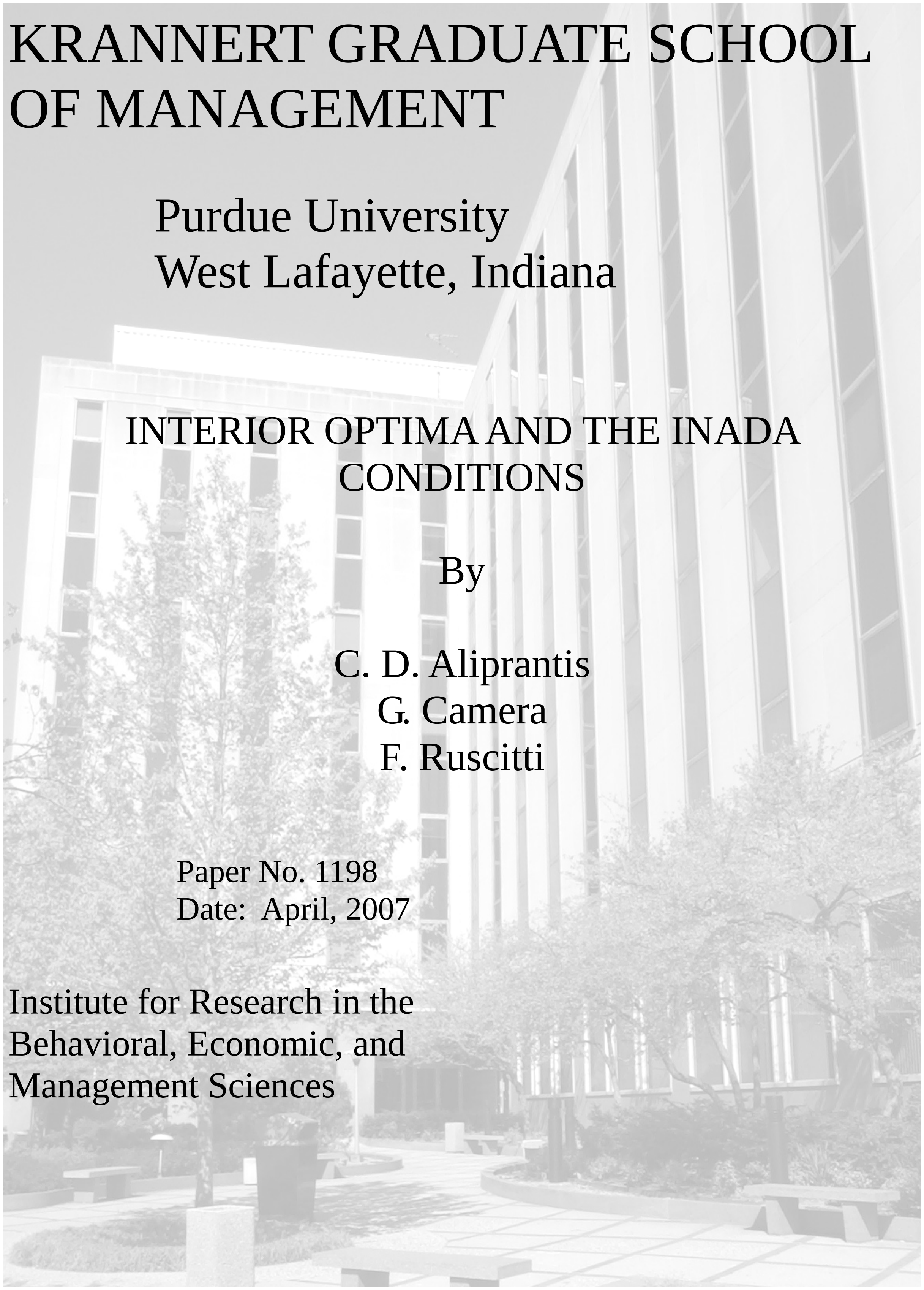

KRANNERT GRADUATE SCHOOL OF MANAGEMENT

Purdue University
West Lafayette, Indiana

INTERIOR OPTIMA AND THE INADA CONDITIONS

By

C. D. Aliprantis
G. Camera
F. Ruscitti

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INTERIOR OPTIMA AND THE INADA CONDITIONS★

C. D. ALIPRANTIS, G. CAMERA, AND F. RUSCITTI

Department of Economics, Krannert School of Management, Purdue University, 403 West State Street, W. Lafayette, IN 47907-2056, USA.

Abstract. We present a new proof of the interiority of the policy function based on the Inada conditions. It is based on supporting properties of concave functions.

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1. Introduction: A Typical Growth Model

The standard formulation of a typical one-sector growth model is as follows.[1] There is a single commodity which is used as capital, along with labor, to produce output. In the simplest formulation, labor is presumed to be supplied in fixed amounts and there is a representative agent. In each period $t = 0, 1, 2, \dots$ a part c_t of the output is consumed and a part x_{t+1} is saved as capital for next period, which fully depreciates after its use. This process repeats *ad infinitum*. The quantities c_t and x_{t+1} satisfy the standard resource constraint

$$c_t + x_{t+1} = f(x_t)\,,$$

where $f\colon [0,\infty) \to [0,\infty)$ is the production function and x_0, the initial capital stock, is given. The function f is assumed to satisfy the **Inada conditions** [5, p. 120]:

(1) f is twice differentiable on $(0,\infty)$,
(2) $f'(x) > 0$ and $f''(x) < 0$ for each $0 < x < \infty$,

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[1] For more details see [4, Chapters 2 and 3] or [8, Chapter 5].

(3) $f'(0)=\infty$ [2] and $\lim\limits_{x\to\infty} f'(x)=0$.

Replacing f by $f-f(0)$, we can assume without loss of generality that the production function satisfies $f(0)=0$ (and still satisfies the Inada conditions). Clearly, the functions $f(x)=x^\alpha$, where $0<\alpha<1$, are all production functions satisfying the Inada conditions. The key point is that every production function satisfying the Inada conditions is strictly increasing and strictly concave.

The following lemma will be used later.

Lemma 1.1. *Assume that a function $f\colon [0,\infty)\to[0,\infty)$ satisfies:*

(1) $f(0)=0$,

(2) *is twice differentiable on $(0,\infty)$,*

(3) $f'(x)>0$ *and* $f''(x)<0$ *for each* $0<x<\infty$.

Then we have $f'(0)=\infty$ if and only if $\lim\limits_{x\to 0^+} f'(x)=\infty$.

Proof. Assume first that $f'(0)=\infty$. Since $f''(x)<0$ holds for all $x>0$, it follows that f' is a (strictly) decreasing function on $(0,\infty)$. This easily implies that $\ell=\lim_{x\to 0^+} f'(x)$ exists in $(0,\infty]$. Now for each $x>0$ there exists (by the Mean Value Theorem) some $0<c_x<x$ such that $\frac{f(x)-f(0)}{x-0}=f'(c_x)$. Clearly, $c_x\to 0^+$ as $x\to 0^+$, and so

$$\ell=\lim_{x\to 0^+} f'(c_x)=\lim_{x\to 0^+}\tfrac{f(x)-f(0)}{x-0}=f'(0)=\infty\,.$$

For the converse, assume that $\lim_{x\to 0^+} f'(x)=\infty$. Using the L'Hôpital's classical rule appropriately (see, for instance [2, Theorem 7.9, p. 292]), we see that

$$f'(0)=\lim_{x\to 0^+}\tfrac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\tfrac{f(x)}{x}=\lim_{x\to 0^+}\tfrac{f'(x)}{x'}=\lim_{x\to 0^+} f'(x)=\infty\,,$$

and the proof is finished. ■

Any sequence $\boldsymbol{x}=(x_0,x_1,x_2,\ldots)$ satisfying $0\le x_{t+1}\le f(x_t)$ for all $t=0,1,2,\ldots$ is called a plan. The collection of all plans starting with x_0 is denoted $\Pi(x_0)$. With each plan $\boldsymbol{x}\in\Pi(x_0)$ we associate the consumption plan $\boldsymbol{c_x}=(c_1,c_2,\ldots)$ defined recursively for each $t=0,1,2,\ldots$ by

$$c_t=f(x_t)-x_{t+1}\,.$$

Clearly, $0\le c_t\le f(x_t)$ for each $t=0,1,2,\ldots$.

The objective is to find a plan $\boldsymbol{x}^*\in\Pi(x_0)$ that maximizes the infinite horizon time-separable utility function defined by

$$U(\boldsymbol{x})=\sum_{t=0}^{\infty}\beta^t u(c_t)=\sum_{t=0}^{\infty}\beta^t u\big(f(x_t)-x_{t+1}\big)\,, \tag{1.1}$$

[2] Here, as usual, we define $f'(0)=\lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}$.

where $u\colon [0,\infty) \to [0,\infty)$ is a bounded function satisfying the Inada conditions and normalized so that $u(0) = 0$ [3] and $0 < \beta < 1$ is a constant discount factor. Since u is bounded, notice that U as given by (1.1) is a well-defined function. As a matter of fact, if $0 \le u(x) \le M$ holds true for all $x \ge 0$, then for every plan $\boldsymbol{x} = (x_0, x_1, \ldots) \in \Pi(x_0)$ we have $0 \le U(\boldsymbol{x}) \le \frac{M}{1-\beta}$.

Let $X = [0,\infty)$, define the constraint correspondence $\Gamma\colon X \twoheadrightarrow X$ by $\Gamma(x) = \big[0, f(x)\big]$, and then introduce the return function $F\colon G_\Gamma \to \mathbb{R}$ defined by $F(x,y) = u\big(f(x) - y\big)$, and assumed to be bounded, continuous, and strictly concave. Here, G_Γ is the graph of the constraint correspondence Γ defined by $G_\Gamma = \{(x,y) \in \mathbb{R}^2_+ \colon\ 0 \le y \le f(x)\}$, which is a convex set.

Given this terminology, we have

$$U(\boldsymbol{x}) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})\,.$$

The function U may be interpreted either as a social welfare function of a central planner or as the lifetime utility function of a representative agent who owns the "backyard" production technology f. Given this, we wish to solve the following optimization problem:

$$(\mathbf{P}) \qquad \text{Maximize:} \qquad U(\boldsymbol{x})$$
$$\text{Subject to:} \qquad \boldsymbol{x} \in \Pi(x_0)\,.$$

The optimization problem $(\mathbf{P})$ gives rise to a function $v\colon [0,\infty) \to \mathbb{R}$ defined for each $x_0 \ge 0$ by

$$v(x_0) = \sup_{\boldsymbol{x} \in \Pi(x_0)} U(\boldsymbol{x})\,.$$

This function v is called the value function and it has been studied extensively (e.g., see [8]). Here, we simply state two well known properties of the value function:

(1) v is strictly increasing, concave and continuous,[4] and

(2) it satisfies the Bellman equation, i.e., for each $x \ge 0$ we have

$$v(x) = \sup_{y \in \Gamma(x)} [F(x,y) + \beta v(y)] = \max_{0 \le y \le f(x)} [F(x,y) + \beta v(y)]\,.$$

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[3] The boundedness of u together with property (2) of the Inada conditions, guarantee that $\lim_{x\to\infty} u'(x) = 0$. To see this, note first that since u' is a decreasing function over $(0,\infty)$, the limit $\lambda = \lim_{x\to\infty} u'(x)$ exists in $[0,\infty)$. By virtue of the Mean Value Theorem, for each $x > 0$ there exists some $x < c_x < 2x$ such that $\frac{u(2x)-u(x)}{2x-x} = u'(c_x)$. Clearly, $\lim_{x\to\infty} c_x = \infty$, and from this it follows that $\lambda = \lim_{x\to\infty} u'(c_x) = \lim_{x\to\infty} \frac{u(2x)-u(x)}{x} = 0$.

[4] Clearly, the concavity of v implies its continuity over $(0,\infty)$, the interior of X. For the point 0 notice first that $v(0) = 0$. Put $\lim_{x\to 0^+} v(x) = \ell \ge 0$ and let $x_n \downarrow 0$. This implies $f(x_n) \downarrow 0$. From $v(x_n) \le u(f(x_n)) + \beta v(f(x_n))$, taking limits, we get $\ell \le 0 + \beta\ell$ or $0 \le (1-\beta)\ell \le 0$. This implies $\ell = 0$, so that the value function v is also continuous at 0. For a different proof of the continuity of v at zero see [6, Proposition 3.1, p. 49].

For a fixed $x > 0$, we consider the function $\phi\colon [0, f(x)] \to \mathbb{R}$ defined for each $0 \le y \le f(x)$ by

$$\phi(y) = u\big(f(x) - y\big) + \beta v(y)\,. \tag{1.2}$$

Using the continuity and strict concavity of u, the continuity of f, and the concavity and continuity of v, it is not difficult to see that ϕ is continuous and strictly concave. In particular, it follows that there exists a unique maximizer of ϕ over the interval $[0, f(x)]$. We shall denote this unique maximizer by $g(x)$. This implies that a function $g\colon [0,\infty) \to [0,\infty)$ can be defined via the identity

$$v(x) = u\big(f(x) - g(x)\big) + \beta v\big(g(x)\big)\,.$$

This new function $g\colon [0,\infty) \to [0,\infty)$ is known as the **policy function**.

For many classes of models (including the typical growth model) it is helpful to set conditions in place such that the policy function lies in the interior of the constraint correspondence. Broadly speaking, this property ensures that the representative agent would always want to split the output into two components, consumption and savings, so the economy can grow. The reason why the Inada conditions are so extensively used in the literature is that they guarantee this property holds along the equilibrium path. The available proofs of this interiority property of the policy function are, to our knowledge, based on the existence of an optimal plan (see for instance [8, Exercise 6.1, p. 134], [3, Lemma 1, p. 158] or [10]). Ideally, however, one would want to demonstrate that the interiority property holds even if one is not given an optimal plan. We accomplish this task in the next section.

2. The Main Result

In this section we demonstrate that for each $x > 0$ the value of the policy function $g(x)$ lies in the interior of the constraint correspondence $\Gamma(x)$. Our contribution is to present a proof that does not require the existence of an optimal plan.

Theorem 2.1. *For each $x > 0$ the value $g(x)$ in an interior point of $\Gamma(x)$. That is, for each $x > 0$ we have $0 < g(x) < f(x)$.*

Proof. Fix $x > 0$. We shall establish first that $g(x) > 0$. To see this, assume by way of contradiction that $g(x) = 0$. This implies that

$$v(x) = \max_{0\le y\le f(x)} \big[u\big(f(x) - y\big) + \beta v(y)\big] = u\big(f(x)\big)\,.$$

Now consider the function $\psi\colon [0, f(x)] \to \mathbb{R}$ defined for each $0 \le y \le f(x)$ by

$$\psi(y) = u\big(f(x) - y\big) + \beta u\big(f(y)\big) = U(x, y, 0, 0, 0, \ldots)\,.$$

We claim that this function is maximized when $y = 0$. To see this, assume that 0 is not a maximizer of ψ over $[0, f(x)]$. This means that there exists some $0 < y^* \le f(x)$ satisfying

$$\psi(y^*) = u\big(f(x) - y^*\big) + \beta u\big(f(y^*)\big) > u\big(f(x)\big)\,.$$

In this case, the plan $\boldsymbol{x} = \big(x, y^*, 0, 0, 0, \ldots\big) \in \Pi(x)$ must satisfy

$$v(x) \ge U(\boldsymbol{x}) = u\big(f(x) - y^*\big) + \beta u\big(f(y^*)\big) = \psi(y^*) > u\big(f(x)\big) = v(x)\,,$$

which is impossible. This contradiction shows that 0 maximizes ψ over $[0, f(x)]$.

Next, notice that for each $0 < y < f(x)$ we have

$$\psi'(y) = -u'\big(f(x) - y\big) + \beta u'\big(f(y)\big)f'(y)\,.$$

Using that both f and u satisfy the Inada conditions and $f(0) = 0$, we get $\lim_{y\to 0^+} \psi'(y) = \infty$. In particular, there is some $0 < y_0 < f(x)$ such that $\psi'(y) > 0$ holds for all $0 < y \le y_0$. This easily implies that ψ is strictly increasing in the interval $[0, y_0]$. But then 0 is not the maximizer of ψ over $[0, f(x)]$, a contradiction. Hence $g(x) > 0$ must be the case.

Next, we establish that $g(x) < f(x)$. To this end, assume by way of contradiction that $g(x) = f(x)$. That is, for each $0 \le y < f(x)$ we have

$$\phi(y) < \phi\big(f(x)\big)\,.$$

In this case, we claim that the function ϕ, as defined in (1.2), is also a strictly increasing function. To see this, let $0 \le x_1 < x_2 < f(x)$. Choose $0 < \lambda < 1$ such that $x_2 = \lambda x_1 + (1-\lambda)f(x)$ and note that by the strict concavity of ϕ we have

$$\phi(x_2) > \lambda\phi(x_1) + (1-\lambda)\phi\big(f(x)\big) > \lambda\phi(x_1) + (1-\lambda)\phi(x_1) = \phi(x_1)\,.$$

Now pick a sequence $\{y_n\} \subseteq (0, f(x))$ such that $y_n \to f(x)$. Let $\hat{v}\colon [0, f(x)] \to \mathbb{R}$ denote the restriction of v to the closed interval $[0, f(x)]$, and also let $G\colon [0, f(x)] \to \mathbb{R}$ be the function defined by $G(y) = u\big(f(x) - y\big)$. Clearly, both $\hat{v}$ and G are concave functions and $\phi = G + \beta\hat{v}$.

According to Lemma 3.3, the superdifferential of ϕ is nonempty at each y_n. If $\tau_n \in \partial\phi(y_n)$, then (by Theorem 3.6) there exist $t_n \in \partial G(y_n)$ and $s_n \in \partial\hat{v}(y_n)$ such that

$$\tau_n = t_n + \beta s_n\,. \tag{2.1}$$

Because ϕ is increasing, we obtain that $\tau_n \ge 0$ (see Part 2 of Lemma 3.2). Since y_n is an interior point of $[0, f(x)]$, it follows that t_n is the derivative of G at y_n. That is, we have $t_n = -u'\big(f(x) - y_n\big)$, and from (2.1) we get

$$u'\big(f(x) - y_n\big) \le \beta s_n\,. \tag{2.2}$$

Now consider the value function $v\colon [0, \infty) \to \mathbb{R}$ and notice that s_n is a supergradient of v at y_n when restricted to $[0, f(x)]$. Since y_n is an interior point of $[0, f(x)]$, it follows from Lemma 3.4 that s_n is a supergradient of v over $[0, \infty)$, i.e., $s_n \in \partial v(y_n)$. Also, since $f(x)$ is an interior point of $[0,\infty)$, it follows from Theorem 3.5 that the supergradient correspondence $x \mapsto\!\!\!\to \partial v(x)$ is upper hemicontinuous at $f(x)$. Taking into account that $\partial v(f(x))$ is a compact set (see Lemma 3.3) and $y_n \to f(x)$, we infer that $\{s_n\}$ has a subsequence that converges to some point in $\partial v(f(x))$. By relabeling, we can assume without loss of generality that $s_n \to s \in \partial v(f(x))$.

Now using the Inada Conditions, Lemma 1.1 and (2.2), we see that

$$\infty = \lim_{n\to\infty} u'\big(f(x) - y_n\big) \le \beta \lim_{n\to\infty} s_n = \beta s \in \mathbb{R}\,,$$

which is impossible. This contradiction establishes that $g(x) < f(x)$ is also true. ■

3. Appendix: Concave Functions

In this section we state a few the basic properties of concave functions that are used in this note. Recall that a function $f\colon C \to \mathbb{R}$, where C is a nonempty convex subset of a vector space, is said to be **concave** if for each $x, y \in C$ and each $0 \le \alpha \le 1$ we have

$$f(\alpha x + (1-\alpha)y) \ge \alpha f(x) + (1-\alpha) f(y)\,.$$

A concave function $f\colon C \to \mathbb{R}$ is called **strictly concave** if $x, y \in C$ with $x \ne y$ and $0 < \alpha < 1$ imply $f(\alpha x + (1-\alpha)y) > \alpha f(x) + (1-\alpha) f(y)$.

Lemma 3.1. *For a concave function $f\colon C \to \mathbb{R}$ we have the following.*

(1) *The (possibly empty) set of maximizers of f is a convex set.*
(2) *If f is also strictly concave, then the (possibly empty) set of maximizers of f is at most a singleton.*

Next we shall discuss some basic supporting properties of concave functions that we need for our work. Let $f\colon D \to \mathbb{R}$ be a function, where D is a subset of a vector space X. We say that a linear functional $x^* \in X^*$ [5] is a **supergradient** of the function at a point $c \in D$, if for all $x \in D$ we have

$$f(x) \le f(c) + x^*(x - c)\,.$$

The collection of all supergradients linear functionals of f at c is denoted $\partial f(c)$ and is called the **superdifferential** of f at c. That is,

$$\partial f(c) = \{x^* \in X^*\colon\; f(x) \le f(c) + x^*(x-c) \text{ for all } x \in D\}\,.$$

It is easy to see that when the function is differentiable at an interior point c of some subset of a Euclidean space, then the only possible supergradient of f at c is the gradient of f at c. In particular, if f is defined on a subset of the real numbers, then the superdifferential of f consists of real numbers—that can be thought of as substituting the notion of the derivative.

[5] As usual, X^* denotes the algebraic dual of X, i.e., the vector space of all linear functionals on X.

The proof of the next lemma is straightforward and is omitted.

Lemma 3.2. *For a function $f\colon D \to \mathbb{R}$, where D is a subset of $\mathbb{R}$, we have the following.*

(1) *If $c \in D$ and $\alpha > 0$, then $\partial[\alpha f](c) = \alpha[\partial f(c)]$.*
(2) *If f is a non-decreasing (resp. non-increasing) function, then at point c in the interior of D we have $\partial f(c) \subseteq [0, \infty)$ (resp. $\partial f(c) \subseteq (-\infty, 0]$).*

For the proof of the following result see [1, Theorem 7.22, p. 272].

Lemma 3.3. *For a concave function $f\colon I \to \mathbb{R}$ defined on an interval of $\mathbb{R}$ we have the following.*

(1) *The left- and right-derivatives f_ℓ and f_r of f exist at each interior point of I.*
(2) *Both functions f_r and f_ℓ (defined on the interior of I) are decreasing functions.*
(3) *If a, b are in the interior of I and satisfy $a < b$, then*

$$f_\ell(a) \geq f_r(a) \geq f_\ell(b) \geq f_r(b)\,.$$

Moreover, at each interior point c of I the superdifferential of f at c is nonempty and

$$\partial f(c) = [f_r(c), f_\ell(c)]\,.$$

In particular, if f is differentiable at an interior point c of I, then $\partial f(c) = \{f'(c)\}$.

For concave functions, local supportability implies global supportability.

Lemma 3.4. *Let $f\colon I \to \mathbb{R}$ be a concave function, where I is an interval of $\mathbb{R}$. Assume that at some interior point c of I there is a neighborhood V of c such that the real number m is a supergradient of f at c for the restriction of f to V. Then, m is a supergradient of f at c.*

Proof. Let $c \in V = (a, b) \subseteq I$ and assume that for each $x \in (a, b)$ we have

$$f(x) \leq f(c) + m(x - c)\,.$$

It follows that $m_r \leq m \leq m_\ell$, where $m_r = f_r(c)$ and $m_\ell = f_\ell(c)$. In particular, for $x \in I$ with $x \geq c$ we have

$$f(x) \leq f(c) + m_r(x - c) \leq f(c) + m(x - c)\,.$$

Similarly, for $x \in I$ with $x < c$ we have

$$f(x) \leq f(c) + m_\ell(x - c) \leq f(c) + m(x - c)\,.$$

From the above conclusions, we easily infer that $m \in \partial f(c)$. ■

The superdifferential correspondence is upper hemicontinuous.

Theorem 3.5. *If $f\colon I \to \mathbb{R}$ is a concave function, where I is an interval of $\mathbb{R}$, then the superdifferential correspondence $\partial f\colon I \twoheadrightarrow \mathbb{R}$ is upper hemicontinuous at every interior point of I.*

Proof. Let c be an interior point of I. If m_ℓ and m_r are the left and right derivatives of f at c, respectively, then we know that $\partial f(c) = [m_r, m_\ell]$. In particular, $\partial f(c)$ is a compact subset of $\mathbb{R}$.

Now assume that a sequence $\{x_n\}$ of I satisfies $x_n \to c$ and a sequence $\{m_n\}$ of $\mathbb{R}$ satisfies $m_n \in \partial f(x_n)$ for each n. We must show that the sequence $\{m_n\}$ has a subsequence that converges to some point of $\partial f(c)$; see [1, Theorem 17.20, p. 565].

Since c is an interior point of I, we can assume without loss of generality that there exist two points $s, t \in I$ and some $\epsilon > 0$ such that $s + \epsilon \le x_n \le t - \epsilon$ holds true for all n.

Since $m_n \in \partial f(x_n)$ we have

$$f(x) \le f(x_n) + m_n(x - x_n) \tag{3.1}$$

for all $x \in I$. In particular, letting $x = s$ and $x = t$ consecutively, it follows from (3.1) that

$$\tfrac{f(t)-f(x_n)}{t-x_n} \le m_n \le \tfrac{f(s)-f(x_n)}{s-x_n} \tag{3.2}$$

holds for each n. Now recalling that (since c is an interior point of I) the function f is continuous at c, we have $\lim f(x_n) = f(c)$ and so

$$\lim_{n\to\infty} \frac{f(s)-f(x_n)}{s-x_n} = \frac{f(s)-f(c)}{s-c} \quad \text{and} \quad \lim_{n\to\infty} \frac{f(t)-f(x_n)}{t-x_n} = \frac{f(t)-f(c)}{t-c}\,.$$

In particular, it follows that the two sequences $\{\frac{f(s)-f(x_n)}{s-x_n}\}$ and $\{\frac{f(t)-f(x_n)}{t-x_n}\}$ are bounded. But then, a glance at (3.2) shows that the sequence $\{m_n\}$ is bounded and so it has a convergent subsequence. By passing to a subsequence and relabeling, we can assume without loss of generality $m_n \to m$. Now letting $n \to \infty$ in (3.1) we get $f(x) \le f(c) + m(x - c)$ for all $x \in I$. This shows that $m \in \partial f(c)$, and the proof is finished. ■

For a proof of the next result see [7, Theorem 23.8, p. 223] or [9, Theorem 5.38, p. 77].

Theorem 3.6 (Moreau–Rockafellar)**.** *If I is a non-trivial interval of $\mathbb{R}$ and $f, g\colon I \to \mathbb{R}$ are two concave functions, then for each $c \in I$ and all $\alpha > 0$ and $\beta > 0$ we have*

$$[\partial(\alpha f + \beta g)](c) = \alpha[\partial f(c)] + \beta[\partial g(c)]\,.$$

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