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Testing for Multiple Structural Changes in Cointegrated Regression Models*

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Abstract

This paper considers issues related to testing for multiple structural changes in cointegrated systems. We derive the limiting distribution of the Sup-Wald test under mild conditions on the errors and regressors for a variety of testing problems. We show that even if the coefficients of the integrated regressors are held fixed but the intercept is allowed to change, the limit distributions are not the same as would prevail in a stationary framework. Including stationary regressors whose coefficients are not allowed to change does not affect the limiting distribution of the tests under the null hypothesis. We also propose a procedure that allows one to test the null hypothesis of, say, k changes, versus the alternative hypothesis of $k+1$ changes. This sequential procedure is useful in that it permits consistent estimation of the number of breaks present. We show via simulations that our tests maintain the correct size in finite samples and are much more powerful than the commonly used LM tests, which suffer from important problems of non-monotonic power in the presence of serial correlation in the errors.

Keywords: Change-point, Sequential procedure, Wald tests, Unit Roots, Cointegration.

JEL Classification: C22

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1 Introduction

Issues related to structural change have received a considerable amount of attention in the statistics and econometrics literature. Andrews (1993) and Andrews and Ploberger (1994) provide a comprehensive treatment of the problem of testing for structural change assuming that the change point is unknown. Bai (1997) studies the least squares estimation of a single change point in regressions involving stationary and/or trending regressors. He derives the consistency, rate of convergence and the limiting distribution of the change point estimator under general conditions on the regressors and the errors. Perron and Zhu (2005) analyze the properties of parameter estimates in models where the trend function exhibits a slope change at an unknown date and the errors can be either stationary, $I(0)$, or have a unit root, $I(1)$, where here, and throughout the text, we refer to an $I(0)$ process as one whose partial sums satisfies a Functional Central Limit Theorem with a Brownian motion as the limit random variable, and an $I(1)$ as the partial sums of an $I(0)$ series.

With integrated variables, the case of interest is when the variables are cointegrated. Accounting for parameter shifts is crucial in cointegration analysis since it normally involves long spans of data which are more likely to be affected by structural breaks. Bai, Lumsdaine and Stock (1998) consider a single break in a multi-equations system. They show consistency of the maximum likelihood estimates and obtain a limit distribution of the break date estimate under a shrinking shifts scenario. Kejriwal and Perron (2008b) study the properties of the estimates of the break dates and parameters in a linear regression with multiple structural changes involving $I(1)$, $I(0)$ and trending regressors.

With respect to testing, Hansen (1992b) develops tests of the null hypothesis of no change in cointegrated models where all coefficients are allowed to change. An extension to partial changes has been analyzed by Kuo (1998). The tests considered are the Sup and Mean LM tests directed against an alternative of a one time change in parameters. Hao (1996) also suggests the use of the exponential LM test. Seo (1998) considers the Sup, Mean and Exp versions of the LM test within a cointegrated VAR setup. However, these test procedures are based on the fully modified estimation method (Phillips and Hansen, 1990) which has been shown to lead to tests with very poor finite sample properties (Carrion-i-Silvestre and Sansó-i-Rosselló, 2006). The results in Quintos and Phillips (1993) also suggest that the LM tests are likely to suffer from the problem of low power in finite samples. Moreover, simulation experiments in Hansen (2000) show that the LM test is quite poorly behaved in the presence of structural changes in the marginal distribution of the regressors. On the other hand, the Sup-Wald test is shown to be reasonably robust to such shifts. Hansen (2003) considers multiple structural changes in a cointegrated system, though his analysis is restricted to the case of known break dates. Finally, Qu (2007) proposes a procedure to detect whether cointegration is present when the cointegrating vector changes at some unknown possibly multiple dates.

The literature on testing for multiple structural changes is relatively sparse. It is, however, practically important since single break tests can suffer from non-monotonic power when the alternative involves more than one break. As stressed by Perron (2006), most tests may exhibit non-monotonic power functions if the number of breaks present is greater than the number explicitly accounted for in the construction of the tests. The aim of this paper is to provide a comprehensive treatment of issues related to testing for multiple structural changes occurring at unknown dates in cointegrated regression models. Our work builds on Bai and Perron (1998) who undertake a similar treatment in a stationary context. Our framework is general enough to allow both I(0) and I(1) variables in the regression. The assumptions about the distribution of the error processes are mild enough to allow for general forms of serial correlation. Moreover, we analyze both pure and partial structural change models. A partial change model is useful in allowing potential savings in the number of degrees of freedom, an issue particularly relevant for multiple changes. It is also important in empirical work since it helps to isolate the variables which are responsible for the failure of the null hypothesis. We derive the limiting distribution of the sup-Wald test under the null hypothesis of no structural change against the alternative hypothesis of a given number of cointegrating regimes. We also consider the double maximum tests proposed in Bai and Perron (1998). We provide critical values for a wide variety of models that are relevant in practice. Our asymptotic results have important implications for inference. We show that in models with both I(1) and I(0) variables, inference is possible as long as the intercept is allowed to change across regimes. Otherwise, the limiting distributions of the tests depend on nuisance parameters. Finally, our simulation experiments show that with serially correlated errors, the commonly used Sup, Mean and Exp-LM tests suffer from non-monotonic power problems. This is true for cases with a single break as well as with multiple breaks. We propose a modified sup Wald test that exhibits a power function which is monotonic with respect to the magnitude of the break(s) while maintaining reasonable size properties.

The paper is organized as follows. Section 2 presents the model and assumptions. In Section 3, we describe the testing problems and the test statistics used. Section 4 contains the theoretical results of this paper about the limit distributions of the tests for a wide variety of cases. This is first done for models involving non-trending regressors, no serial correlation in the errors and exogenous regressors. These restrictions are relaxed in Section 4.2, 5.1 and 5.2, respectively. Asymptotic critical values are presented in Section 4.3. Section 6 presents simulation experiments that address issues related to the size and power of the tests including a comparison with the often used LM tests. Section 7 offers concluding remarks and all technical derivations are included in a mathematical appendix.

2 The model and assumptions

Consider the following linear regression model with m breaks ($m+1$ regimes):

$$y_t = c_j + z'_{ft}\delta_f + z'_{bt}\delta_{bj} + x'_{ft}\beta_f + x'_{bt}\beta_{bj} + u_t \qquad (t = T_{j-1}+1, ..., T_j) \tag{1}$$

for $j = 1, ..., m+1$, where $T_0 = 0$, $T_{m+1} = T$ and T is the sample size. In this model, y_t is a scalar dependent $I(1)$ variable, x_{ft} ($p_f \times 1$) and x_{bt} ($p_b \times 1$) are vectors of $I(0)$ variables while z_{ft} ($q_f \times 1$) and z_{bt} ($q_b \times 1$) are vectors of $I(1)$ variables defined by: $z_{ft} = z_{f,t-1} + u^f_{zt}$, $z_{bt} = z_{b,t-1} + u^b_{zt}$, $x_{ft} = \mu_f + u^f_{xt}$ and $x_{bt} = \mu_b + u^b_{xt}$, where z_{f0} and z_{b0} are assumed, for simplicity, to be either $O_p(1)$ random variables or fixed finite constants. For ease of reference, the subscript b on the error term stands for "break" and the subscript f stands for "fixed" (across regimes). The break points $(T_1, ..., T_m)$ are treated as unknown. This is a partial structural change model in which the coefficients of only a subset of the regressors are subject to change. When $p_f = q_f = 0$, we have a pure structural change model with all coefficients allowed to change across regimes. It will be useful to express (1) in matrix form as:

$$Y = G\alpha + \bar{W}\gamma + U$$

where $Y = (y_1, ..., y_T)'$, $G = (Z_f, X_f)$, $Z_f = (z_{f1}, ..., z_{fT})'$, $X_f = (x_{f1}, ..., x_{fT})'$, $U = (u_1, ..., u_T)'$, $W = (w_1, ..., w_T)'$, $w_t = (1, z'_{bt}, x'_{bt})'$, $\gamma = (\delta'_{b1}, \beta'_{b1}, ..., \delta'_{b,m+1}, \beta'_{b,m+1})'$, $\alpha = (\delta'_f, \beta'_f)'$ and $\bar{W}$ is the matrix which diagonally partitions W at the $m-$partition $(T_1, ..., T_m)$, that is, $\bar{W} = diag(W_1, ..., W_{m+1})$ with $W_i = (w_{T_{i-1}+1}, ..., w_{T_i})'$ for $i = 1, ..., m+1$. Kejriwal and Perron (2008b) analyze the properties of the estimates of the break dates and the other parameters of the model under general conditions on the regressors and the errors. In this paper, the interest is in testing the null hypothesis of no structural change versus the alternative hypothesis of m changes as specified by the model (1). Hence, the data generating process is assumed to be given by (1) with $p_b = q_b = 0$.

As a matter of notation, "$\xrightarrow{p}$" denotes convergence in probability, "$\xrightarrow{d}$" convergence in distribution and "$\Rightarrow$" weak convergence in the space $D[0,1]$ under the Skorohod metric.

Also, $x_t = (x'_{ft}, x'_{bt})'$, $u_{xt} = (u^{f\prime}_{xt}, u^{b\prime}_{xt})'$, $z_t = (z'_{ft}, z'_{bt})'$, $\mu = (\mu'_f, \mu'_b)'$ and $\lambda = \{\lambda_1, ..., \lambda_m\}$ is the vector of break fractions defined by $\lambda_i = T_i/T$ for $i = 1, ..., m$. We make the following assumptions on $\xi_t = (u_t, u^{f\prime}_{zt}, u^{b\prime}_{zt}, u^{f\prime}_{xt}, u^{b\prime}_{xt})'$, a vector of dimension $n = q_f + p_f + q_b + p_b + 1$.
Assumption A1: The vector ξ_t satisfies the following multivariate Functional Central Limit Theorem (FCLT): $T^{-1/2}\sum_{t=1}^{[Tr]}\xi_t \Rightarrow B(r)$, with $B(r) = (B_1(r), B^f_z(r)', B^b_z(r)', B^f_x(r)', B^b_x(r)')'$ is a n vector Brownian motion with symmetric covariance matrix

$$\Omega = \begin{pmatrix} \sigma^2 & \Omega^f_{1z} & \Omega^b_{1z} & \Omega^f_{1x} & \Omega^b_{1x} \\ \Omega^f_{z1} & \Omega^{ff}_{zz} & \Omega^{fb}_{zz} & \Omega^{fb}_{zx} & \Omega^{ff}_{zx} \\ \Omega^b_{z1} & \Omega^{bf}_{zz} & \Omega^{bb}_{zz} & \Omega^{bf}_{zx} & \Omega^{bb}_{zx} \\ \Omega^f_{x1} & \Omega^{ff}_{xz} & \Omega^{fb}_{xz} & \Omega^{ff}_{xx} & \Omega^{fb}_{xx} \\ \Omega^b_{x1} & \Omega^{bf}_{xz} & \Omega^{bb}_{xz} & \Omega^{bf}_{xx} & \Omega^{bb}_{xx} \end{pmatrix} \begin{matrix} 1 \\ q_f \\ q_b \\ p_f \\ p_b \end{matrix}$$

$$= \lim_{T\to\infty} T^{-1}E(S_T S'_T) = \Sigma + \Lambda + \Lambda'$$

where $S_T = \sum_{t=1}^{T}\xi_t$, $\Sigma = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}E(\xi_t\xi'_t)$ and $\Lambda = \lim_{T\to\infty} T^{-1}\sum_{j=1}^{T-1}\sum_{t=1}^{T-j}E(\xi_t\xi'_{t+j})$. We also assume $\sigma^2 > 0$ and $p\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}u_t^2 = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}E[u_t^2] \equiv \sigma_u^2$.
Assumption A2: The vector $\{x_t u_t\}$ satisfies Assumption A4 in Qu and Perron (2007) so that $T^{-1/2}\sum_{t=1}^{[Tr]}(u^f_{xt}, u^b_{xt})u_t \Rightarrow \sigma Q^{*1/2}W^*_x(r)$, where $W^*_x(r) = (W^*_{xf}(r)', W^*_{xb}(r)')'$ is a $(p_f + p_b)$ vector of independent Wiener processes and

$$Q^* = \begin{bmatrix} Q^{ff*}_x & Q^{fb*}_x \\ Q^{bf*}_x & Q^{bb*}_x \end{bmatrix}$$

Assumption A3: For all t and s: a) $E(u_{xt}u_t z_s) = 0$; b) $E(u_{xt}u_t u_s) = 0$; c) $E(u_{xt}u_t u_{xs}) = 0$.
Assumption A4: The matrix $\begin{pmatrix} \Omega^{ff}_{zz} & \Omega^{fb}_{zz} \\ \Omega^{bf}_{zz} & \Omega^{bb}_{zz} \end{pmatrix}$ is positive definite.
Assumption A5: $T^{-1}\sum_{t=1}^{[Ts]}x_t x'_t \xrightarrow{p} sQ$ and, $T^{-1}\sum_{t=1}^{[Ts]}u_{xt}u'_{xt} \xrightarrow{p} sQ^*$, uniformly in $s \in [0, 1]$, for some positive definite matrices Q and Q^*.

Assumption A1 requires that the errors satisfy a multivariate FCLT. The conditions for this to hold are very general (see, e.g., Davidson, 1994). It can be shown to apply to a large class of linear processes including those generated by all stationary and invertible ARMA models. A2 guarantees that a $FCLT$ also holds for the sequence $\{u_{xt}u_t\}$. Assumption A3 restricts somewhat the class of models applicable but is quite mild. Sufficient conditions for it to hold are: for (a) that the $I(0)$ regressors are uncorrelated with the errors contemporaneously even conditional on the $I(1)$ variables; for (b) that the autocovariance structure of the $I(0)$ regressors be independent of the errors and, similarly, for (c) that the autocovariance structure of the errors be independent of the $I(0)$ regressors. This assumption is needed to guarantee that $W_x^*(\cdot)$ and $B(\cdot)$ are uncorrelated and, being Gaussian, are therefore independent. Without this condition, the analysis would be much more complex. A4 rules out cointegration among the $I(1)$ regressors. A5 is standard for $I(0)$ regressors but rules out trending regressors, which we shall relax in Section 4.2.

Under the alternative hypothesis, the estimates of the parameters are obtained by minimizing the global sum of squared residuals. For each m-partition $(T_1, ..., T_m)$, denoted $\{T_j\}$, the associated least squares estimates of α and γ are obtained by minimizing

$$SSR_T(T_1, ..., T_m) = \sum_{i=1}^{m+1} \sum_{t=T_{i-1}+1}^{T_i} [y_t - c_i - z'_{ft}\delta_f - x'_{ft}\beta_f - z'_{bt}\delta_{bi} - x'_{bt}\beta_{bi}]^2 \qquad (2)$$

Let $\hat{\alpha}(\{T_j\})$ and $\hat{\gamma}(\{T_j\})$ be the resulting estimates. Substituting these into the objective function and denoting the resulting sum of squared residuals as $S_T(T_1, ..., T_m)$, the estimate of the break points are $(\hat{T}_1, ..., \hat{T}_m) = \arg\min_{T_1,...,T_m} S_T(T_1, ..., T_m)$, where the minimization is taken over all partitions $(T_1, ..., T_m)$ such that $T_i - T_{i-1} \geq \epsilon T$ for some $\epsilon > 0$. The estimates of the regression coefficients are then $\hat{\alpha} = \hat{\alpha}(\{\hat{T}_j\})$ and $\hat{\gamma} = \hat{\gamma}(\{\hat{T}_j\})$. Such estimates can be obtained using the algorithm of Bai and Perron (2003). Finally, consistent estimates of the matrices Σ and Λ (and, hence, Ω) are $\hat{\Sigma} = T^{-1}\sum_{t=1}^{T} \hat{\xi}_t \hat{\xi}_t'$ and $\hat{\Lambda} = T^{-1}\sum_{j=1}^{T-1} w(j/l) \sum_{t=1}^{T-j} \hat{\xi}_t \hat{\xi}_{t+j}$, where $\hat{\xi}_t = (\hat{u}_t, \Delta z'_{ft}, \Delta z'_{bt}, (x_{ft} - \bar{x}_f)', (x_{bt} - \bar{x}_b)')'$ with $\hat{u}_t$ the OLS residuals from regression (1), $\bar{x}_i = T^{-1}\sum_{t=1}^{T} x_{it}$ $(i = f, b)$ and $w(j/l)$ is a kernel function that is continuous and even with $w(0) = 1$ and $\int_{-\infty}^{\infty} w^2(x)dx < \infty$. Also, $l \to \infty$ as $T \to \infty$ and $l = o(T^{1/2})$. Consistency of these covariance matrix estimates has been shown in Hansen (1992c).

3 The testing problem and the test statistics

The data generating process (1) is the most general and in practice restricted versions may be used. This gives rise to a variety of possible cases for the testing problems considered. We classify them in two categories: a) models with only $I(1)$ regressors; b) models with both $I(1)$ and $I(0)$ regressors. This classification in two categories is useful since oftentimes researchers are faced with only $I(1)$ variables. For this category (a), the testing problems considered are the following (for ease of reference, we list the relevant regression under the alternative hypothesis):

Testing problems, Category (a), Models with I(1) variables only $(p_f = p_b = 0$, for all cases)**:** Let H_0^a denotes the restrictions $\{c_j = c,\ \delta_{bj} = \delta_b$ for all $j = 1, .., m + 1\}$.

1. $H_0^a(1) = \{H_0^a,\, q_f = 0\}$ versus $H_1^a(1) = \{q_f = 0\}$ $(y_t = c_j + z'_{bt}\delta_{bj} + u_t)$;

2. $H_0^a(2) = \{H_0^a,\, q_b = 0\}$ versus $H_1^a(2) = \{q_b = 0\}$ $(y_t = c_j + z'_{ft}\delta_f + u_t)$;

3. $H_0^a(3) = \{H_0^a,\, q_f = 0\}$ versus $H_1^a(3) = \{c_j = c \text{ for all } j = 1, .., m+1,\, q_f = 0\}$ $(y_t = c + z'_{bt}\delta_{bj} + u_t)$;

4. $H_0^a(4) = \{H_0^a\}$ versus $H_1^a(4) = \{\text{no restriction}\}$ $(y_t = c_j + z'_{ft}\delta_f + z'_{bt}\delta_{bj} + u_t)$;

5. $H_0^a(5) = \{H_0^a\}$ versus $H_1^a(5) = \{c_j = c \text{ for all } j = 1, .., m+1\}$ $(y_t = c + z'_{ft}\delta_f + z'_{bt}\delta_{bj} + u_t)$.

Testing problems, Category (b), Models with both I(1) and I(0) variables: Let H_0^b denotes the restrictions $\{c_j = c,\, \delta_{bj} = \delta_b,\, \beta_{bj} = \beta_b \text{ for all } j = 1, .., m+1\}$.

1. $H_0^b(1) = \{H_0^b,\, p_f = q_b = 0\}$ versus $H_1^b(1) = \{c_j = c \text{ for all } j = 1, .., m+1,\, p_f = q_b = 0\}$ $(y_t = c + z'_{ft}\delta_f + x'_{bt}\beta_{bj} + u_t)$;

2. $H_0^b(2) = \{H_0^b,\, p_b = q_f = 0\}$ versus $H_1^b(2) = \{c_j = c \text{ for all } j = 1, .., m+1,\, p_b = q_f = 0\}$ $(y_t = c + z'_{bt}\delta_{bj} + x'_{ft}\beta_f + u_t)$;

3. $H_0^b(3) = \{H_0^b,\, p_f = q_f = 0\}$ versus $H_1^b(3) = \{c_j = c \text{ for all } j = 1, .., m+1,\, p_f = q_f = 0\}$ $(y_t = c + z'_{bt}\delta_{bj} + x'_{bt}\beta_{bj} + u_t)$;

4. $H_0^b(4) = \{H_0^b,\, p_f = q_f = 0\}$ versus $H_1^b(4) = \{p_f = q_f = 0\}$ $(y_t = c_j + z'_{bt}\delta_{bj} + x'_{bt}\beta_{bj} + u_t)$;

5. $H_0^b(5) = \{H_0^b,\, p_b = q_b = 0\}$ versus $H_1^b(5) = \{p_b = q_b = 0\}$ $(y_t = c_j + z'_{ft}\delta_f + x'_{ft}\beta_f + u_t)$;

6. $H_0^b(6) = \{H_0^b,\, p_b = q_f = 0\}$ versus $H_1^b(6) = \{p_b = q_f = 0\}$ $(y_t = c_j + z'_{bt}\delta_{bj} + x'_{ft}\beta_f + u_t)$;

7. $H_0^b(7) = \{H_0^b,\, p_f = q_b = 0\}$ versus $H_1^b(7) = \{p_f = q_b = 0\}$ $(y_t = c_j + z'_{ft}\delta_f + x'_{bt}\beta_{bj} + u_t)$;

8. $H_0^b(8) = \{H_0^b,\, q_f = 0\}$ versus $H_1^b(8) = \{q_f = 0\}$ $(y_t = c_j + z'_{bt}\delta_{bj} + x'_{ft}\beta_f + x'_{bt}\beta_{bj} + u_t)$;

9. $H_0^b(9) = \{H_0^b,\, q_b = 0\}$ versus $H_1^b(9) = \{q_b = 0\}$ $(y_t = c_j + z'_{ft}\delta_f + x'_{ft}\beta_f + x'_{bt}\beta_{bj} + u_t)$;

10. $H_0^b(10) = \{H_0^b\}$ versus $H_1^b(10) = \{\text{no restriction}\}$ $(y_t = c_j + z'_{ft}\delta_f + z'_{bt}\delta_{bj} + x'_{ft}\beta_f + x'_{bt}\beta_{bj} + u_t)$;

11. $H_0^b(11) = \{H_0^b\}$ versus $H_1^b(10) = \{c_j = c \text{ for all } j = 1, .., m+1\}$ $(y_t = c + z'_{ft}\delta_f + z'_{bt}\delta_{bj} + x'_{ft}\beta_f + x'_{bt}\beta_{bj} + u_t)$.

We now give a brief description of each of the models in the two categories. First consider Category (a). Case 1 is a pure structural change model which allows for a change in the intercept as well. Case 2 is a partial change model in which only the intercept is allowed to change. Case 3 is again a partial change model where the intercept is not allowed to change. Cases 4 and 5 are block partial models in which a subset of the $I(1)$ coefficients is allowed to change. In Category (b), Cases 1 to 3 are partial change models where the intercept is not allowed to change across regimes. Case 4 is a pure change model where all $I(1)$ and $I(0)$ coefficients as well as the intercept are allowed to change. Case 5 is a partial change model, which involves only an intercept shift. Case 6 is a partial change model where the $I(0)$ coefficients are not allowed to change. Similarly, Case 7 is a partial change model where the $I(1)$ coefficients are not allowed to change. Cases 8-11 are block partial models in which a subset of coefficients of at least one type of regressor is not allowed to change.

We consider three types of tests. The first applies when the alternative hypothesis involves a fixed value $m = k$ of changes. We consider the Wald test, scaled by the number of regressors whose coefficient are allowed to change, defined by

$$F_T(\lambda, k) = (\frac{T - (k+1)(q_b + p_b) - (p_f + q_f)}{k})\frac{\hat{\gamma}' R' (R(\bar{W}' M_G \bar{W})^{-1} R')^{-1} R\hat{\gamma}}{SSR_k} \tag{3}$$

where R is the conventional matrix such that $(R\gamma)' = (\gamma_1' - \gamma_2', ..., \gamma_k' - \gamma_{k+1}')$ and $M_G = I - G(G'G)^{-1}G'$. Here SSR_k is the sum of squared residuals under the alternative hypothesis. As in Bai and Perron (1998), we define the following set for some arbitrary small positive number ϵ, $\Lambda_\epsilon^k = \{\lambda : |\lambda_{i+1} - \lambda_i| \geq \epsilon, \lambda_1 \geq \epsilon, \lambda_k \leq 1 - \epsilon\}$. The sup-Wald test is then defined as sup-$F_T(k) = sup_{\lambda \in \Lambda_\epsilon^k} F_T(\lambda, k)$. Since, in the current cases, the estimates $\hat{\lambda} = \{\hat{\lambda}_1, ..., \hat{\lambda}_k\}$ with $\hat{\lambda}_i = \hat{T}_i / T$ (for $i = 1, ..., k$) obtained by minimizing the global sum of squared residuals correspond to those that maximize the test $F_T(\lambda, k)$, we have sup-$F_T(k) = F_T(\hat{\lambda}, k)$.

The second procedure applies when the alternative hypothesis involves an unknown number of changes between 1 and some upper bound M. As in Bai and Perron (1998), we consider a double maximum test based on the maximum of the individual tests for the null of no break versus m breaks ($m = 1, ..., M$), defined by $UD\max F_T(M) = \max_{1 \leq m \leq M} sup_{\lambda \in \Lambda_\epsilon^m} F_T(\lambda, m)$. This test is arguably the most useful to apply when trying to determine if structural changes are present. Simulations presented in Bai and Perron (2006) show that with multiple changes, the power of tests for a single break can be quite low in finite samples, especially for certain types of multiple changes; e.g., two breaks with identical first and third regimes. Also tests for a particular number of changes may have non-monotonic power when the number of changes is greater than specified. Finally, in their simulations they found the power of

$UD\max$ to be nearly as high as that of the sup-F_T test based on the true number of changes.

The third testing procedure is a sequential one based on the estimates of the break dates obtained from a global minimization of sum of squared residuals, as in Bai and Perron (1998). Consider a model with k breaks, with estimates denoted by $(\hat{T}_1, ..., \hat{T}_k)$, which are obtained by a global minimization of the sum of squared residuals. The procedure to test the null hypothesis of k breaks versus the alternative hypothesis of $k+1$ breaks is to perform a one break test for each of the $(k+1)$ segments defined by the partition $(\hat{T}_1, ..., \hat{T}_k)$ and to assess whether the maximum of the tests is significant. More precisely, the test is defined by

$$SEQ_T\left(k+1|k\right) = \max_{1\le j\le k+1} \sup_{\tau\in\Lambda_{j,\varepsilon}} T\{SSR_T(\hat{T}_1, ..., \hat{T}_k) - SSR_T(\hat{T}_1, ...\hat{T}_{j-1}, \tau, \hat{T}_j, ..., \hat{T}_k)\}/SSR_{k+1}$$

where $\Lambda_{j,\varepsilon} = \{\tau; \hat{T}_{j-1} + (\hat{T}_j - \hat{T}_{j-1})\varepsilon \le \tau \le \hat{T}_j - (\hat{T}_j - \hat{T}_{j-1})\varepsilon\}$. Note that this is different from a purely sequential procedure since for each value of k the break dates are re-estimated to get those that correspond to the global minimizers of the sum of squared residuals.

4 The asymptotic distributions of the tests

With integrated regressors, an important issue that arises is the correlation between the regressors and the errors. We first consider the case where all $I(1)$ regressors are strictly exogenous. Later, we deal with the case of endogenous regressors and show that if the regression is augmented with leads and lags of the the first differences of the $I(1)$ regressors, the limiting distribution of the tests is the same as that obtained when all $I(1)$ regressors are strictly exogenous. Hence, for now, we assume $\Omega^f_{1z} = \Omega^b_{1z} = 0$, which will be relaxed in Section 5.2. We also start with the following assumption that imposes serially uncorrelated errors in the cointegrating regression to be relaxed in Section 5.1:

Assumption A6: Let $\xi^*_t = (u^{f\prime}_{zt}, u^{b\prime}_{zt}, u^{f\prime}_{xt}, u^{b\prime}_{xt})'$, the errors $\{u_t\}$ form an array of martingale differences relative to $\{\mathcal{F}_t\} = \sigma$-field$\{\xi^*_{t-s}, u_{t-1-s};\ s > 0\}$.

4.1 The main theoretical results

As a matter of notation, we define the following functionals, where $W_1 = \sigma^{-1}B_1$:

$$\begin{aligned} h(G,a,b) &= (\textstyle\int_a^b GdW_1)'(\int_a^b GG')^{-1}(\int_a^b GdW_1), \\ f(G) &= (\sum_{i=1}^{k+1}\int_{\lambda_{i-1}}^{\lambda_i} GdW_1)'(\sum_{i=1}^{k+1}\int_{\lambda_{i-1}}^{\lambda_i} GG')^{-1}(\sum_{i=1}^{k+1}\int_{\lambda_{i-1}}^{\lambda_i} GdW_1), \end{aligned}$$

$g(G,a,b) = (aG(b) - bG(a))'(aG(b) - bG(a))/ba(b-a)$ and $G^{(a,b)}(r) = G(r) - (\lambda_b - \lambda_{a-1})^{-1}\int_{\lambda_{a-1}}^{\lambda_b} G$. Also, by convention $\lambda_0 = 0$ and $\lambda_{k+1} = 1$. The limit distributions of the tests when only $I(1)$ variables are involved are stated in the following Theorem.

Theorem 1 *Assume A1-A6 and* $\Omega_{1z}^f = \Omega_{1z}^b = 0$. *For the testing problems in Category (a), the limit distribution of* $sup_{\lambda\in\Lambda_\epsilon^k} F_T(\lambda,k)$ *is* $\sup_{\lambda\in\Lambda_\epsilon^k} F(\lambda,k)/k$ *with* $F(\lambda,k)$ *defined as follows for the various cases. For Case (1),*

$$F(\lambda,k) = \sum_{i=1}^{k}[h(W_z^{b(1,i)},0,\lambda_i) - h(W_z^{b(1,i+1)},0,\lambda_{i+1}) + h(W_z^{b(i+1,i+1)},\lambda_i,\lambda_{i+1}) + g(W_1,\lambda_i,\lambda_{i+1})]$$

For Case (2), $F(\lambda,k) = f(W_z^{f(i,i)}) - h(W_z^{f(1,k+1)},0,1) + \sum_{i=1}^{k} g(W_1,\lambda_i,\lambda_{i+1})$, *where* $W_z^f(r) = (\Omega_{zz}^{ff})^{-1/2}B_z^f(r)$. *For Case (3),*

$$F(\lambda,k) = f(P_{zi}^b) - h(W_z^{b(1,k+1)},0,1) - W_1(1)^2 + \sum_{i=1}^{k+1} h(W_z^b,\lambda_{i-1},\lambda_i)$$

where $P_{zi}^b(r) = 1 - (\int_{\lambda_{i-1}}^{\lambda_i} W_z^{b\prime})(\int_{\lambda_{i-1}}^{\lambda_i} W_z^b W_z^{b\prime})^{-1} W_z^b(r)$, *for* $r \in [\lambda_{i-1},\lambda_i]$. *For Case (4),*

$$F(\lambda,k) = f(W_z^{M(i,i)}) - h(W_z^{fb(1,k+1)},0,1) + \sum_{i=1}^{k+1} h(W_z^{b(i,i)},\lambda_{i-1},\lambda_i) + \sum_{i=1}^{k} g(W_1,\lambda_i,\lambda_{i+1})$$

with $W_z^{fb}(r) = (W_z^f(r), W_z^b(r))$, *and where*

$$W_z^{M(i,i)}(r) = W_z^{f(i,i)}(r) - \int_{\lambda_{i-1}}^{\lambda_i} W_z^{f(i,i)} W_z^{b(i,i)\prime}(\int_{\lambda_{i-1}}^{\lambda_i} W_z^{b(i,i)} W_z^{b(i,i)\prime})^{-1} W_z^{b(i,i)}(r).$$

For Case (5), $F(\lambda,k) = f(P_{zi}) - h(W_z^{fb(1,k+1)},0,1) - W_1(1)^2 + \sum_{i=1}^{k+1} h(W_z^b,\lambda_{i-1},\lambda_i)$, *where* $P_{zi}(r)' = (P_{zi}^b(r)', P_{zi}^{fb}(r)')$ *with* $P_{zi}^{fb}(r) = W_z^f(r) - (\int_{\lambda_{i-1}}^{\lambda_i} W_z^f W_z^{b\prime})(\int_{\lambda_{i-1}}^{\lambda_i} W_z^b W_z^{b\prime})^{-1} W_z^b(r)$.

Theorem 1 shows that it is possible to make inference in models involving $I(1)$ variables using the sup-Wald test. Also, the limiting distributions are different depending on whether the intercept and/or the $I(1)$ coefficients are allowed to change. Note that for Cases 2, 4 and 5 the limit distributions depend on the number of $I(1)$ coefficients that are not allowed to change. This is different from a stationary framework where the limit distribution is independent of the number of regressors whose coefficients are not allowed to change. We now consider the limit distributions of the test for the various cases in Category (b) where both I(1) and I(0) regressors are present.

Theorem 2 *Assume A1-A6 and* $\Omega_{1z}^f = \Omega_{1z}^b = 0$ *and let* $W_{xb(1)}^* = (W_{xb}^{*\prime}, W_1)'$. *For cases in Category (b), the limiting distributions of* $sup_{\lambda\in\Lambda_\epsilon^k} F_T(\lambda,k)$ *under the null hypothesis* *are given by* $\sup_{\lambda \in \Lambda_\epsilon^k} F(\lambda, k)/k$ *with* $F(\lambda, k)$ *defined as follows. For case (1),* $F(\lambda, k) = \sum_{i=1}^{k} g(W_{xb}^*, \lambda_i, \lambda_{i+1})$*. For Case (2), the limit distribution is the same as for Case (3) in Category (a). For Case (3),*

$$F(\lambda, k) = f(P_{zi}^b) - h(W_z^{b(1,k+1)}, 0, 1) - W_1(1)^2 + \sum_{i=1}^{k+1} h(W_z^b, \lambda_{i-1}, \lambda_i) + \sum_{i=1}^{k} g(W_{xb}^*, \lambda_i, \lambda_{i+1}).$$

For Cases (4) and (8),

$$F(\lambda, k) = \sum_{i=1}^{k} [h(W_z^{b(1,i)}, 0, \lambda_i) - h(W_z^{b(1,i+1)}, 0, \lambda_{i+1}) + h(W_z^{b(i+1,i+1)}, \lambda_i, \lambda_{i+1}) + g(W_{xb(1)}^*, \lambda_i, \lambda_{i+1})]$$

For Cases (5) and (6), the limit distributions are the same as for Cases (2) and (1), respectively, in Category (a). For Case (7) and (9),

$$F(\lambda, k) = f(W_z^{f(i,i)}) - h(W_z^{f(1,k+1)}, 0, 1) + \sum_{i=1}^{k} g(W_{xb(1)}^*, \lambda_i, \lambda_{i+1}).$$

For Case (10),

$$F(\lambda, k) = f(W_z^{M(i,i)}) - h(W_z^{fb(1,k+1)}, 0, 1) + \sum_{i=1}^{k+1} h(W_z^{b(i,i)}, \lambda_{i-1}, \lambda_i) + \sum_{i=1}^{k} g(W_{xb(1)}^*, \lambda_i, \lambda_{i+1}).$$

And, for Case (11),

$$F(\lambda, k) = f(P_{zi}) - h(W_z^{fb(1,k+1)}, 0, 1) - W_1(1)^2 + \sum_{i=1}^{k+1} h(W_z^b, \lambda_{i-1}, \lambda_i) + \sum_{i=1}^{k} g(W_{xb}^*, \lambda_i, \lambda_{i+1}).$$

The practical implications of Theorem 2 are as follows. As shown in Case (1), if the intercept and the $I(1)$ variables are held fixed and only the coefficients on the $I(0)$ variables are allowed to change, the same limit distribution as in Bai and Perron (1998) applies. However, this equivalence with the case of stationary regressors only holds if the constant is not allowed to change. As shown in Case (7), the limit distribution is different when the intercept is allowed to change and depends on the number of $I(1)$ variables present. The effect of allowing the intercept to change or not can also be seen by comparing Cases (3) and (4). The limit distributions are different and, as expected, both depend on the number of $I(1)$ and $I(0)$ variables whose coefficients are allowed to change. A similar feature also applies when the regression involves $I(1)$ and $I(0)$ variables whose coefficients are not allowed to change, as shown in Cases (10) and (11). Comparing these with Cases (3) and (4) again shows that having $I(1)$ variables whose coefficients are not allowed to change alters the limit distributions. Finally, comparing Cases (a-1) and (b-6), (a-2) and (b-5), (a-3) and (b-2), (b-4) and (b-8), and (b-7) and (b-9), shows that including $I(0)$ regressors whose coefficients are not allowed to change does not alter the limit distribution.

Remark 1 *For Case (4) in Category (b), the limit distribution of* $sup_{\lambda \in \Lambda_\epsilon^k} F_T(\lambda, k)$ *is:*

$$\sup_{(\lambda_1,\dots,\lambda_k)\in\Lambda_\epsilon^k} \{\sum_{i=1}^{k}(S^*(\lambda_i,\lambda_{i+1})'V(\lambda_i,\lambda_{i+1})^{-1}S^*(\lambda_i,\lambda_{i+1})) + \sum_{i=1}^{k}\frac{(\lambda_i W_{xb}^*(\lambda_{i+1})-\lambda_{i+1}W_{xb}^*(\lambda_i))'(\lambda_i W_{xb}^*(\lambda_{i+1})-\lambda_{i+1}W_{xb}^*(\lambda_i))}{\lambda_{i+1}\lambda_i(\lambda_{i+1}-\lambda_i)}\}$$

with $S^*(\lambda_i,\lambda_{i+1}) = S(\lambda_i) - M(\lambda_i)M(\lambda_{i+1})^{-1}S(\lambda_{i+1})$, $V(\lambda_i,\lambda_{i+1}) = M(\lambda_i) - M(\lambda_i)M(\lambda_{i+1})^{-1}M(\lambda_i)$, $S(\lambda_i) = \int_0^{\lambda_i} Z^* dW_1$, $M(\lambda_i) = \int_0^{\lambda_i} Z^* Z^{*\prime}$ *and* $Z^* = (1, W_z^{b\prime})'$. *The first summation corresponds to the distribution in Case 1 of Category (a), while the second corresponds to the* p_b $I(0)$ *regressors whose coefficients are allowed to change.*

With these theoretical results for the $sup\text{-}F_T(\lambda,k)$, we can obtain the limit distribution of the $UD\max$ and $SEQ_T(k+1|k)$ tests. These are stated in the following Corollary.

Corollary 1 *Under A1-A6 and* $\Omega_{1z}^f = \Omega_{1z}^b = 0$, *for a particular testing problem denote the limit distribution of the test* $sup_{\lambda\in\Lambda_\epsilon^k} F_T(\lambda,k)$ *by* $sup_{\lambda\in\Lambda_\epsilon^k} F(\lambda,k)/k$, *then: a)* $UD\max F_T(M) = \max_{1\le m\le M} sup_{\lambda\in\Lambda_\epsilon^m} F_T(\lambda,m) \Rightarrow \max_{1\le m\le M} sup_{\lambda\in\Lambda_\epsilon^m} F(\lambda,m)/m$, *b)* $\lim_{T\to\infty} P(SEQ_T(k+1|k) \le x) = G_\varepsilon(x)^{k+1}$, *with* $G_\varepsilon(x)$ *the distribution function of* $\sup_{\lambda\in\Lambda_\varepsilon^1} F(\lambda,1)$.

4.2 Trends in regressors

Suppose now that the $I(1)$ regressors have a trending non-stochastic component, i.e., are generated by $z_{ft}^* = \rho_f t + z_{ft}$ and $z_{bt}^* = \rho_b t + z_{bt}$ with $q_b > 1$ and $\rho_b \neq 0$. The limiting distributions of the tests are then different from the non-trending case. The derivation of the required modifications follow the treatment of Hansen (1992a). Consider a $q_b \times (q_b - 1)$ matrix ρ_b^* which spans the null space of ρ_b and let $C_2 = [C_{12}, C_{22}] = (\rho_b(\rho_b'\rho_b)^{-1}, \rho_b^*(\rho_b^{*\prime}\Omega_{zz}^{bb}\rho_b^*)^{-1/2})$. Note that $C_2' z_{bt}^* = (C_{12}' z_{bt} + t, C_{22}' z_{bt})'$. With $\bar{W}_{2T} = diag\left(T, I_{q_b-1}T^{1/2}\right)$, we have

$$\bar{W}_{2T}^{-1}C_2' z_{b[Tr]} = \begin{pmatrix} T^{-1}C_{12}' z_{b[Tr]} + T^{-1}[Tr] \\ T^{-1/2}C_{22}' z_{b[Tr]} \end{pmatrix} \Rightarrow \begin{pmatrix} r \\ W_{z(-1)}^b(r) \end{pmatrix} \equiv J_z^b(r) \tag{4}$$

where $W_{z(-1)}^b(r)$ is a $(q_b - 1)$ dimensional vector of independent Wiener processes (a linear combination of $W_z^b(r)$). Note that when $q_b = 1$, $W_{z(-1)}^b(r) = r$. It then follows that

$$T^{-1}\bar{W}_{2T}^{-1}C_2'\sum_{t=1}^{[Tr]} z_{bt}^* z_{bt}^{*\prime} C_2 \bar{W}_{2T}^{-1} \Rightarrow \int_0^r J_z^b J_z^{b\prime} \tag{5}$$

$$T^{-1/2}\bar{W}_{2T}^{-1}C_2'\sum_{t=1}^{[Tr]} z_{bt}^* u_t \Rightarrow \sigma\int_0^r J_z^b dW_1 \tag{6}$$

Note that (4) through (6) also hold for z^*_{ft} with $W^b_{z(-1)}(r)$ replaced by $W^f_{z(-1)}(r)$, a $(q_f - 1)$ dimensional vector of independent Wiener processes (a linear combination of $W^f_z(r)$). Here also, when $q_f = 1$, $W^f_{z(-1)}(r) = r$. Therefore, with trending regressors, the limiting distributions of the tests are not the same as that without trends. However, we can obtain them by simply replacing W^f_z and W^b_x by J^f_z and J^b_z, respectively.

4.3 Asymptotic critical values

Since the asymptotic distributions are non-standard, critical values are obtained through simulations. These are provided for models with and without trends in regressors. We approximate the Wiener processes by partial sums of *i.i.d.* Normal random variables with $N = 500$ steps. The number of replications is 2000. For each replication, the supremum of $F(\lambda, k)$ with respect to $(\lambda_1, ..., \lambda_k)$ over the set Λ^k_ϵ is obtained via a dynamic programming algorithm (see Bai and Perron, 2003, for details). The $I(0)$ regressors are simulated as independent sequences of *i.i.d.* $N(0,1)$ random variables, and the $I(1)$ regressors as independent random walks with *i.i.d.* $N(0,1)$ errors (also independent of the $I(0)$ regressors). The values of the trimming used are $\epsilon = .05, .10, .15, .20$ and $.25$. Critical values are presented for up to 9 breaks and 4 regressors. The maximum number of breaks allowed is 8 when $\epsilon = 0.10$, 5 when $\epsilon = 0.15$, 3 when $\epsilon = 0.20$ and 2 when $\epsilon = 0.25$. For the UDmax test, M is set to 5 or the maximum number of breaks possible. For models involving both $I(1)$ and $I(0)$ variables, critical values are provided for all possible permutations up to 2 regressors of each type. For the limit distributions of the tests when the regressors contain trends and for the sequential tests, the critical values are tabulated for $\epsilon = .15$, $.20$ and $.25$. Given the large number of results, we present critical values only for $\epsilon = 0.15$ in Tables 1 through 4. For other trimming values, tables of critical values are available on the authors' website.

5 Extensions

We now extend the analysis of the previous Section to the cases where we can have either a) serially correlated errors in the cointegrating regression; b) endogenous regressors. We show that simple modifications yield tests with the same limit distributions as stated above.

5.1 Serially correlated errors: a modified sup-Wald test

With serially correlated errors, we use the following robust version of the scaled F test

$$F^*_T(\lambda, k) = \frac{(T - (k+1)(q_b + p_b) - (q_f + p_f))}{k} \hat{\gamma}' R' (RT\hat{V}(\hat{\gamma})R')^{-1} R\hat{\gamma} \tag{7}$$

where $\hat{V}(\hat{\gamma})$ is an estimate of the covariance matrix of $\hat{\gamma}$ that is robust to serial correlation and heteroskedasticity; see Bai and Perron (1998) for details. Note that when testing for the stability of coefficients associated with $I(1)$ variables, whether $I(0)$ variables are included or not, we can simply apply the following transformation to the test in (3): $F_T^*(\lambda; k) = \left(\hat{\sigma}_u^2/\hat{\sigma}^2\right) F_T(\lambda, k)$, where $\hat{\sigma}_u^2 = T^{-1}\sum_{t=1}^{T} \hat{u}_t^2$ and $\hat{\sigma}^2$ is a consistent estimate of σ^2. Since the break fractions are consistent even with serially correlated errors, we can first take the supremum of the original F test to obtain the break points. The robust version of the test is then obtained by evaluating $F_T^*(\lambda; k)$ at these estimated break dates, i.e., the test considered is $sup_{\lambda\in\Lambda_\epsilon^k} F_T^*(\lambda, k) = F_T^*(\hat{\lambda}, k)$ where $\hat{\lambda} = (\hat{\lambda}_1, ..., \hat{\lambda}_k)$ are the estimates of the break fractions obtained by minimizing the global sum of squared residuals (2).

A problem with the Sup-Wald test is that with persistent errors, the size distortions can be substantial. The reason for this is the estimation of the long run variance using residuals under the alternative hypothesis. On the other hand, Vogelsang (1999) shows through simulation experiments that the estimation of the long run variance under the null hypothesis leads to the problem of non-monotonic power in finite samples. In a related paper, Crainiceanu and Vogelsang (2007) show that commonly used data dependent bandwidths for the estimation of the long run variance (based on the misspecified null model) are too large under the alternative hypothesis. This in turn leads to a decrease in power as the magnitude of the change increases. As a solution to this size-power trade-off, we use a new estimator of the long run variance constructed using a hybrid method that involves residuals computed under both the null and alternative hypotheses. In particular, the data dependent bandwidth is selected based on the residuals obtained under the alternative hypothesis. With this particular value of the bandwidth, the estimate is computed using residuals obtained under the null hypothesis of no structural change. Specifically, the proposed estimator is

$$\hat{\sigma}^2 = T^{-1}\sum_{t=1}^{T} \widetilde{u}_t^2 + 2T^{-1}\sum_{j=1}^{T-1} w(j/\hat{h}) \sum_{t=j+1}^{T} \widetilde{u}_t \widetilde{u}_{t-j} \tag{8}$$

where $\widetilde{u}_t$ are the residuals obtained imposing the null hypothesis. The kernel function $w(\cdot)$ is the Quadratic Spectral and the estimate of the bandwidth is, following Andrews (1991), given by $\hat{h} = 1.3221(\hat{a}(2)T)^{1/5}$ where $\hat{a}(2) = [4\hat{\rho}^2/(1-\hat{\rho})^4]$ and $\hat{\rho} = \sum_{t=2}^{T} \hat{u}_t\hat{u}_{t-1}/\sum_{t=2}^{T} \hat{u}_{t-1}^2$, with $\hat{u}_t$ the residuals from the model estimated under the alternative hypotheses. As we later demonstrate, the sup-Wald test based on this estimator is able to bypass the problem of non-monotonic power while maintaining an exact size close to the nominal size. For more details on the merits of this approach, see Kejriwal (2008).

5.2 Endogenous $I(1)$ regressors

Generally, the assumption of strict exogeneity is too restrictive and the test statistics developed in the previous section are not robust to the problem of endogenous regressors. In this section, we use the linear leads and lags estimator (dynamic OLS) as proposed by Saikkonen (1991) and Stock and Watson (1993) and prove that the limiting distributions of the tests based on this estimator are the same as those obtained with the static regression under strict exogeneity. The modified regression is given by

$$y_t = \hat{c}_i + z'_{ft}\hat{\delta}_f + x'_{ft}\hat{\beta}_f + z'_{bt}\hat{\delta}_{bi} + x'_{bt}\hat{\beta}_{bi} + \sum_{j=-\ell_T}^{\ell_T} \Delta z'_{t-j}\hat{\Pi}_j + \hat{v}^*_t \tag{9}$$

where $z_t = (z'_{ft}, z'_{bt})'$. Note that the number of leads and lags of Δz_t need not be the same. We specify the same value for simplicity. Alternatively, one can interpret ℓ_T as the maximum of the number of leads and lags. In order to prove our results, we need a few additional assumptions, which are the same that are required to show the consistency of the estimate of the cointegrating vector in the case of a model with no structural change.

Assumption A7: Let $\zeta_t = (u_t, u^{f\prime}_{zt}, u^{b\prime}_{zt})'$ and $\zeta_{zt} = (u^{f\prime}_{zt}, u^{b\prime}_{zt})'$. The spectral density matrix $f_{\zeta\zeta}(w)$ is bounded away from zero so that $f_{\zeta\zeta}(w) \geq \alpha I_n$ $(n = q_f + q_b + 1)$ for $w \in [0, \pi]$ where $\alpha > 0$. Also, the covariance function of ζ_t is absolutely summable, i.e., denoting $E(\zeta_t\zeta'_{t+k}) = \Gamma(k)$, we require that $\sum_{k=-\infty}^{\infty} ||\Gamma(k)|| < \infty$ where $||\cdot||$ is the standard Euclidean norm. Denoting the fourth order cumulants of ζ_t by $\kappa_{ijkl}(m_1, m_2, m_3)$, it is assumed that $\sum_{m_1}\sum_{m_2}\sum_{m_3} |\kappa_{ijkl}(m_1, m_2, m_3)| < \infty$ (where the summations run from $-\infty$ to $+\infty$).

Assumption A7 states the same conditions used by Saikkonen (1991) and allows to represent the error u_t as follows: $u_t = \sum_{j=-\infty}^{\infty} \zeta'_{zt-j}\Pi_j + v_t$, with $\sum_{k=-\infty}^{\infty} ||\Pi_j|| < \infty$ and where v_t is a stationary process such that $E(\zeta_{zt}v_{t+k}) = 0$, for all k, and $f_{vv}(w) = f_{uu}(w) - f_{u\zeta_z}(w)f_{\zeta_z\zeta_z}(w)^{-1}f_{\zeta_z u}(w)$. The DGP under the null hypothesis is then

$$y_t = c + z'_{ft}\delta_f + x'_{ft}\beta_f + \sum_{j=-\ell_T}^{\ell_T} \Delta z'_{t-j}\Pi_j + v^*_t$$

where $v^*_t = v_t + \sum_{|j|>\ell_T} \zeta'_{z,t-j}\Pi_j \equiv v_t + e_t$. The last requirements pertain to the possible rate of increase of ℓ_T as T increases. Following Kejriwal and Perron (2008a), these are given by:

Assumption A8: As $T \to \infty$, $\ell_T \to \infty$, $\ell_T^2/T \to 0$ and $\ell_T \sum_{|j|>\ell_T} ||\Pi_j|| \to 0$.

Note that A8 allows the use of information criteria such as the AIC or BIC. Since there can be serial correlation in the errors v_t, we need to apply a correction for its presence. Hence, we consider the statistic $sup_{\lambda \in \Lambda^k_\epsilon}\ F^D_T(\lambda, k) = F^D_T(\hat{\lambda}, k)$ where $\hat{\lambda} = (\hat{\lambda}_1, ..., \hat{\lambda}_k)$ are the estimates of the break fractions obtained by minimizing the global sum of squared residuals (2), and $F_T^D(\lambda, k) = T^{-1}(SSR_k/\hat{\sigma}_v^2)F_T(\lambda, k)$ with $F_T(\lambda, k)$ as defined in (3). We consider an estimate $\hat{\sigma}_v^2$ based on a weighted sum of the sample autocovariances of $\widetilde{v}_t^*$, the residuals obtained imposing the null, as defined by (8) with $\widetilde{v}_t^*$ instead of $\widetilde{u}_t$ (and using the unrestricted residuals to obtain the bandwidth as discussed in the previous section). The relevant result is stated in the following Proposition.

Theorem 3 *Under A1-A5 and A7-A8, for all testing problems the limit distributions of the test* $\sup_{\lambda \in \Lambda_\epsilon^k} F_T^D(\lambda, k)$, *based on regression (9), are the same as those that apply to the test* $\sup_{\lambda \in \Lambda_\epsilon^k} F_T(\lambda, k)$ *under the added assumption of A6 and strict exogeneity with* $\Omega_{1z}^f = \Omega_{1z}^b = 0$.

6 Simulation experiments

We now present the results of simulation experiments that pertain to the size and power of the tests, including a comparison with the often used LM tests. Hansen's (2000) method based on a "fixed regressors bootstrap" is also a possible avenue to provide valid large sample inference in some of the models considered. In theory, an advantage of his method is that it remains valid in the presence of changes in the marginal distributions of the regressors. We conducted extensive simulations and found that the Wald tests considered here are very robust to changes in the drift of the I(1) regressors and changes in the variance of the innovations driving them (as in the stationary case as reported by Hansen, 2000). Our asymptotic results provide tests with exact sizes close to nominal size, as we shall show.

6.1 The size of the tests

We start with the case where the DGP exhibits no structural change and hence analyze the size of the tests. The sample sizes considered are $T = 120$ and $T = 240$. The value of the trimming ϵ is set to .20. The maximum number of breaks (M) considered is 3. Depending on whether we correct for serial correlation and/or endogeneity, we have the following four specifications: (i) S_Corr=0, C_Corr=0: no correction for serial correlation or endogeneity; (ii) S_Corr=1, C_Corr=0: correction for serial correlation but not for endogeneity; (iii) S_Corr=0, C_Corr=1: correction for endogeneity but not for serial correlation; and (iv) S_Corr=1, C_Corr=1: correction for both endogeneity and serial correlation. To correct for serial correlation, we use the method discussed in Section 5.1. To correct for endogeneity, we use the dynamic OLS estimator, discussed in Section 5.2, with $\ell_T = 2$. The various DGPs considered include the following basic components: $y_t = z_t + u_t$ with $z_t = z_{t-1} + \eta_t$, where

$\eta_t \sim i.i.d.\ N(0,1)$. The DGPs considered are, where $e_t \sim i.i.d.\ N(0,1)$ and $Cov(\eta_t, e_t) = 0$: DGP-1 (i.i.d. errors, exogenous regressor): $u_t = e_t$; DGP-2 (AR(1) errors, exogenous regressor): $u_t = \rho u_{t-1} + e_t$; DGP-3 (MA(1) errors, exogenous regressor): $u_t = e_t - \theta e_{t-1}$; DGP-4 (i.i.d. errors, endogenous regressor): $u_t = 0.8\eta_t + e_t$; DGP-5 (MA(1) errors, endogenous regressor): $u_t = 0.5v_t + \eta_t,\ v_t = e_t - 0.5e_{t-1}$.

For each DGP, we consider the case where the regressors are $\{1, z_t\}$ and both the intercept and the cointegrating coefficient are allowed to change across regimes. In all experiments, 1000 replications are used. All rejection frequencies are calculated at the nominal 5% level. Table 5 reports the empirical size, with $T = 120$ and 240 and $\rho = \theta = 0.5$. Consider first the base case represented by DGP-1 where the regressor is strictly exogenous and the errors are *i.i.d.*. With S_Corr=0, C_Corr=0, the size is adequate for all the tests irrespective of the specification used. For DGP-2 with AR(1) errors, all tests show substantial distortions when we do not correct for serial correlation. However, using our proposed long run variance estimator, the size distortions are no longer present and the tests become somewhat conservative. With MA(1) errors (DGP-3), the tests have zero size when no correction for serial correlation is made. Again, the size is accurate once we use S_Corr=1. With endogeneity but no serial correlation (DGP-4), we see that all the tests have good size for S_Corr=0, C_Corr=1. Otherwise, size distortions up to 20% may occur. This shows that the correction for endogeneity based on the dynamic OLS estimator is quite effective. When both serial correlation and endogeneity are present (DGP-5), the tests have adequate size when we use S_Corr=1, C_Corr=1, although some mild distortions persist when testing for multiple breaks. When $T = 240$, for the DGP-5 and S_Corr=1, C_Corr=1, the rejection frequencies are reduced and even the multiple break tests become conservative.

We also considered the case where the regressors are $\{1, z_t, x_t\}$, with $x_t \sim i.i.d.\ N(1,1)$, $Cov(x_t, u_t) = Cov(x_t, \eta_t) = 0$, and the model allows the intercept and the cointegrating coefficient to change across regimes but the coefficient of x_t is held fixed. The results were similar to those in Table 5. Hence, including an irrelevant $I(0)$ regressor does not lead to any size inaccuracies over and above the case when they are not included.

6.2 A power comparison with the LM type tests

In this section, we analyze the power of the sup-Wald test and compare it with the sup, mean and exp-LM tests proposed in Hansen (1992b) and Hao (1996). Vogelsang (1999) and Crainiceanu and Vogelsang (2007) show that the power function of a wide variety of tests for a shift in the mean of a dynamic time series is non-monotonic with respect to the magnitude of the break. One cause is the behavior of the estimate of the error variance in the presence of a shift in mean. In particular, they find that if the error variance is estimated under the null hypothesis, non-monotonic power can result. We show that the LM type tests suffer from the same problem in the cointegration setup and in certain cases, the power can go to zero as the magnitude of the break increases. Since the main issue pertains to the presence of serial correlation in the errors, we consider the case where the regressor is strictly exogenous and the trimming is set at $\epsilon = 0.15$ (we also performed simulation of the power of our tests with a DGP involving endogenous regressors and, actually, the power is enhanced relative to the exogenous regressor case). For the case with one break, the DGP is $y_t = z_t + u_t$, if $t \leq [T/2]$ and $y_t = (1+\delta)z_t + u_t$, if $t > [T/2]$, where $\eta_t \sim i.i.d.\ N(0,1)$, $Cov(u_t, \eta_t) = 0$. The sample size is $T = 240$. We consider DGP 2 (AR(1) errors) and 3 (MA(1) errors). The specification S_Corr=1, C_Corr=0 is used. We analyze the pure structural change model in which both the intercept and the cointegrating coefficient are allowed to change. The power functions are plotted in Figure 1. Consider first the case with AR(1) errors. The non-monotonicity of the power function of the LM tests is evident even at moderate values of δ. For very small values of δ, the power of the mean LM test is slightly higher than the modified Wald test. This is due to the fact that the mean LM test is particularly suited to detect small changes (see Andrews and Ploberger, 1994). Surprisingly, however, the mean LM test performs better than the exp-LM test even for large changes. The sup-LM test is dominated by all tests irrespective of the sample size and the degree of persistence. With MA(1) errors, the picture is quite different. All tests have higher power compared to the autoregressive case although non-monotonicity is still evident for the LM tests. The performance of the LM tests is quite similar and no clear ranking emerges between them.

Next, we consider the case where the DGP involves 2 breaks and 3 regimes, specified by $y_t = z_t + u_t$, if $t \leq [T/3]$, $y_t = (1+\delta)z_t + u_t$ if $[T/3] < t \leq [2T/3]$ and $y_t = z_t + u_t$ if $[2T/3] < t \leq T$, where $z_t = z_{t-1} + \eta_t$, $z_t = z_{t-1} + \eta_t$, $\eta_t \sim i.i.d.\ N(0,1)$ and $Cov(u_t, \eta_t) = 0$. The power functions are plotted in Figure 2. Consider first the case with AR(1) errors. Given that single break tests have difficulty in detecting such parameter changes, it is not surprising that all tests exhibit non-monotonic power. The modified sup-Wald test dominates all the LM tests regardless of the sample size and the extent of persistence. With MA(1) errors, again all tests display non-monotonicity although the power function of the modified Wald test is much higher than that of the LM tests. What is quite remarkable is the fact that the $UD\max$ test has, in all cases, a monotonic power function that is much higher than any of the other tests. This provides clear evidence to its usefulness.

Finally, it is useful to comment on what happens when the regression is spurious, i.e., there is no cointegration. Hansen (1992b) showed that the LM test designed to detect a martingale specification in the intercept, in the spirit of Nyblom's (1989) test, can be viewed as a test for the null of cointegration against the alternative of no cointegration. Although the sup-Wald test is not specifically targeted for the alternative of random variation in the intercept, it still has power against spurious regressions (i.e., no cointegration). This means that it will also reject when no structural change is present and there is no cointegration (the errors are $I(1)$). However, we can use the following approach to determine if the data suggest structural changes in a cointegrating relationship or a spurious regression. Suppose that one is willing to put an upper bound M (say 5) on the number of breaks. Then if the system is cointegrated with less than M breaks, the sequential testing procedure can be used to consistently estimate the number of breaks. On the other hand, if the regression is spurious, the number of breaks selected will always (in large samples) be the maximum number of breaks allowed. Thus, selecting the maximum allowable number of breaks can be indicative of the presence of $I(1)$ errors. The same is true when information criteria are used to select the number of breaks. We verified via simulations that this is indeed the case.

7 Conclusion

We presented a comprehensive treatment of issues related to testing in cointegrated regression models with multiple structural changes. We analyzed models with $I(1)$ variables only as well as models which incorporate both $I(0)$ and $I(1)$ regressors. The breaks are allowed to occur either in the intercept, the cointegrating coefficients, the parameters of the $I(0)$ regressors or any combination of these. Our simulation experiments show that the commonly used LM tests are plagued with the problem of non-monotonic power in finite samples. The sup-Wald test however is able to avoid such non-monotonicity while maintaining adequate size. Our asymptotic results allow us to devise a sequential procedure to select the number of breaks. Finally, we provide the asymptotic critical values of our tests for a wide range of models that are expected to be useful in practice. The simulation experiments demonstrate the favorable properties of our test and the proposed long run variance estimator. It is important to note that the idea of constructing the estimate of the long run variance using information under both the null and alternative hypothesis is quite general and is applicable even in regression models which do not involve structural change.

References

Andrews, D.W.K. (1991), "Heteroskedasticity and Autocorrelation Consistent Covariance Matrix Estimation," *Econometrica*, 59, 817-858.

Andrews, D.W.K. (1993), "Tests for Parameter Instability and Structural Change with Unknown Change Point," *Econometrica*, 61, 821-856.

Andrews, D.W.K., and Ploberger, W. (1994), "Optimal Tests when a Nuisance Parameter is present only under the Alternative," *Econometrica,* 62, 1383-1414.

Bai, J. (1997), "Estimation of a Change Point in Multiple Regression Models," *Review of Economics and Statistics,* 79, 551-63.

Bai, J., Lumsdaine, R.L., and Stock, J.H. (1998), "Testing for and Dating Breaks in Multivariate Time Series," *Review of Economic Studies,* 65, 395-432.

Bai, J., and Perron, P. (1998), "Estimating and Testing Linear Models with Multiple Structural Changes," *Econometrica,* 66, 47-78.

Bai, J., and Perron, P. (2003), "Computation and Analysis of Multiple Structural Change Models," *Journal of Applied Econometrics,* 18, 1-22.

Bai, J., and Perron, P. (2006), "Multiple Structural Change Models: A Simulation Analysis," in *Econometric Theory and Practice: Frontiers of Analysis and Applied Research,* D. Corbea, S. Durlauf and B. E. Hansen (eds.), Cambridge University Press, 212-237.

Carrion-i-Silvestre, J.L., and Sansó-i-Rosselló, A.S. (2006), "Testing the Null Hypothesis of Cointegration with Structural Breaks," *Oxford Bulletin of Economics and Statistics,* 68, 623-646.

Crainiceanu, C.M., and Vogelsang, T.J. (2007), "Nonmonotonic Power for Tests of a Mean Shift in a Time Series," forthcoming in J*ournal of Statistical Computation and Simulation.*

Davidson, J. (1994), *Stochastic Limit Theory.* Oxford University Press: Oxford.

Hansen, B.E. (1992a), "Efficient Estimation and Testing of Cointegrating Vectors in the presence of Deterministic Trends," *Journal Econometrics,* 53, 87-121.

Hansen, B.E. (1992b), "Tests for Parameter Instability in Regressions with I(1) Processes," *Journal of Business and Economic Statistics,* 10, 321-335.

Hansen, B.E. (1992c), "Consistent Covariance Matrix Estimation for Dependent Heterogeneous Processes," *Econometrica,* 60, 967-972.

Hansen, B.E. (2000), "Testing for Structural Change in Conditional Models," *Journal of Econometrics,* 97, 93-115.

Hansen, P.R. (2003), "Structural Changes in the Cointegrated Vector Autoregressive Model," *Journal of Econometrics,* 114, 261-295.

Hao, K. (1996), "Testing for Structural Change in Cointegrated Regression Models: Some Comparisons and Generalizations," *Econometric Reviews,* 15, 401-429.

Kejriwal, M. (2008): "Tests for a Mean Shift with Good Size and Monotonic Power," forthcoming in *Economics Letters*.

Kejriwal, M., and Perron, P. (2008a): "Data Dependent Rules for the Selection of the Number of Leads and Lags in the Dynamic OLS Cointegrating Regression," *Econometric Theory,* 24, 1425-1441.

Kejriwal, M., and Perron, P. (2008b): "The Limit Distribution of the Estimates in Cointegrated Regression Models with Multiple Structural Changes," *Journal of Econometrics,* 146, 59-73..

Kuo, B-S. (1998), "Test for Partial Parameter Instability in Regressions with I(1) Processes," *Journal of Econometrics,* 86, 337-368.

Ng, S., and Perron, P. (1997), "Estimation and Inference in Nearly Unbalanced and Nearly Cointegrated Systems," *Journal of Econometrics, 79,* 53-81.

Nyblom, J. (1989), "Testing the Constancy of Parameters Over Time," *Journal of the American Statistical Association*, 84, 223-230.

Phillips, P.C.B., and Hansen, B.E. (1990), "Statistical Inference in Instrumental Variables Regression with I(1) Processes," *Review of Economic Studies,* 57, 99-125.

Perron, P. (2006), "Dealing with Structural Breaks," in *Palgrave Handbook of Econometrics*, K. Patterson and T.C. Mills (eds.), Palgrave Macmillan, 278-352.

Perron, P., and Zhu, X. (2005), "Structural Breaks with Deterministic and Stochastic Trends," *Journal of Econometrics,* 129, 65-119.

Qu, Z. (2007), "Searching for Cointegration in a Dynamic System," *Econometrics Journal*, 10, 580-604.

Qu, Z., and Perron, P. (2007), "Estimating and Testing Structural Changes in Multivariate Regressions," *Econometrica,* 75, 459-502.

Quintos, C.E., and Phillips, P.C.B. (1993), "Parameter Constancy in Cointegrated Regressions," *Empirical Economics,* 18, 675-706.

Saikkonen, P. (1991), "Asymptotically Efficient Estimation of Cointegration Regressions," *Econometric Theory,* 7, 1-21.

Seo, B. (1998), "Tests for Structural Change in Cointegrated Systems," *Econometric Theory,* 14, 222-259.

Stock, J.H., and Watson, M.W. (1993), "A Simple Estimator of Cointegrated Vectors in Higher Order Integrated Systems," *Econometrica,* 61, 783-820.

Vogelsang, T.J. (1999), "Sources of Nonmonotonic Power when Testing for a Shift in Mean of a Dynamic Time Series," *Journal of Econometrics,* 88, 283-299.

Appendix

We use $\|.\|$ to denote the Euclidean norm, i.e., $\|x\| = (\sum_{i=1}^{p} x_i^2)^{1/2}$ for $x \epsilon R^p$. For a matrix A, we use the vector-induced norm, i.e., $\|A\| = \sup_{x \neq 0} \|Ax\| / \|x\|$. We have $\|A\| \leq [tr(A'A)]^{1/2}$. Also, for a projection matrix P, $\|PA\| \leq \|A\|$. We use the notation $\widetilde{A}_{i,j} = A_{(i,j)} - \bar{A}_{(i,j)}$, where $A_{(i,j)}$ is the matrix of observations from regime i to regime j (both inclusive), i.e., $A_{(i,j)} = (a_{T_{i-1}+1}, ..., a_{T_j})'$ while $\bar{A}_{(i,j)}$ is the matrix (conformable to $A_{i,j}$) of means, i.e., $\bar{A}_{(i,j)} = (\bar{a}_{i,j}, ..., \bar{a}_{i,j})'$ where $\bar{a}_{i,j} = (T_j - T_{i-1})^{-1} \sum_{t=T_{i-1}+1}^{T_j} a_t$. Also, we use $A^*_{(i,j)} = A_{(i,j)} - \bar{A}^{(i,j)}$, where $\bar{A}^{(i,j)}$ is the matrix (conformable to $A_{(i,j)}$) of sample averages, i.e., $\bar{A}^{(i,j)} = (\bar{x}, ..., \bar{x})'$, where $\bar{x} = T^{-1} \sum_{t=1}^{T} x_t$. Let $1_{(i,j)}$ be a $(T_j - T_{i-1}) \times 1$ vector of ones. To ease notation, we will write $\widetilde{A}_{(i,i)}$ as $\widetilde{A}_i$, $A^*_{(i,i)}$ as A^*_i, $\bar{A}_{(i,i)}$ as $\bar{A}_i$, $\bar{A}^{(i,i)}$ as $\bar{A}^i$ and $1_{(i,j)}$ as 1_i, $(W_1, W_z^f, W_z^b, W_x^f, W_x^b)$ are independent Wiener processes with dimensions corresponding to those of $(B_1, B_z^f, B_z^b, B_x^f, B_x^b)$. We also use the notation $W_z = (W_z^{f\prime}, W_z^{b\prime})'$. We start with a Lemma about the weak convergence of various sample moments whose proof is standard given the results in Qu and Perron (2007).

Lemma A.1 *Under A1-A5, the following weak convergence results hold (for $i = 1, ..., m+1$): a)* $T^{-3/2} \sum_{t=1}^{[T\lambda_i]} z_{ft} \Rightarrow \int_0^{\lambda_i} B_z^f$, $T^{-3/2} \sum_{t=1}^{[T\lambda_i]} z_{bt} \Rightarrow \int_0^{\lambda_i} B_z^b$, $T^{-1/2} \sum_{t=1}^{[T\lambda_i]} u_{xt}^f \Rightarrow B_x^f(\lambda_i)$, $T^{-1/2} \sum_{t=1}^{[T\lambda_i]} u_{xt}^b \Rightarrow B_x^b(\lambda_i)$, $T^{-1/2} \sum_{t=1}^{[T\lambda_i]} u_t \Rightarrow B_1(\lambda_i)$; *b)* $T^{-2} \sum_{t=1}^{[T\lambda_i]} z_{ft} z'_{ft} \Rightarrow \int_0^{\lambda_i} B_z^f B_z^{f\prime}$, $T^{-2} \sum_{t=1}^{[T\lambda_i]} z_{bt} z'_{bt} \Rightarrow \int_0^{\lambda_i} B_z^b B_z^{b\prime}$; *c)* $T^{-1} \sum_{t=1}^{[T\lambda_i]} z_{ft} u_t \Rightarrow \int_0^{\lambda_i} B_z^f dB_1 + \lambda_i(\Sigma_{z1}^f + \Lambda_{z1}^f)$, $T^{-1} \sum_{t=1}^{[T\lambda_i]} z_{bt} u_t \Rightarrow \int_0^{\lambda_i} B_z^b dB_1 + \lambda_i(\Sigma_{z1}^b + \Lambda_{z1}^b)$; *d)* $T^{-1} \sum_{t=1}^{[T\lambda_i]} z_{ft} u_{xt}^{f\prime} \Rightarrow \int_0^{\lambda_i} B_z^f dB_x^{f\prime} + \lambda_i(\Sigma_{zx}^{ff} + \Lambda_{zx}^{ff})$, $T^{-1} \sum_{t=1}^{[T\lambda_i]} z_{ft} u_{xt}^{b\prime} \Rightarrow \int_0^{\lambda_i} B_z^f dB_x^{b\prime} + \lambda_i(\Sigma_{zx}^{fb} + \Lambda_{zx}^{fb})$, $T^{-1} \sum_{t=1}^{[T\lambda_i]} z_{bt} u_{xt}^{f\prime} \Rightarrow \int_0^{\lambda_i} B_z^b dB_x^{f\prime} + \lambda_i(\Sigma_{zx}^{bf} + \Lambda_{zx}^{bf})$, $T^{-1} \sum_{t=1}^{[T\lambda_i]} z_{bt} u_{xt}^{b\prime} \Rightarrow \int_0^{\lambda_i} B_z^b dB_x^{b\prime} + \lambda_i(\Sigma_{zx}^{bb} + \Lambda_{zx}^{bb})$.

The next Lemma will also be useful in subsequent developments.

Lemma A.2 *Let* $\bar{X}_{i(T_i - T_{i-1}) \times p)} = (\bar{x}_i, ..., \bar{x}_i)'$, $\bar{x}_i = (T_i - T_{i-1})^{-1} \sum_{t=T_{i-1}+1}^{T_i} x_t$ *and* $\mu^i_{((T_i - T_{i-1}) \times p)} = (\mu, ..., \mu)'$. *Then under A1-A4, we have for* $i = 1, ..., m+1$*: (i)* $\mu^i - \bar{X}_i \xrightarrow{p} 0$*; (ii)* $T^{-1/2}(X_i - \bar{X}_i)'U_i = T^{-1/2}(X_i - \mu^i)'U_i + o_p(1)$*; (iii)* $T^{-1}(X_i - \bar{X}_i)'(X_i - \bar{X}_i) = T^{-1}(X_i - \mu^i)'(X_i - \mu^i) + o_p(1)$*; (iv)* $T^{-3/2} Z_i'(X_i - \bar{X}_i) = T^{-3/2} Z_i'(X_i - \mu^i) + o_p(1)$.

Proof of Lemma A.2: Part (i) follows trivially. To prove (ii), note that $T^{-1/2}(X_i - \bar{X}_i)'U_i = T^{-1/2}(X_i - \mu^i)'U_i + T^{-1/2}(\mu^i - \bar{X}_i)'U_i$. We have $T^{-1/2}(\mu^i - \bar{X}_i)'U_i = (\mu - \bar{x}_i) T^{-1/2} \sum_{t=T_{i-1}+1}^{T_i} u_t = o_p(1)$, using part (i). For (iii), note that

$$\begin{aligned} T^{-1}(X_i - \bar{X}_i)'(X_i - \bar{X}_i) &= T^{-1}(X_i - \mu^i)'(X_i - \mu^i) + T^{-1}(X_i - \mu^i)'(\mu^i - \bar{X}_i) \\ &\quad + T^{-1}(\mu^i - \bar{X}_i)'(X_i - \mu^i) + T^{-1}(\mu^i - \bar{X}_i)'(\mu^i - \bar{X}_i) \end{aligned}$$

Now $T^{-1}(X_i - \mu^i)'(\mu^i - \bar{X}_i) = T^{-1} \sum_{t=T_{i-1}+1}^{T_i} (x_t - \mu)(\mu - \bar{x}_i)' = -(\lambda_i - \lambda_{i-1})(\mu - \bar{x}_i)(\mu - \bar{x}_i)' = o_p(1)$. Similarly, $T^{-1}(\mu^i - \bar{X}_i)'(X_i - \mu^i) = o_p(1)$. Finally, $T^{-1}(\mu^i - \bar{X}_i)'(\mu^i - \bar{X}_i) = (\lambda_i - \lambda_{i-1})(\mu - \bar{x}_i)(\mu - \bar{x}_i)' = o_p(1)$. To prove (iv), note that $T^{-3/2} Z_i'(\mu^i - \bar{X}_i) = T^{-3/2}(\sum_{t=T_{i-1}+1}^{T_i} z_t)(\mu - \bar{x}_i) = o_p(1)$ and the result follows immediately.

Proof of Theorem 1: We only consider Cases (1) and (4). The details for the other cases can be found in the working paper version. We have

$$F_T(\lambda,k)=\frac{SSR_0-SSR_k}{k(T-(k+1)(q_b+p_b)-q_f-p_f)^{-1}SSR_k}$$

where SSR_0 and SSR_k are the sum of squared residuals under the null and alternative hypotheses, respectively. In all cases, we have $k(T-(k+1)(q_b+p_b)-q_f-p_f)^{-1}SSR_k\xrightarrow{p}k\sigma^2$.
Case 1: The regression under H_1 is $y_t=c_i+z_{bt}'\delta_{bi}+u_t$ and for SSR_0 we have

$$\begin{aligned}SSR_0 &= (Y^*_{(1,k+1)}-Z^*_{b(1,k+1)}\widetilde{\delta}_b)'(Y^*_{(1,k+1)}-Z^*_{b(1,k+1)}\widetilde{\delta}_b)\\ &= (Z^*_{b(1,k+1)}(\delta_b-\widetilde{\delta}_b)+U^*_{(1,k+1)})'(Z^*_{b(1,k+1)}(\delta_b-\widetilde{\delta}_b)+U^*_{(1,k+1)})\\ &= U^{*\prime}_{(1,k+1)}U^*_{(1,k+1)}-(Z^{*\prime}_{b(1,k+1)}U^*_{(1,k+1)})'(Z^{*\prime}_{b(1,k+1)}Z^*_{b(1,k+1)})^{-1}(Z^{*\prime}_{b(1,k+1)}U^*_{(1,k+1)}) \qquad (A.1)\end{aligned}$$

$$\begin{aligned}SSR_k &= \sum_{i=1}^{k+1}(\widetilde{Y}_i-\widetilde{Z}_{bi}\hat{\delta}_{bi})'(\widetilde{Y}_i-\widetilde{Z}_{bi}\hat{\delta}_{bi})=\sum_{i=1}^{k+1}(\widetilde{Z}_{bi}(\delta_b-\hat{\delta}_{bi})+\widetilde{U}_i)'(\widetilde{Z}_{bi}(\delta_b-\hat{\delta}_{bi})+\widetilde{U}_i)\\ &= \sum_{i=1}^{k+1}\{-(\widetilde{Z}_{bi}'\widetilde{U}_i)'(\widetilde{Z}_{bi}'\widetilde{Z}_{bi})^{-1}(\widetilde{Z}_{bi}'\widetilde{U}_i)+\widetilde{U}_i'\widetilde{U}_i\}\end{aligned}$$

Therefore,

$$\begin{aligned}&SSR_0-SSR_k\Rightarrow-\sigma^2(\textstyle\int_0^1 W_z^{b(1,k+1)}dW_1)'(\int_0^1 W_z^{b(1,k+1)}W_z^{b(1,k+1)\prime})^{-1}(\int_0^1 W_z^{b(1,k+1)}dW_1)\\ &+\sigma^2\sum_{i=1}^{k+1}\{(\textstyle\int_{\lambda_{i-1}}^{\lambda_i}W_z^{b(i,i)}dW_1)'(\int_{\lambda_{i-1}}^{\lambda_i}W_z^{b(i,i)}W_z^{b(i,i)\prime})^{-1}(\int_{\lambda_{i-1}}^{\lambda_i}W_z^{b(i,i)}dW_1)\}\\ &+\sigma^2\sum_{i=1}^{k}\frac{(\lambda_iW_1(\lambda_{i+1})-\lambda_{i+1}W_1(\lambda_i))^2}{\lambda_{i+1}\lambda_i(\lambda_{i+1}-\lambda_i)}\end{aligned}$$

and the result stated follows. **Case 4**: The regression under the alternative hypothesis is $y_t=c_i+z_{ft}'\delta_f+z_{bt}'\delta_{bi}+u_t$. Let $Z^*_{(1,k+1)}=(Z^*_{f(1,k+1)},Z^*_{b(1,k+1)})$ and $\widetilde{\delta}=(\widetilde{\delta}_f',\widetilde{\delta}_b')'$. We have

$$\begin{aligned}SSR_0 &= (Y^*_{(1,k+1)}-Z^*_{(1,k+1)}\widetilde{\delta})'(Y^*_{(1,k+1)}-Z^*_{(1,k+1)}\widetilde{\delta})\\ &= -(Z^{*\prime}_{(1,k+1)}U^*_{(1,k+1)})'(Z^{*\prime}_{(1,k+1)}Z^*_{(1,k+1)})^{-1}(Z^{*\prime}_{(1,k+1)}U^*_{(1,k+1)})+U^{*\prime}_{(1,k+1)}U^*_{(1,k+1)}\end{aligned}$$

$$\begin{aligned}SSR_k &= \sum_{i=1}^{k+1}(\widetilde{Y}_i-\widetilde{Z}_{fi}\hat{\delta}_f-\widetilde{Z}_{bi}\hat{\delta}_{bi})'(\widetilde{Y}_i-\widetilde{Z}_{fi}\hat{\delta}_f-\widetilde{Z}_{bi}\hat{\delta}_{bi})\\ &= \sum_{i=1}^{k+1}(\widetilde{Z}_{fi}(\delta_f-\hat{\delta}_f)+\widetilde{Z}_{bi}(\delta_b-\hat{\delta}_{bi})+\widetilde{U}_i)'(\widetilde{Z}_{fi}(\delta_f-\hat{\delta}_f)+\widetilde{Z}_{bi}(\delta_b-\hat{\delta}_{bi})+\widetilde{U}_i)\end{aligned}$$

After considerable algebra, we can show that

$$\begin{aligned}SSR_k &= -(\sum_{i=1}^{k+1}\widetilde{Z}_{1i}'M_{bi}\widetilde{U}_i)'(\sum_{i=1}^{k+1}\widetilde{Z}_{1i}'M_{bi}\widetilde{Z}_{1i})^{-1}(\sum_{i=1}^{k+1}\widetilde{Z}_{1i}'M_{bi}\widetilde{U}_i)\\ &\quad-\sum_{i=1}^{k+1}(\widetilde{Z}_{bi}'\widetilde{U}_i)'(\widetilde{Z}_{bi}'\widetilde{Z}_{bi})^{-1}(\widetilde{Z}_{bi}'\widetilde{U}_i)+\sum_{i=1}^{k+1}(\widetilde{U}_i'\widetilde{U}_i)\end{aligned}$$

where $M_{bi} = I_i - \widetilde{Z}_{bi}(\widetilde{Z}'_{bi}\widetilde{Z}_{bi})^{-1}\widetilde{Z}'_{bi}$ and I_i the $(T_i - T_{i-1}) \times (T_i - T_{i-1})$ identity matrix. Thus,

$$\begin{aligned} SSR_0 - SSR_k &= -(Z^{*\prime}_{(1,k+1)}U^*_{(1,k+1)})'(Z^{*\prime}_{(1,k+1)}Z^*_{(1,k+1)})^{-1}(Z^{*\prime}_{(1,k+1)}U^*_{(1,k+1)}) \\ &\quad +(\sum_{i=1}^{k+1}\widetilde{Z}'_{fi}M_{bi}\widetilde{U}_i)'(\sum_{i=1}^{k+1}\widetilde{Z}'_{fi}M_{bi}\widetilde{Z}_{fi})^{-1}(\sum_{i=1}^{k+1}\widetilde{Z}'_{fi}M_{bi}\widetilde{U}_i) \\ &\quad +\sum_{i=1}^{k+1}(\widetilde{Z}'_{bi}\widetilde{U}_i)'(\widetilde{Z}'_{bi}\widetilde{Z}_{bi})^{-1}(\widetilde{Z}'_{bi}\widetilde{U}_i) + U^{*\prime}_{(1,k+1)}U^*_{(1,k+1)} - \sum_{i=1}^{k+1}(\widetilde{U}'_i\widetilde{U}_i) \end{aligned}$$

and, with $B^{fb}_z(r) = (B^f_z(r)', B^b_z(r)')'$,

$$\begin{aligned} SSR_0 - SSR_k \Rightarrow &-(\int_0^1 B_z^{fb(1,k+1)}dB_1)'(\int_0^1 B_z^{fb(1,k+1)}B_z^{fb(1,k+1)\prime})^{-1}(\int_0^1 B_z^{fb(1,k+1)}dB_1) \\ &+(\sum_{i=1}^{k+1}\int_{\lambda_{i-1}}^{\lambda_i}B_z^{M(i,i)}dB_1)'(\sum_{i=1}^{k+1}\int_{\lambda_{i-1}}^{\lambda_i}B_z^{M(i,i)}B_z^{M(i,i)\prime})^{-1}(\sum_{i=1}^{k+1}\int_{\lambda_{i-1}}^{\lambda_i}B_z^{M(i,i)}dB_1) \\ &+\sum_{i=1}^{k+1}(\int_{\lambda_{i-1}}^{\lambda_i}B_z^{b(i,i)}dB_1)'(\int_{\lambda_{i-1}}^{\lambda_i}B_z^{b(i,i)}B_z^{b(i,i)\prime})^{-1}(\int_{\lambda_{i-1}}^{\lambda_i}B_z^{b(i,i)}dB_1) \\ &+\sum_{i=1}^{k}\frac{(\lambda_i B_1(\lambda_{i+1}) - \lambda_{i+1}B_1(\lambda_i))'(\lambda_i B_1(\lambda_{i+1}) - \lambda_{i+1}B_1(\lambda_i))}{\lambda_{i+1}\lambda_i(\lambda_{i+1}-\lambda_i)} \end{aligned}$$

where $B_z^{M(i,i)}(r) = B_z^{f(i,i)}(r) - \int_{\lambda_{i-1}}^{\lambda_i}B_z^{f(i,i)}B_z^{b(i,i)\prime}(\int_{\lambda_{i-1}}^{\lambda_i}B_z^{b(i,i)}B_z^{b(i,i)\prime})^{-1}B_z^{b(i,i)}(r)$. Note that each element of $B_z^{M(i,i)}(r)$ is the residual from the projection of the corresponding element of $B_z^{f(i,i)}(r)$ onto the space spanned by $\{B_{zj}^{b(i,i)}\}_{j=1}^{q_b}$ for a given realization of these stochastic processes. We also have $B_z^{M(i,i)}(r) = (\Omega_{zz}^{ff})^{1/2}W_z^{M(i,i)}(r)$, so that

$$\begin{aligned} kF_T(\lambda,k) \Rightarrow &-(\int_0^1 W_z^{fb(1,k+1)}dW_1)'(\int_0^1 W_z^{fb(1,k+1)}W_z^{fb(1,k+1)\prime})^{-1}(\int_0^1 W_z^{fb(1,k+1)}dW_1) \\ &+(\sum_{i=1}^{k+1}\int_{\lambda_{i-1}}^{\lambda_i}W_z^{M(i,i)}dW_1)'(\sum_{i=1}^{k+1}\int_{\lambda_{i-1}}^{\lambda_i}W_z^{M(i,i)}W_z^{M(i,i)\prime})^{-1}(\sum_{i=1}^{k+1}\int_{\lambda_{i-1}}^{\lambda_i}W_z^{M(i,i)}dW_1) \\ &+\sum_{i=1}^{k+1}(\int_{\lambda_{i-1}}^{\lambda_i}W_z^{b(i,i)}dW_1)'(\int_{\lambda_{i-1}}^{\lambda_i}W_z^{b(i,i)}W_z^{b(i,i)\prime})^{-1}(\int_{\lambda_{i-1}}^{\lambda_i}W_z^{b(i,i)}dW_1) \\ &+\sum_{i=1}^{k}\frac{(\lambda_i W_1(\lambda_{i+1}) - \lambda_{i+1}W_1(\lambda_i))'(\lambda_i W_1(\lambda_{i+1}) - \lambda_{i+1}W_1(\lambda_i))}{\lambda_{i+1}\lambda_i(\lambda_{i+1}-\lambda_i)} \end{aligned}$$

Proof of Theorem 2. We give the details only for cases 4 to 6. **Case 4**. The regression under H_1 is $y_t = c_i + z'_{bt}\delta_{bi} + x'_{bt}\beta_{bi} + u_t$. We have,

$$SSR_0 = [Y^*_{(1,k+1)} - Z^*_{b(1,k+1)}\widetilde{\delta}_b - X^*_{b(1,k+1)}\widetilde{\beta}_b]'[Y^*_{(1,k+1)} - Z^*_{b(1,k+1)}\widetilde{\delta}_b - X^*_{b(1,k+1)}\widetilde{\beta}_b]$$

By Lemmas A.1 and A.2, $T^{-3/2}Z^{*\prime}_{b(1,k+1)}X^*_{b(1,k+1)} = o_p(1)$. Thus,

$$\begin{aligned} SSR_0 &= [Z^*_{b(1,k+1)}(\delta_b - \widetilde{\delta}_b) + X^*_{b(1,k+1)}(\beta_b - \widetilde{\beta}_b) + U^*_{(1,k+1)}]' \times \\ &\quad\ [Z^*_{b(1,k+1)}(\delta_b - \widetilde{\delta}_b) + X^*_{b(1,k+1)}(\beta_b - \widetilde{\beta}_b) + U^*_{(1,k+1)}] \\ &= (\delta_b - \widetilde{\delta}_b)'Z^{*\prime}_{b(1,k+1)}Z^*_{b(1,k+1)}(\delta_b - \widetilde{\delta}_b) + 2(\delta_b - \widetilde{\delta}_b)'Z^{*\prime}_{b(1,k+1)}U^*_{(1,k+1)} + U^{*\prime}_{(1,k+1)}U^*_{(1,k+1)} \\ &\quad +(\beta_b - \widetilde{\beta}_b)'X^{*\prime}_{b(1,k+1)}X^*_{b(1,k+1)}(\beta_b - \widetilde{\beta}_b) + 2(\beta_b - \widetilde{\beta}_b)'X^{*\prime}_{b(1,k+1)}U^*_{(1,k+1)} + o_p(1) \end{aligned}$$

$$\begin{aligned} &= -(T^{-1}U^{*\prime}_{(1,k+1)}Z^*_{b(1,k+1)})(T^{-2}Z^{*\prime}_{b(1,k+1)}Z^*_{b(1,k+1)})^{-1}(T^{-1}Z^{*\prime}_{b(1,k+1)}U^*_{(1,k+1)}) \\ &\quad -(T^{-1/2}U^{*\prime}_{(1,k+1)}X^*_{b(1,k+1)})(T^{-1}X^{*\prime}_{b(1,k+1)}X^*_{b(1,k+1)})^{-1}(T^{-1/2}X^{*\prime}_{b(1,k+1)}U^*_{(1,k+1)}) \\ &\quad +U^{*\prime}_{(1,k+1)}U^*_{(1,k+1)} + o_p(1) \end{aligned}$$

We have $SSR_k = \sum_{i=1}^{k+1}[\widetilde{Y}_i - \widetilde{X}_{bi}\hat{\beta}_{bi} - \widetilde{Z}_{bi}\hat{\delta}_{bi}]'[\widetilde{Y}_i - \widetilde{X}_{bi}\hat{\beta}_{bi} - \widetilde{Z}_{bi}\hat{\delta}_{bi}]$. Using Lemmas A.1-A.2, $T^{-3/2}\widetilde{Z}'_{bi}\widetilde{X}_{bi} = o_p(1)$ and under H_0, $\widetilde{Y}_i = \widetilde{X}_{bi}\beta_b + \widetilde{Z}_{bi}\delta_b + \widetilde{U}_i$, so that

$$\begin{aligned} SSR_k &= \sum_{i=1}^{k+1}[\widetilde{X}_{bi}(\beta_b - \hat{\beta}_{bi}) + \widetilde{Z}_{bi}(\delta_b - \hat{\delta}_{bi}) + \widetilde{U}_i]'[\widetilde{X}_{bi}(\beta_b - \hat{\beta}_{bi}) + \widetilde{Z}_{bi}(\delta_b - \hat{\delta}_{bi}) + \widetilde{U}_i] \\ &= \sum_{i=1}^{k+1}[-(T^{-1}\widetilde{U}'_i\widetilde{Z}_{bi})(T^{-2}\widetilde{Z}'_{bi}\widetilde{Z}_{bi})^{-1}(T^{-1}\widetilde{Z}'_{bi}\widetilde{U}_i) \\ &\quad -(T^{-1/2}\widetilde{U}'_i\widetilde{X}_{bi})(T^{-1}\widetilde{X}'_{bi}\widetilde{X}_{bi})^{-1}(T^{-1/2}\widetilde{X}'_{bi}\widetilde{U}_i) + \widetilde{U}'_i\widetilde{U}_i] + o_p(1) \end{aligned}$$

Therefore,

$$\begin{aligned} kF_T(\lambda,k) \;\Rightarrow\; & -(\textstyle\int_0^1 W_z^{b(1,k+1)}dW_1)'(\int_0^1 W_z^{b(1,k+1)}W_z^{b(1,k+1)\prime})^{-1}(\int_0^1 W_z^{b(1,k+1)}dW_1) \\ & -W^*_{xb}(1)'W^*_{xb}(1) - W_1(1)^2 + \sum_{i=1}^{k+1}\left\{(\lambda_i - \lambda_{i-1})^{-1}(W_1(\lambda_i) - W_1(\lambda_{i-1}))^2\right\} \\ & +\sum_{i=1}^{k+1}(\lambda_i - \lambda_{i-1})^{-1}(W^*_{xb}(\lambda_i) - W^*_{xb}(\lambda_{i-1}))'(W^*_{xb}(\lambda_i) - W^*_{xb}(\lambda_{i-1}) \\ & +\sum_{i=1}^{k+1}[(\textstyle\int_{\lambda_{i-1}}^{\lambda_i} W_z^{b(i,i)}dW_1)'(\int_{\lambda_{i-1}}^{\lambda_i} W_z^{b(i,i)}W_z^{b(i,i)\prime})^{-1}(\int_{\lambda_{i-1}}^{\lambda_i} W_z^{b(i,i)}dW_1)] \end{aligned}$$

which reduces to the expression stated in the Theorem. **Case 5:** The model under H_1 is $y_t = c_i + z'_{ft}\delta_f + x'_{ft}\beta_f + u_t$. We have $SSR_k = \sum_{i=1}^{k+1}[\widetilde{Y}_i - \widetilde{X}_{fi}\hat{\beta}_f - \widetilde{Z}_{fi}\hat{\delta}_f]'[\widetilde{Y}_i - \widetilde{X}_{fi}\hat{\beta}_f - \widetilde{Z}_{fi}\hat{\delta}_f]$. Under H_0, $\widetilde{Y}_i = \widetilde{X}_{fi}\beta_f + \widetilde{Z}_{fi}\delta_f + \widetilde{U}_i$, so that

$$SSR_k = \sum_{i=1}^{k+1}[\widetilde{X}_{fi}(\beta_f - \hat{\beta}_f) + \widetilde{Z}_{fi}(\delta_f - \hat{\delta}_f) + \widetilde{U}_i]'[\widetilde{X}_{fi}(\beta_f - \hat{\beta}_f) + \widetilde{Z}_{fi}(\delta_f - \hat{\delta}_f) + \widetilde{U}_i]$$

Furthermore, $T(\hat{\delta}_f - \delta_f) = (T^{-2}\sum_{i=1}^{k+1}\widetilde{Z}'_{fi}\widetilde{Z}_{fi})^{-1}(T^{-1}\sum_{i=1}^{k+1}\widetilde{Z}'_{fi}\widetilde{U}_i) + o_p(1)$ and

$$T^{1/2}(\hat{\beta}_f - \beta_f) = (T^{-1}\sum_{i=1}^{k+1}\widetilde{X}'_{fi}\widetilde{X}_{fi})^{-1}(T^{-1/2}\sum_{i=1}^{k+1}\widetilde{X}'_{fi}\widetilde{U}_i) + o_p(1).$$

Hence, after some algebra,

$$\begin{aligned} SSR_k &= -(T^{-1}\sum_{i=1}^{k+1}\widetilde{U}'_i\widetilde{Z}_{fi})(T^{-2}\sum_{i=1}^{k+1}\widetilde{Z}'_{fi}\widetilde{Z}_{fi})^{-1}(T^{-1}\sum_{i=1}^{k+1}\widetilde{Z}'_{fi}\widetilde{U}_i) \\ &\quad -(T^{-1/2}\sum_{i=1}^{k+1}\widetilde{U}'_i\widetilde{X}_{fi})(T^{-1}\sum_{i=1}^{k+1}\widetilde{X}'_{fi}\widetilde{X}_{fi})^{-1}(T^{-1/2}\sum_{i=1}^{k+1}\widetilde{X}'_{fi}\widetilde{U}_i) + \sum_{i=1}^{k+1}\widetilde{U}'_i\widetilde{U}_i + o_p(1) \end{aligned}$$

and

$$
\begin{aligned}
kF_T(\lambda,k) \;\Rightarrow\; & -(\int_0^1 W_z^{f(1,k+1)}dW_1)'(\int_0^1 W_z^{f(1,k+1)}W_z^{f(1,k+1)\prime})^{-1}(\int_0^1 W_z^{f(1,k+1)}dW_1) \\
& +(\sum_{i=1}^{k+1}\int_{\lambda_{i-1}}^{\lambda_i} W_z^{f(i,i)}dW_1)'(\sum_{i=1}^{k+1}\int_{\lambda_{i-1}}^{\lambda_i} W_z^{f(i,i)}W_z^{f(i,i)\prime})^{-1}(\sum_{i=1}^{k+1}\int_{\lambda_{i-1}}^{\lambda_i} W_z^{f(i,i)}dW_1) \\
& +\sum_{i=1}^{k}\frac{(\lambda_i W_1(\lambda_{i+1})-\lambda_{i+1}W_1(\lambda_i))^2}{\lambda_{i+1}\lambda_i(\lambda_{i+1}-\lambda_i)}
\end{aligned}
$$

Case 6: The model under H_1 is $y_t = c_i + z_{bt}'\delta_{bi} + x_{ft}'\beta_f + u_t$. In this case, $SSR_k = \sum_{i=1}^{k+1}[\widetilde{Y}_i - \widetilde{X}_{fi}\hat{\beta}_f - \widetilde{Z}_{bi}\hat{\delta}_{bi}]'[\widetilde{Y}_i - \widetilde{X}_{fi}\hat{\beta}_f - \widetilde{Z}_{bi}\hat{\delta}_{bi}]$. Under H_0, $\widetilde{Y}_i = \widetilde{X}_{fi}\beta_f + \widetilde{Z}_{bi}\delta_b + \widetilde{U}_i$, so that

$$
SSR_k = \sum_{i=1}^{k+1}[\widetilde{X}_{fi}(\beta_f-\hat{\beta}_f) + \widetilde{Z}_{bi}(\delta_b - \hat{\delta}_{bi}) + \widetilde{U}_i]'[\widetilde{X}_{fi}(\beta_f-\hat{\beta}_f) + \widetilde{Z}_{bi}(\delta_b - \hat{\delta}_{bi}) + \widetilde{U}_i]
$$

We also have $T(\hat{\delta}_{bi} - \delta_b) = (T^{-2}\widetilde{Z}_{bi}'\widetilde{Z}_{bi})^{-1}T^{-1}\widetilde{Z}_{bi}'\widetilde{U}_i + o_p(1)$ and

$$
T^{1/2}(\hat{\beta}_f - \beta_f) = (T^{-1}\sum_{i=1}^{k+1}\widetilde{X}_{fi}'\widetilde{X}_{fi})^{-1}(T^{-1/2}\sum_{i=1}^{k+1}\widetilde{X}_{fi}'\widetilde{U}_i) + o_p(1).
$$

Hence,

$$
\begin{aligned}
SSR_k \;=\; & -\sum_{i=1}^{k+1}(T^{-1}\widetilde{U}_i'\widetilde{Z}_{bi})(T^{-2}\widetilde{Z}_{bi}'\widetilde{Z}_{bi})^{-1}(T^{-1}\widetilde{Z}_{bi}'\widetilde{U}_i) \\
& -(T^{-1/2}\sum_{i=1}^{k+1}\widetilde{U}_i'\widetilde{X}_{fi}')(\sum_{i=1}^{k+1}T^{-1}\widetilde{X}_{fi}'\widetilde{X}_{fi})^{-1}(T^{-1/2}\sum_{i=1}^{k+1}\widetilde{X}_{fi}'\widetilde{U}_i) + \sum_{i=1}^{k+1}\widetilde{U}_i'\widetilde{U}_i
\end{aligned}
$$

so that

$$
\begin{aligned}
kF_T(\lambda,k) \;\Rightarrow\; & \sum_{i=1}^{k+1}[-(\int_0^{\lambda_{i+1}} W_z^{b(1,i+1)}dW_1)'(\int_0^{\lambda_{i+1}} W_z^{b(1,i+1)}W_z^{b(1,i+1)\prime})^{-1}(\int_0^{\lambda_{i+1}} W_z^{b(1,i+1)}dW_1) \\
& +(\int_0^{\lambda_i} W_z^{b(1,i)}dW_1)'(\int_0^{\lambda_i} W_z^{b(1,i)}W_z^{b(1,i)\prime})^{-1}(\int_0^{\lambda_i} W_z^{b(1,i)}dW_1) \\
& +(\int_{\lambda_i}^{\lambda_{i+1}} W_z^{b(i+1,i+1)}dW_1)'(\int_{\lambda_i}^{\lambda_{i+1}} W_z^{b(i+!,i+1)}W_z^{b(i+1,i+1)\prime})^{-1}(\int_{\lambda_i}^{\lambda_{i+1}} W_z^{b(i+1,i+1)}dW_1)] \\
& +\sum_{i=1}^{k}\frac{(\lambda_i W_1(\lambda_{i+1})-\lambda_{i+1}W_1(\lambda_i))^2}{\lambda_{i+1}\lambda_i(\lambda_{i+1}-\lambda_i)}
\end{aligned}
$$

Proof of Theorem 3: We provide a proof for the testing problem (2) in Category (a), a pure structural change model with only $I(1)$ regressors and a constant. The proofs for the other cases are very similar. We first let $\widetilde{B}_T = T^{-1/2}\sum_{j=1}^{[Tr]}\widetilde{\zeta}_j$,where $\widetilde{\zeta}_t = (v_t, u_{zt}^{b\prime})'$. Under the stated conditions, $\widetilde{B}_T \Rightarrow \widetilde{B} \equiv (B_{1.z}, B_z^b)$ as $T\to\infty$, where $B_{1.z}^b = B_1 - \Omega_{1z}^b(\Omega_{zz}^{bb})^{-1}B_z^b$. Note that $B_{1.z}^b$ is independent of B_z^b. Thus, $\widetilde{B}$ denotes a vector Brownian motion with block diagonal covariance matrix $\widetilde{\Omega} = diag((\sigma_{1.z}^b)^2, \Omega_{zz}^{bb})$, where $(\sigma_{1.z}^b)^2 = \sigma^2 - \Omega_{1z}^b(\Omega_{zz}^{bb})^{-1}\Omega_{z1}^b$. The relevant regression under the alternative hypothesis is

$$
y_t = c_i + z_{bt}'\hat{\delta}_{bi} + \sum_{j=-\ell_T}^{\ell_T}\Delta z_{b,t-j}'\hat{\Pi}_j + \hat{v}_t^*
$$

As a matter of notation, let $\eta_{bt}^* = (\Delta z_{bt-\ell_T}', ..., \Delta z_{bt+\ell_T}')'$, $\eta_b^* = (\eta_{b1}^*, ..., \eta_{bT}^*)'$, $E = (e_1, ..., e_T)'$, $V = (v_1, ..., v_T)'$, and $\Pi = (\Pi_{-\ell_T}', ..., \Pi_{\ell_T}')'$. Also, define $M_\eta = I_T - \eta_b^*(\eta_b'^*\eta_b^*)^{-1}\eta_b'^*$, $z_t = (1, z_{bt})'$, $Z = (z_1, ..., z_T)'$, $Z_i = (z_{T_{i-1}+1}, ..., z_{T_i})$, $\bar{Z} = diag(Z_1, ..., Z_{k+1})$, $\delta = (c, \delta_b')'$ and the $[(k+1)(q_b+1) \times 1]$ vector $\bar{\delta} = (\delta, \delta, ..., \delta)$. The vectors of estimates under the null and the alternative are $\bar{\bar{\delta}}$ and $\hat{\delta}$, respectively. The vector of residuals is $\widetilde{v}^* = M_\eta Y - M_\eta \bar{Z}\bar{\bar{\delta}}$ under the null and $\hat{v}^* = M_\eta Y - M_\eta \bar{Z}\hat{\delta}$ under the alternative. We have $\widetilde{v}^* = \hat{v}^* + M_\eta \bar{Z}(\hat{\delta} - \bar{\bar{\delta}})$, so that

$$
\begin{aligned}
SSR_0 - SSR_k &= \widetilde{v}^{*\prime}\widetilde{v}^* - \hat{v}^{*\prime}\hat{v}^* = (\hat{\delta} - \bar{\bar{\delta}})'\bar{Z}'M_\eta \bar{Z}(\hat{\delta} - \bar{\bar{\delta}}) \\
&= (\hat{\delta} - \bar{\bar{\delta}})'\bar{Z}'\bar{Z}(\hat{\delta} - \bar{\bar{\delta}}) - (\hat{\delta} - \bar{\bar{\delta}})'\bar{Z}'\eta_b^*(\eta_b'^*\eta_b^*)^{-1}\eta_b'^*\bar{Z}(\hat{\delta} - \bar{\bar{\delta}})
\end{aligned}
$$

Now note that

$||(\hat{\delta}-\bar{\bar{\delta}})'\bar{Z}'\eta_b^*(\eta_b'^*\eta_b^*)^{-1}\eta_b'^*\bar{Z}(\hat{\delta}-\bar{\bar{\delta}})|| \leq ||(\hat{\delta}-\bar{\bar{\delta}})'D_T||\,||D_T^{-1}\bar{Z}'\eta_b^*||\,||(\eta_b'^*\eta_b^*)^{-1}||\,||\eta_b'^*\bar{Z}D_T^{-1}||\,||D_T(\hat{\delta}-\bar{\bar{\delta}})||$

where the $[(k+1) \times (q_b+1)]$ diagonal matrix $D_T = diag(T^{1/2}, T, T, ..., T, ..., T^{1/2}, T, ..., T)$. We have $||D_T(\hat{\delta} - \bar{\bar{\delta}})|| = O_p(1)$, $||(\eta_b'^*\eta_b^*)^{-1}|| = O_p(T^{-1})$, $||D_T^{-1}\bar{Z}'\eta_b^*|| = O_p(l_T^{1/2})$, since $||T^{-1}\sum_{t=1}^T Z_{bt}\eta_{bt}^{*\prime}|| = O_p(l_T^{1/2})$, $||T^{-1/2}\sum_{t=1}^T \eta_{bt}^{*\prime}|| = O_p(l_T^{1/2})$ (Saikkonen, 1991, Kejriwal and Perron, 2008a). Hence, $||(\hat{\delta} - \bar{\bar{\delta}})'\bar{Z}'\eta_b^*(\eta_b'^*\eta_b^*)^{-1}\eta_b'^*\bar{Z}(\hat{\delta} - \bar{\bar{\delta}})|| = O_p(l_T/T) = o_p(1)$. Next,

$$
\begin{aligned}
(\hat{\delta} - \bar{\bar{\delta}})'\bar{Z}'\bar{Z}(\hat{\delta} - \bar{\bar{\delta}}) &= -(Z'V)'(Z'Z)^{-1}Z'V + \sum_{i=1}^{k+1}(Z_i'V_i)'(Z_i'Z_i)^{-1}(Z_i'V_i) + o_p(1) \\
&= -(Z_{b(1,k+1)}^{*\prime}V_{(1,k+1)}^*)'(Z_{b(1,k+1)}^{*\prime}Z_{b(1,k+1)}^*)^{-1}(Z_{b(1,k+1)}^{*\prime}V_{(1,k+1)}^*) \\
&\quad + V_{(1,k+1)}^{*\prime}V_{(1,k+1)}^* + \sum_{i=1}^{k+1}\{(\widetilde{Z}_{bi}'\widetilde{V}_i)'(\widetilde{Z}_{bi}'\widetilde{Z}_{bi})^{-1}(\widetilde{Z}_{bi}'\widetilde{V}_i) - \widetilde{V}_i'\widetilde{V}_i\} + o_p(1)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&SSR_0 - SSR_k \Rightarrow -(\textstyle\int_0^1 B_z^{b(1,k+1)}dB_{1.z}^b)'(\int_0^1 B_z^{b(1,k+1)}B_z^{b(1,k+1)\prime})^{-1}(\int_0^1 B_z^{b(1,k+1)}dB_{1.z}^b) \\
&+ \sum_{i=1}^{k+1}\{(\textstyle\int_{\lambda_{i-1}}^{\lambda_i} B_z^{b(i,i)}dB_{1.z}^b)'(\int_{\lambda_{i-1}}^{\lambda_i} B_z^{b(i,i)}B_z^{b(i,i)\prime})^{-1}(\int_{\lambda_{i-1}}^{\lambda_i} B_z^{b(i,i)}dB_{1.z}^b)\} \\
&+ \sum_{i=1}^{k}\frac{(\lambda_i B_{1.z}^b(\lambda_{i+1}) - \lambda_{i+1}B_{1.z}^b(\lambda_i))^2}{\lambda_{i+1}\lambda_i(\lambda_{i+1} - \lambda_i)}
\end{aligned}
$$

Since $B_{1.z}^b$ and B_z are independent, $B_{1.z}^b = \sigma_{1.z}^b W_1$ and $B_z^b = (\Omega_{zz}^b)^{1/2}W_z^b$, so that

$$
\begin{aligned}
&SSR_0 - SSR_k \Rightarrow -(\sigma_{1.z}^b)^2(\textstyle\int_0^1 W_z^{b(1,k+1)}dW_1)'(\int_0^1 W_z^{b(1,k+1)}W_z^{b(1,k+1)\prime})^{-1}(\int_0^1 W_z^{b(1,k+1)}dW_1) \\
&+(\sigma_{1.z}^b)^2\sum_{i=1}^{k+1}\{(\textstyle\int_{\lambda_{i-1}}^{\lambda_i} W_z^{b(i,i)}dW_1)'(\int_{\lambda_{i-1}}^{\lambda_i} W_z^{b(i,i)}W_z^{b(i,i)\prime})^{-1}(\int_{\lambda_{i-1}}^{\lambda_i} W_z^{b(i,i)}dW_1)\} \\
&+(\sigma_{1.z}^b)^2\sum_{i=1}^{k}\frac{(\lambda_i W_1(\lambda_{i+1}) - \lambda_{i+1}W_1(\lambda_i))^2}{\lambda_{i+1}\lambda_i(\lambda_{i+1} - \lambda_i)}
\end{aligned}
$$

It can be shown, using arguments as in Kejriwal and Perron (2008a) that $\hat{\sigma}_v$ is a consistent estimate of $\sigma_{1.z}^b$ under the stated conditions (the proof is quite tedious and omitted). This proves the theorem.

Table 1.a: Asymptotic Critical Values for Category (a) Case 1, $\epsilon = .15$.

(The entries are quantiles x such that $P(\sup F(\lambda, k)/k \leq x) = \alpha$)

		Non Trending Case						Trending Case					
		Number of Breaks, k						Number of Breaks, k					
q_b	α	1	2	3	4	5	$UD\max$	1	2	3	4	5	$UD\max$
1	.90	10.34	8.85	7.66	6.66	5.30	10.53	11.18	9.25	8.09	6.95	5.53	11.33
	.95	12.11	9.96	8.60	7.36	5.90	12.25	13.03	10.39	8.94	7.60	6.12	13.07
	.975	13.85	11.41	9.40	7.99	6.42	13.91	15.08	11.49	9.66	8.28	6.67	15.13
	.99	17.03	12.41	10.40	8.71	7.08	17.40	16.86	12.73	10.82	8.95	7.32	16.86
2	.90	12.36	11.01	9.60	8.45	6.96	12.64	11.88	10.31	9.00	7.98	6.62	12.13
	.95	14.30	12.11	10.41	9.19	7.64	14.47	13.63	11.34	9.94	8.68	7.31	13.99
	.975	15.72	13.37	11.26	9.75	8.15	15.90	15.51	12.57	10.86	9.37	7.92	15.53
	.99	17.67	14.73	12.21	10.77	8.82	17.67	17.31	14.63	12.10	10.51	8.73	17.31
3	.90	14.88	12.84	11.49	10.19	8.53	15.09	14.39	12.14	10.79	9.61	8.22	14.65
	.95	16.66	14.11	12.38	10.94	9.12	16.71	16.50	13.22	11.66	10.33	8.92	16.61
	.975	18.32	15.24	13.01	11.52	9.61	18.35	18.08	14.45	12.54	11.04	9.44	18.24
	.99	20.78	16.29	14.36	12.37	10.23	20.78	20.28	15.55	13.80	12.02	10.10	20.28
4	.90	16.87	14.72	13.20	11.75	9.90	17.05	16.27	13.80	12.41	11.17	9.62	16.46
	.95	19.08	15.90	14.15	12.68	10.72	19.16	18.36	15.08	13.38	12.07	10.28	18.46
	.975	20.81	17.15	15.21	13.38	11.43	20.89	20.52	17.01	14.33	12.98	10.93	20.52
	.99	22.59	18.85	16.44	14.25	11.98	22.59	23.12	18.71	15.77	13.87	11.72	23.12

Table 1.b: Asymptotic Critical Values for Category (a) Case 2, $\epsilon = .15$.

(The entries are quantiles x such that $P(\sup F(\lambda, k)/k \leq x) = \alpha$)

		Non Trending Case						Trending Case					
		Number of Breaks, k						Number of Breaks, k					
q_f	α	1	2	3	4	5	$UD\max$	1	2	3	4	5	$UD\max$
1	.90	7.52	6.38	5.37	4.54	3.49	7.79	8.67	6.84	6.07	5.31	4.01	8.90
	.95	9.26	7.30	6.21	5.19	3.98	9.38	10.29	7.89	6.85	5.97	4.49	10.44
	.975	10.63	8.25	6.98	5.67	4.40	10.87	12.18	8.99	7.57	6.66	5.02	12.18
	.99	12.57	10.01	7.77	6.42	4.88	12.60	14.21	10.19	8.45	7.10	5.62	14.27
2	.90	8.48	6.70	5.66	4.77	3.63	8.66	8.32	6.49	5.65	4.98	3.84	8.60
	.95	10.13	7.66	6.43	5.36	4.10	10.25	10.06	7.45	6.42	5.67	4.36	10.11
	.975	11.69	8.85	7.34	5.99	4.62	11.82	11.47	8.59	7.21	6.29	5.02	11.52
	.99	13.66	10.20	8.09	6.91	5.35	13.66	13.21	9.86	8.29	7.01	5.49	13.30
3	.90	8.47	6.51	5.59	4.77	3.58	8.74	8.40	6.53	5.64	5.03	3.91	8.66
	.95	10.08	7.61	6.26	5.49	4.07	10.26	10.08	7.48	6.35	5.65	4.35	10.10
	.975	11.27	8.51	7.21	6.12	4.49	11.43	11.68	8.55	6.90	6.15	4.83	11.68
	.99	12.88	9.95	7.88	6.70	5.13	12.93	13.72	9.53	7.51	6.72	5.34	13.72
4	.90	8.56	6.59	5.71	4.87	3.81	8.85	8.57	6.49	5.69	4.94	3.85	8.69
	.95	10.07	7.66	6.52	5.55	4.30	10.17	10.22	7.34	6.51	5.59	4.46	10.36
	.975	11.69	8.61	7.10	6.09	4.70	11.69	11.90	8.33	7.22	6.26	4.88	11.95
	.99	13.88	9.64	7.83	6.58	5.33	13.88	14.53	9.68	8.33	6.97	5.53	14.53

Table 1.c: Asymptotic Critical Values for Category (a) Case 3, $\epsilon = .15$.

(The entries are quantiles x such that $P(\sup F(\lambda, k)/k \leq x) = \alpha$)

		Non Trending Case						Trending Case					
		Number of Breaks, k						Number of Breaks, k					
q_b	α	1	2	3	4	5	$UD\max$	1	2	3	4	5	$UD\max$
1	.90	7.90	6.37	5.36	4.49	3.46	8.21	7.21	5.34	4.54	3.81	3.02	7.43
	.95	9.50	7.36	6.08	5.01	3.90	9.75	8.98	6.32	5.29	4.42	3.54	9.07
	.975	10.83	8.44	6.75	5.66	4.34	10.93	10.74	7.54	5.97	4.97	3.89	10.74
	.99	12.34	9.73	7.82	6.31	4.96	12.34	13.10	8.76	7.38	5.86	4.58	13.10
2	.90	10.59	9.13	7.94	6.81	5.43	10.83	10.33	8.90	7.70	6.68	5.35	10.61
	.95	12.49	10.36	8.72	7.52	5.94	12.69	12.01	9.93	8.57	7.28	5.91	12.08
	.975	14.33	11.31	9.56	8.13	6.45	14.40	13.48	10.80	9.32	7.89	6.27	13.51
	.99	16.56	12.78	10.45	8.94	7.03	16.56	15.61	11.97	10.10	8.55	6.85	15.62
3	.90	12.74	10.98	9.71	8.56	6.98	12.94	13.15	11.11	9.77	8.57	7.04	13.23
	.95	14.53	12.18	10.62	9.30	7.49	14.61	14.85	12.22	10.82	9.32	7.61	14.97
	.975	16.14	13.24	11.43	9.96	8.17	16.14	16.32	13.20	11.57	10.02	8.17	16.32
	.99	17.97	14.64	12.58	10.87	8.83	17.97	18.70	14.76	12.15	10.60	8.76	18.70
4	.90	14.85	12.81	11.44	10.13	8.44	14.95	15.21	13.05	11.57	10.24	8.50	15.33
	.95	16.77	14.00	12.35	10.82	9.12	16.99	17.23	14.09	12.54	11.03	9.29	17.31
	.975	18.77	15.27	13.17	11.50	9.71	18.79	19.10	15.22	13.12	11.81	9.82	19.10
	.99	20.76	16.15	14.43	12.28	10.35	20.87	21.14	16.73	14.24	12.60	10.57	21.14

Table 1.d: Asymptotic Critical Values for Category (a) Case 4, $\epsilon = .15$.

(The entries are quantiles x such that $P(\sup F(\lambda, k)/k \leq x) = \alpha$)

		Non Trending Case						Trending Case					
		Number of Breaks, k						Number of Breaks, k					
q_f, q_b	α	1	2	3	4	5	$UD\max$	1	2	3	4	5	$UD\max$
1,1	.90	10.19	8.77	7.74	6.60	5.26	10.53	10.81	9.18	7.99	6.89	5.48	10.98
	.95	12.03	9.78	8.53	7.18	5.81	12.30	12.27	10.30	8.87	7.61	6.09	12.34
	.975	14.05	11.03	9.28	7.92	6.30	14.07	14.43	11.39	9.54	8.28	6.72	14.45
	.99	16.02	12.33	10.33	8.67	6.99	16.09	16.65	12.56	10.45	9.02	7.14	16.65
1,2	.90	12.89	11.03	9.70	8.60	7.02	13.16	12.57	10.62	9.17	8.17	6.80	12.76
	.95	14.88	12.27	10.76	9.38	7.68	14.97	14.19	11.69	10.12	8.93	7.43	14.27
	.975	16.72	13.67	11.63	10.03	8.48	16.75	15.86	12.73	10.78	9.51	7.85	15.89
	.99	18.48	14.72	12.48	10.89	9.06	18.48	17.89	13.79	11.76	10.18	8.39	18.16
2,1	.90	10.99	9.08	7.91	6.82	5.46	11.15	11.33	9.36	8.07	7.04	5.66	11.45
	.95	13.04	10.09	8.71	7.43	6.02	13.06	13.18	10.46	9.09	7.73	6.21	13.26
	.975	14.80	10.84	9.46	8.01	6.60	14.80	15.22	11.55	9.80	8.33	6.71	15.22
	.99	16.46	12.08	10.43	8.87	7.04	16.46	17.85	12.48	10.49	9.08	7.32	17.85
2,2	.90	12.87	11.04	9.71	8.58	7.12	13.07	12.58	10.41	9.15	8.15	6.78	12.78
	.95	14.81	12.25	10.75	9.44	7.74	15.01	14.65	11.78	10.04	8.85	7.48	14.72
	.975	16.74	13.48	11.57	10.15	8.34	16.74	15.95	12.92	10.94	9.57	8.04	16.12
	.99	19.36	14.78	12.29	10.83	8.78	19.36	17.94	13.91	11.83	10.32	8.91	18.08

Table 1.e: Asymptotic Critical Values for Category (a) Case 5, $\epsilon = .15$.

(The entries are quantiles x such that $P(\sup F(\lambda, k)/k \leq x) = \alpha$)

		Non Trending Case						Trending Case					
		Number of Breaks, k						Number of Breaks, k					
q_f, p_b	α	1	2	3	4	5	$UD\max$	1	2	3	4	5	$UD\max$
1,1	.90	7.97	6.43	5.34	4.52	3.48	8.17	9.06	6.84	5.72	4.77	3.70	9.23
	.95	9.46	7.48	6.11	5.14	3.98	9.62	10.43	7.75	6.36	5.30	4.14	10.47
	.975	11.36	8.49	6.88	5.76	4.52	11.47	11.82	8.61	7.14	5.97	4.54	11.87
	.99	13.44	9.89	7.70	6.56	4.96	13.47	14.03	9.54	8.09	6.60	5.06	14.03
1,2	.90	10.80	8.91	7.79	6.73	5.39	11.04	10.23	8.06	6.95	6.10	4.99	10.47
	.95	12.41	9.96	8.57	7.31	5.89	12.47	11.83	9.19	7.70	6.87	5.67	11.93
	.975	13.63	11.01	9.49	8.03	6.40	13.68	13.85	10.27	8.63	7.47	6.25	14.01
	.99	15.97	12.20	10.46	8.70	7.03	16.10	15.75	11.42	9.61	8.22	6.94	16.04
2,1	.90	7.89	6.47	5.46	4.63	3.55	8.10	8.82	6.91	5.90	4.96	3.83	8.96
	.95	9.54	7.47	6.18	5.18	3.99	9.68	10.84	7.87	6.72	5.52	4.39	11.01
	.975	10.96	8.44	6.83	5.68	4.43	11.20	13.05	8.84	7.33	6.02	4.76	13.05
	.99	12.44	9.39	7.50	6.31	4.91	12.44	15.58	10.50	8.35	6.91	5.31	15.58
2,2	.90	10.83	8.89	7.83	6.74	5.41	10.98	10.16	8.16	7.18	6.23	5.11	10.36
	.95	12.76	10.11	8.55	7.31	5.96	12.76	11.88	9.30	8.03	6.84	5.64	12.02
	.975	14.26	10.90	9.28	7.93	6.50	14.26	13.26	10.32	8.79	7.59	6.21	13.40
	.99	15.56	11.83	10.21	8.63	6.98	15.56	14.91	11.58	9.80	8.19	6.77	15.03

Table 2.a: Asymptotic Critical Values for Category (b) Case 3, $\epsilon = .15$.

(The entries are quantiles x such that $P(\sup F(\lambda, k)/k \leq x) = \alpha$)

		Non Trending Case						Trending Case					
		Number of Breaks, k						Number of Breaks, k					
q_b, p_b	α	1	2	3	4	5	$UD\max$	1	2	3	4	5	$UD\max$
1,1	.90	10.08	8.61	7.30	6.38	5.15	10.40	10.88	8.76	7.62	6.66	5.37	10.99
	.95	11.94	9.42	8.28	6.93	5.74	12.11	12.44	10.17	8.61	7.28	5.94	12.44
	.975	13.40	10.70	9.35	7.97	6.18	13.58	14.93	11.15	9.48	8.03	6.38	14.93
	.99	14.96	12.30	10.70	8.94	6.85	15.11	16.90	12.12	10.58	8.82	7.24	16.90
1,2	.90	12.24	10.80	9.53	8.37	6.82	12.53	12.88	11.03	9.61	8.39	6.82	12.97
	.95	14.53	11.94	10.38	9.28	7.51	14.79	14.90	12.32	10.62	9.20	7.41	14.97
	.975	15.91	13.22	11.40	9.89	8.28	16.14	16.60	13.44	11.59	10.16	8.27	16.60
	.99	19.33	14.92	12.70	11.03	8.91	19.33	19.60	15.02	12.95	11.32	9.09	19.60
2,1	.90	12.87	11.04	9.96	8.63	7.14	13.05	13.15	11.23	9.93	8.64	7.15	13.32
	.95	14.55	12.21	10.73	9.38	7.74	14.90	15.04	12.42	10.76	9.51	7.78	15.10
	.975	16.74	13.25	11.59	10.21	8.42	16.91	16.72	13.74	11.98	10.23	8.38	16.73
	.99	19.05	14.74	12.88	11.08	8.94	19.05	19.49	14.63	12.61	11.14	9.06	19.49
2,2	.90	14.77	12.94	11.56	10.25	8.54	14.97	14.86	13.05	11.73	10.31	8.58	14.99
	.95	16.30	14.07	12.42	11.10	9.02	16.80	16.89	14.23	12.79	11.19	9.27	17.02
	.975	17.92	15.06	13.75	11.77	9.77	18.13	18.46	15.80	13.82	12.52	10.24	18.50
	.99	19.89	17.19	14.60	12.84	10.73	19.89	21.17	17.41	15.24	13.17	10.87	21.17

Table 2.b: Asymptotic Critical Values for Category (b) Cases 4 and 8, $\epsilon = .15$.

(The entries are quantiles x such that $P(\sup F(\lambda, k)/k \leq x) = \alpha$)

		Non Trending Case						Trending Case					
		Number of Breaks, k						Number of Breaks, k					
q_b, p_b	α	1	2	3	4	5	$UD\max$	1	2	3	4	5	$UD\max$
1,1	.90	11.69	9.88	8.63	7.52	6.27	11.99	11.98	10.29	8.96	7.83	6.63	12.27
	.95	13.24	10.96	9.62	8.29	6.87	13.43	13.74	11.64	9.92	8.66	7.28	14.06
	.975	14.78	12.10	10.54	8.99	7.56	14.87	15.86	12.85	10.87	9.30	7.87	15.91
	.99	17.28	13.40	11.53	9.75	8.11	17.39	17.99	14.27	11.87	10.20	8.44	17.99
1,2	.90	12.88	11.06	9.55	8.53	7.52	13.26	13.24	11.17	9.79	8.85	7.69	13.51
	.95	15.10	12.13	10.53	9.42	8.16	15.25	15.16	12.19	10.85	9.61	8.29	15.20
	.975	17.51	13.04	11.30	9.98	8.71	17.60	16.89	13.33	11.59	10.48	8.87	16.89
	.99	19.10	14.68	12.35	11.07	9.51	19.10	18.95	14.43	12.79	11.23	9.90	18.95
2,1	.90	13.85	12.05	10.48	9.35	7.99	14.23	13.42	11.33	10.06	9.00	7.73	13.64
	.95	15.91	13.45	11.50	10.23	8.64	16.07	15.42	12.76	11.03	9.86	8.44	15.47
	.975	17.68	14.60	12.44	11.06	9.30	18.06	17.50	13.95	12.05	10.58	8.97	17.50
	.99	19.89	16.02	13.80	11.88	10.14	20.03	19.61	15.23	13.05	11.38	9.59	19.61
2,2	.90	14.82	13.09	11.64	10.40	9.04	15.24	14.91	12.50	11.14	10.06	8.83	15.28
	.95	17.02	14.49	12.51	11.19	9.73	17.33	17.17	14.02	12.23	10.91	9.59	17.22
	.975	19.59	15.57	13.39	11.85	10.29	19.59	19.48	15.41	13.18	11.57	10.23	19.48
	.99	21.66	17.07	14.35	12.81	10.85	21.66	21.46	16.50	14.18	12.60	10.82	21.46

Table 2.c: Asymptotic Critical Values for Category (b) Cases 7 and 9, $\epsilon = .15$.

(The entries are quantiles x such that $P(\sup F(\lambda, k)/k \leq x) = \alpha$)

		Non Trending Case						Trending Case					
		Number of Breaks, k						Number of Breaks, k					
q_f, p_b	α	1	2	3	4	5	$UD\max$	1	2	3	4	5	$UD\max$
1,1	.90	8.72	7.48	6.23	5.41	4.52	9.12	8.38	6.72	5.82	5.15	4.29	8.64
	.95	10.65	8.59	6.97	6.13	5.06	10.87	10.16	7.93	6.82	5.76	4.73	10.34
	.975	12.13	9.61	7.92	6.68	5.50	12.39	11.95	9.18	7.52	6.32	5.34	11.99
	.99	14.37	10.75	9.10	7.76	6.32	14.95	13.88	10.40	8.26	6.99	6.09	13.88
1,2	.90	9.95	8.17	7.17	6.50	5.63	10.31	9.35	7.38	6.58	5.93	5.31	9.62
	.95	11.58	9.54	8.25	7.23	6.25	11.93	10.98	8.60	7.32	6.61	5.92	11.07
	.975	12.99	10.74	9.23	7.83	6.85	13.68	12.76	9.59	8.24	7.35	6.48	12.83
	.99	15.66	12.19	10.30	8.65	7.71	15.68	15.22	10.92	9.55	8.20	7.16	15.22
2,1	.90	9.03	7.51	6.45	5.70	4.66	9.49	8.96	6.80	5.94	5.19	4.41	9.08
	.95	10.70	8.77	7.34	6.32	5.22	10.85	10.56	7.90	6.84	5.85	5.00	10.73
	.975	11.98	9.77	7.98	6.98	5.70	12.30	12.50	8.99	7.48	6.53	5.46	12.55
	.99	15.29	10.80	8.95	7.71	6.32	15.29	14.98	9.87	8.53	7.08	6.03	14.98
2,2	.90	10.58	8.52	7.36	6.64	5.78	10.88	9.82	7.95	7.00	6.31	5.50	10.33
	.95	12.32	9.72	8.23	7.45	6.39	12.53	11.82	9.26	7.88	7.09	6.20	12.09
	.975	14.09	11.05	9.36	8.23	6.95	14.22	13.76	10.64	8.79	7.87	6.85	13.99
	.99	16.23	12.04	10.43	9.13	7.67	16.23	15.75	12.06	10.23	8.68	7.70	16.09

Table 2.d: Asymptotic Critical Values for Category (b) Case 10, $\epsilon = .15$.

(The entries are quantiles x such that $P(\sup\ F(\lambda,k)/k \le x) = \alpha$)

		Non Trending Case						Trending Case					
		Number of Breaks, k						Number of Breaks, k					
q_f, q_b, p_b	α	1	2	3	4	5	$UD\max$	1	2	3	4	5	$UD\max$
1,1,1	.90	11.83	10.06	8.74	7.79	6.47	12.04	12.30	10.39	9.18	8.10	6.61	12.68
	.95	13.95	11.26	9.76	8.47	7.15	14.02	14.55	11.71	10.14	8.97	7.32	14.66
	.975	15.76	12.31	10.61	9.30	7.76	15.79	16.70	12.97	11.17	9.73	7.97	16.70
	.99	17.98	13.55	11.36	9.85	8.56	17.98	18.68	14.61	12.38	10.45	8.61	18.68
1,1,2	.90	12.87	10.93	9.59	8.68	7.52	13.22	13.45	11.50	10.17	8.88	7.75	13.83
	.95	15.07	12.24	10.78	9.46	8.28	15.20	15.70	12.78	11.14	9.78	8.38	15.72
	.975	16.68	13.17	11.62	10.23	8.94	17.10	18.41	14.04	11.86	10.55	8.97	18.41
	.99	19.17	14.71	12.61	11.03	9.64	19.26	20.75	15.09	12.98	11.23	9.71	20.75
1,2,1	.90	14.06	12.05	10.51	9.48	8.05	14.30	13.80	11.59	10.44	9.08	7.83	14.05
	.95	15.99	13.20	11.61	10.23	8.77	15.99	15.79	12.99	11.44	9.83	8.56	15.95
	.975	17.72	14.58	12.38	11.02	9.36	17.78	17.60	14.03	12.25	10.51	9.05	17.67
	.99	19.77	16.16	13.80	12.00	10.09	19.77	20.69	15.52	13.13	11.66	9.77	20.69
1,2,2	.90	15.06	12.97	11.51	10.40	9.05	15.47	14.61	12.22	11.07	10.17	8.95	15.10
	.95	17.60	14.32	12.47	11.19	9.62	17.79	16.75	13.64	12.17	10.96	9.63	16.98
	.975	19.42	15.75	13.55	12.09	10.37	19.57	18.67	15.03	13.34	12.00	10.37	18.88
	.99	22.29	17.48	14.77	13.10	11.18	22.29	20.94	16.52	14.94	13.02	11.27	20.96
2,1,1	.90	12.06	10.02	8.85	7.81	6.55	12.29	12.39	10.56	9.10	8.06	6.66	12.71
	.95	13.80	11.36	9.70	8.57	7.21	13.92	14.37	11.87	10.17	8.75	7.31	14.76
	.975	16.14	12.50	10.57	9.28	7.77	16.16	16.04	13.33	11.18	9.65	7.82	16.36
	.99	18.68	14.40	11.75	10.21	8.50	18.76	19.23	14.56	12.18	10.48	8.67	19.23
2,1,2	.90	13.13	10.91	9.72	8.72	7.50	13.49	13.56	11.44	10.16	9.06	7.85	13.78
	.95	15.23	12.41	10.68	9.53	8.24	15.46	15.74	12.62	11.05	9.71	8.43	15.79
	.975	17.23	13.51	11.56	10.13	8.92	17.36	17.56	13.76	11.97	10.47	8.85	17.62
	.99	19.37	15.19	12.63	11.23	9.49	19.37	20.26	15.23	12.82	11.26	9.56	20.26
2,2,1	.90	14.50	12.16	10.69	9.58	8.06	14.72	13.78	11.55	10.22	9.25	7.99	14.05
	.95	16.78	13.46	11.88	10.35	8.74	16.80	15.64	12.81	11.18	9.98	8.62	15.81
	.975	18.50	14.64	12.76	11.11	9.37	18.50	17.22	14.22	12.07	10.67	9.21	17.24
	.99	20.83	16.28	13.77	11.82	9.92	20.83	19.20	15.48	13.49	11.61	10.04	19.20
2,2,2	.90	15.29	13.03	11.64	10.49	9.09	15.70	14.82	12.52	11.15	10.17	8.92	15.23
	.95	17.00	14.47	12.88	11.42	9.75	17.22	16.86	13.94	12.33	11.07	9.70	17.06
	.975	18.87	15.49	13.72	12.12	10.43	19.08	18.99	15.48	13.30	11.79	10.27	19.24
	.99	22.03	16.89	14.50	12.96	11.20	22.03	21.22	16.91	14.75	12.67	11.18	21.22

Table 2.e: Asymptotic Critical Values for Category (b) Case 11, $\epsilon = .15$.

(The entries are quantiles x such that $P(\sup\ F(\lambda, k)/k \leq x) = \alpha$)

		Non Trending Case						Trending Case					
		Number of Breaks, k						Number of Breaks, k					
q_f, q_b, p_b	α	1	2	3	4	5	$UD\max$	1	2'	3	4	5	$UD\max$
1,1,1	.90	10.72	8.86	7.68	6.64	5.44	10.86	10.62	9.04	7.85	6.83	5.47	10.85
	.95	12.44	10.02	8.51	7.37	6.06	12.57	12.25	10.12	8.71	7.44	6.06	12.30
	.975	14.10	10.97	9.62	8.27	6.55	14.19	14.05	10.96	9.48	8.10	6.67	14.05
	.99	16.41	12.74	10.78	8.98	7.28	16.57	15.82	12.20	10.36	8.98	7.37	15.82
1,1,2	.90	12.34	10.73	9.36	8.26	6.62	12.57	12.77	10.83	9.64	8.41	6.91	12.94
	.95	14.20	11.88	10.32	8.96	7.46	14.49	14.27	11.97	10.61	9.09	7.51	14.28
	.975	16.06	12.70	11.00	9.87	8.11	16.23	15.58	13.03	11.36	9.64	8.10	15.58
	.99	17.71	13.73	12.02	10.61	8.82	17.71	18.04	13.89	11.95	10.49	8.61	18.04
1,2,1	.90	12.93	11.05	9.78	8.72	7.06	13.04	13.37	11.14	9.92	8.62	7.05	13.47
	.95	15.38	12.46	10.99	9.59	7.82	15.51	15.18	12.37	10.82	9.55	7.57	15.38
	.975	17.33	13.87	11.94	10.22	8.43	17.43	16.67	13.74	11.88	10.20	8.29	16.69
	.99	19.61	15.57	12.91	11.40	9.22	19.61	18.56	15.92	12.79	11.23	9.09	18.56
1,2,2	.90	14.75	12.83	11.46	10.18	8.59	14.95	14.92	12.93	11.59	10.49	8.69	15.12
	.95	17.39	13.96	12.36	11.04	9.28	17.39	16.97	13.99	12.78	11.35	9.44	16.98
	.975	18.82	15.12	13.12	11.55	9.75	19.01	18.56	15.31	13.75	12.06	10.23	18.65
	.99	21.70	16.41	14.47	12.44	10.59	21.70	22.02	16.66	14.56	12.88	11.00	22.02
2,1,1	.90	10.77	8.88	7.78	6.87	5.29	11.05	11.41	9.36	8.15	7.02	5.52	11.49
	.95	12.43	10.31	8.71	7.51	5.96	12.65	13.22	10.43	9.01	7.83	6.09	13.47
	.975	13.65	11.33	9.74	8.30	6.40	13.77	15.53	11.59	10.04	8.37	6.63	15.64
	.99	16.69	13.49	11.27	9.05	7.18	16.81	17.99	13.38	10.48	9.10	7.18	17.99
2,1,2	.90	12.52	10.70	9.64	8.48	7.03	12.78	13.04	11.12	9.83	8.62	7.20	13.18
	.95	14.51	12.04	10.65	9.38	7.71	14.60	15.05	12.34	10.89	9.51	7.85	15.17
	.975	16.21	13.01	11.37	10.04	8.29	16.21	16.91	13.68	11.95	10.51	8.64	17.15
	.99	18.75	14.38	12.24	10.99	9.10	18.75	18.72	15.19	13.05	11.41	9.33	18.72
2,2,1	.90	13.12	11.25	10.02	8.81	7.21	13.63	14.01	11.40	10.21	8.88	7.46	14.07
	.95	15.53	12.68	11.01	9.47	7.78	15.69	15.93	12.61	11.04	9.71	7.98	16.03
	.975	17.63	13.78	11.80	10.19	8.45	18.24	17.34	13.85	11.81	10.32	8.59	17.34
	.99	20.25	15.55	12.93	11.29	9.40	20.25	19.27	14.83	12.94	11.08	9.13	19.27
2,2,2	.90	14.58	12.90	11.68	10.29	8.70	14.71	14.65	12.91	11.56	10.29	8.64	14.95
	.95	16.33	14.08	12.51	11.17	9.22	16.40	16.84	13.97	12.40	11.13	9.44	16.87
	.975	18.37	14.66	13.32	11.97	10.06	18.46	18.21	15.16	13.28	11.98	9.98	18.21
	.99	21.47	16.24	14.27	13.18	10.57	21.47	20.29	15.90	14.59	12.81	10.82	20.29

Table 3.a: Asymptotic Critical Values of the Sequential Test $SEQ_T(k+1|k)$ for Category (a) Case 1, $\epsilon = .15$.

		Non Trending Case					Trending Case				
		k					k				
q_b	α	1	2	3	4	5	1	2	3	4	5
1	.90	12.00	12.94	13.74	14.53	15.23	12.94	13.99	14.93	15.50	15.73
	.95	13.78	15.25	16.38	17.02	17.70	15.01	15.85	16.53	16.86	17.04
	.975	16.38	17.70	18.24	18.53	19.18	16.53	17.04	17.17	17.43	18.04
	.99	18.53	19.33	19.92	20.50	21.34	17.43	18.58	19.11	19.22	19.54
2	.90	14.26	15.02	15.64	16.02	16.51	13.57	14.78	15.40	15.87	16.12
	.95	15.65	16.61	17.12	17.66	17.85	15.51	16.18	17.08	17.31	17.50
	.975	17.12	17.85	18.22	19.04	19.27	17.08	17.50	19.27	19.62	19.70
	.99	19.04	19.35	19.90	19.99	20.01	19.62	19.79	21.52	22.58	22.75
3	.90	16.64	17.57	18.28	18.86	19.53	16.38	17.30	17.92	18.40	18.62
	.95	18.30	19.58	20.21	20.77	21.45	17.99	18.74	19.77	20.28	20.89
	.975	20.21	21.45	22.67	23.36	23.48	19.77	20.89	21.56	22.11	22.28
	.99	23.36	23.52	24.13	24.43	25.16	22.11	22.37	22.83	23.98	24.54
4	.90	18.96	19.91	20.68	21.13	21.51	18.29	19.54	20.43	20.97	21.32
	.95	20.80	21.59	22.36	22.58	23.12	20.51	21.81	22.40	23.12	23.78
	.975	22.36	23.12	24.10	25.73	26.11	22.40	23.78	25.10	25.75	25.84
	.99	25.73	27.01	27.43	27.47	27.75	25.75	26.36	26.66	26.86	27.71

Table 3.b: Asymptotic Critical Values of the Sequential Test $SEQ_T(k+1|k)$ for Category (a) Case 2, $\epsilon = .15$.

		Non Trending Case					Trending Case				
		k					k				
q_f	α	1	2	3	4	5	1	2	3	4	5
1	.90	9.14	10.09	10.61	11.04	11.45	10.22	11.21	12.02	12.33	12.75
	.95	10.63	11.54	12.09	12.57	12.86	12.15	12.77	13.48	14.21	14.32
	.975	12.09	12.86	13.25	14.01	14.19	13.48	14.32	14.66	15.41	15.72
	.99	14.01	14.33	14.80	15.33	16.43	15.41	15.96	16.23	16.48	16.62
2	.90	10.06	11.18	11.68	12.21	12.52	9.92	10.73	11.41	11.79	12.18
	.95	11.69	12.62	13.33	13.66	14.07	11.41	12.18	12.80	13.21	13.69
	.975	13.33	14.07	14.61	15.22	15.31	12.80	13.69	14.19	14.68	14.94
	.99	15.22	15.40	16.51	17.02	18.13	14.68	15.00	15.96	16.37	17.09
3	.90	9.97	10.74	11.25	11.73	12.17	9.95	11.05	11.64	11.92	12.76
	.95	11.27	12.18	12.60	12.88	12.94	11.66	12.77	13.26	13.72	14.15
	.975	12.60	12.94	13.24	14.33	14.49	13.26	14.15	14.70	14.83	15.71
	.99	14.33	15.14	15.32	15.56	16.12	14.83	15.86	16.59	16.66	16.91
4	.90	10.01	10.81	11.55	12.09	12.37	10.19	11.19	11.79	12.67	13.05
	.95	11.59	12.40	12.80	13.88	14.23	11.90	13.08	13.68	14.53	15.03
	.975	12.80	14.23	15.59	15.74	16.03	13.68	15.03	15.62	16.08	16.70
	.99	15.74	16.10	16.61	16.93	17.05	16.08	16.80	17.48	17.48	17.80

Table 3.c: Asymptotic Critical Values of the Sequential Test $SEQ_T(k+1|k)$ for Category (a) Case 3, $\epsilon = .15$.

		Non Trending Case					Trending Case				
		k					k				
q_b	α	1	2	3	4	5	1	2	3	4	5
1	.90	9.46	10.27	10.63	11.03	11.31	8.84	10.09	10.69	11.13	11.80
	.95	10.68	11.37	11.80	12.34	12.65	10.73	11.87	12.58	13.10	13.83
	.975	11.80	12.65	12.97	13.12	13.50	12.58	13.83	14.73	15.04	15.30
	.99	13.12	15.33	16.54	16.68	16.83	15.04	15.37	15.74	16.31	16.71
2	.90	12.43	13.59	14.14	14.85	15.33	11.92	12.90	13.42	13.77	14.38
	.95	14.29	15.42	15.87	16.56	17.02	13.44	14.39	15.05	15.61	15.94
	.975	15.87	17.02	17.41	17.50	18.09	15.05	15.94	16.33	16.59	16.85
	.99	17.50	19.35	19.50	20.73	21.08	16.59	17.64	18.14	18.15	18.71
3	.90	14.48	15.51	16.11	16.53	16.72	14.83	15.61	16.24	17.15	17.47
	.95	16.12	16.78	17.66	17.97	18.14	16.26	17.48	17.97	18.70	19.01
	.975	17.66	18.14	18.85	19.45	20.30	17.97	19.01	19.79	20.11	20.22
	.99	19.45	20.34	21.65	21.66	22.84	20.11	20.64	21.23	21.27	21.39
4	.90	16.74	17.81	18.75	19.22	19.53	17.17	18.19	19.08	19.46	19.84
	.95	18.77	19.73	20.53	20.76	21.10	19.10	19.93	20.62	21.14	21.51
	.975	20.53	21.10	22.15	22.50	23.20	20.62	21.51	21.85	22.31	22.58
	.99	22.50	23.24	23.36	23.53	23.95	22.31	22.61	24.20	24.99	25.19

Table 3.d: Asymptotic Critical Values of the Sequential Test $SEQ_T(k+1|k)$ for Category (a) Case 4, $\epsilon = .15$.

		Non Trending Case					Trending Case				
		k					k				
q_f, q_b	α	1	2	3	4	5	1	2	3	4	5
1,1	.90	11.98	13.02	14.03	14.73	14.94	12.20	13.51	14.26	14.63	15.21
	.95	14.05	14.94	15.48	16.02	16.50	14.30	15.25	16.28	16.65	17.05
	.975	15.48	16.50	17.10	17.57	17.92	16.28	17.05	17.85	18.17	18.46
	.99	17.57	18.68	20.20	20.26	20.63	18.17	18.54	20.88	22.23	22.35
1,2	.90	14.77	15.85	16.63	17.17	17.35	14.09	15.20	15.77	16.04	16.38
	.95	16.64	17.36	18.10	18.48	18.70	15.82	16.44	17.19	17.89	18.19
	.975	18.10	18.70	19.48	20.38	20.61	17.19	18.19	18.76	19.21	19.61
	.99	20.38	21.05	21.57	22.36	22.54	19.21	19.69	20.34	20.48	20.66
2,1	.90	12.87	13.78	14.72	15.06	15.47	13.11	14.03	15.14	15.73	16.22
	.95	14.77	15.55	16.14	16.46	16.70	15.22	16.45	17.21	17.85	18.15
	.975	16.14	16.70	16.99	17.19	18.20	17.21	18.15	18.79	18.96	19.01
	.99	17.19	18.36	18.55	18.58	18.91	18.96	19.48	20.33	20.49	20.86
2,2	.90	14.70	15.67	16.70	17.04	17.56	14.48	15.40	15.93	16.37	16.70
	.95	16.71	17.65	18.63	19.36	19.49	15.93	16.72	17.56	17.94	18.10
	.975	18.63	19.49	20.02	20.55	21.07	17.56	18.10	18.78	19.01	19.64
	.99	20.55	21.38	22.89	23.16	24.18	19.01	20.34	21.05	21.28	21.30

Table 3.e: Asymptotic Critical Values of the Sequential Test $SEQ_T(k+1|k)$ for Category (a) Case 5, $\epsilon = .15$.

		Non Trending Case					Trending Case				
		k					k				
q_f, p_b	α	1	2	3	4	5	1	2	3	4	5
1,1	.90	9.40	10.30	11.14	11.80	12.42	10.39	11.15	11.69	12.10	12.40
	.95	11.24	12.45	13.00	13.44	13.52	11.70	12.44	13.17	14.03	14.10
	.975	13.00	13.52	14.23	14.75	15.12	13.17	14.10	14.76	14.98	15.17
	.99	14.75	15.19	15.85	16.30	16.40	14.98	15.52	15.87	15.89	16.16
1,2	.90	12.29	13.11	13.49	14.09	14.51	11.80	12.78	13.80	14.39	14.92
	.95	13.52	14.61	15.70	15.97	16.56	13.81	14.94	15.43	15.75	16.24
	.975	15.70	16.56	16.96	17.72	18.29	15.43	16.24	16.43	16.67	16.98
	.99	17.72	18.41	19.13	19.21	20.07	16.67	17.67	18.06	18.48	19.13
2,1	.90	9.49	10.38	10.88	11.34	11.65	10.76	12.02	12.82	13.44	13.86
	.95	10.95	11.74	12.27	12.44	13.07	12.95	13.88	15.17	15.58	15.97
	.975	12.27	13.07	13.58	14.31	15.40	15.17	15.97	16.42	16.65	17.37
	.99	14.31	15.57	15.78	15.79	15.96	16.65	17.63	18.64	18.75	19.30
2,2	.90	12.64	13.53	14.14	14.59	14.73	11.79	12.66	13.15	13.57	14.15
	.95	14.19	14.74	15.34	15.56	16.55	13.18	14.25	14.77	14.91	15.27
	.975	15.34	16.55	16.69	17.53	17.82	14.77	15.27	15.95	16.76	17.14
	.99	17.53	18.04	18.21	18.49	19.00	16.76	17.18	17.59	19.83	19.89

Table 4.a: Asymptotic Critical Values of the Sequential Test $SEQ_T(k+1|k)$ for Category (b) Case 3, $\epsilon = .15$.

		Non Trending Case					Trending Case				
		k					k				
q_b, p_b	α	1	2	3	4	5	1	2	3	4	5
1,1	.90	11.93	12.95	13.38	13.62	14.18	12.47	13.83	14.82	15.19	15.58
	.95	13.40	14.18	14.59	14.96	15.02	14.93	15.58	16.44	16.90	17.19
	.975	14.59	15.11	15.99	16.54	17.22	16.44	17.43	17.70	18.61	18.88
	.99	16.54	17.54	17.54	17.93	17.93	18.61	18.98	18.98	19.20	19.20
1,2	.90	14.47	15.21	15.83	16.47	17.36	14.81	15.68	16.46	17.21	18.30
	.95	15.91	17.36	18.19	19.33	19.34	16.60	18.30	19.07	19.60	19.65
	.975	18.19	19.49	20.39	20.48	21.15	19.07	19.93	21.46	21.96	22.00
	.99	20.48	21.52	21.52	23.28	23.28	21.96	23.85	23.85	24.68	24.68
2,1	.90	14.54	15.95	16.68	17.07	17.24	14.99	15.99	16.61	17.22	17.43
	.95	16.74	17.24	17.40	19.05	19.15	16.72	17.43	18.24	19.49	19.56
	.975	17.40	19.44	20.71	21.24	21.81	18.24	19.59	21.89	22.00	24.38
	.99	21.24	22.01	22.01	23.54	23.54	22.00	24.84	24.84	26.36	26.36
	.90	16.28	17.34	17.90	18.13	18.64	16.89	17.75	18.42	19.03	19.60
	.95	17.92	18.64	19.05	19.89	20.30	18.46	19.60	20.28	21.17	21.21
	.975	19.05	20.33	20.46	21.40	21.80	20.28	21.45	21.60	21.90	21.94
	.99	21.40	23.39	23.39	23.64	23.64	21.90	22.58	22.58	24.08	24.08

Table 4.b: Asymptotic Critical Values of the Sequential Test $SEQ_T(k+1|k)$ for Category (b) Cases 4 and 8, $\epsilon = .15$.

		Non Trending Case					Trending Case				
		k					k				
q_b, p_b	α	1	2	3	4	5	1	2	3	4	5
1,1	.90	13.18	13.92	14.70	15.08	15.79	13.72	15.14	15.72	16.44	16.75
	.95	14.72	15.82	16.60	17.28	17.61	15.73	16.83	17.54	17.99	18.17
	.975	16.60	17.61	19.20	19.43	19.85	17.54	18.17	19.27	19.97	20.53
	.99	19.43	20.02	21.38	21.43	22.10	19.97	21.13	22.77	23.42	23.98
1,2	.90	15.06	16.32	17.39	17.83	18.22	15.09	16.21	16.85	17.33	17.85
	.95	17.44	18.25	18.65	19.10	19.96	16.86	17.87	18.81	18.95	19.28
	.975	18.65	19.96	20.06	20.37	20.69	18.81	19.28	19.66	21.10	21.43
	.99	20.37	20.73	21.96	23.13	23.22	21.10	21.61	22.74	23.70	24.12
2,1	.90	15.82	16.69	17.59	18.15	18.39	15.21	16.54	17.44	17.98	18.46
	.95	17.68	18.63	19.37	19.89	20.39	17.49	18.49	19.26	19.61	20.27
	.975	19.37	20.39	21.48	22.63	22.84	19.26	20.27	20.76	21.69	22.03
	.99	22.63	23.82	24.73	25.40	25.62	21.69	22.37	22.94	24.08	24.08
2,2	.90	16.95	18.69	19.46	20.06	20.44	17.12	18.56	19.40	19.92	20.75
	.95	19.48	20.44	21.33	21.66	21.97	19.45	20.42	21.16	21.46	22.33
	.975	21.33	21.97	22.39	23.52	24.03	21.16	21.86	22.89	23.41	23.85
	.99	23.52	24.11	24.75	25.05	25.12	23.41	23.85	25.06	25.94	26.32

Table 4.c: Asymptotic Critical Values of the Sequential Test $SEQ_T(k+1|k)$ for Category (b) Cases 7 and 9, $\epsilon = .15$.

		Non Trending Case					Trending Case				
		k					k				
q_f, p_b	α	1	2	3	4	5	1	2	3	4	5
1,1	.90	10.56	11.68	12.06	12.70	13.25	10.14	11.07	11.81	12.31	12.90
	.95	12.08	13.26	14.04	14.37	14.95	11.85	13.01	13.57	13.88	13.99
	.975	14.04	14.95	15.11	15.68	16.31	13.57	13.99	14.63	15.19	15.95
	.99	15.68	17.70	18.33	19.01	20.20	15.19	16.15	16.24	16.25	16.34
1,2	.90	11.52	12.51	12.96	13.57	14.28	10.95	11.83	12.70	12.92	13.89
	.95	12.98	14.45	15.30	15.66	15.93	12.70	13.89	14.90	15.22	16.00
	.975	15.30	15.93	16.30	16.85	16.95	14.90	16.00	16.68	17.33	17.48
	.99	16.85	17.36	17.77	18.54	19.60	17.33	17.91	18.29	18.71	19.21
2,1	.90	10.65	11.45	11.95	12.68	13.14	10.49	11.45	12.34	12.86	13.34
	.95	11.97	13.47	14.57	15.29	15.85	12.36	13.69	14.55	14.98	15.07
	.975	14.57	15.85	16.64	17.43	17.92	14.55	15.07	15.29	15.72	15.86
	.99	17.43	18.13	18.71	19.52	19.64	15.72	15.96	16.44	16.48	17.43
2,2	.90	12.22	13.22	14.03	14.56	14.93	11.76	12.88	13.46	14.31	14.75
	.95	14.03	15.05	15.56	16.23	16.54	13.51	14.97	15.19	15.75	16.10
	.975	15.56	16.54	17.38	17.82	18.18	15.19	16.10	16.45	17.06	17.27
	.99	17.82	18.46	19.61	19.65	20.18	17.06	17.40	18.55	19.65	20.08

Table 4.d: Asymptotic Critical Values of the Sequential Test $SEQ_T(k+1|k)$ for Category (b) Case 10, $\epsilon = .15$.

		Non Trending Case					Trending Case				
		k					k				
q_f, q_b, p_b	α	1	2	3	4	5	1	2	3	4	5
1,1,1	.90	13.72	15.13	16.24	16.68	17.11	14.32	15.97	16.59	17.08	17.31
	.95	15.75	16.74	17.98	18.34	18.44	16.60	17.35	18.07	18.68	18.99
	.975	17.42	18.34	19.12	19.85	20.15	18.07	18.99	19.73	20.26	20.71
	.99	19.12	20.15	21.12	21.21	21.21	20.26	21.24	22.52	22.55	22.81
1,1,2	.90	14.85	15.95	17.30	17.86	18.46	15.58	16.90	18.31	18.92	19.14
	.95	16.64	18.01	19.17	19.55	19.72	18.39	19.14	19.98	20.75	21.50
	.975	18.69	19.55	21.44	21.63	22.04	19.98	21.50	21.94	22.54	22.86
	.99	21.44	22.04	23.51	24.20	24.20	22.54	23.07	23.18	23.35	23.85
1,2,1	.90	15.94	16.98	17.99	18.30	18.46	15.72	17.04	17.59	17.75	18.16
	.95	17.69	18.31	19.77	20.07	20.32	17.59	18.47	19.63	20.69	21.06
	.975	19.01	20.07	20.93	21.42	21.81	19.63	21.06	21.76	22.59	22.70
	.99	20.93	21.81	22.83	22.88	22.88	22.59	22.83	23.91	24.31	24.81
1,2,2	.90	17.43	18.53	19.99	20.17	20.75	16.70	17.70	18.60	19.20	19.82
	.95	19.42	20.21	22.29	22.49	22.57	18.63	19.83	20.60	20.94	21.27
	.975	21.40	22.49	23.20	24.51	24.63	20.60	21.27	21.71	23.06	23.19
	.99	23.20	24.63	25.82	26.26	26.26	23.06	23.23	23.52	23.54	25.67
2,1,1	.90	13.75	14.86	16.55	17.03	18.10	14.31	15.26	15.96	16.71	17.40
	.95	16.09	17.16	18.68	18.90	20.00	16.00	17.60	18.26	19.23	19.96
	.975	18.12	18.90	20.85	21.25	22.27	18.26	19.96	21.00	22.20	22.30
	.99	20.85	22.09	22.93	23.14	23.14	22.20	22.61	24.61	24.76	25.10
2,1,2	.90	15.22	16.26	17.60	18.14	18.99	15.68	16.62	17.43	17.99	18.50
	.95	17.14	18.23	19.37	20.03	20.79	17.52	18.50	19.57	20.26	20.44
	.975	19.03	20.03	21.76	22.33	23.09	19.57	20.44	21.28	21.79	22.42
	.99	21.76	22.83	23.18	23.46	23.46	21.79	22.50	22.82	23.61	23.91
2,2,1	.90	16.70	17.48	18.82	19.34	20.49	15.54	16.36	17.08	17.44	17.96
	.95	18.34	19.38	20.83	21.46	21.70	17.11	18.01	18.62	19.20	19.60
	.975	20.52	21.46	22.00	23.35	23.69	18.62	19.60	20.74	21.16	22.04
	.99	22.00	23.59	24.22	25.41	25.41	21.16	22.35	22.92	23.90	24.39
2,2,2	.90	16.93	18.15	19.27	19.87	20.73	16.76	17.94	18.93	19.82	20.09
	.95	18.87	19.92	22.03	22.30	22.85	18.97	20.11	20.59	21.22	21.83
	.975	21.29	22.30	23.24	23.62	23.70	20.59	21.83	22.31	22.75	23.49
	.99	23.24	23.64	24.24	24.36	24.36	22.75	23.58	25.33	25.76	26.04

Table 4.e: Asymptotic Critical Values of the Sequential Test $SEQ_T(k+1|k)$ for Category (b) Case 11, $\epsilon = .15$.

		Non Trending Case					Trending Case				
		k					k				
q_f, q_b, p_b	α	1	2	3	4	5	1	2	3	4	5
1,1,1	.90	12.42	13.24	14.08	14.40	15.10	12.21	13.08	14.02	14.74	14.94
	.95	14.10	15.10	15.60	16.41	16.57	14.05	14.94	15.08	15.82	16.06
	.975	15.60	16.61	17.46	17.79	18.58	15.08	16.42	16.52	16.54	17.87
	.99	17.79	18.90	18.90	21.38	21.38	16.54	19.39	19.39	20.59	20.59
1,1,2	.90	14.17	15.46	16.06	16.53	17.09	14.20	15.10	15.55	16.07	16.49
	.95	16.06	17.09	17.34	17.71	18.06	15.58	16.49	17.32	18.04	18.24
	.975	17.34	18.09	18.57	19.03	19.08	17.32	18.65	19.02	19.12	19.69
	.99	19.03	19.51	19.51	20.33	20.33	19.12	19.70	19.70	20.43	20.43
1,2,1	.90	15.38	16.40	17.29	17.98	18.29	15.18	16.23	16.61	16.99	17.36
	.95	17.33	18.29	19.09	19.61	20.04	16.67	17.36	17.69	18.56	19.40
	.975	19.09	20.57	21.29	21.64	21.85	17.69	19.66	20.44	21.68	23.43
	.99	21.64	22.22	22.22	23.43	23.43	21.68	23.63	23.63	25.79	25.79
1,2,2	.90	17.28	18.20	18.80	19.30	19.92	16.87	17.73	18.45	18.83	19.78
	.95	18.82	19.92	21.25	21.70	22.15	18.56	19.78	21.03	22.02	22.23
	.975	21.25	22.28	22.80	24.44	25.09	21.03	22.69	23.56	23.72	23.98
	.99	24.44	25.23	25.23	25.95	25.95	23.72	24.03	24.03	25.78	25.78
2,1,1	.90	12.43	13.03	13.57	14.25	14.98	13.21	14.44	15.52	16.04	16.63
	.95	13.65	14.98	15.88	16.69	16.82	15.53	16.63	17.42	17.99	18.08
	.975	15.88	16.94	18.74	18.83	19.15	17.42	18.86	19.27	19.34	20.07
	.99	18.83	20.20	20.20	22.79	22.79	19.34	20.08	20.08	21.03	21.03
2,1,2	.90	14.42	15.34	16.01	16.89	17.49	14.97	15.91	16.90	17.29	17.71
	.95	16.21	17.49	18.53	18.75	19.26	16.91	17.71	17.90	18.72	19.44
	.975	18.53	19.42	19.84	20.11	20.60	17.90	19.53	20.62	20.91	21.98
	99	20.11	21.74	21.74	22.13	22.13	20.91	22.55	22.55	23.93	23.93
2,2,1	.90	15.45	16.28	17.60	18.68	19.01	15.90	16.74	17.34	17.58	17.96
	.95	17.63	19.01	19.81	20.25	20.42	17.34	17.96	18.11	19.27	19.52
	.975	19.81	20.44	21.19	21.62	21.68	18.11	19.54	19.59	20.30	20.53
	.99	21.62	22.46	22.46	22.67	22.67	20.30	20.69	20.69	20.81	20.81
2,2,2	.90	16.27	17.06	17.98	19.43	19.91	16.75	17.86	18.20	18.51	19.06
	.95	18.37	19.91	20.55	21.47	21.70	18.21	19.06	19.74	20.29	20.30
	.975	20.55	22.16	22.79	22.90	23.80	19.74	20.53	20.90	21.29	21.94
	.99	22.90	23.94	23.94	24.12	24.12	21.29	23.17	23.17	23.65	23.65

Table 5: Empirical Size.

		$T = 120$					$T = 240$				
Specification	Test\DGP	1	2	3	4	5	1	2	3	4	5
S_Corr=0,C_Corr=0	$SupF_T^*(1)$	.04	.55	.00	.15	.20	.04	.63	.00	.14	.19
	$SupF_T^*(2)$	.05	.73	.00	.19	.27	.04	.82	.00	.20	.31
	$SupF_T^*(3)$	.04	.75	.00	.20	.27	.04	.85	.00	.22	.33
	$UDmax$	.04	.65	.00	.16	.21	.05	.72	.00	.16	.21
S_Corr=1,C_Corr=0	$SupF_T^*(1)$	.04	.03	.02	.14	.25	.03	.03	.02	.12	.28
	$SupF_T^*(2)$	.03	.02	.05	.13	.29	.03	.02	.02	.17	.43
	$SupF_T^*(3)$	.02	.01	.05	.12	.29	.02	.00	.02	.18	.45
	$UDmax$	.04	.03	.03	.14	.26	.03	.02	.02	.14	.32
S_Corr=0,C_Corr=1	$SupF_T^*(1)$	.06	.58	.00	.05	.00	.05	.64	.00	.05	.00
	$SupF_T^*(2)$	.07	.76	.00	.07	.00	.05	.82	.00	.05	.00
	$SupF_T^*(3)$	.06	.77	.00	.06	.00	.05	.86	.00	.05	.00
	$UDmax$	.06	.67	.00	.05	.00	.05	.73	.00	.05	.00
S_Corr=1,C_Corr=1	$SupF_T^*(1)$	.05	.04	.03	.04	.04	.04	.04	.01	.04	.01
	$SupF_T^*(2)$	.03	.02	.06	.03	.07	.03	.02	.02	.04	.03
	$SupF_T^*(3)$	.03	.02	.07	.02	.07	.02	.01	.02	.04	.03
	$UDmax$	.05	.04	.04	.04	.05	.04	.04	.01	.05	.02

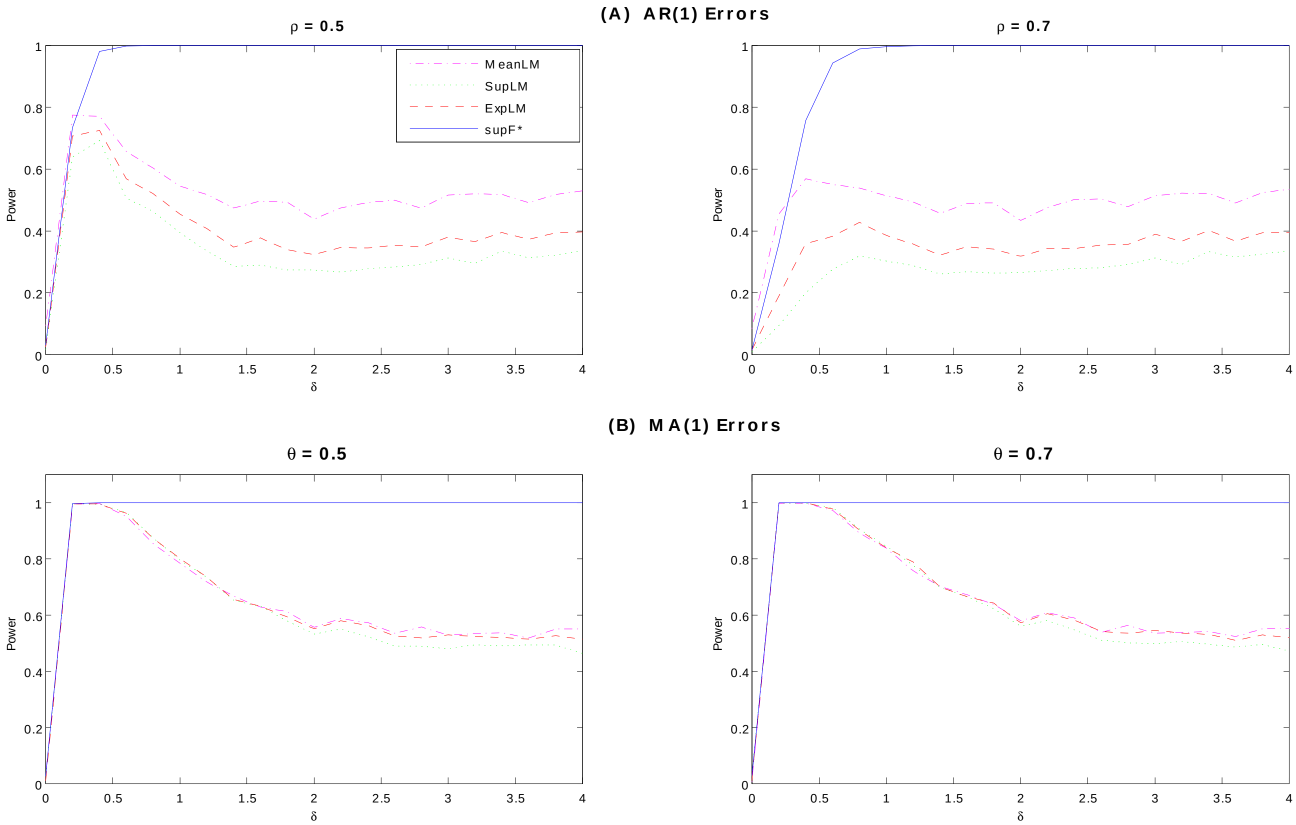

Figure 1: Power Functions: The Case with One Break

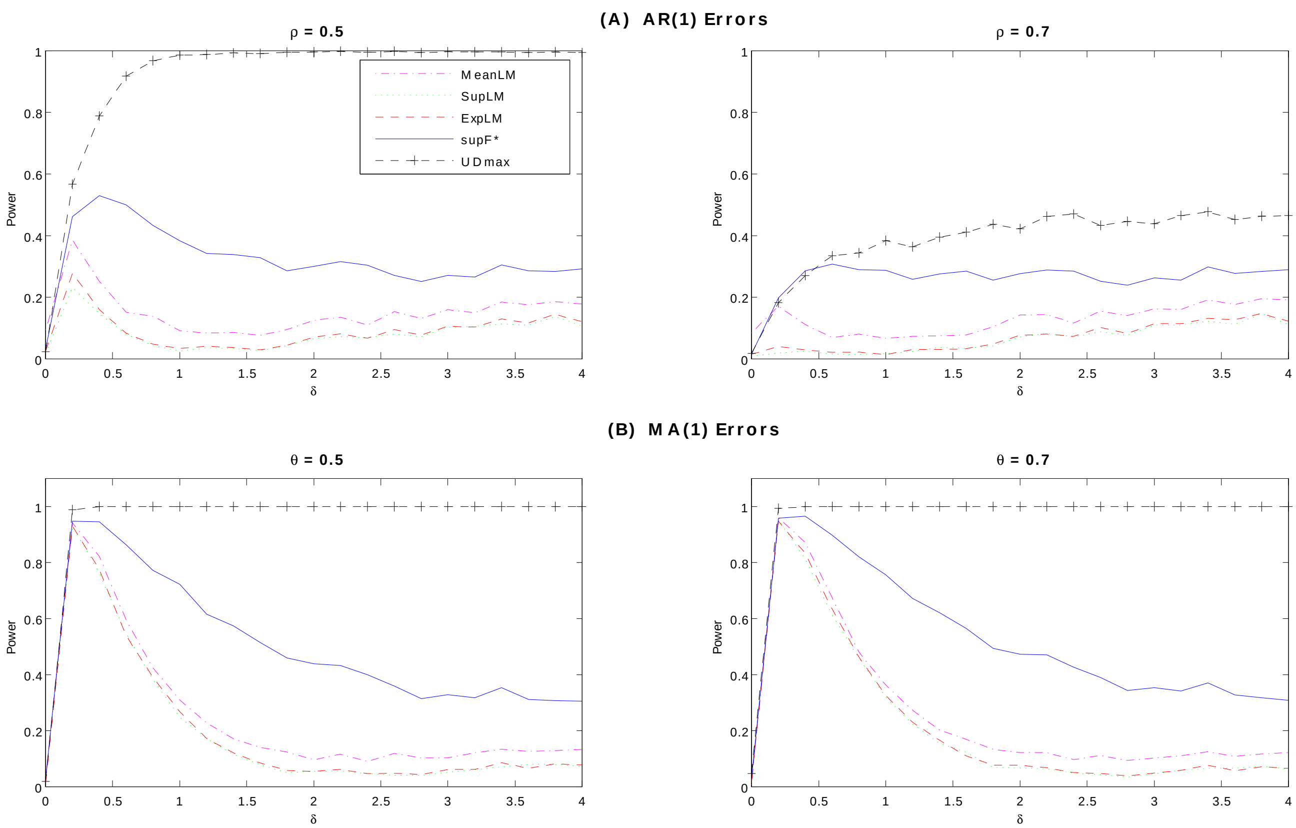

Figure 2: Power Functions: The Case with Two Breaks