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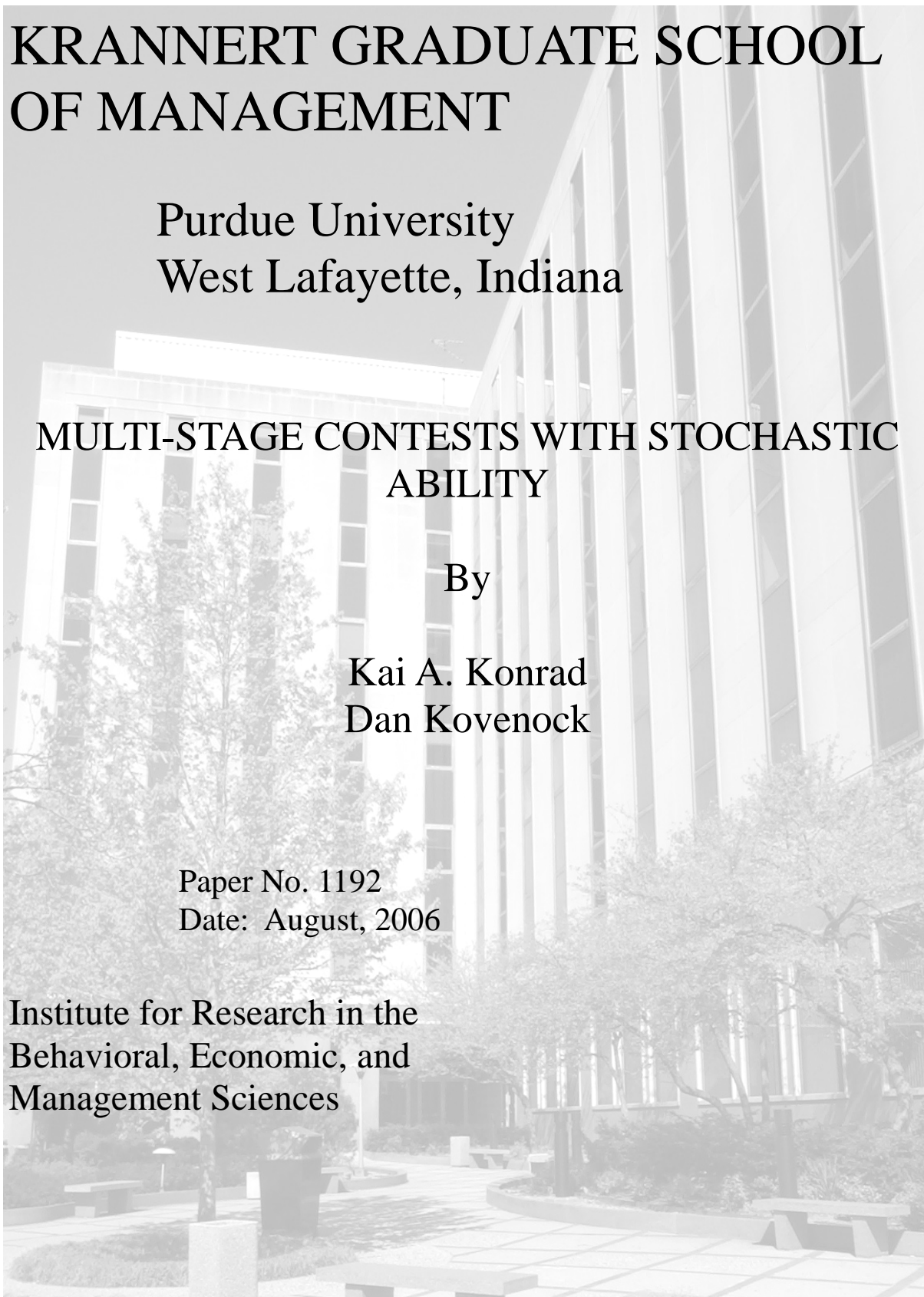
## MULTI-STAGE CONTESTS WITH STOCHASTIC ABILITY

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# Multi-stage contests with stochastic ability\*

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## Abstract

We consider the properties of perfectly discriminating contests in which players' abilities are stochastic, but become common knowledge before efforts are expended. Players whose expected ability is lower than that of their rivals may still earn a positive expected payoff from participating in the contest, which may explain why they participate. We also show that an increase in the dispersion of a player's own ability generally benefits this player. It may benefit or harm his rival, but cannot benefit the rival more than it benefits himself. We also explore the role of stochastic ability for sequential contests with the same opponent (multi-battle contests) and with varying opponents (elimination tournaments) and show that it reduces the strong discouragement effects and hold-up problems that may otherwise emerge in such games. High own ability dispersion selects such players into the contest and favors them in elimination contests.

Keywords: all-pay auction, elimination tournament, contest, race, conflict, multi-stage, random ability, discouragement.

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## 1 Introduction

Many tournaments are characterized by multiple rounds, with or without the elimination of some candidates in earlier stages of the process of determining

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a final winner. Internal career competition has features of an elimination tournament in which the number of participants may shrink gradually.<sup>1</sup> In patent races several firms may take part in what Harris and Vickers (1987) have termed a race: a multi-battle contest in which the competitor who first accumulates a given number of successes wins. In the political context, many election processes consist of multiple contest stages. In the race for US presidency several competition stages gradually narrow down the number of competitors.<sup>2</sup> Many sports disciplines provide more obvious examples.<sup>3</sup> Also violent fighting for turf, for successorship in the context of autocratic governance systems, or for military victory in wars typically consist of multiple battles with victory being a function of the outcomes in these battles, and with some of the competitors being eliminated at some stage of the process.

Contests with multiple rounds or tournaments in which the outcome of previous battles determines whether players are allowed to enter into or win something in later stages of the game have an important hold-up feature in common: successful participation in the future stages of the game may require substantial effort, and this may make it less attractive to expend effort in preliminary rounds of the game. Similarly, once a player has accumulated a sufficiently large disadvantage in the game, he may simply want to give up, even though success in later rounds may bring him back into play. Returning to a state in which the competition becomes more balanced may not be worth much effort, because the economic rents from winning the competition at this state may be dissipated by the efforts expended in the state. Wärneryd (1998), McAfee (2000), Müller and Wärneryd (2001), Klumpp and Polborn (2006), Konrad (2004) and Konrad and Kovenock (2005, 2006) illustrate discouragement effects of this type.<sup>4</sup>

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<sup>1</sup>Such competition has been analysed by Rosen (1986) and an early literature survey is by Lazear (1995). Rosen (1986) distinguishes between heterogenous contestants with common knowledge of all players' talents, and an elimination tournament with two stages in which there are different types of players but all players share the same symmetric priors about themselves and about all other players.

<sup>2</sup>Several dynamic aspects of the presidential nomination campaigns have been analysed by Aldrich (1980), Strumpf (2002) and Klumpp and Polborn (2006). The latter emphasize that outcomes of early rounds may lead to what could be called a discouragement effect for the player who lost in early rounds.

<sup>3</sup>Szymanski (2003) discusses a large set of design issues in this context. The structure of an elimination contest has been analysed in the context of sports, e.g., by Abrevaya (2002), Groh et al. (2003), Harbaugh and Klumpp (2005) and Horen and Riezman (1985).

<sup>4</sup>The possibility of conflict in future periods and its implications for the resolution of conflict in earlier periods has also received considerable attention in the literature on the economics of conflict (see Garfinkel and Skaperdas 2000, Mehlum and Moene 2004).

In this paper we identify an important reason why the discouragement effect of future conflict may be less severe than current theory would imply. A player’s ability, measured, for instance, by his or her cost of expending effort, may be random. Empirically, the existence of transitory changes in a player’s ability is seemingly a very reasonable assumption for all of the examples mentioned. Athletes obviously have transitory ups and down in their ability. The same should apply to managers and workers in firms, to researchers in laboratories and the managers who hire and supervise them, and to politicians and their advisors in the different stages of a campaign. Moreover, many aspects of a player’s actual ability or effectiveness in a given battle, match, or campaign may be easily observed by an adversary, so that it is not unreasonable to model these transitory realizations as common knowledge at the start of the battle.

Such randomness has important implications. Shocks to the unit cost of effort ameliorate the effects of cutthroat competition in single and multi-stage perfectly discriminating contests.<sup>5</sup> More precisely, despite the fact that, all else equal, less able players earn a zero expected utility in such a contest, stochastic ability means that “on any given day” an underdog may be more able than a favorite. This turns participation in such a contest into an option: in perfectly discriminating contests in which a player is less able than his rival he earns a zero expected payoff, but earns a positive payoff, linearly decreasing in his own cost of effort, in contests in which he is more able. Hence, players benefit from mean preserving spreads of their own cost distribution. Mean preserving spreads of a rival’s distribution of cost may benefit or harm a player, but never benefit the player more than the rival himself. All of this implies that with transient ability shocks dissipation will be lower than in the absence of shocks and the value that players attribute ex ante to participating in such a contest will be higher.

With transient ability, typically all players earn a positive expected payoff from the contest ex ante (in contrast to the deterministic cost case). Therefore the cutthroat nature of later stage contests is moderated and does not completely discourage effort in earlier stage contests. Moreover, the “reversion to the mean” arising in later stage contests means that even if a player in a given contest is less able than his rival, if he is more able on average, his continuation value from winning the contest will be greater, and therefore his imputed value of the prize from the present contest will be

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<sup>5</sup>Randomness of ability has qualitatively similar implications if the contest success function is not perfectly discriminatory. However, we do not address this issue in this paper.

greater. This leads to greater effort, at least partially offsetting his transient ability disadvantage. Hence, our analysis has implications for the interaction of favorites and underdogs initiated by Dixit (1987) and further elaborated upon by Baik and Shogren (1992) and Leininger (1993). In this line of research, ability is some property that remains fixed for the duration of the contest despite any noise that might be generated by the imperfectly discriminating nature of the contest success function. In contrast, we concentrate on an environment in which players differ over time in their abilities. A player's ability in each round of the contest is a random draw from a player (and possibly time) specific distribution.

Our results have far reaching implications for both naturally arising and mechanism-induced selection processes. First, we demonstrate that, given two rival players with identical mean abilities, the player with the greater dispersion in ability achieves higher payoffs in the contest against his rival. Moreover, the “riskier” player also obtains a higher expected payoff than does his rival against any third player, regardless of that player's distribution of ability. Hence, all else equal, we would expect evolutionary forces to lead to greater fitness of players with “riskier” distributions of abilities. Such players would also be more willing to expend whatever entry costs might be required to participate in perfectly discriminating contests. In addition to this naturally occurring selection, within mechanism selection also arises. All else equal, players with more dispersed abilities have higher continuation values from winning at early or intermediate stages of multistage contests, which increases their cost contingent incentive to expend effort in the current stage-contest faced. This leads to both higher effort and an increased probability of advancement. Hence not only does an elimination contest have a tendency to select individuals with higher dispersion, the dispersion among participating individuals should increase in the later stages of an elimination tournament.

A roadmap for the remainder of the paper is as follows. In section 2 we develop the formal framework and analyse the role of cost dispersion for the payoffs of players in a static, perfectly discriminating contest. We consider how these results are reinforced in a dynamic elimination contest in section 3, and in a race in section 4. Section 5 concludes.

## 2 Cost dispersion and the contest

In this section we study a static contest with two players  $i = 1, 2$ . A prize is awarded to the winner. The value of the prize is normalized to unity. The

competition for this prize is organized as a perfectly discriminating contest (all-pay auction), in which the two players 1 and 2 simultaneously expend effort  $e_1 \geq 0$  and  $e_2 \geq 0$  and have costs of effort that are equal to  $c_1 e_1 \geq 0$  and  $c_2 e_2 \geq 0$ . Here,  $c_1$  and  $c_2$  are the per-unit-of-bid effort costs of players 1 and 2, with  $c_1, c_2 \in [\underline{c}, \bar{c}]$ , and randomness of these unit cost parameters will be our main concern. However, at the point of time when the efforts are chosen, each player knows his own and the rival player's unit cost; hence, at this stage, the problem describes a perfectly discriminating contest with complete information, with payoffs of the players characterized as

$$\begin{aligned}\pi_1(c_1, e_1, c_2, e_2) &= p_1(e_1, e_2) \cdot 1 - c_1 e_1 \\ \pi_2(c_1, e_1, c_2, e_2) &= p_2(e_1, e_2) \cdot 1 - c_2 e_2\end{aligned}\tag{1}$$

where  $p_i(e_1, e_2) = 1$  if  $e_i > e_j$  for  $i, j \in \{1, 2\}$ , and  $p_1 = p_2 = 1/2$  if  $e_1 = e_2$ . This game has been carefully analysed by Hillman and Riley (1989) and Baye, Kovenock and deVries (1996). As they show, the equilibrium of the perfectly discriminating contest for given values of  $c_1$  and  $c_2$  is unique and described as follows:

**Proposition 1** (Hillman and Riley 1989)<sup>6</sup> *The unique equilibrium of the two-player all-pay auction with complete information is in mixed strategies. Let  $c_1 < c_2$ . Then bids are described by the following cumulative distribution functions:*

$$\begin{aligned}G_1(e) &= \begin{cases} c_2 e & \text{for } e \in [0, \frac{1}{c_2}) \\ 1 & \text{for } e \geq \frac{1}{c_2} \end{cases} \\ G_2(e) &= \begin{cases} 1 - \frac{c_1}{c_2} + e c_1 & \text{for } e \in [0, \frac{1}{c_2}) \\ 1 & \text{for } e \geq \frac{1}{c_2} \end{cases}\end{aligned}$$

*The payoffs are  $1 - \frac{c_1}{c_2}$  for player 1 and 0 for player 2, and win probabilities are equal to  $1 - \frac{c_1}{2c_2}$  for player 1 and  $\frac{c_1}{2c_2}$  for player 2.*

We now turn to the point in time at which the players have not learned their actual unit cost of expending effort in the perfectly discriminating contest. We assume that these costs are random variables. The main focus of this section is how this randomness affects the expected equilibrium payoffs

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<sup>6</sup>The original characterization of equilibrium in two-player perfectly discriminating contests assumed identical cost of effort. However, as noted by Baye, Kovenock and de Vries (1996, p. 292), behavior is invariant with respect to affine transformations of utility, so that by dividing each player's payoff by his own constant unit cost of effort one can transform a game with different unit costs of effort into one with identical unit costs of effort but transformed valuations of the prize.

of the players at the stage when they do not yet know the realization of their unit costs.

Assume that unit costs  $c_i$  are independent random variables that are absolutely continuous with finite support  $[\underline{c}, \bar{c}]$  with  $\bar{c} > \underline{c} > 0$ . The cumulative distribution functions of  $c_1$  and  $c_2$  are  $F_1(c_1)$  and  $F_2(c_2)$  with corresponding densities  $f_1$  and  $f_2$ , which we assume to be positive on  $[\underline{c}, \bar{c}]$ . As is seen from Proposition 1, not the absolute values of  $c_1$  and  $c_2$ , but rather their ratio is of importance for the equilibrium payoffs. Let  $\alpha \equiv c_1/c_2$ . Then  $F_1$  and  $F_2$  induce a cumulative distribution function  $Z(\alpha)$ ,<sup>7</sup> which we assume is absolutely continuous with density function  $z(\alpha)$ , defined on  $[\underline{\alpha}, \bar{\alpha}]$ , where  $\underline{\alpha} = \underline{c}/\bar{c}$  and  $\bar{\alpha} = \bar{c}/\underline{c}$ . The expected payoffs are

$$\pi_1(Z) = \int_{\underline{\alpha}}^1 (1 - \alpha)z(\alpha)d\alpha \text{ and } \pi_2(Z) = \int_1^{\bar{\alpha}} (1 - \frac{1}{\alpha})z(\alpha)d\alpha. \quad (2)$$

In this perfectly discriminating contest only the player who has an actual cost advantage receives a positive payoff, and the player with the cost disadvantage receives a zero payoff. Accordingly, if cost is dispersed, and if both players' cost parameters are independent random variables with the same support, then each player has a cost advantage with positive probability, and therefore earns a positive expected payoff.

Changes in the distributions of cost parameters affect the expected payoffs. We first study generalizations of mean-shifts and then turn to changes in the riskiness in the sense of second order stochastic dominance.

**Proposition 2** *Consider two distributions  $Z(\alpha)$  and  $\hat{Z}(\alpha)$  with  $Z(\alpha) \geq \hat{Z}(\alpha)$  for all  $\alpha$ . Then  $\pi_1(Z(\alpha)) \geq \pi_1(\hat{Z}(\alpha))$  and  $\pi_2(Z(\alpha)) \leq \pi_2(\hat{Z}(\alpha))$ .*

**Proof.** Integrating by parts,

$$\begin{aligned} & \int_{\underline{\alpha}}^1 (1 - \alpha)(z(\alpha) - \hat{z}(\alpha))d\alpha & (3) \\ &= \left[ (1 - \alpha)(Z(\alpha) - \hat{Z}(\alpha)) \right]_{\underline{\alpha}}^1 + \int_{\underline{\alpha}}^1 (Z(\alpha) - \hat{Z}(\alpha))d\alpha \\ &= 0 + \int_{\underline{\alpha}}^1 (Z(\alpha) - \hat{Z}(\alpha))d\alpha \geq 0, \end{aligned}$$

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<sup>7</sup>Unless otherwise noted, our results stated on  $Z(\alpha)$  will hold for more general joint distributions  $F(c_1, c_2)$  that induce the respective cumulative distribution function  $Z(\alpha)$  and density function  $z(\alpha)$ .

where the last inequality uses the assumption that  $Z(\alpha) - \hat{Z}(\alpha) \geq 0$  for all  $\alpha$ .

For player 2, let  $T(1/\alpha)$  and  $\hat{T}(1/\alpha)$  be the cumulative distributions for  $1/\alpha$  if  $\alpha$  is distributed according to  $Z(\alpha)$  and  $\hat{Z}(\alpha)$ , respectively. Note that  $T(1/\alpha)$  dominates  $\hat{T}(1/\alpha)$  in the sense of first-order stochastic dominance if and only if  $Z(\alpha)$  is dominated by  $\hat{Z}(\alpha)$  in the sense of first-order stochastic dominance. Moreover,

$$\begin{aligned} \pi_2(T(1/\alpha)) - \pi_2(\hat{T}(1/\alpha)) &= \\ \int_{\underline{\alpha}}^1 (1 - \frac{1}{\alpha})(t(\frac{1}{\alpha}) - \hat{t}(\frac{1}{\alpha}))d\frac{1}{\alpha}. \end{aligned} \quad (4)$$

From here, the proof for player 2 follows by integrating by parts and using the definition of first-order stochastic dominance. ■

Proposition 2 considers a generalized shift in the mean of  $c_1/c_2$ , in the sense of first-order stochastic dominance. Intuitively speaking, if it becomes more likely that  $c_1/c_2$  is higher, then this shifts probability mass from states with cost ratios for which the payoff of player 1 is high to states with cost ratios for which the payoff of player 1 is smaller, or even zero. The expected payoff is, therefore, reduced.

We now turn to changes in the dispersion of cost. To symbolize the property that  $\int_{\underline{\alpha}}^x [\hat{Z}(\alpha) - Z(\alpha)]d\alpha \leq 0$  for all  $x$ , i.e.,  $Z$  is dominated by  $\hat{Z}$  in the sense of second-order stochastic dominance, we use  $Z \leq_{SSD} \hat{Z}$ . The following holds:

**Proposition 3** *Consider two distributions  $Z$  and  $\hat{Z}$  of  $\alpha$ , such that  $Z \leq_{SSD} \hat{Z}$ . Then  $\pi_1(Z) \geq \pi_1(\hat{Z})$ .*

**Proof.**

$$\begin{aligned} \pi_1(Z) - \pi_1(\hat{Z}) &= \int_{\underline{\alpha}}^1 (1 - \alpha)(z(\alpha) - \hat{z}(\alpha))d\alpha \\ &= \left[ (1 - \alpha)(Z(\alpha) - \hat{Z}(\alpha)) \right]_{\underline{\alpha}}^1 + \int_{\underline{\alpha}}^1 (Z(\alpha) - \hat{Z}(\alpha))d\alpha \\ &= \int_{\underline{\alpha}}^1 (Z(\alpha) - \hat{Z}(\alpha))d\alpha \geq 0. \end{aligned} \quad (5)$$

The second line follows from the first line by integration by parts, and the last inequality holds by the definition of SSD. ■



Proposition 3 has implications for the ex-ante benefits of uncertainty of own strength in the perfectly discriminating contest. These implications can be spelled out easily with the help of the following lemma:

**Lemma 1** *Consider three positive random variables  $c_i$ ,  $c_j$ , and  $c_k$ , with cumulative distribution functions  $F_i(c_i)$ ,  $F_j(c_j)$  and  $F_k(c_k)$ . Let  $c_i$  and  $c_k$  and  $c_j$  and  $c_k$  be pairwise stochastically independent. Then, if  $F_i(c_i) \leq_{SSD} F_j(c_j)$ , then  $Z(\frac{c_i}{c_k}) \leq_{SSD} Z(\frac{c_j}{c_k})$ .*

The proof of Lemma 1 has been relegated to the Appendix. The Lemma states an intuitive result. Suppose  $c_1$  is dominated by  $\tilde{c}_1$  in the sense of SSD. Then, for any given  $c_2 > 0$ , it holds that  $c_1/c_2$  is dominated by  $\tilde{c}_1/c_2$  in the sense of SSD. But if this holds for all  $c_2 > 0$ , it should also hold for the weighted sum over  $c_2$  of  $c_1/c_2$  and  $\tilde{c}_1/c_2$ . Proposition 4 together with Lemma 1 can be used to make the following observation.

**Corollary 1** *If  $F_1(c_1) \leq_{SSD} \hat{F}_1(c_1)$ , then, at the stage where  $c_1$  and  $c_2$  are not known to the players, player 1 with a cost distribution  $F_1(c_1)$  has the higher expected equilibrium payoff than player 1 with a cost distribution  $\hat{F}_1(c_1)$ .*

**Proof.** By Lemma 1,  $Z(\alpha) \leq_{SSD} \hat{Z}(\alpha)$  follows from  $F_1(c_1) \leq_{SSD} \hat{F}_1(c_1)$ . Hence, by Proposition 3,  $\pi_1(Z) \geq \pi_1(\hat{Z})$ . ■

Corollary 1 suggests that a player benefits from a higher dispersion in his own ability. In many other areas of economics a higher dispersion is associated with higher risk, and generally disliked. In a contest environment, a higher dispersion of own ability is beneficial. This property may have implications for players' decisions to enter into games which can be characterized as all-pay auctions or contests. Players with a high variability in their ability earn larger expected rents when entering into such games. Hence, they are likely to be willing to expend a higher entry cost. If there are deterministic entry fees into such games or an opportunity cost of participating, we should therefore expect some self-selection of players: for given entry cost, players with high variability in their ability benefit more from entering into such games and should be more likely to participate, whereas players with the same average ability, but less variability are more likely to stay out.

Intuitively, the corollary can also be interpreted from a competition point of view. If  $c_1 = c_2$ , the rules of the perfectly discriminating contest make players compete very strongly. Competition is so strong that, as is shown in Proposition 1, the players dissipate the full value of the prize. Dissipation is less than complete if competitors differ from each other, i.e., if

their costs are not symmetric. More randomness will generally mean that, in the actual perfectly discriminating contest, the realizations of the cost parameters typically differ. Hence, randomness will cause some differentiation between players, and this will relax competition. The result parallels results on randomness and diversity in competition theory more generally. For instance, in both Bertrand and Cournot competition with constant unit cost, randomness of own unit cost typically benefits a firm.

The effect of changes in the cost distribution of player 2 on player 1's payoff (or vice versa) is less straightforward. Let  $c_1$  and  $c_2$  be stochastically independent of each other and distributed according to  $F_1$  and  $F_2$  and let  $F_2(c_2)$  be a mean preserving spread of some distribution  $\tilde{F}_2(c_2)$ , (so that  $\int_c^x (F_2(c) - \tilde{F}_2(c))dc \geq 0$  for all  $x$  and  $\int_c^{\bar{c}} (F_2(c) - \tilde{F}_2(c))dc = 0$ ). Such a spread does not leave the mean of  $\alpha = \frac{c_1}{c_2}$  unchanged and the implication of such a spread for the payoff of player 1 is not well determined.

For the most simple case in which  $c_1$  is constant, the difference in player 1's profit is:

$$\begin{aligned}
& \pi_1(\tilde{Z}) - \pi_1(Z) & (6) \\
&= \int_{c_1}^{\bar{c}} \frac{c - c_1}{c} (f_2(c) - \tilde{f}_2(c)) dc \\
&= \left[ \frac{c - c_1}{c} (\tilde{F}_2 - F_2) \right]_{c_1}^{\bar{c}} - \int_{c_1}^{\bar{c}} \frac{c_1}{c^2} (\tilde{F}_2 - F_2) dc \\
&= 0 - \int_{c_1}^{\bar{c}} \frac{c_1}{c^2} (\tilde{F}_2 - F_2) dc.
\end{aligned}$$

The indeterminacy of the sign of this expression can be illustrated by considering two very simple distributions of  $c_2$ . For this purpose, assume that  $\tilde{F}_2$  is degenerate, with  $\tilde{c}_2 = \frac{c_1}{\Delta}$ . Moreover, let  $F_2$  be equal to  $\tilde{F}_2$  plus some very simple and symmetric noise. More precisely, let  $c_2$  have two possible outcomes in this case,  $c_2 = \frac{1}{2} \frac{c_1}{\Delta}$  with probability 1/2 and  $c_2 = \frac{3}{2} \frac{c_1}{\Delta}$ , also with probability 1/2. Calculating the expected profit of player 1 yields

$$\pi_1(\tilde{F}) = \begin{cases} 1 - \Delta & \text{if } \Delta < 1 \\ 0 & \text{if } \Delta \geq 1 \end{cases}$$

and

$$\pi_1(F) = \begin{cases} \frac{1}{2}(1 - 2\Delta) + \frac{1}{2}(1 - \frac{2\Delta}{3}) & \text{if } \Delta < \frac{1}{2} \\ \frac{1}{2}(1 - \frac{2\Delta}{3}) & \text{if } \frac{1}{2} \leq \Delta \leq \frac{3}{2} \\ 0 & \text{if } \Delta > \frac{3}{2} \end{cases}$$

Accordingly, whether the mean preserving spread in  $c_2$  increases or decreases player 1's payoff depends here on this player's advantage in the degenerate case. If  $\Delta$  was larger than 1 but smaller than  $3/2$ , then the payoff of player 1 increases from zero to something positive. If, for instance,  $\Delta$  was between  $1/2$  and  $3/4$ , the payoff of player 1 actually decreases due to the mean preserving spread in  $c_2$ .

These results refer to the implications of a mean preserving spread of the cost distribution of one player for this player and for the other player.

If two contestants' cost parameters are identically distributed with cumulative distribution functions  $F_1(c_1) = F_2(c_2) = F(c)$ , symmetry implies  $\pi_1(Z) = \pi_2(Z)$ . We now compare payoffs of the two players who compete with each other if their cost distributions are ranked by second-order stochastic dominance. We can state the following result:

**Proposition 4** *Let  $Z(\alpha) \leq_{SSD} \tilde{Z}(\alpha)$ . Then  $\pi_1(Z) - \pi_2(Z) \geq \pi_1(\tilde{Z}) - \pi_2(\tilde{Z})$ .*

**Proof.** By (2),

$$\pi_1 - \pi_2 = \int_{\underline{\alpha}}^{\bar{\alpha}} \left[ (1 - \alpha)I_{\{\alpha \leq 1\}} + \left(\frac{1}{\alpha} - 1\right)I_{\{\alpha > 1\}} \right] z(\alpha) d\alpha \quad (7)$$

with  $I_{\{\alpha \leq 1\}}$  an indicator function that takes on the value 1 if  $\alpha \leq 1$  and zero otherwise, and  $I_{\{\alpha > 1\}}$  an indicator function that takes on the value 1 if  $\alpha > 1$  and zero otherwise. Define

$$\Psi(\alpha) \equiv (1 - \alpha)I_{\{\alpha \leq 1\}} + \left(\frac{1}{\alpha} - 1\right)I_{\{\alpha > 1\}}. \quad (8)$$

This function is depicted in Figure 1. It is continuously differentiable with  $\Psi'(\alpha) < 0$  and  $\Psi''(\alpha) \geq 0$  for all  $\alpha$ . Applying Theorem 2 in Hadar and Russel (1969),

$$\pi_1 - \pi_2 = \int_{\underline{\alpha}}^{\bar{\alpha}} \Psi(\alpha) z(\alpha) d\alpha \quad (9)$$

is higher for  $Z$  than for  $\tilde{Z}$  if  $Z(\alpha) \leq_{SSD} \tilde{Z}(\alpha)$ .<sup>8</sup> ■

Proposition 4 holds for distributions of  $\alpha$  ranked by second-order stochastic dominance, which may be generated by different combinations of

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<sup>8</sup>The result is obtained directly by twice integrating  $[\pi_1(Z) - \pi_2(Z)] - [\pi_1(\tilde{Z}) - \pi_2(\tilde{Z})]$  by parts. This yields  $\int_{\underline{\alpha}}^{\bar{\alpha}} \Psi''(\alpha) \int_{\underline{\alpha}}^{\alpha} (Z(x) - \tilde{Z}(x)) dx d\alpha - \Psi'(\bar{\alpha}) \int_{\underline{\alpha}}^{\bar{\alpha}} (Z(x) - \tilde{Z}(x)) dx$  and this expression is non-negative if  $\Psi'(\alpha) \leq 0$  and  $\Psi''(\alpha) \geq 0$  by the definition of SSD.

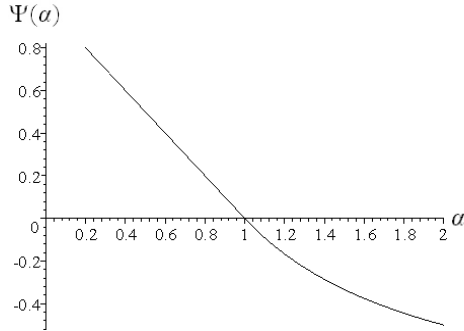


Figure 1: The graph of  $\Psi(\alpha)$

changes in  $F_1$  and  $F_2$ . We are mostly interested in the implications of one player's cost distribution and the change in this distribution. If  $F_1, \tilde{F}_1$  and  $F_2$  are stochastically independent, we know from Lemma 1 that  $F_1 \leq_{SSD} \tilde{F}_1$  implies  $Z(\alpha) \leq_{SSD} \tilde{Z}(\alpha)$ . This yields the following result:

**Corollary 2** *If  $F_1, \tilde{F}_1$  and  $F_2$  are stochastically independent and  $F_1 \leq_{SSD} \tilde{F}_1$  then, for the difference in expected payoffs,  $\pi_1(F_1, F_2) - \pi_2(F_1, F_2) \geq \pi_1(\tilde{F}_1, F_2) - \pi_2(\tilde{F}_1, F_2)$  holds.*

The corollary 2 states a seemingly natural property: as has been seen from Corollary 1, an increase in a player's cost dispersion directly increases the payer's payoff. This increase in the dispersion may also increase the other player's payoff. Corollary 2 suggests that the direct effect of own cost dispersion is stronger than the potentially positive effect for the competing player. The next corollary follows from Proposition 4 and allows us to compare the players' payoffs directly.

**Corollary 3** *Suppose  $c_1$  and  $c_2$  are independent and  $F_1(c_1) \leq_{SSD} F_2(c_2)$ . Then  $\pi_1(F_1, F_2) \geq \pi_2(F_1, F_2)$ .*

**Proof.** If  $F_1 = F_2$ , then  $\pi_1(F_1, F_2) - \pi_2(F_1, F_2) = 0$  by symmetry. Now suppose  $F_1 \leq_{SSD} F_2$ . Then, by Corollary 2,  $\pi_1(F_1, F_2) - \pi_2(F_1, F_2) \geq \pi_1(F_2, F_2) - \pi_2(F_2, F_2) = 0$ . ■

As a special case, Corollary 3 may be used to compare two players who have the same expected ability, but differ in their abilities by a mean preserving spread (see Rothschild and Stiglitz 1970). It shows that the higher

dispersion of ability will generally favor the player who has this higher dispersion. This difference in payoffs should also affect individuals' decisions whether to participate in this competition. Individuals with a higher variability (in the mean-preserving spread sense) of their ability have a stronger incentive to participate in this game.

### 3 Elimination tournaments

If players anticipate that the prize from winning a perfectly discriminating contest is to enter into another perfectly discriminating contest, it may be surprising that players expend considerable effort in the semi-final, even if their rival in the final is expected to be a very strong player. Particularly if the contest is adequately described by a perfectly discriminating contest, the fact that weaker players do not receive a rent in the equilibrium of the future perfectly discriminating contest should strongly discourage most of the players from entering into early rounds of a such multiple-round elimination tournaments. Moreover, if they do enter, it should reduce their incentives to expend effort in early rounds. This discouragement effect has been noted for multi-battle contests by Harris and Vickers (1987), Klumpp and Polborn (2006) and Konrad and Kovenock (2006), and for elimination tournaments by Rosen (1986), Gradstein and Konrad (1999) and Groh et al. (2003).

In this section we show that ex-ante uncertainty about players' actual ability in any of a series of perfectly discriminating contests provides a possible explanation why players are willing to expend considerable effort in any round of a contest architecture with sequential perfectly discriminating contests, even if they are weaker than their future opponents in terms of expected unit cost of effort. We also show that a player whose cost distribution is more dispersed has a genuine advantage compared to the other player; his expected payoff is higher, and he wins with a higher probability.

We first consider the least complex dynamic structure, which, however, proves to be a useful building block for the analysis of more complex structures. The structure is depicted in Figure 2. There are three players,  $i = 1, 2, k$ . In a first round players 1 and 2 compete against each other in a perfectly discriminating contest, which will be called the "semi-final". The winner in the semi-final will be admitted to the final, where this winner will compete against player  $k$  in a perfectly discriminating contest. The timing of the game is as follows. In stage 1 players  $i = 1, 2$  learn their own and their opponent's unit-cost parameters  $c_1$  and  $c_2$ , which, for now, are draws from stochastically independent and time invariant distributions  $F_1$

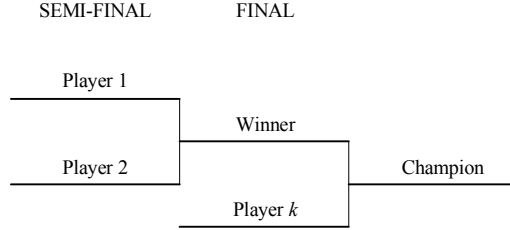


Figure 2: Semi-final

and  $F_2$ . In stage 2 the players simultaneously choose their efforts  $e_1$  and  $e_2$ , which cost  $c_1e_1$  and  $c_2e_2$ , respectively. The player with the higher effort wins and enters into the final. In the final, the winner of the semi-final has to play against player  $k$  in a perfectly discriminating contest. Player  $k$ 's characteristics are known to players  $i = 1, 2$ , already in the semi-final, and are described by a cumulative distribution function  $F_k$  of player  $k$ 's cost of effort that is also stochastically independent of  $F_1$  and  $F_2$ .

Given that there are now more than 2 players, it helps to define

$$\frac{c_i}{c_j} \equiv \alpha_{ij} \quad (10)$$

and the cumulative distribution functions and density functions of  $\alpha_{ij}$  that are induced by  $F_1$ ,  $F_2$  and  $F_k$  by  $Z_{ij}$  and  $z_{ij}$ , respectively. From the analysis in section 2 we know the two players' expected payoffs  $\pi_{1k}$  and  $\pi_{2k}$  from entering the final, given the characteristics of player  $k$ , and their own characteristics. Using (2) they are given by

$$\pi_{1k} = \int_{\underline{\alpha}}^1 (1 - \alpha) z_{1k}(\alpha) d\alpha \quad \text{and} \quad \pi_{2k} = \int_{\underline{\alpha}}^1 (1 - \alpha) z_{2k}(\alpha) d\alpha. \quad (11)$$

In the final the winner of the semi-final between 1 and 2 enters into a perfectly discriminating contest with player  $k$ , and all this is common knowledge in the semi-final. We can now consider the equilibrium payoffs for players 1 and 2 in the semi-final:

**Proposition 5** *Let  $c_1$  and  $c_2$  be the realization of the unit costs of effort of players 1 and 2, respectively, in the semi-final. Then their overall equilibrium payoffs when playing the semi-finals are*

$$\begin{aligned} \pi_{12}^S(c_1, c_2) &= \max\left\{0, \pi_{1k} - \frac{c_1 \pi_{2k}}{c_2}\right\} \\ \pi_{21}^S(c_1, c_2) &= \max\left\{0, \pi_{2k} - \frac{c_2 \pi_{1k}}{c_1}\right\}. \end{aligned} \quad (12)$$

**Proof.** When players 1 and 2 compete in the perfectly discriminating contest in stage 1 and have cost parameters  $c_1$  and  $c_2$ , they maximize

$$\begin{aligned}\pi_1(e_1, e_2, c_1, c_2) &= \text{prob}(e_1 > e_2)\pi_{1k} - c_1e_1, \text{ and} \\ \pi_2(e_1, e_2, c_1, c_2) &= \text{prob}(e_2 > e_1)\pi_{2k} - c_2e_2.\end{aligned}\tag{13}$$

These objective functions are strategically equivalent to a situation in which players 1 and 2 maximize

$$\begin{aligned}\frac{\pi_1}{\pi_{1k}} &= \text{prob}(e_1 > e_2) - \frac{c_1}{\pi_{1k}}e_1, \text{ and} \\ \frac{\pi_2}{\pi_{2k}} &= \text{prob}(e_2 > e_1) - \frac{c_2}{\pi_{2k}}e_2,\end{aligned}\tag{14}$$

respectively. For this problem, from Proposition 1, player 2 earns a payoff of zero and player 1 earns a positive payoff equal to

$$\pi_1 = \pi_{1k} - \frac{c_1}{c_2}\pi_{2k}\tag{15}$$

if  $\frac{\pi_{1k}}{c_1} > \frac{\pi_{2k}}{c_2}$ , and similarly for player 2 and player 1 switching roles if the reverse inequality holds. Accordingly, if

$$\frac{\pi_{1k}}{\pi_{2k}} = \frac{c_1}{c_2},\tag{16}$$

the two players are symmetric and dissipate all rent in expectation. If  $\frac{\pi_{1k}}{\pi_{2k}} > \frac{c_1}{c_2}$ , then player 1 has a strictly positive expected payoff that is equal to  $\pi_{1k} - \frac{c_1}{c_2}\pi_{2k}$ , and if  $\frac{\pi_{1k}}{\pi_{2k}} < \frac{c_1}{c_2}$ , then player 2 has a strictly positive expected payoff that is equal to  $\pi_{2k} - \frac{c_2}{c_1}\pi_{1k}$ . ■

This result can be used to state the ex-ante expected payoffs of players 1 and 2 in the semi-final for given cost distributions. Recall that  $Z_{12}(\alpha)$  is the cumulative distribution of  $\alpha$  obtained for  $\alpha = c_1/c_2$ , with  $c_1$  and  $c_2$  independent draws from distributions  $F_1(c_1)$  and  $F_2(c_2)$ , respectively. The equilibrium payoffs from simultaneous optimization of these objective functions follow from Proposition 1:

$$\pi_1^S(Z_{12}) = \int_{\frac{\pi_{1k}}{\pi_{2k}}}^{\frac{\pi_{1k}}{\pi_{2k}}} [\pi_{1k} - \alpha\pi_{2k}]z_{12}(\alpha)d\alpha\tag{17}$$

and

$$\pi_2^S(Z_{12}) = \int_{\frac{\pi_{1k}}{\pi_{2k}}}^{\bar{\alpha}} [\pi_{2k} - \frac{1}{\alpha}\pi_{1k}]z_{12}(\alpha)d\alpha\tag{18}$$

We may now compare players 1 and 2 with different distributions of their unit costs, if these distributions are ordered by SSD.

**Proposition 6** *If  $F_1 \leq_{SSD} F_2$ , then  $\pi_1^S \geq \pi_2^S$ .*

**Proof.** Consider  $Z_{12}(\alpha; F_1, F_2)$  the cumulative distribution of  $\alpha$  obtained for  $\alpha = c_1/c_2$ , with  $c_1$  and  $c_2$  independent draws from distributions  $F_1(c_1)$  and  $F_2(c_2)$ . Let  $F_1 \leq_{SSD} F_2$ . Then  $Z_{12}(\alpha; F_1, F_2) \leq_{SSD} Z_{12}(\alpha; F_2, F_2)$  by Lemma 1. We claim that under  $Z_{12}(\alpha; F_1, F_2)$ ,  $\pi_2^S \leq \pi_1^S$ .

First note that  $\pi_{1k}(F_1, F_k) = \pi_{2k}(F_2, F_k)$  if  $F_1 = F_2$  by symmetry, and  $\pi_{1k}(F_1, F_k) \geq \pi_{2k}(F_2, F_k)$  by Proposition 3 and Lemma 1 if  $F_1 \leq_{SSD} F_2$ . From (17) and (18),

$$\begin{aligned}
& \pi_1^S(F_1, F_2) - \pi_2^S(F_1, F_2) \tag{19} \\
&= \int_{\underline{\alpha}}^{\frac{\pi_{1k}}{\pi_{2k}}} (\pi_{1k} - \alpha\pi_{2k})z_{12}(\alpha)d\alpha - \int_{\frac{\pi_{1k}}{\pi_{2k}}}^{\bar{\alpha}} (\pi_{2k} - \frac{1}{\alpha}\pi_{1k})z_{12}(\alpha)d\alpha \\
&= \pi_{2k} \left[ \int_{\underline{\alpha}}^{\frac{\pi_{1k}}{\pi_{2k}}} \left( \frac{\pi_{1k}}{\pi_{2k}} - \alpha \right) z_{12}(\alpha) d\alpha - \int_{\frac{\pi_{1k}}{\pi_{2k}}}^{\bar{\alpha}} \left( 1 - \frac{1}{\alpha} \frac{\pi_{1k}}{\pi_{2k}} \right) z_{12}(\alpha) d\alpha \right] \\
&\geq \pi_{2k} \left[ \int_{\underline{\alpha}}^1 (1 - \alpha) z_{12}(\alpha) d\alpha - \int_1^{\bar{\alpha}} \left( 1 - \frac{1}{\alpha} \right) z_{12}(\alpha) d\alpha \right] \\
&= \pi_{2k} \int_{\underline{\alpha}}^{\bar{\alpha}} \left[ (1 - \alpha) I_{\{\alpha \leq 1\}} + \left( \frac{1}{\alpha} - 1 \right) I_{\{\alpha > 1\}} \right] z_{12}(\alpha) d\alpha \\
&\equiv \pi_{2k} \int_{\underline{\alpha}}^{\bar{\alpha}} \Psi(\alpha) z_{12}(\alpha) d\alpha.
\end{aligned}$$

Since  $\pi_{2k}$  is constant with respect to a change in  $F_1$ , and it follows from Proposition 4 that  $\int_{\underline{\alpha}}^{\bar{\alpha}} \Psi(\alpha) z_{12}(\alpha) d\alpha$  is non-negative. To complete the proof, we confirm that the inequality used in (19) holds. Define  $s \equiv \frac{\pi_{1k}}{\pi_{2k}}$ , replace this definition in the third line of (19) to obtain

$$\pi_{2k} \left[ \int_{\underline{\alpha}}^s (s - \alpha) z_{12}(\alpha) d\alpha - \int_s^{\bar{\alpha}} \left( 1 - \frac{1}{\alpha} s \right) z_{12}(\alpha) d\alpha \right], \tag{20}$$

and note that

$$\begin{aligned}
& \frac{\partial}{\partial s} \left[ \int_{\underline{\alpha}}^s (s - \alpha) z_{12}(\alpha) d\alpha - \int_s^{\bar{\alpha}} \left( 1 - \frac{1}{\alpha} s \right) z_{12}(\alpha) d\alpha \right] \tag{21} \\
&= 0 + \int_{\underline{\alpha}}^s z_{12}(\alpha) d\alpha + 0 - \int_s^{\bar{\alpha}} -\frac{1}{\alpha} z_{12}(\alpha) d\alpha \\
&= \int_{\underline{\alpha}}^s z_{12}(\alpha) d\alpha + \int_s^{\bar{\alpha}} \frac{1}{\alpha} z_{12}(\alpha) d\alpha \geq 0.
\end{aligned}$$



Accordingly, replacing all  $\frac{\pi_{1k}}{\pi_{2k}} > 1$  by 1 will not increase the value of the expression, and this confirms the weak inequality. ■

Proposition 6 shows that the benefits that a player receives from a more dispersed cost distribution in the static game carry over to the dynamic game. The benefit of a higher dispersion in the structure here is two-fold. First, the player with the more dispersed cost distribution benefits from this dispersion if he makes it to the final and has a higher expected payoff if he is admitted to the final. But also in the semi-final, the higher dispersion benefits a player and increases the player's expected payoff. As the proof shows, the two effects compound.

We also consider the implications of dispersion of the cost parameter for players' probabilities of winning the contest. The following proposition holds.

**Proposition 7** *If  $F_1 \leq_{SSD} F_2$ , then player 1 wins the semi-final with a (weakly) higher probability than player 2.*

**Proof.** Since the objective functions of the competition between 1 and 2 for given  $\frac{c_1}{c_2}$  can be stated equivalently by (14) we can apply Proposition 1 to characterize the probability that player 1 wins the contest between 1 and 2 as

$$p_1(c_1, c_2) = \begin{cases} \frac{\pi_{1k}/c_1}{2(\pi_{2k}/c_2)} & \text{if } \frac{\pi_{1k}}{\pi_{2k}} < \frac{c_1}{c_2} \\ 1 - \frac{\pi_{2k}/c_2}{2(\pi_{1k}/c_1)} & \text{if } \frac{\pi_{1k}}{\pi_{2k}} \geq \frac{c_1}{c_2} \end{cases} \quad (22)$$

and  $p_2(c_1, c_2) = 1 - p_1(c_1, c_2)$ .

Consider  $Z_{12}(\alpha; F_1, F_2)$  the cumulative distribution of  $\alpha$  obtained for  $\alpha = c_1/c_2$ , with  $c_1$  and  $c_2$  independent draws from distributions from  $F_1(c_1)$  and  $F_2(c_2)$ . Integrating over all possible  $\frac{c_1}{c_2} = \alpha$ , we can write

$$p_1(Z_{12}) = \int_{\underline{\alpha}}^{\frac{\pi_{1k}}{\pi_{2k}}} \left(1 - \frac{\alpha}{2\frac{\pi_{1k}}{\pi_{2k}}}\right) z_{12}(\alpha) d\alpha + \int_{\frac{\pi_{1k}}{\pi_{2k}}}^{\bar{\alpha}} \frac{\pi_{1k}}{2\alpha} z_{12}(\alpha) d\alpha \quad (23)$$

for a given distribution  $Z_{12}(\alpha; F_1, F_2)$ .

Let  $F_1 \leq_{SSD} F_2$ . Then  $Z_{12}(\alpha; F_1, F_2) \leq_{SSD} Z_{12}(\alpha; F_2, F_2)$  by Lemma 1. We claim that  $p_1(Z_{12}(\alpha; F_1, F_2)) \geq p_1(Z_{12}(\alpha; F_2, F_2))$ . Note that

$$p_1(Z_{12}(\alpha; F_2, F_2)) = \int_{\underline{\alpha}}^1 \left(1 - \frac{\alpha}{2}\right) z_{12}(\alpha; F_2, F_2) d\alpha + \int_1^{\bar{\alpha}} \frac{1}{2\alpha} z_{12}(\alpha; F_2, F_2) d\alpha = 1/2 \quad (24)$$

which follows from symmetry. Consider now

$$\begin{aligned}
& p_1(Z_{12}(\alpha; F_1, F_2)) \tag{25} \\
&= \int_{\underline{\alpha}}^{\frac{\pi_{1k}}{\pi_{2k}}} \left(1 - \frac{\alpha}{2\frac{\pi_{1k}}{\pi_{2k}}}\right) z_{12}(\alpha; F_1, F_2) d\alpha + \int_{\frac{\pi_{1k}}{\pi_{2k}}}^{\bar{\alpha}} \frac{\frac{\pi_{1k}}{\pi_{2k}}}{2\alpha} z_{12}(\alpha; F_1, F_2) d\alpha \\
&\geq \int_{\underline{\alpha}}^1 \left(1 - \frac{\alpha}{2}\right) z_{12}(\alpha; F_1, F_2) d\alpha + \int_1^{\bar{\alpha}} \frac{1}{2\alpha} z_{12}(\alpha; F_1, F_2) d\alpha \\
&= \int_{\underline{\alpha}}^{\bar{\alpha}} \left[ \frac{1}{2\alpha} I_{\{\alpha > 1\}} + \left(1 - \frac{\alpha}{2}\right) I_{\{\alpha \leq 1\}} \right] z_{12}(\alpha; F_1, F_2) d\alpha.
\end{aligned}$$

The win probability of player 1 is monotonically increasing in  $\frac{\pi_{1k}}{\pi_{2k}}$  and  $\frac{\pi_{1k}}{\pi_{2k}} \geq 1$  by  $F_1 \leq_{SSD} F_2$  and Corollary 1, which is used for the inequality in line 3 of (25). To confirm this monotonicity, denote  $s \equiv \frac{\pi_{1k}}{\pi_{2k}}$  and consider first derivatives:

$$\begin{aligned}
& \frac{\partial}{\partial s} \left( \int_{\underline{\alpha}}^s \left(1 - \frac{\alpha}{2s}\right) z_{12}(\alpha) d\alpha + \int_s^{\bar{\alpha}} \frac{s}{2\alpha} z_{12}(\alpha) d\alpha \right) \tag{26} \\
&= \left(1 - \frac{s}{2s}\right) z_{12}(s) - \frac{s}{2s} z_{12}(s) \\
&\quad + \int_{\underline{\alpha}}^s \frac{\alpha}{2s^2} z_{12}(\alpha) d\alpha + \int_s^{\bar{\alpha}} \frac{1}{2\alpha} z_{12}(\alpha) d\alpha \\
&= \int_{\underline{\alpha}}^s \frac{\alpha}{2s^2} z_{12}(\alpha) d\alpha + \int_s^{\bar{\alpha}} \frac{1}{2\alpha} z_{12}(\alpha) d\alpha > 0
\end{aligned}$$

Define the bracketed expression in the integrand in the last line of (25) to be  $\Phi(\alpha)$ , so that

$$p_1(Z_{12}(\alpha; F_1, F_2)) = \int_{\underline{\alpha}}^{\bar{\alpha}} \Phi(\alpha) z_{12}(\alpha; F_1, F_2) d\alpha. \tag{27}$$

Note that  $\Phi(\alpha)$  is continuously differentiable everywhere on  $(\underline{\alpha}, \bar{\alpha})$  and decreasing in  $\alpha$  with  $\frac{\partial \Phi(\alpha)}{\partial \alpha} = -\frac{1}{2\alpha^2} < 0$  for  $\alpha > 1$  and  $\frac{\partial \Phi(\alpha)}{\partial \alpha} = -\frac{1}{2} < 0$  for  $\alpha \leq 1$ . Furthermore,  $\Phi(\alpha)$  is convex since  $\frac{\partial^2 \Phi(\alpha)}{(\partial \alpha)^2} = \frac{1}{\alpha^3} > 0$  for  $\alpha > 1$  and  $\frac{\partial^2 \Phi(\alpha)}{(\partial \alpha)^2} = 0$  for  $\alpha \leq 1$ . Accordingly, we can again apply Theorem 2 in Hadar

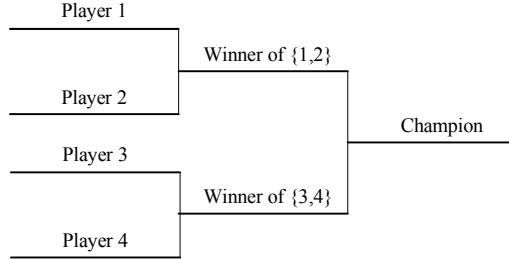


Figure 3: Elimination tournament

and Russel (1969) to find that

$$\begin{aligned}
 & p_1(Z_{12}(\alpha; F_1, F_2)) & (28) \\
 & \geq \int_{\underline{\alpha}}^{\bar{\alpha}} \Phi(\alpha) z_{12}(\alpha; F_1, F_2) d\alpha \\
 & \geq \int_{\underline{\alpha}}^{\bar{\alpha}} \Phi(\alpha) z_{12}(\alpha; F_2, F_2) d\alpha \\
 & = p_1(Z_{12}(\alpha; F_2, F_2)).
 \end{aligned}$$

This concludes the proof. ■

These insights can now be applied to the simplest example of a self-contained elimination tournament. Consider four players  $i \in \{1, 2, 3, 4\}$  in the elimination tournament that is depicted in Figure 3. The tournament consists of a series of elimination matches. Player 1 plays against player 2 in one of the semi-finals, and players 3 and 4 play against each other in a parallel semi-final. The winner from each of these semi-finals is admitted to the final. Both the semi-finals and the final follow the rules of a perfectly discriminating contest similar to the perfectly discriminating contest that was considered in section 2. Each of the respective two participants expends effort and the contestant with the higher effort wins the respective stage game, with the winner determined by a random draw in the case of a tie in effort. In each stage game the cost parameters  $c_i$  of the two contestants are independent draws (across players and time) from a probability distribution with cumulative distribution functions  $F_i$  for  $i \in \{1, 2, 3, 4\}$ .

We assume here that these distribution functions are time invariant; i.e.,  $c_i$  of player  $i$  in the semi-final and in the final are independent draws from the same distribution  $F_i$ . Changes in the distributions over time will be considered in section 4. For tractability, we consider the problem for players

1 and 2 assuming that players 3 and 4 have identical cost distributions  $F_3 = F_4 \equiv F_k$ . For any given distribution of player  $i \in \{1, 2\}$ ,  $F_i$ , the payoff from taking part in the final is  $\pi_{ik}(F_i, F_k)$  and determined by (2) with  $Z_{ik}(\alpha)$  being the distribution of  $\frac{c_i}{c_k}$  that is induced by  $F_i$  and  $F_k$ , where  $F_3 = F_4 = F_k$  makes it a matter of irrelevance for players 1 and 2 whether 3 or 4 is the other finalist.

Given this game, (17) and (18) determine the equilibrium payoffs in the semi-final for players 1 and 2. We can apply Propositions 6 and 7 to conclude that, starting at the semi-final stage, the expected equilibrium payoff and the win probability of player 1 are higher than the payoff and the win probability of player 2 if  $F_1 \leq_{SSD} F_2$ .

This example reveals that a higher cost dispersion also benefits a player in a dynamic contest. It makes it more likely that the player succeeds and is not eliminated in an earlier round of the tournament and it also increases the player's payoff from participating in the tournament. Thinking about selection properties of repeated elimination tournaments, this result suggests that individuals with a higher variability in their ability have a two-fold advantage in such competition structures. The prize from winning in earlier stages is higher, and for given prize levels, the probability of winning is higher. For the population of potential participants in such competition structures, the self-selection of types in the entry stage and the selection forces in the course of the elimination tournament compound in their effects. Participants from a larger population who self-select into such competition structures should have an ability that is more dispersed than average, and this dispersion should increase in the later stages of an elimination tournament due to the selection properties of the elimination contest.

## 4 Multi-battle contests

The conclusion that dispersed ability benefits players also holds for problems in which the same players compete with each other in multi-battle contests. Consider two players 1 and 2 in a simplified and symmetric multi-battle contest as described Konrad and Kovenock (2006). The two players take part in a game which is comprised of a sequence of similar one shot simultaneous move perfectly discriminating contests which we refer to as battles. A prize of size 1 is awarded to the one player who is the first to win two battles; the loser receives a prize of zero. The problem is depicted in Figure 4. Starting from the initial state (2, 2) the first battle takes place. If player 1 wins, they move to state (1, 2). From there, player 1 wins the prize if he wins the

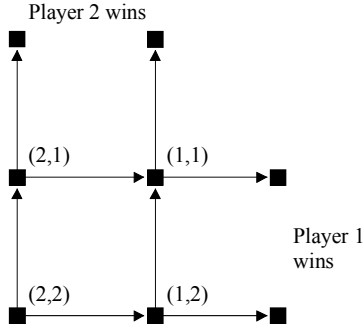


Figure 4: Multi-battle contest

subsequent battle. If player 2 wins the subsequent battle, they move to state  $(1, 1)$ . Similarly, if player 2 wins the battle at  $(2, 2)$ , they enter into state  $(2, 1)$ . From there, 2 can win the prize in the next battle, or, if 1 wins at  $(2, 1)$ , they move to  $(1, 1)$ . Finally, the subgame at state  $(1, 1)$  is equivalent with the static perfectly discriminating contest that was studied in section 2.

Konrad and Kovenock (2006) consider a more general version of this game with asymmetric players, with more than two required battle wins, and with intermediate prizes that are allocated to the winner of any battle. However, they assume that the ability of players is exogenous, invariant across all states, and known to both players. Applying their framework to the simple symmetric case, they show that the symmetric game has the following interesting features: at  $(2, 2)$ , the sum of both players' efforts is equal to the unit value of the prize. From there, players move to state  $(2, 1)$  or  $(1, 2)$ . At this asymmetric state the advantaged player wins without expending any further effort and the perfectly discriminating contest at  $(2, 1)$  or  $(1, 2)$  becomes trivial. The key for understanding this result is the following fact. Suppose the players are in state  $(2, 1)$ . Player 1 could expend some positive effort and try to win the perfectly discriminating contest in this state. But if he does this and wins, the players will enter into state  $(1, 1)$ , at which they will dissipate all rent fighting over the unit prize in a symmetric perfectly discriminating contest with complete information. It is this anticipated outcome that leads to hold-up and prevents player 1 from trying to get back into play and to win, once the contest becomes asymmetric.

We consider how ability uncertainty at each state changes the result. For this purpose, let  $F_1^{(i,i)}(c_1) = F_2^{(i,i)}(c_2) \equiv F^{(i,i)}$  in state  $(i, i)$  and let

$F_1^{(i,j)}(c_1) = F_2^{(j,i)}(c_2) \equiv F^{(i,j)}$  in states  $(i, j)$  and  $(j, i)$ , in the sense that the actual cost parameters  $c_k^{(i,j)}$  at state  $(i, j)$  are draws from  $F^{(i,j)}$ , and stochastically independent over players and time. Further, let  $Z^{(i,j)}(\alpha)$  be the distribution of  $\frac{c_1}{c_2}$  that is induced by these distribution functions. We solve the multi-battle contest recursively, starting with state  $(1, 1)$ .

At  $(1, 1)$ , a perfectly discriminating contest takes place. The expected payoffs of 1 and 2 at this state prior to the resolution of  $c_1^{(1,1)}$  and  $c_2^{(1,1)}$  are  $\pi_1^{(1,1)} = \pi_2^{(1,1)} = \pi^{(1,1)}$  and are given in (2). Turn now to  $(1, 2)$ . Player 1's payoff from winning at  $(1, 2)$  is equal to 1 (the unit prize), and 1's payoff from losing is determined by the payoff in the continuation game at  $(1, 1)$ , i.e., equal to  $\pi^{(1,1)}$ . For player 2, the payoff from losing at  $(1, 2)$  is zero. The payoff from winning is the equilibrium payoff in the continuation game at  $(1, 1)$ , i.e., equal to  $\pi^{(1,1)}$ .

From Proposition 1, for given  $c_1$  and  $c_2$ , it holds that the equilibrium payoff for player 1 is  $1 - \frac{c_1}{c_2}\pi^{(1,1)}$  for  $\frac{c_1}{c_2} < \frac{1-\pi^{(1,1)}}{\pi^{(1,1)}}$  and  $\pi^{(1,1)}$  for  $\frac{c_1}{c_2} \geq \frac{1-\pi^{(1,1)}}{\pi^{(1,1)}}$ . Accordingly, the expected payoff of player 1 is

$$\pi_1^{(1,2)} = \int_{\underline{\alpha}}^{\frac{1-\pi^{(1,1)}}{\pi^{(1,1)}}} (1 - \alpha\pi^{(1,1)})z^{(1,2)}(\alpha)d\alpha + \int_{\frac{1-\pi^{(1,1)}}{\pi^{(1,1)}}}^{\bar{\alpha}} \pi^{(1,1)}z^{(1,2)}(\alpha)d\alpha. \quad (29)$$

Analogous reasoning for player 2 yields an equilibrium payoff equal to  $\pi^{(1,1)} - \frac{c_2}{c_1}(1 - \pi^{(1,1)})$  for given cost parameters with  $\frac{c_1}{c_2} > \frac{1-\pi^{(1,1)}}{\pi^{(1,1)}}$ , and a payoff of zero for  $\frac{c_1}{c_2} \leq \frac{1-\pi^{(1,1)}}{\pi^{(1,1)}}$ . Hence, the expected equilibrium payoff at  $(1, 2)$  prior to the revelation of the actual cost parameters at this state is

$$\pi_2^{(1,2)} = \int_{\frac{1-\pi^{(1,1)}}{\pi^{(1,1)}}}^{\bar{\alpha}} (\pi^{(1,1)} - \frac{1}{\alpha}(1 - \pi^{(1,1)}))z^{(1,2)}(\alpha)d\alpha. \quad (30)$$

By symmetry, the payoffs at  $(2, 1)$  are  $\pi_1^{(2,1)} = \pi_2^{(1,2)}$  and  $\pi_2^{(2,1)} = \pi_1^{(1,2)}$ .

Turn now to state  $(2, 2)$ . Player 1's gain from winning the perfectly discriminating contest at  $(2, 2)$  equals  $\pi_1^{(1,2)} - \pi_1^{(2,1)} = \pi_1^{(1,2)} - \pi_2^{(1,2)}$  and, by symmetry, the same applies for player 2. This difference can be calculated further and turns out to be

$$\pi_1^{(1,2)} - \pi_2^{(1,2)} = \int_{\underline{\alpha}}^{\frac{1-\pi^{(1,1)}}{\pi^{(1,1)}}} (1 - \alpha\pi^{(1,1)})z^{(1,2)}(\alpha)d\alpha + \int_{\frac{1-\pi^{(1,1)}}{\pi^{(1,1)}}}^{\bar{\alpha}} \frac{1}{\alpha}(1 - \pi^{(1,1)})z^{(1,2)}(\alpha)d\alpha. \quad (31)$$

This difference is strictly positive. Note also that the function

$$(1 - \alpha\pi^{(1,1)})I_{\{\alpha \leq \frac{1-\pi^{(1,1)}}{\pi^{(1,1)}}\}} + \frac{1}{\alpha}(1 - \pi^{(1,1)})I_{\{\alpha > \frac{1-\pi^{(1,1)}}{\pi^{(1,1)}}\}} \quad (32)$$

is convex in  $\alpha$  for a given value of  $\pi^{(1,1)}$ . Accordingly, making again use of (2), but using that the value of winning is not equal to 1, but equal to (31), the equilibrium payoff of player 1 or player 2 from winning at (2, 2) is

$$\pi^{(2,2)} = \int_{\alpha}^1 [(\pi_1^{(1,2)} - \pi_2^{(1,2)})(1 - \alpha)]z^{(2,2)}(\alpha)d\alpha + \pi_1^{(2,1)}. \quad (33)$$

This payoff is typically strictly positive. Of course, the payoff is bounded from above, as the difference  $\pi_1^{(1,2)} - \pi_2^{(1,2)} \leq 1$ .

Consider also changes in the distribution of  $Z^{(i,j)}(\alpha)$  in the sense of SSD. If  $Z^{(2,2)} \leq_{SSD} \hat{Z}^{(2,2)}$ , then by Proposition 3 player 1's payoff is higher under  $Z^{(2,2)}$  than under  $\hat{Z}^{(2,2)}$ . Similarly, if  $Z^{(1,1)} \leq_{SSD} \hat{Z}^{(1,1)}$ , then  $\pi^{(1,1)}$  under  $Z^{(1,1)}$  is higher than  $\pi^{(1,1)}$  under  $\hat{Z}^{(1,1)}$ . Also,  $\pi_1^{(1,2)}$  is convex in  $\alpha$ , and, hence, a mean preserving spread in  $Z^{(1,2)}(\alpha)$  will increase  $\pi_1^{(1,2)}$ .

It is conceptually straightforward and notationally cumbersome to generalize this outcome for multi-battle contests that do not start at (2, 2), but at some state  $(n, m)$ . But it is clear from this example that uncertainty about actual ability in each single battle will partially resolve the hold-up problem in this game. Players will not dissipate the value of the prize if they start in a symmetric state  $(n, n)$  in which they have symmetric, but random abilities. Also, in contrast to the case of deterministic ability (Konrad and Kovenock 2006), effort will generally not drop to zero in the perfectly discriminating contest in asymmetric states. Intuitively, starting in an asymmetric state, returning to a state of symmetry will not imply that all rent will be dissipated in expectation at this state, and this provides incentives for the disadvantaged player to try and catch up to the advantaged player.

## 5 Conclusions

In this paper we show that transient ability shocks or, more precisely, shocks to the unit cost of effort, ameliorate the effects of cutthroat competition in single and multi-stage perfectly discriminating contests. More precisely, despite the fact that, all else equal, less able players earn a zero expected utility in such a contest, stochastic ability means that “on any given day” an

underdog may be more able than a favorite. This turns participation in such a contest into an option: a player earns a zero expected payoff in perfectly discriminatory contests in which he is less able than his rival, but earns a positive payoff, linearly decreasing in his own unit cost of effort, in contests in which he is more able. Hence, players benefit from mean preserving spreads of their own cost distribution. Mean preserving spreads of a rival's distribution of cost may benefit or harm a player, but never benefit the player more than the rival himself. This has important implications for the hold-up problem arising in multi-stage contests. First, because players earn a positive expected payoff from the contest *ex ante* (in contrast to the deterministic cost case), the cutthroat nature of later stage contests does not completely discourage effort in earlier stage contests. Second, the “reversion to the mean” arising in later stage contests means that even if a player in a given contest is less able than his rival, if he is more able on average, his continuation value from winning the contest will be greater, and therefore his imputed value of the prize from the present contest will be greater. This leads to greater effort, at least partially offsetting his transient ability disadvantage.

Our results have far reaching implications for both naturally arising and mechanism-induced selection processes. First, we demonstrate that, given two rival players with identical mean abilities, the player with the greater dispersion in ability achieves higher payoffs in the contest against his rival. Moreover, the “riskier” player also obtains a higher expected payoff than does his rival against any third player, regardless of that player's distribution of ability. Hence, all else equal, we would expect evolutionary forces to lead to greater fitness of players with “riskier” distributions of abilities. Such players would also be more willing to expend whatever entry costs might be required to participate in perfectly discriminating contests. In addition to this naturally occurring selection, within mechanism selection also arises. All else equal, players with more dispersed abilities have, higher continuation values from winning at early or intermediate stages of multistage contests, which increases their cost contingent incentive to expend effort in the current stage-contest faced. This leads to both higher effort and an increased probability of advancement.

## 6 Appendix

In this Appendix we formally prove Lemma 1 for absolutely continuous distributions. The proof for more general distributions is similar. Define



$\alpha_s = \frac{c_s}{c_k}$  for  $s \in \{i, j\}$ , and let  $Z_s(\alpha)$  be the cumulative distribution function of  $\alpha_s$ . Note that

$$\begin{aligned} Z_s(\alpha) &= \text{prob}\left(\frac{c_s}{c_k} \leq \alpha\right) \\ &= \text{prob}(c_s \leq \alpha c_k) \\ &= \int_{\underline{c}}^{\bar{c}} F_s(\alpha c_k) f_k(c_k) dc_k \end{aligned}$$

where use is made of the assumption that  $c_k$  is positive. Hence,

$$\begin{aligned} Z_i(\alpha) - Z_j(\alpha) &= \int_{\underline{c}}^{\bar{c}} F_i(\alpha c_k) f_k(c_k) dc_k - \int_{\underline{c}}^{\bar{c}} F_j(\alpha c_k) f_k(c_k) dc_k \\ &= \int_{\underline{c}}^{\bar{c}} (F_i(\alpha c_k) - F_j(\alpha c_k)) f_k(c_k) dc_k \end{aligned}$$

Accordingly,

$$\begin{aligned} &\int_{\underline{\alpha}}^x (Z_i(\alpha) - Z_j(\alpha)) d\alpha \tag{A3} \\ &= \int_{\underline{\alpha}}^x \left[ \int_{\underline{c}}^{\bar{c}} (F_i(\alpha c_k) - F_j(\alpha c_k)) f_k(c_k) dc_k \right] d\alpha \\ &= \int_{\underline{c}}^{\bar{c}} \left[ \int_{\underline{\alpha}}^x (F_i(\alpha c_k) - F_j(\alpha c_k)) d\alpha \right] f_k(c_k) dc_k \\ &= \int_{\underline{c}}^{\bar{c}} \frac{1}{c_k} \left[ \int_{\underline{\alpha}}^{c_k x} (F_i(\gamma) - F_j(\gamma)) d\gamma \right] f_k(c_k) dc_k, \end{aligned}$$

where  $\gamma \equiv \alpha c_k$ . Now, since  $c_j$  dominates  $c_i$  in the sense of second-order stochastic dominance,  $\left[ \int_{\underline{c}}^{c_k x} (F_i(\gamma) - F_j(\gamma)) d\gamma \right] \geq 0$  for all  $x c_k \geq 0$ , and this, in turn, implies that the last line in (34) is non-negative for all  $x$  and this completes the proof.

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