidity problem of the entrepreneur may distort the banker's behavior in the direction of *increased* investment. ■

## 4.2 Equilibrium Liquidation of the Banker's Assets

In the model, liquidation of a project circumvents the need to provide incentives. Therefore, even though the overall return from the project is lower when it is liquidated, it is still possible that liquidation may put more value in the hands of outside claims holders than could be achieved by carrying the project through to maturity. Indeed, Lemma 3 shows

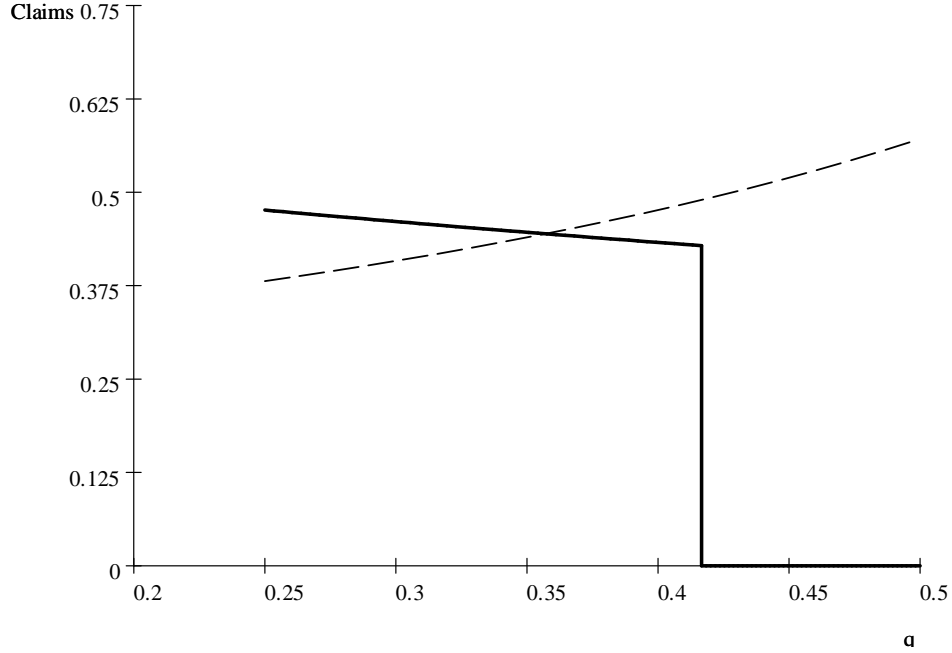


Figure 2: Supply and Demand of Liquidity for the Parameterization in Example 2

that this is precisely the necessary and sufficient condition for the banker to be willing to liquidate when the price of liquidity becomes extreme. In such an environment, his private objective of expected payoff maximization is served by selling state  $1H$  claims based on the liquidation value of his investment, rather than their value at maturity.

My last example illustrates this phenomenon numerically.

**Example 3.** Modify the example of the previous subsection by setting  $L = \frac{6}{7} > \tilde{R}^b$ . In this case, Lemma 3 shows that liquidation will be optimal for the banker whenever  $q_{1H}$  is at least  $q^b = \frac{1}{3}$ .

Figure 3 depicts the demand correspondence of the entrepreneur, analyzed in Example 2, together with the implied supply correspondence of the banker with the higher liquidation value. As is apparent from the figure, the new equilibrium price is  $\frac{1}{3}$ , which is lower than it was in the previous example, and the banker now chooses to liquidate a small fraction of his project in the bad state. More precisely, the fraction of the banker's project to be liquidated

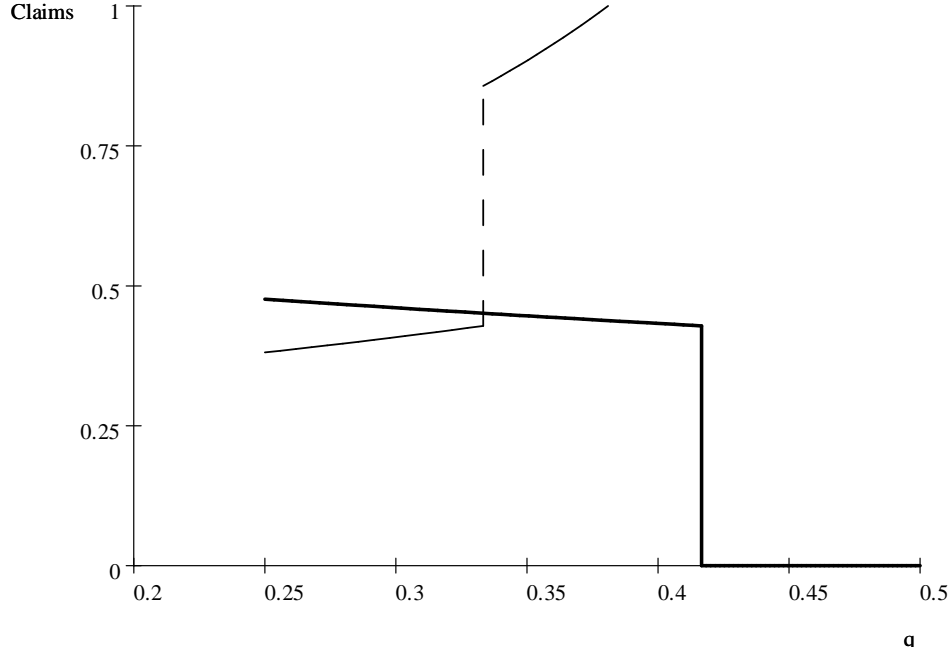


Figure 3: Supply and Demand for Liquidity for the Parameterization in Example 3

in equilibrium satisfies

$$\begin{aligned}
 -\beta^b \left( \frac{1}{3}, \lambda_H^b \right) &= \frac{\frac{2}{7} [\lambda_H^b (\frac{6}{7}) + (1 - \lambda_H^b) (\frac{4}{7})]}{1 - \frac{3}{4} (\frac{4}{7}) - \frac{1}{3} [\lambda_H^b (\frac{6}{7}) + (1 - \lambda_H^b) (\frac{4}{7})]} \\
 &= \beta^e \left( \frac{1}{3}, 0 \right) = \frac{20}{35 + 28 (\frac{1}{3})},
 \end{aligned}$$

and it may be calculated that  $\lambda_H^b = \frac{2}{29}$ . ■

Documenting the Savings and Loan Crisis in the U.S., White (1991) and others have observed banks in financial distress “gambling for resurrection” by expanding their asset portfolios beyond a prudent limit. While such behavior may at first appear to be related to that described here, closer observation reveals important differences.<sup>12</sup> In particular, the environment described by these authors pre-supposes private information possessed by the banks about the quality of their loan portfolios prior to the investment decision. Gambling is then adverse to the interest of the holders of the banks’ liabilities. In the world described

<sup>12</sup>Dewatripont and Tirole (1994) offer a good interpretation of the S&L Crisis in terms of economic theory.

here, there is perfect information, and liquidation, when it occurs, represents a constrained optimum.

A plausible historical analogy is that to episodes of liquidation by institutions under the National Banking System of the nineteenth century. In that era, liquidity crises were relatively frequent occurrences, and liquidation of banks' assets often coincided with them. This was true even while the crises did not necessarily affect the banks' assets directly. The closure of this analogy implies that, in times of the most severe crises, banks' liabilities did not garner sufficient faith that they could be circulated, and redemptions necessitated asset liquidations. Given the frequency and systematic nature of these crises, it seems likely that the gamble of over-issue of liabilities by banks was undertaken consciously and widely understood by other actors in the economy. That is, these lapses of faith seem to have been anticipated *ex ante*.

### 4.3 Real Bills

The dual role of the banker in the model recalls the “real bills doctrine” that private financial instruments backed by appropriate assets should be allowed to supplement other media of exchange. The real bills doctrine holds that creation and circulation of “bills of trade” backed by the proceeds of commerce imparts a beneficial elasticity to the supply of liquidity.

At the level of abstraction of the model, it is unclear whether the liabilities that provide liquidity services in my model should be interpreted as bank deposits or simply corporate debt. Gorton and Pennacchi (1990) face a similar question in their paper. Comparing the transactions velocity of bank deposits and corporate debt, they find unsupported the empirical conclusion that corporate debt serves the role ascribed to the liquid liabilities in their model. But the users of the liquidity services in their model are consumers, whose needs may not be comparable to those of the entrepreneur as modeled here. In other words, the frequency of the liquidity shocks that affect *corporations* may be such that corporate debt is

an appropriate instrument with which to meet those shocks. Moreover, it seems likely that financially savvy firms would find access to the corporate debt market less an obstacle than would ordinary consumers.

The models of currency elasticity of Sargent and Wallace (1982) and Champ, Smith and Williamson (1996) each exhibit institutions in which money creation under *laissez faire* conditions impart a beneficial elasticity to the supply of money. There is no scarcity of money *per se*, because these models abstract from commitment frictions that might disqualify certain assets from serving as security for a note. In Sargent and Wallace, for example, each intertemporal trading opportunity may be assumed to give rise to a risk-free bill of exchange for the full value of the desired transaction. Allocations are never constrained by the quantity of money that can be created, because the issue of a note is assumed to induce an obligation that does not admit default. The model of the present paper contributes to the understanding of the elasticity of the transactions medium by showing how, when contracting frictions exist, valuable commercial opportunities may be missed due to a shortage of liabilities suitable for circulation. Whereas money creation and credit creation are identical in these other environments, my model implies an endogenous separation between the assets and the liabilities of the banker even in a *laissez faire* setting.

#### 4.4 Liquidity Supply and Balance Sheet Effects

The effect of a reduction of the endowment of the banker on the market for liquidity is evident by reference to the example in Subsection 3.2. In Figure 1, it is shown how the decrease in  $\omega^b$  shifts the liquidity supply curve down, increasing the equilibrium price of liquidity. Thus, the rate of return on liquid assets, the reciprocal of the price of liquidity, falls as the supply contracts due to a decrease in the banker's endowment.

Without specifically interpreting the banker in terms of the banking sector in the real world, the model shows how sectors of the economy may be financially interdependent, and how a shock to the balance sheet of one sector of the economy can spillover to affect



investment in another. In particular, it does not seem necessary that the sector that issues instruments of liquidity have a “special” role in finance, as real world banks undoubtedly have. It is only necessary that the value of the liabilities issued by firms in this sector offer an appropriate hedge against shocks that affect another sector.

Of course, by interpreting the banker as a “bank”, the model implies a finding complementary to the literature, following Holmström and Tirole (1997), that distinguishes between shocks to banks’ capital and shocks to corporate balance sheets.<sup>13</sup> And, while it remains an empirical question the extent to which the liabilities of non-financial corporations provide these liquidity services to other corporations, it is clear that banks perform this function by offering deposits and credit lines.

## 5 Conclusions

In the previous sections, I have presented a model of entrepreneurial finance in which a liquidity need is generated by the confluence of two factors. First, moral hazard induces a wedge between market and private valuations of an entrepreneur’s project. Second, the project faces the possibility of a cost shock at a time when cash flow is low. In this case, the entrepreneur will need to hoard liquid securities to avoid having to liquidate his still-valuable project. In the model, all securities must be backed by productive assets; there is no government, and the promises of workers can be reneged with impunity.

The innovation of the paper comes through the introduction of a second entrepreneur, whom I label the “banker”, whose project is not susceptible to the liquidity shock. It is shown that the liabilities of the banker can provide the liquidity valued by the entrepreneur. In this case, the supply of liquidity inherits the elasticity properties of the banker’s investment project. The upward-sloping liquidity supply curve in my model stands in contrast to the vertical one in Kiyotaki and Moore (2005) and the horizontal one in Holmström and Tirole (1998). When liquidity is scarce, the price of these liabilities is high, and the entrepreneur

---

<sup>13</sup>See also Santomero and Seater (2000) and Chen (2001).

effectively subsidizes the investment of the banker in demanding them. Thus, cost volatility (i.e., the prospect of a random liquidity shock) in one sector may affect investment in the other sector beneficially.

The idea that the issue and circulation of liabilities backed by commercial projects facilitates production in other sectors recalls the “real bills doctrine” of classical monetary theory. Under this interpretation, the presence of moral hazard frictions offers an explanation for scarcity of the liquid medium. As in the doctrine itself, it seems unnecessary that the issuers of liquid claims be true “banks”, but only that their liabilities serve as a hedge against the risks faced by another sector. The degree to which liabilities like corporate bonds serve the function analogous to the liquid claims in the model remains an empirical question of some interest.

## 6 Appendix: Proofs of the Results

**Proof of Proposition 1.** The result is obvious from the discussion in the text and inspection of the first-order conditions of the worker’s problem. ■

**Proof of Proposition 2.** The result is obvious from the discussion in the text. ■

**Proof of Lemma 1.** First suppose that  $(c^b, I^b, \lambda^b, B^b)$  is a feasible policy, and suppose that  $c_0^b = \Delta > 0$ . Now construct the alternative policy  $(\tilde{c}^b, \tilde{I}^b, \tilde{\lambda}^b, B^b)$  as follows. Let  $\tilde{c}_0^b = 0$ ,  $\tilde{I}^b = I^b + \Delta$ , and  $\tilde{\lambda}_s^b = \lambda_s^b I^b / \tilde{I}^b$  and  $\tilde{c}_{2s}^b = c_{2s}^b + \Delta R^b$  for each  $s$ . Let the remaining elements of the new policy be identical to the old. Now it is easy to check that the new policy is feasible if the old one is. Moreover, subtracting the payoff under the old policy from that generated by the new gives  $\Delta (R^b - 1) > 0$ , so that the new one is an improvement, a contradiction.

To see that the banker’s incentive compatibility constraints must bind, let  $(c^b, I^b, \lambda^b, B^b)$  be a feasible policy for the banker, and suppose that

$$c_{2\sigma}^b = I^b (1 - \lambda_\sigma^b) \gamma^b + \Delta$$

for some  $\Delta > 0$ . Now construct the alternative policy  $(\tilde{c}^b, \tilde{I}^b, \tilde{\lambda}^b, B^b)$  as follows. Let  $\tilde{I}^b = I^b + q_{1\sigma}q_{2\sigma}\Delta$ , and  $\tilde{\lambda}_s^b = \lambda_s^b I^b / \tilde{I}^b$  for each  $s$ ; and let  $\tilde{c}_{2\sigma}^b = c_{2\sigma}^b - \Delta + q_{1\sigma}q_{2\sigma}\Delta R^b$  and  $\tilde{c}_{2s}^b = c_{2s}^b + q_{1\sigma}q_{2\sigma}\Delta R^b$  for  $s \neq \sigma$ . Let the remaining elements of the new policy be identical to the old. Now it is easy to check (using  $q_{1\sigma}q_{2\sigma} \geq \phi_\sigma$ ) that the new policy is feasible if the old one is. Moreover, subtracting the payoff under the old policy from that generated by the new gives

$$\Delta (q_{1\sigma}q_{2\sigma}R^b - \phi_\sigma) \geq \Delta\phi_\sigma (R^b - 1) > 0.$$

Finally, suppose that the feasible policy has  $c_{1\sigma}^b = \Delta > 0$ . Then construct the tilde policy with  $\tilde{c}_{1\sigma}^b = 0$ ,  $\tilde{I}^b = I^b + q_{1\sigma}\Delta$ ,  $\tilde{\lambda}_s^b = \lambda_s^b I^b / \tilde{I}^b$ , and  $\tilde{c}_{2s}^b = c_{2s}^b + q_{1\sigma}R^b$ ; and let the other elements be as in the original policy. Again the feasibility and superiority of the new policy can be verified. ■

**Proof of Lemma 2.** Suppose that  $(c^e, I^e, \lambda^e, B^e)$  is a feasible policy, and that  $c_0^e = \Delta > 0$ . Construct an alternative policy  $(\tilde{c}^e, \tilde{I}^e, \tilde{\lambda}^e, B^e)$  as follows. Let  $\tilde{c}_0^e = 0$ ,  $\tilde{c}_{2L}^e = c_{2L}^e + \Delta R^e$ ,  $\tilde{I}^e = I^e + \Delta$ ,  $\tilde{\lambda}_L^e = I^e \lambda_L^e / \tilde{I}^e$ , and  $\tilde{\lambda}_H^e = 1 - (1 - \lambda_H^e) I^e / \tilde{I}^e$ . Define the remaining elements of the new policy as in the old. Now it is easy to check that the new policy is feasible if the old one is. Moreover, subtracting the payoff to the entrepreneur of the old policy from that generated by the new one gives

$$\Delta [(1 - \phi) R^e - 1] > 0,$$

where the inequality follows from (20).

(Surpluses for the other states can be excluded in the manner of the proof of Lemma 1.)

■

**Proof of Corollary 1.** From the budget constraints of the banker, one has the following

expressions for optimal claims holding:

$$\begin{aligned} B_{1s}^b &= -I^b \left[ \lambda_s^b L + q_{2s} (1 - \lambda_s^b) \tilde{R}^b \right] < 0 \\ B_{2s}^b &= -I^b (1 - \lambda_s^b) \tilde{R}^b \leq 0. \end{aligned}$$

(The strict inequality follows from the fact that  $I^b > 0$ .) Suppose by way of contradiction that  $q_{1L} > 1 - \phi$ . It is immediate from Proposition 1 that  $B_{1L}^w = 0$ , and from the budget constraints of the entrepreneur and Lemma 2, it can be seen that

$$B_{1L}^e = -q_{2L} I^e (1 - \lambda_L^e) \tilde{R}^e \leq 0.$$

But now  $\sum_s B_{1L}^i < 0$ , contradicting the market clearing conditions. Thus, it must be that  $q_{1L} = 1 - \phi$ .

Next suppose that  $q_{2s} > 1$ ; then Proposition 1 gives  $B_{2s}^w = 0$ , and from the budget constraints of the entrepreneur and Lemma 2, it can be seen that

$$B_{2s}^e = -I^e (1 - \lambda_s^e) \tilde{R}^e \leq 0. \quad (24)$$

Since  $B_{2s}^i \leq 0$ ,  $i \in \{b, e\}$ , market clearing implies that  $B_{2s}^i = 0$  for  $i \in \{b, e\}$ . Since  $I^b > 0$ , this implies that  $\lambda_s^b = 1$ . From the first-order (n.s.) conditions for (22), it is straightforward to show that  $\lambda_s^b > 0$  only if  $q_{1\sigma} > \phi_\sigma$  for some  $\sigma \in \{H, L\}$ .<sup>14</sup> Suppose first that  $\sigma = s$ ; that is,  $q_{1s} > \phi_s$ . Then Proposition 1 gives  $B_{1s}^w = 0$ , so that  $B_{1s}^e = -B_{1s}^b > 0$ . Now from the budget constraints of the entrepreneur and Lemma 2, one has

$$B_{1s}^e = -q_{2s} I^e (1 - \lambda_s^e) \left( \tilde{R}^e - \rho_s \right),$$

and it must be that  $I^e (1 - \lambda_s^e) > 0$ . But now (24) contradicts the implication shown above

---

<sup>14</sup>Writing the Kuhn-Tucker condition for  $\lambda_s^b$  and plugging in  $q_{1\sigma} = \phi_\sigma$  for each  $\sigma$ , the criterion reduces to  $L - 1$ ; this quantity is always negative, implying that  $\lambda_s^b = 0$  by the Kuhn-Tucker theorem.

that  $B_{2s}^e = 0$ .

Next suppose that  $\sigma$  is not the same as  $s$ . Again Proposition 1 gives that  $B_{1\sigma}^w = 0$ , and  $I^e(1 - \lambda_\sigma^e) > 0$ . This implies that the entrepreneur liquidates his project in the good state, which can easily be seen (e.g., from the first-order conditions for the problem) to contradict optimality. ■

**Proof of Lemma 3.** The necessary and sufficient first-order Kuhn-Tucker conditions for a maximum of the strictly quasiconcave objective function (22) are

$$\frac{q_{1\sigma}}{\phi_\sigma} \left( L - \tilde{R}^b \right) \sum_s \phi_s (1 - \lambda_s^b) - \left\{ 1 - \sum_s q_{1s} \left[ \lambda_s^b L + (1 - \lambda_s^b) \tilde{R}^b \right] \right\} \geq 0 \text{ if } \lambda_\sigma^b > 0$$

$$\leq 0 \text{ if } \lambda_\sigma^b < 1,$$

where I have imposed the result of Corollary 1 that  $q_{2s} = 1$  in equilibrium. I have already argued that the term in braces must be positive for all  $\lambda^b \in [0, 1]^2$  at equilibrium prices, so result 1 is obvious by inspection.

Now consider the case that  $\tilde{R}^b < L$ . using the result of Corollary 1 that  $q_{1L} = 1 - \phi$ , the critical condition for  $\lambda_L^b > 0$  to be optimal can be written as  $q_{1H} \geq Q_L(\lambda_H^b)$  where

$$Q_L(\lambda_H^b) := \frac{1 - (1 - \phi)L - \phi(L - \tilde{R}^b)(1 - \lambda_H^b)}{\lambda_H^b L + (1 - \lambda_H^b)\tilde{R}^b};$$

and that for  $\lambda_H^b > 0$  can be written as  $q_{1H} \geq Q_H(\lambda_L^b)$  where

$$Q_H(\lambda_L^b) := \frac{\phi \left\{ 1 - (1 - \phi) \left[ \lambda_L^b L + (1 - \lambda_L^b) \tilde{R}^b \right] \right\}}{\phi L + (L - \tilde{R}^b)(1 - \phi)(1 - \lambda_L^b)}.$$

It is straightforward to show that  $Q_H - Q_L < 0$ ,<sup>15</sup> so that  $q_{1H} \geq Q_L(\lambda_H^b)$  implies that  $q_{1H} > Q_H(\lambda_L^b)$ . Therefore  $q_{1H} \geq Q_L(\lambda_H^b)$  implies that  $\lambda_H^b = 1$ . But inspection of the

---

<sup>15</sup>To show this, first show that  $Q_L$  is decreasing in  $\lambda_H^b$  and  $Q_H$  is increasing in  $\lambda_L^b$ . Thus  $Q_H(\lambda_L^b) - Q_L(\lambda_H^b) \leq Q_H(1) - Q_L(0)$ . Then evaluating the RHS, it is easy to see that it must be negative.

relation  $q_{1H} \geq Q_L(1)$  reveals a contradiction to the equilibrium condition

$$1 - \sum_s q_{1s} \left[ \lambda_s^b L + (1 - \lambda_s^b) \tilde{R}^b \right] > 0.$$

Thus, it cannot be that  $\lambda_L^b > 0$  in equilibrium, proving the first part of result 2. Now evaluating  $Q_H(0)$ , the critical condition for  $\lambda_H^b > 0$  can be restated as  $q_{1H} \geq q^b$ , and the optimality of the rule  $\lambda_H^b \in \Lambda^b(q_{1H})$  follows directly. ■

**Proof of Lemma 4.** The function to be maximized by  $\lambda^e$  is strictly quasiconcave, so that the first-order Kuhn-Tucker conditions are necessary and sufficient for optimality. Using the result that  $q_{2s} = 1$  (Corollary 1), the first-order condition for  $\lambda_\sigma^e$  is

$$-\frac{q_{1\sigma}}{\phi_\sigma} \left( \tilde{R}^e - \rho_\sigma \right) \sum_s \phi_s (1 - \lambda_s^e) - \left[ 1 - \sum_s q_{1s} (1 - \lambda_s^e) \left( \tilde{R}^e - \rho_s \right) \right] \begin{array}{l} \geq 0 \text{ if } \lambda_\sigma^e > 0 \\ \leq 0 \text{ if } \lambda_\sigma^e < 1. \end{array}$$

I have already argued that the expression in square brackets is positive, so that  $\rho_L = 0$  implies immediately that  $\lambda_L^e = 0$ . Imposing this result in the condition for  $\lambda_H^b$  and using the fact that  $q_{1L} = 1 - \phi$  (Corollary 1), the cutoff price  $\bar{q}$  may be derived by simplifying and solving for the value of  $q_{1H}$  that makes the left-hand side criterion exactly equal to zero. ■

**Proof of Proposition 3.** The result follows easily from the banker's budget constraints in light of previous results. ■

**Proof of Proposition 4.** The result follows easily from the entrepreneur's budget constraints, equation (23), and Corollary 1. ■

**Proof of Proposition 5.** The result follows from the previous ones and the definition of equilibrium. ■

## References

- [1] Champ, Bruce, Bruce D. Smith, and Stephen D. Williamson (1996), “Currency Elasticity and Banking Panics: Theory and Evidence,” *Canadian Journal of Economics* 24, 828-864.
- [2] Chen, Nan-Kuang (2001), “Bank Net Worth, Asset Prices, and Economic Activity,” *Journal of Monetary Economics* 48, 415-436.
- [3] Cooley, Thomas J., and Gary D. Hansen (1995), “Money and the Business Cycle,” in Thomas J. Cooley, ed. *Frontiers of Business Cycle Research*, Princeton University Press.
- [4] Dewatripont, Mathias, and Jean Tirole (1994), *The Prudential Regulation of Banks*, MIT Press.
- [5] Friedman, Benjamin M., and Kenneth M. Kuttner (1993), “Another Look at the Evidence on Money-income Causality,” *Journal of Econometrics* 57, 189-203.
- [6] Friedman, Milton, and Anna Schwartz (1963), *A Monetary History of the United States*, Princeton.
- [7] Gorton, Gary, and George Pennacchi (1990), “Financial Intermediaries and Liquidity Creation,” *Journal of Finance*, 49-71.
- [8] Holmström, Bengt, and Jean Tirole (1997), “Financial Intermediation, Loanable Funds, and the Real Sector,” *Quarterly Journal of Economics* 112, 663-691.
- [9] Holmström, Bengt, and Jean Tirole (1998), “Private and Public Supply of Liquidity,” *Journal of Political Economy* 106, 1-40.
- [10] Holmström, Bengt, and Jean Tirole (2001), “LAPM: A Liquidity-Based Assets Pricing Model,” *Journal of Finance* 56, 1837-1867.

- [11] Kiyotaki, Nobuhiro, and John Moore (1997), “Credit Cycles,” *Journal of Political Economy* 105, 211-248.
- [12] Kiyotaki, Nobuhiro, and John Moore (2005), “Liquidity and Asset Prices,” *International Economic Review* 46, 317-349.
- [13] Mints, Lloyd J. (1945), *A history of Banking Theory*, University of Chicago Press.
- [14] Opler, Tim, Lee Pinkowitz, René Stulz, and Rohan Williamson (1999), “The Determinants and Implications of Corporate Cash Holdings,” *Journal of Financial Economics*, 53, 3-46.
- [15] Santomero, Anthony M., and John J. Seater (2000), “Is there an Optimal Size for the Financial Sector?” *Journal of Banking and Finance* 24, 945-965.
- [16] Sargent, Thomas J., and Neil Wallace (1982), “The Real-Bills Doctrine versus the Quantity Theory: A Reconsideration,” *Journal of Political Economy* 90, 1212-1236.
- [17] Van den Heuvel, Skander (2002), “The Bank capital Channel of Monetary Policy,” University of Pennsylvania working paper.
- [18] White, Lawrence J. (1991), *The S&L Debacle*, Oxford University Press.