Faculty of Economics and Commerce

Australian National University

Working Paper in Economics and Econometrics Number 444

A high-rise on Main Street: Hotelling with mobile consumers

Martin Richardson[*]

School of Economics
Faculty of Economics and Commerce
Australian National University
Canberra ACT 022
Australia

This version: July 2004

Abstract

This note considers Hotelling's (1929) model of locational choices by two firms and subsequent price competition in a setting where atomistic consumers locate first. It is shown that any equilibrium in pure strategies involves either one or two mass points with all surplus captured either by the consumers or by firms, respectively.

Key words: Hotelling, location, price competition
JEL subject codes:

1. Introduction

Harold Hotelling's 1929 classic "Stability in competition" has spawned an extensive literature on firms' locational choices, both spatial and otherwise. Hotelling himself suggested that two firms locating along a line segment populated by a uniform distribution of consumers would have an incentive to locate together – the principle of minimal differentiation – and this 'equilibrium' is often cited to explain the clustering of firms.[1] This conjecture is correct if firms' only choices are locations i.e. if prices are fixed, as firms' locational decisions are then driven by maximizing market share alone, given that there is no price competition, and any location of a rival is best met by moving in closer to them.

When the game is enriched by adding price competition, however, things change dramatically. Suppose that firms first (simultaneously and non-cooperatively) choose locations and subsequently (simultaneously and non-cooperatively) choose prices. If transport costs (disutility) are sufficiently convex[2] in distance – say quadratic – then there is a pure strategy equilibrium to the price game for every pair of locations, but the optimal locations are as far from each other as possible[3]: the principle of maximal differentiation. Essentially firms now recognize that closer locations lead to more intense price competition in the next stage of the game.

---

[1] "Clustering" here refers not just to locations in physical space. Hotelling himself suggested that the, "tremendous standardisation of our furniture, our houses, our clothing, our automobiles and our education are due in part to the economies of large-scale production, in part to fashion and imitation. But over and above these forces is...the tendency to make only slight deviations in order to have for the new commodity as many buyers of the old as possible, to get, so to speak, *between* one's competitors and a mass of customers." (Hotelling, 1929 p.54.)

[2] See Economides (1986).

[3] This is true so long as each consumer's reservation price is 'sufficiently' high. Under the Hotelling assumption of transport costs *linear* in distance it has been shown (see d'Aspremont, Gabszewicz and Thisse (1979))) that there is no pure strategy equilibrium to the price-setting game if firms locations are 'too close' – roughly, each firm's best response to the other's 'optimal' price is either to undercut it and capture the entire market or to charge a higher price and settle for a smaller market share. Osborne and Pitchik (1987) identify mixed-strategy equilibria to the price game for some locations where there is no pure strategy price equilibrium.

As a guiding principle for firms' locational decisions the principle of maximal differentiation is, of course, subject to a number of caveats. In particular, an offsetting desideratum is to "be where the demand is": if consumers were to be clustered at certain points rather than uniformly distributed this would naturally mitigate incentives for maximal differentiation.

This is the starting point for the present short paper, along with the following observation. In the two-stage game just considered, consumer welfare is minimised, across the set of symmetric locations, with maximal differentiation – aggregate transport costs are the same as with minimal differentiation but firms' prices are increasing in their spatial separation. If consumers could choose their locations, one might then anticipate that the maximal differentiation result would collapse. In this paper we take the two-stage version of Hotelling but add an extra stage at the beginning in which atomistic consumers choose their locations optimally. Our conjecture is that this will re-establish minimal differentiation, even for non-strategic atomistic consumers, as they will cluster thus inducing the firms to locate more closely and hence to engage in fiercer price competition, to the consumers' benefit. We demonstrate that this intuition is borne out, at least in the sense that there exists an equilibrium in which it holds. While we make the argument in the context of a two firm model, it will be clear that it applies, mutatis mutandis, to the n firm case.

2. The Model

For reasons that we explain below, whether transport costs are quadratic or linear makes no difference at all here so we shall take the simplest case of linear transport costs. So consider a line segment of unit length along which atomistic consumers individually choose to locate, where locations are indexed on the interval [0,1]. The

density of consumers is L. Once they have committed to locations, two firms then each choose a location and, following that, compete in prices for their identical products. Each consumer is willing to pay a reservation price of $\underline{v}$ for a single unit of the good in question and will incur a transport cost of t for every unit of distance they travel to a vendor. So a consumer travelling a distance s_i to purchase at money price p_i from vendor i would derive utility of $\underline{v}\text{-}ts_i\text{-}p_i$ where we shall refer to ts_i+p_i as the *full* price of the good to the consumer – the money price plus transport costs. We normalise each firm's costs to zero. We assume, by way of a tie-breaker, that, if a consumer is exactly indifferent between purchasing from the two firms, they will purchase from the closer of the two; only if the firms are the same distance and set the same price will they share the market equally. We seek a subgame-perfect Nash equilibrium in pure strategies at each stage.[4] Accordingly, we wish to solve the game backwards, but it turns out to be sensible to first consider stage one of the game.

Stage One: the consumers' locational decisions

We suppose, as in the standard Hotelling framework, that consumers are atomistic. Accordingly, each consumer alone can do nothing to influence firms' locations, given the locations of all other consumers. We are interested in a non-cooperative outcome so we do not consider collusive behaviour by consumers. We make the following claim:

Proposition One: in a 2-firm setting, the equilibrium distribution of consumers will consist of (at most) 2 distinct mass points.

---

[4] The results of Osborne and Pitchik (1987) suggest that this may be a considerable restriction – we show later that many locational choices lead to non-existence of pure-strategy equilibria in the price-setting subgame but their results indicate that mixed strategy equilibria are likely to exist in many of these cases. We do not consider such equilibria henceforth.

To see this, suppose the Proposition was incorrect and we had some equilibrium to the full game in which some consumers were not located at the same point as a firm (which would be the case whatever locations the two firms actually chose if the claim were incorrect.) Consider one of these consumers. They could not then have chosen locations optimally initially – choosing to locate where the nearest firm locates would have no effect on the firms' choices (either in terms of locations or prices), as each consumer is atomistic, but would reduce the consumer's transport costs and thus raise their welfare.

Stage Three: the firms' pricing decisions

By Proposition One we know that, in the case of two firms, consumers will be massed in either one or two mass points. As a consequence there is a potential new reason for non-existence of a pure strategy equilibrium in the firms' price-setting game. To see this, suppose there are two mass points of consumers and suppose initially that they are equal ($\frac{1}{2}L$). Figure One illustrates the nature of competition here. Suppose the consumers are massed at points c_1 and c_2 and the two firms are located at f_1 and f_2.[5] We portray the density at each mass point by the height of each thick line, measured off the right-hand axis. If each firm chooses its price so as to maximise profits from its closest consumer group only, they will price at p_1* and p_2* respectively, measured on the left-hand axis, where these are chosen so that all consumers face a price, including transport costs, that leaves them zero surplus.

---

[5] Throughout the paper we set the location notation so that 2 is to the right of 1, whether we are considering firms or consumers. We also refer occasionally to a firm selling to its *own* consumers; by this we mean the firm at f_i selling to consumers located at c_i.
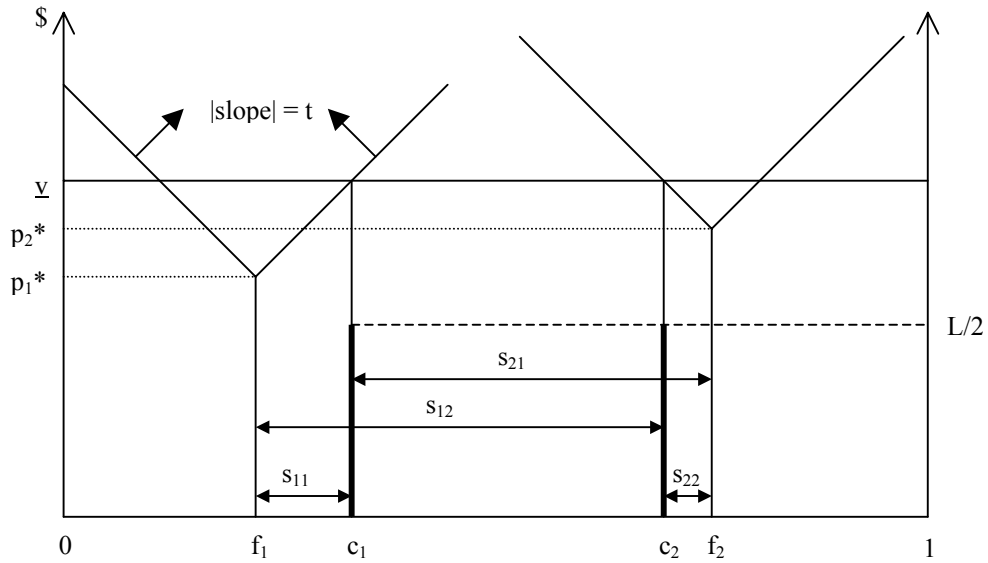
Figure One

Clearly these optimal prices are $p_i^*=\underline{v}-ts_{ii}$, $i=1,2$, where $s_{ij}=|f_i-c_j|$ denotes the distance from firm i to consumer mass point j. Profits are then $\tfrac{1}{2}Lp_i^*=\tfrac{1}{2}L(\underline{v}-ts_{ii})$ and both firms would prefer to locate at the consumer mass points, thus reducing s_{ii} to zero.

However, if firms' locations were closer firm i could capture the entire market by charging a price that just undercuts firm j at consumer mass point j. This requires a price infinitesimally less than $p_i'=\underline{v}-ts_{ij}$ thus yielding profits of $\pi_i'=Lp_i'$ which exceeds $\tfrac{1}{2}L(\underline{v}-ts_{ii})$ if $\underline{v}\geq t(2s_{ij}-s_{ii})$. If this latter condition is satisfied then the possibility arises of there being no pure strategy equilibrium in the price game, for the same reason as in the standard Hotelling case: each firm's profit function is discontinuous at the price at which it takes the entire market.

While similar to the Hotelling non-existence problem note that, in contrast, quadratic transport costs would not avoid the non-existence problem here[6] as the discrete nature of the consumers' distribution would still give a discontinuity at that

---

[6] See d'Aspremont *et al* (1979) for a discussion of this in the Hotelling case.

critical price.[7] Consequently, there are a number of cases in which no pure-strategy

equilibrium exists in the price-setting subgame, depending on the proximity of the

consumers' mass points, the proximity of firms and the distribution of consumers

across the two mass points.[8] Where an equilibrium does exist for the case of two

distinct mass points, it is one of two types: either each firm serves only consumers at

its closest mass point and charges a price that extracts all their surplus, or firm i

charges a price of zero, firm j a positive price and firm i sells nothing for zero profit

while firm j makes a positive profit capturing the entire market. In the case of a single

mass point of consumers, the only equilibrium is that the most distant firm charges a

money price of zero and sells nothing, the closest firm sets a money price such that

the full price equals that of the distant firm and captures the entire market. If both

firms locate equidistantly from the mass point (including a distance of zero), however,

both set zero prices and share the market.

Stage Two: the firms' location decisions

In stage two, firms are aware of the consequences of their locational choices for

subsequent price competition. As we seek a pure-strategy equilibrium, we confine

our attention to those locational choices that yield a pure-strategy equilibrium in the

third stage and, as noted, these are of two kinds when there are two distinct mass

points of consumers. In the first, where each firm serves only consumers at its closest

mass point and charges the price p_i* that extracts all their surplus, a profit-maximising

---

[7] In the usual case with a given continuous distribution of consumers, a small reduction in price can
yield a discontinuity in a firm's profit function when costs are linear, as the other firm is undercut in
every location, but not when they are quadratic. Here, even with quadratic costs, a firm's profit
function will be discontinuous for a small price cut when it erodes the other firm's market because
there is only a single location for each firm's consumers so undercutting there *is* effectively
undercutting at every location, just as in the standard Hotelling with linear costs.
[8] The Appendix details the conditions under which a pure-strategy equilibrium does exist for different
locational choices.

firm will find it pays to move closer to those consumers (so long as it is not then tempted to serve the other mass point as well.[9]) This enables the firm to capture surplus otherwise dissipated as transport costs: $p_i^*=\underline{v}-ts_{ii}$, $i=1,2$, yielding profits of $\pi_i=L_i(\underline{v}-ts_{ii})$ where L_i denotes the mass of consumers at c_i. Reducing s_{ii} thus increases the money price that can be charged and hence raises profits.

The second type of price equilibrium occurs when both firms lie to the same side of both consumer locations (or one firm is so distant from all consumers that the other firm can capture the whole market even when the distant firm charges a zero money price.) For example, suppose $f_1<f_2<c_1<c_2$. Now firm 2 is closer to all consumers than is firm 1 and firm 1 will make zero profit if it does not undercut any p_2. Accordingly, the only equilibrium is that $p_1=0$ and firm 2 sets either the identical full price, $p_2=t(s_{11}-s_{21})$ or, if f_1 is sufficiently distant (if $s_{11}>\underline{v}/t$), then either $p_2=\underline{v}-ts_{22}$ (if the full market is served) or $p_2=\underline{v}-ts_{21}$ if firm 2 serves only its own consumers (which will occur if c_1 and c_2 are sufficiently far apart.) Thus $\pi_1=0$ and $\pi_2=Lt(s_{11}-s_{21})$ or $\pi_2=max\{L(\underline{v}-ts_{22}), L_1(\underline{v}-ts_{21})\}$. In such a case firm 1's location is clearly not optimal: given f_2 firm 1 would do better to locate such that $f_1=f_2+\varepsilon$ or at $f_1=c_2$ if that yielded local monopolies and higher profit. The same is true when firm i is distant and sets $p_i=0$ in the price equilibrium. In sum, this second type of price equilibrium cannot be supported as an equilibrium in location choices.

Finally, if all consumers have the same single location then only the closest firm makes positive profits; accordingly, the only equilibrium locations are for both firms to locate exactly at the consumer mass point.[10]

---

[9] That is, if the firms move too close together they will enter the locations in which there is no equilibrium in the subsequent price-setting subgame.

[10] Technically, this requires a further 'tie-breaker' assumption that a firm with a number of alternative locations all yielding zero profit chooses the one that maximises its sales. Otherwise a single firm at the mass point could capture the entire market with price $\underline{v}$ and the other firm, not at c, would make

Stage One: the consumers' location decisions revisited

So far we have established equilibria in the location-choice stage that yield equilibria in the price-setting subgame. However, the initial locational choices of consumers further restrict the locational choices of firms that might be equilibria. Suppose that c_1 and c_2 are sufficiently close together that $f_i = c_i$ would yield no equilibrium in the price-setting game. Nevertheless, if the firms located such that moving any closer would induce non-equilibrium, then we have an equilibrium for the location-choice subgame: for each firm, given the location of the other firm, there is no location that yields greater profits.

However, this configuration also cannot be an equilibrium to the entire game as it involves non-optimal location choices by consumers. As noted earlier, each atomistic consumer here would be better relocating to one of the firms' locations.

Letting Δ denote the distance between two consumer mass points (i.e. $\Delta = c_2 - c_1$) and noting that when each firm locates at its own consumers' location $s_{ii} = 0$ and $s_{ji} = \Delta$, we get a pure strategy equilibrium if $\dfrac{L_i}{L_j} < \dfrac{t\Delta}{\underline{v} - t\Delta}$.

To summarise the forgoing discussion, we have the following Proposition.

Proposition Two: in a 2- firm setting, any equilibrium in pure strategies to this game involves either (i) a single mass point of consumers located at the same point as both firms, each of whom charges a zero money price and makes zero profits thus yielding surplus of $\underline{v}$ to all consumers, or (ii) two distinct mass points of consumers with

---

zero profit for any price but would have no reason to move to c, as it will make zero profit there as well. Hence this could be an equilibrium too without this further tie-breaker.

$\frac{L_i}{L_j} < \frac{t\Delta}{\underline{v} - t\Delta}$ and each firm located at one mass point charging a money price of $\underline{v}$ and

making profits of $\underline{v}$ per consumer served thus yielding zero surplus to consumers.

3. Discussion and conclusion

Our initial conjecture – that mobile consumers might lead to the collapse of the literature's maximal differentiation result – has (sort of) been borne out here. The suspicion was that, by locating close together, consumers could induce the firms to locate together and thereby price more aggressively, to the consumers' benefit. However, we have shown that there are two sorts of generic equilibria to this game and only in one, when all consumers locate at the same point, does this intuition hold as the firms will locate there too and charge zero prices. In the other, where there are two mass points of consumers, the firms act as local monopolists and extract all surplus from the consumers.

There is, of course, an infinite number of equilibria to this game and the usual question of selecting amongst this plethora of possibilities arises here. All of the equilibria are efficient – there are no transport costs incurred in any of them – but the two broad types (single or double locations) differ in terms of who captures the surplus (consumers or firms.) A simple modification to the game that has some appeal is to adjust the first stage so that consumers arrive in sequence. They are still atomistic but they each choose location in turn. In such a setting the location chosen by the first consumer will be 'focal' for other consumers and we would observe only a single location being chosen by consumers.[11]

While we have cast this argument in the context of consumers' locations in a linear city, it might also be argued in the context of city locations in a linear country.

---

[11] This game still has an infinite number of equilibria, however, as any single location on [0,1] will do.

In this sense the paper provides a rationale for the existence of cities (a strategic rationale, rather than the usual arguments given in the agglomeration literature) as well as for observing a high-rise on Hotelling's Main Street.

Finally, while we have cast this argument in the setting of only two firms, it is reasonably apparent that it will generalise to the case of n firms. In particular, the equilibria with a single mass point of consumers will be unaffected by increasing the number of firms but the case of n mass points will be less likely to support a subgame perfect equilibrium in pure strategies the greater is n for the same reasons as in the two firm case when the mass points are too close together.

Hotelling's paper closes with a lament of conformity triggered by minimal differentiation: "[o]ur cities become uneconomically large and the business districts within them are too concentrated. Methodist and Presbyterian churches are too much alike; cider is too homogeneous." We close with the observation that at least the cider is likely to be cheap.

References

d'Aspremont, C., J. Gabszewicz and J.-F. Thisse, 1979, On Hotelling's "Stability in Competition". *Econometrica* 47#5, 1145-1150.

Economides, N., 1986, Minimal and maximal product differentiation in Hotelling's duopoly. *Economics Letters* 21, 67-71.

Fujita, M., and J.-F. Thisse, 1986, Spatial competition with a land market: Hotelling and Von Thunen unified. *Review of Economic Studies* 53, 819-841.

Hotelling, H., 1929, Stability in competition. *Economic Journal* 39#153, 41-57.

Osborne, M. and C. Pitchik, 1987, Equilibrium in Hotelling's model of spatial competition. *Econometrica* 55#4, 911-922.

Appendix

Pure strategy equilibria in the price-setting subgame

First, suppose consumers are at two distinct mass points on $[0,1]$ where, without loss of generality, $c_1 < c_2$. Labelling the firms' locations such that $f_1 < f_2$ we have six possible arrays of locations which we discuss in order. Throughout we use the notation that $s_{ij} = |f_i - c_j|$ denotes the distance from firm i to consumer mass point j and we let the density of consumers at point c_i be L_i, $i=1,2$, where $L_1 + L_2 = L$. Let p_i^* denote the money price charged by firm i such that the full price at c_i is $\underline{v}$; that is, it is the local monopoly price for a firm to its 'own' consumers. Finally, by way of a tie-breaker, suppose that, if a firm is indifferent between charging a price that serves only its "own" consumers and charging a lower price that captures the entire market, then it will choose the former.

(1) $f_1 < c_1 < c_2 < f_2$. Suppose firm 1 charges p_1^*. If firm 2 charges $p_2^* = \underline{v} - ts_{22}$ then it will earn profits of $L_2(\underline{v} - ts_{22})$ because, with $p_i = p_i^*$ and $c_2 > c_1$ each will sell only to its own market and p_i^* is its optimal price. Alternatively, firm 2 might levy a price that catches the entire market – this requires a price just less than or equal to (the latter iff $s_{21} \leq s_{11}$) $p_2' = \underline{v} - ts_{21}$ – in which case its profit is $Lp_2' \cong L(\underline{v} - ts_{21})$.

Diagrammatically,

Page A1
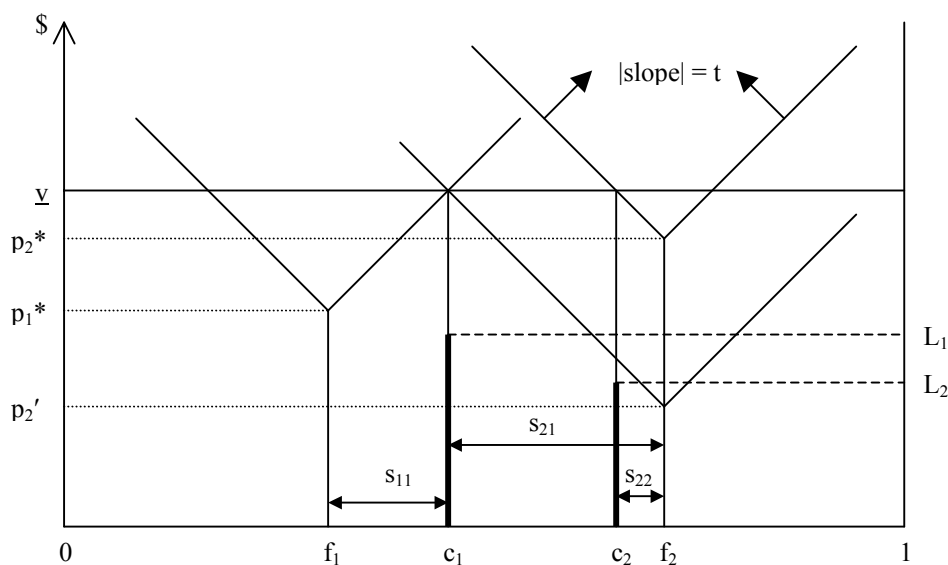
So p_2* is a best response to p_1* iff $L(\underline{v}\text{-}ts_{21})\leq L_2(\underline{v}\text{-}ts_{22})$ i.e. iff. $\dfrac{L_1}{L_2}\leq\dfrac{t(s_{21}-s_{22})}{(\underline{v}-ts_{21})}$

Intuitively, firm 2 is more tempted to undercut firm 1 and serve the whole market the greater is the density of consumers at c_1 relative to that at c_2, the closer are the two consumer locations (the smaller is s_{21}-s_{22}) and the greater is the surplus to be had from consumers at c_1 (the greater is $\underline{v}$-ts_{21}.) By symmetry, p_1* is a best response to p_2* iff

$$\frac{L_2}{L_1}\leq\frac{t(s_{12}-s_{11})}{(\underline{v}-ts_{12})}.$$

But if firm 2 sets p_2' then firm 1 makes zero profit and p_1* is not a best response. Firm 1's best response is to undercut 2's full price at c_1 by epsilon (or, if a similar calculation to that done above for firm 2 also holds for firm 1, to set p_1 such that firm 1 serves the whole market) to which 2's best response is a slightly lower price and so on and so forth. This argument applies until one of two cases occurs. First, firm i's price is sufficiently low that firm j makes greater profit by giving up on serving the whole market, increasing its price and serving only its own consumers. In such a case firm i would also wish to raise its price to just undercut firm j and the argument repeats; we have no equilibrium in pure strategies. Second, firm i's price goes to zero but firm j still finds it more profitable to serve the whole market than just its own consumers (at a higher price). This case will arise where s_{ii} is so great compared to s_{ji} that firm j can capture the whole market even when p_i=0 and make higher profits than serving only c_j at p_j*. When p_i=0 consumers at c_i pay ts_{ii} to buy from firm i and so will purchase from firm j at p_j+ts_{ji} so long as p_j+$ts_{ji}\leq ts_{ii}$ i.e. so long as $p_j\leq t(s_{ii}$-$s_{ji})$. In such a case $p_i\geq 0$ and p_j=$t(s_{ii}$-$s_{ji})$ constitute an equilibrium if π_j=$tL(s_{ii}$-$s_{ji})>L_j(\underline{v}$-$ts_{jj})$.

In sum, when $f_1<c_1<c_2\leq f_2$ we either get an equilibrium in which p_i=p_i* and each firm is a local monopoly to its own consumers only or an equilibrium in which

one firm sets any price, even 0, and the other captures the entire market, or we get no equilibrium at all.

(2) $f_1 < c_1 < f_2 < c_2$. Suppose first that $s_{21} < s_{22}$ i.e. firm 2 is closer to c_1 than it is to c_2. In this case, any p_2 that yields positive surplus to consumers at c_2 will also yield positive surplus to consumers at c_1 and firm 1 will wish to undercut by epsilon in order to capture c_1. The argument made in case (1) above then applies here and we are left with either an equilibrium in which firm 1 sets a price of 0 and firm 2 captures the entire market or we get no equilibrium at all. So consider instead the case where $s_{21} > s_{22}$ i.e. firm 2 is closer to c_2 than it is to c_1. By reasoning similar to that above, we can sustain $p_i = p_i^*$ as an equilibrium iff the profits to firm 2 from capturing the entire market are less than those from setting p_2^* in response to p_1^* and this occurs, as

before, iff $\dfrac{L_1}{L_2} < \dfrac{t(s_{21} - s_{22})}{(\underline{v} - ts_{21})}$. In sum, when $f_1 < c_1 < f_2 < c_2$ we again either get an

equilibrium in which $p_i = p_i^*$ and each firm is a local monopoly to its own consumers only or an equilibrium in which one firm sets any price, even 0, and the other captures the entire market, or we get no equilibrium at all.

(3) $f_1 < f_2 < c_1 < c_2$. Now firm 2 is closer to all consumers than is firm 1 and firm 1 will make zero profit if it does not undercut any p_2. Accordingly, the only equilibrium is that $p_1 = 0$ and firm 2 sets either the identical full price, $p_2 = t(s_{11} - s_{21})$ or, if f_1 is sufficiently distant (if $s_{11} > \underline{v}/t$), then either $p_2 = \underline{v} - ts_{22}$ (if the full market is served) or $p_2 = \underline{v} - ts_{21}$ if firm 2 serves only its own consumers (which will occur if c_1 and c_2 are sufficiently far apart.) Thus $\pi_1 = 0$ and $\pi_2 = Lt(s_{11} - s_{21})$ or $\pi_2 = max\{L(\underline{v} - ts_{22}), L_1(\underline{v} - ts_{21})\}$.

(4) $c_1 < f_1 < c_2 < f_2$. As in case (2) above, firm 1 must be closer to c_1 than to c_2. By reasoning that should now be familiar, we can sustain $p_i = p_i*$ as an equilibrium iff.

$$\frac{L_1}{L_2} \leq \frac{t(s_{21} - s_{22})}{(v - ts_{21})}$$

(5) $c_1 < f_1 < f_2 < c_2$. Again we need each firm to be closer to its own consumers than it is to the other consumers for an equilibrium to exist and, by our usual reasoning, we can sustain $p_i = p_i*$ as an equilibrium iff $\dfrac{L_1}{L_2} \leq \dfrac{t(s_{21} - s_{22})}{(v - ts_{21})}$.

(6) $c_1 < c_2 < f_1 < f_2$. As in case (3) above, with both firms to one side of all consumers, the only equilibrium is that $p_2 = 0$ and firm 1 sets either the identical full price, $p_1 = t(s_{22} - s_{12})$ or, if f_2 is sufficiently distant (if $s_{22} > v/t$), then either $p_1 = v - ts_{11}$ (if the full market is served) or $p_1 = v - ts_{12}$ if firm 1 serves only consumers at c_2 (which will occur if c_1 and c_2 are sufficiently far apart.) Thus $\pi_2 = 0$ and $\pi_1 = Lt(s_{22} - s_{12})$ or $\pi_1 = max\{L(v - ts_{11}), L_2(v - ts_{12})\}$.

Second, suppose consumers are all at one mass point. If one firm i is sufficiently distant that $s_i > v/t$ where $s_i \equiv |c - f_i|$ and c denotes the location of all the consumers, then clearly the only equilibrium is that $p_j = v - ts_j$, $\pi_i = 0$ and $\pi_j = L(v - ts_j)$: one firm is too distant to compete, even at a zero money price, and the other captures the entire market and extracts all surplus. If $s_i > v/t$ for $i = 1,2$, however, then price competition will drive the price of the most distant firm down to zero and the closest firm will match the full price and capture the entire market. That is, if $0 \leq s_j < s_i$ then $p_i \geq 0$ and $p_j = t(s_i - s_j)$ so $\pi_i = 0$ and $\pi_j = Lt(s_i - s_j)$. Finally, if $0 \leq s_1 = s_2$ then equilibrium involves $p_i = 0$ and $\pi_i = 0$ for $i = 1,2$.

Pure strategy equilibria in the location subgame

First, again suppose consumers are at two distinct mass points on [0,1] and consider the six possible arrays of locations discussed above.

(1) $f_1 < c_1 < c_2 < f_2$. We noted that p_2* and p_1* constitute a price-setting equilibrium iff

$$\frac{L_1}{L_2} \le \frac{t(s_{21} - s_{22})}{(\underline{v} - ts_{21})} \text{ and } \frac{L_2}{L_1} \le \frac{t(s_{12} - s_{11})}{(\underline{v} - ts_{12})}.$$ If this fails then either there is no price

equilibrium in pure strategies or, if $\pi_j = tL(s_{ii} - s_{jj}) > L_j(\underline{v} - ts_{jj})$, then $p_i \ge 0$ and $p_j = t(s_{ii} - s_{ji})$ constitute an equilibrium. In the first of these cases, let s_i denote the location in [0,1] at which firm i is just indifferent between serving its own consumers and serving the entire market. Note that the set of locations at which, say, firm 1 will not be tempted to serve the entire market, is *[0,s₁]*, a closed set. Note too that in that set $\pi_i = L_i p_i* = L_i(\underline{v} - ts_{ii})$ so each firm's profit is increased by moving closer to its consumers – reducing s_{ii}, subject to the caveat that it does not get so close to the rival as to induce non-equilibrium in the price subgame i.e. get beyond s_i. Accordingly, because of this caveat, equilibrium locations either involve $f_i = c_i$, i=1,2 where $s_1 > c_1$ or $s_2 < c_2$, or $f_i = s_i$ where $s_1 \le c_1$ or $s_2 \ge c_2$. Most significantly, our conditions for a price-setting equilibrium when the firms locate exactly at the consumer mass points become

$$\frac{L_i}{L_j} \le \frac{t\Delta}{\underline{v} - t\Delta}$$ where Δ denotes $c_2 - c_1$. That is, the densities of consumers should not be

"too different", the mass points should not be "too close" and the reservation value should not be "too high".[12] Finally, in the other sort of equilibrium mentioned initially – where the distant firm sells nothing – it is clearly in the interests of such a firm to relocate closer to the consumers. This takes us back into the case where both

---

[12] Conditions for an equilibrium where the firms do not locate at the mass points are uninteresting as such equilibria will be ruled out in stage one of the overall game.

firms have positive sales and thus to the case just discussed. Thus there is no

locational equilibrium in which $f_1 < c_1 < c_2 \leq f_2$ strictly.

(2) $f_1 < c_1 \leq f_2 < c_2$. Here we again either get an equilibrium in which $p_i = p_i^*$ and each

firm is a local monopoly to its own consumers only or an equilibrium in which one

firm sets a price of 0 and the other captures the entire market or we get no equilibrium

at all. In the first and second cases it is again true that a firm's profits are increased

by moving closer to its own consumers, for the same reasons as exposited in case (1):

there is no locational equilibrium in which $f_1 < c_1 \leq f_2 < c_2$ strictly.

(3) $f_1 \leq f_2 < c_1 < c_2$. Now equilibrium involves $p_1 \geq 0$ and either $p_2 = t(s_{11} - s_{21})$ or, if f_1 is

sufficiently distant (if $s_{11} > \underline{v}/t$), then either $p_2 = \underline{v} - ts_{22}$ (if the full market is served) or

$p_2 = \underline{v} - ts_{21}$ if firm 2 serves only its own consumers (which will occur if c_1 and c_2 are

sufficiently far apart.) Thus $\pi_1 = 0$ and $\pi_2 = Lt(s_{11} - s_{21})$ or $\pi_2 = max\{L(\underline{v} - ts_{22}), L_1(\underline{v} - ts_{21})\}$.

Now firm 1's location cannot be optimal – by leapfrogging firm 2 to locate closer to

the consumers it can earn positive profits. The argument then applies to firm 1 etc.;

conclusion: there can be no locational equilibrium in which $f_i \leq f_j < c_1 < c_2$.

(4) $c_1 \leq f_1 < c_2 \leq f_2$. Here we can sustain $p_i = p_i^*$ as an equilibrium iff $\dfrac{L_1}{L_2} \leq \dfrac{t(s_{21} - s_{22})}{(\underline{v} - ts_{21})}$ and

our reasoning from case (2) applies directly: there is no locational equilibrium in

which $c_1 \leq f_1 < c_2 \leq f_2$ strictly.

(5) $c_1 \leq f_1 \leq f_2 < c_2$. Here we can sustain $p_i = p_i^*$ as an equilibrium iff $\dfrac{L_1}{L_2} \leq \dfrac{t(s_{21} - s_{22})}{(\underline{v} - ts_{21})}$ and

again there is no locational equilibrium in which $c_1 \leq f_1 \leq f_2 < c_2$ strictly.

(6) $c_1 < c_2 \leq f_1 \leq f_2$. As in case (3) above, with both firms to one side of all consumers,

there can be no locational equilibrium in which $c_1 < c_2 \leq f_1 \leq f_2$.

The only possible equilibria with two mass points of consumers, then, are (1) where $f_i=c_i$ and $p_i=p_i^*$ which can only occur if the consumer densities and distances are such that $\dfrac{L_i}{L_j} \leq \dfrac{t\Delta}{\underline{v}-t\Delta}$ for $i=1,2$, and (2) where $f_i=s_i$ and at least one $s_i\neq c_i$.

Second, suppose consumers are all at one mass point. We make an important further 'tie-breaker' assumption that a firm with a number of alternative locations all yielding zero profit chooses the one that maximises its sales. If one firm is too distant to compete, even at a zero money price, and the other captures the entire market and extracts all surplus then the distant firm can do better by moving closer to the consumers. If locations are such that price competition drives the price of the most distant firm down to zero while the closest firm matches the full price and captures the entire market then again the distant firm can do better by moving closer to the consumers. Finally, if the firms are equi-distant from the consumers, and so making zero profits, a firm is better moving in closer and earning positive profits. In conclusion, the only equilibrium to the location game is that each firm locates at the same point as the consumers, sets $p_i=0$ and earns $\pi_i=0$.