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Multi-Item Contests

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Multi-Item Contests

Alexander R.W. Robson*

Abstract

Contests are games in which the players compete for a valuable prize by exerting effort or using resources so as to increase their probability of winning. This paper examines two player *multi-item contests*, a class of games in which players are faced with a decision about how much of a given resource to devote to an entire collection or sequence of different contests. Applications include multi-item rent-seeking behavior, multi-good marketing and advertising, multi-jurisdictional political contests. In these games, even when the (uncertain) outcomes in each contest are assumed to be mutually statistically independent, equilibrium efforts can exhibit strong interdependencies. Changes in either the contest success function or value of the prize in one contest usually alter the equilibrium amount of resources devoted to all contests by both players. We unify and extend results from marketing and political science, and also derive conditions under which both players exert zero effort in equilibrium in some subset of contests.

1 Introduction

The properties of equilibrium behavior in single-item contests have been widely studied in the literature, and the models and results have been put to work in many applied situations. In all of these applications and examples players exert effort in order to obtain a single valuable prize. But many interesting situations in economics and political science involve instances in which players are faced with a decision about how much resources to devote to a collection or sequence of different contests.

One example of a such a situation is a national political campaign, where funds are raised by the central party political machine and then distributed across various physical locations or issues. Another example occurs when two potential monopolists attempt to gain the exclusive rights to sell two different goods. If each has identical resources and production technologies, the remaining issue is to determine the efforts that each player puts in to obtaining each license.

This paper relaxes the assumption of a single prize for which the players compete, and instead studies a class of simple two player games in which players must decide how much

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effort to exert in trying to obtain many prizes. We call this class of games *multi-item contests.*

The problem of multi-item contest becomes more interesting if one of the players has an advantage over the other in one of the contests, in the sense that, with an equal amount of resources, one player has a more than even chance of winning the contest. An even more interesting situation is one where one or both of the players find it optimal, in equilibrium, *not to compete at all* in a certain subset of the contests that are initially available to them. It is easy to envisage political campaigns or marketing schemes where the players choose not to participate at all in some contests, and instead focus their resources on other issues, products, or jurisdictions where returns are greatest. In election campaigns one often observes this kind of behavior – in many cases campaign strategists and candidates virtually ignore certain jurisdictions or issues and focus on their strengths, even in the final days of the campaign.

To our knowledge, the earliest study of a situation similar to ours is Tukey (1949) and the subsequent literature on "Blotto" games.[1] In the marketing literature, Friedman (1958) was the first to analyze *market share attraction games* in which firms use advertising expenditures to attract customers. Friedman analyzes the optimal allocation of advertising expenditures by two firms in different product markets, using strictly competitive game theory and saddlepoint arguments. Friedman's analysis has been widely adopted in the subsequent marketing and advertising literature, most notably by Monahan (1987). [2]

Another important paper which analyzes behavior in distinct but simultaneous political contests is Snyder (1989), who models the behavior of two parties engaged in competition over many legislative seats in fashion similar to ours.[3] In Snyder's model, all "prizes" are legislative seats of the same value, and players are assumed to have either of the following two objectives: they maximize the expected number of seats won, or maximize the probability of winning a majority of seats. Under the first behavioral assumption, Snyder obtains the result that players will put identical effort into each contest. This result is a function of his assumptions about resource constraints (he does not explicitly allow for resource constraints, but instead assumes that resources are available in any amount at constant marginal costs), and about candidate behavior (players only care about the number of seats won, so that the value of each "prize" is regarded as identical).

This paper develops a general model of multi-item contests and examines these and other issues in detail. Section 2 develops a model of multi-item contests and analyzes equilibria for the generalized Tullock (1980) functional form. We solve for an characterize the unique pure strategy Nash equilibrium in effort profiles, and also provide some simple examples of multi-item contests. Section 3 analyzes this special case further, by examining the sensitivity

[1] A general version of the Blotto game is described by Blackett (1954). For other analyses of Blotto games, see Karlin (1959), Cooper and Restropo (1967) and Bellman (1968).

[2] Cooper (1993) provides an excellent summary of the market-share attraction literature. Friedman's basic model and approach is also briefly discussed in Luenberger (1968) and Ordeshook (1986). Interestingly, Friedman and later marketing scholars use the simple ratio functional form popularized much later by Tullock (1980) in the rent seeking literature. However, Friedman's paper and the marketing literature appears to have been virtually ignored by researchers studying rent-seeking behavior. The exceptions are Nti (1997) and Snyder (1989).

[3] Similar ideas are explored by Brams and Davis (1974) and Colantoni et al (1975).

of the equilibrium efforts to changes in parameter values. Both sections illustrate one of the main results of the paper – the equilibrium expenditures in multi-item contests exhibit strong interdependence among the various contests, in the sense that changes in the value of the prize or parameters in one contest alters the resources devoted to *all* contests. Section 4 introduces a more general functional form for the contest success function, which allows for the possibility that both players might, in equilibrium, put no effort into a subset of contests. We provide a set of sufficient conditions under which this outcome can occur, and present an example to illustrate the main ideas behind these conditions. Section 5 concludes.

2 A General Model of Multi-Item Contests

Suppose that there are two players labelled x and y with resource endowments $\mathcal{R}_x > 0$ and $\mathcal{R}_y > 0$. There is a finite set $\mathcal{I}$ (where $\mathcal{I}$ is of size $n \geq 2$) of contests. Label these contests $i = 1, \ldots, n$, and let the value of the prize in each contest i be $V_i > 0$ to both players. Player x has the utility function u_x and player y's utility function is u_y. Players devote resources $x_i \geq 0$ and $y_i \geq 0$ to each contest i to influence the probability of obtaining each prize. Resource endowments cannot be consumed. At the outset, we must make some assumption about the probability of obtaining more than one prize. Let $p_i(x_i, y_i)$ be the marginal probability of obtaining the prize in contest i, and let p_S be the marginal probability of obtaining all prizes in some subset $S \subseteq \mathcal{I}$. We follow Snyder in assuming that, for any allocations $\{(x_i, y_i) : i = 1, \ldots, n\}$ the winning probabilities for player x are *mutually statistically independent*:

$$p_S = \prod_{i \in S} p_i \text{ for any } S \subseteq \mathcal{I}$$

The assumption of mutual statistical independence means that player x's probability of winning in contest i, conditional on the outcome in other contests, does not depend *directly* on the winning probabilities in other contests or efforts devoted to those other contests. It is easy to imagine situations in which this assumption may not hold and where there may be some correlation between success or failure in different contests. However, in what follows, because both players face a resource constraint, devoting greater resources to other contests $j \neq i$ leaves fewer resources available for contest i, and so in equilibrium there will be *indirect* effects between outcomes in different contests. In other words, in this paper interdependence between contests works via the budget constraint and emerges as a result of equilibrium behavior, rather than as a result of any assumption about the statistical dependence of the contest probabilities.

Next, we assume a functional form for each p_i. We follow most of the literature here (and, in particular Skaperdas, 1996) and assume that the contest success functions (CSF) in each contest i are:

$$p_i(x_i, y_i) \equiv \begin{cases} \alpha_i x_i^{r_i} / [\alpha_i x_i^{r_i} + y^{r_i}] & \text{if} \quad x_i \text{ and } y_i > 0 \\ \alpha_i / (\alpha_i + 1) & \text{if} \quad x_i + y_i = 0 \end{cases} \tag{1}$$

where $0 < r_i \leq 1$, and where $\alpha_i > 0$ is now a parameter reflecting the possibility that player x has an advantage or disadvantage in particular contests or over a subset of contests.[4] For example, in political campaigns we often observe the phenomenon that parties have intrinsic advantages in some geographical areas and disadvantages in others, or that for historical reasons the parties' platforms naturally appeal to certain classes of voters according to economic status, union membership, gender, racial attributes and so on.

We assume that in the event a player is not successful in a particular contest, he receives a payoff of zero. The expected utility of player x is then:

$$\begin{aligned} EU_x &= \prod_{i=1}^{n} p_i.u_x\left(\sum_{i=1}^{n} V_i\right) + \sum_{i=1}^{n}\left[(1-p_i)\prod_{j\neq i}^{n} p_j u_x\left(\sum_{j\neq i}^{n} V_j\right)\right] \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}(1-p_i)(1-p_j)\prod_{k\neq i,j}^{n} p_k.u_x\left(\sum_{k\neq i,j}^{n} V_k\right) + \ldots + \prod_{i=1}^{n}(1-p_i).u_x(0) \end{aligned} \quad (2)$$

and player y's expected utility is:

$$\begin{aligned} EU_y &= \prod_{i=1}^{n} p_i.u_y(0) + \sum_{i=1}^{n}\left[(1-p_i).\prod_{j\neq i}^{n} p_j.u_y(V_i)\right] \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}(1-p_i)(1-p_j)\prod_{k\neq i,j}^{n} p_k.u_y(V_i+V_j) + \ldots + \prod_{i=1}^{n}(1-p_i).u_y\left(\sum_{i=1}^{n} V_i\right) \end{aligned} \quad (3)$$

We further assume that the agents are risk neutral, leading to the following result, which allows us to considerably simplify the payoff functions and the remaining analysis.

Proposition 1 *If the contest probabilities are mutually statistically independent and if both agents are risk neutral, the expected payoffs are:*

$$EU_x = \sum_{i=1}^{n} p_i V_i$$

for player x and

$$EU_y = \sum_{i=1}^{n} (1-p_i) V_i$$

for player y.

Proof. All proofs are contained in the Appendix. ■

[4] Skaperdas (1996) shows that the functional form in equation (1) is satisfied if and only if certain behavioral axioms hold. Interestingly, the "if" part of Skaperdas' result was proved much earlier in the marketing literature by Bell (1975).

The resource constraints for each player are:

$$\mathcal{R}_x = \sum_{i=1}^{n} x_i \ , \ \mathcal{R}_y = \sum_{i=1}^{n} y_i \tag{4}$$

An effort profile $(x, y) \equiv \{x_i, y_i\}_{i=1}^n$ is *feasible* if $\sum_{i=1}^n x_i = \mathcal{R}_x$ and $\sum_{i=1}^n y_i = \mathcal{R}_y$. A *feasible pure strategy Nash equilibrium* is a collection of feasible effort profiles $(x^*, y^*) \equiv \{x_i^*, y_i^*\}_{i=1}^n$ such that $EU_x(x^*, y^*) \geq EU_x(x, y^*)$ for all $x \neq x^*$ and $EU_y(x^*, y^*) \geq EU_y(x, y^*)$ for all $y \neq y^*$.[5]

In a feasible Nash equilibrium $\{x_i^*, y_i^*\}_{i=1}^n$, player x solves:

$$\begin{aligned} & \max_{\{x_i\}_{i=1}^n} \sum_{i=1}^{n} \frac{\alpha_i x_i^{r_i}}{\alpha_i x_i^{r_i} + (y_i^*)^{r_i}} V_i \\ \text{subject to:} \quad & \mathcal{R}_x = \sum_{i=1}^{n} x_i \end{aligned}$$

and player y solves:

$$\begin{aligned} & \max_{\{y_i\}_{i=1}^n} \sum_{i=1}^{n} \frac{y_i^{r_i}}{\alpha_i (x_i^*)^{r_i} + y_i^{r_i}} V_i \\ \text{subject to:} \quad & \mathcal{R}_y = \sum_{i=1}^{n} y_i \end{aligned}$$

2.1 Equilibrium Conditions, Efforts and Payoffs

For player x, the first order conditions are:

$$\frac{\alpha_i r_i (x_i^*)^{r_i - 1} (y_i^*)^{r_i} V_i}{\left[\alpha_i (x_i^*)^{r_i} + (y_i^*)^{r_i}\right]^2} = \lambda_x \quad \forall \quad i \tag{5}$$

and for player y they are:

$$\frac{\alpha_i r_i (y_i^*)^{r_i - 1} (x_i^*)^{r_i} V_i}{\left[\alpha_i (x_i^*)^{r_i} + (y_i^*)^{r_i}\right]^2} = \lambda_y \quad \forall \quad i \tag{6}$$

where λ_x and λ_y are the multipliers on the resource constraints for each player. Dividing equations (5) and (6) gives:

$$\frac{y_i^*}{x_i^*} = \frac{\lambda_x}{\lambda_y} \quad \forall \quad i$$

or:

$$y_i^* = \frac{\lambda_x}{\lambda_y} x_i^* \quad \forall \quad i$$

[5] It is straighforward to show that in this setup, players never leave any resources unused, so the assumption that $\sum_{i=1}^n x_i = \mathcal{R}_x$ and $\sum_{i=1}^n y_i = \mathcal{R}_y$ involves no loss of generality.

Summing over all i yields:

$$\mathcal{R}_y = \frac{\lambda_x}{\lambda_y}\mathcal{R}_x$$

or:

$$\frac{\lambda_x}{\lambda_y} = \frac{\mathcal{R}_y}{\mathcal{R}_x}$$

Substituting this into the first order conditions (5) for player x yields:

$$\frac{\alpha_i r_i \left(x_i^*\right)^{r_i-1}\left(\dfrac{x_i^*\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i} V_i}{\left[\alpha_i \left(x_i^*\right)^{r_i} + \left(\dfrac{x_i^*\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2} = \lambda_x \quad \forall \quad i$$

This then gives us an expression for the effort x_i of player x in contest i:

$$x_i^* = \frac{V_i\alpha_i r_i \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\lambda_x\left[\alpha_i + \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2} \quad \forall \quad i \tag{7}$$

Similarly, for player y we have:

$$y_i^* = \frac{V_i\alpha_i r_i \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\lambda_y\left[\alpha_i + \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2} \quad \forall \quad i \tag{8}$$

These expressions are incomplete, however, because we have not solved for the multipliers λ_x and λ_y. To this end, adding up equations (7) and (8) over all i and using the resource constraints, we have:

$$\mathcal{R}_x = \sum_{i=1}^{n} x_i^* = \frac{1}{\lambda_x}\sum_{i=1}^{n}\left\{\frac{V_i\alpha_i r_i \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i + \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}\right\} \tag{9}$$

and

$$\mathcal{R}_y = \sum_{i=1}^{n} y_i^* = \frac{1}{\lambda_y}\sum_{i=1}^{n}\left\{\frac{V_i\alpha_i r_i \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i + \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}\right\} \tag{10}$$

We can therefore solve for the multipliers to get:

$$\lambda_x = \frac{1}{\mathcal{R}_x}\sum_{i=1}^{n}\left\{\frac{V_i\alpha_i r_i \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i + \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}\right\} \tag{11}$$

and:

$$\lambda_y = \frac{1}{\mathcal{R}_y} \sum_{i=1}^{n} \left\{ \frac{V_i \alpha_i r_i \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i + \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2} \right\} \tag{12}$$

We therefore have:

Proposition 2 *Suppose that in the two-player, n item contest, each contest success function i takes the functional form in equation (1). Then the feasible equilibrium efforts are:*

$$x_i^* = \mathcal{R}_x \frac{\dfrac{V_i \alpha_i r_i \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i + \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}}{\sum_{i=1}^{n} \left\{ \dfrac{V_i \alpha_i r_i \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i + \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2} \right\}} \quad \forall \ i \tag{13}$$

and

$$y_i^* = \mathcal{R}_y \frac{\dfrac{V_i \alpha_i r_i \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i + \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}}{\sum_{i=1}^{n} \left\{ \dfrac{V_i \alpha_i r_i \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i + \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2} \right\}} \quad \forall \ i \tag{14}$$

Since $y_i^* = \frac{\mathcal{R}_y}{\mathcal{R}_x} x_i^*$ for all i, we have that $y_i^* > x_i^*$ for all i iff $\mathcal{R}_y > \mathcal{R}_x$.[6] An interesting feature of the equilibrium expected payoffs is that, depending on the structure of the α_i's , r_i's and V_i's, it is not necessarily the case that the player with the highest initial resources has the highest equilibrium expected payoffs, even though he puts higher effort into every contest. To see this, note that the equilibrium expected payoffs for each player are :

$$EU_x = \sum_{i=1}^{n} \frac{\alpha_i \left(\mathcal{R}_x\right)^{r_i}}{\alpha_i \left(\mathcal{R}_x\right)^{r_i} + \left(\mathcal{R}_y\right)^{r_i}} V_i \quad \text{and} \quad EU_y = \sum_{i=1}^{n} \frac{\left(\mathcal{R}_y\right)^{r_i}}{\alpha_i \left(\mathcal{R}_x\right)^{r_i} + \left(\mathcal{R}_y\right)^{r_i}} V_i \tag{15}$$

Using the payoffs in equation (15), it is straightforward to derive examples in which this occurs. In other words:

Proposition 3 *There exist situations in which $EU_x < EU_y$, even though $\mathcal{R}_x > \mathcal{R}_y$ and $x_i^* > y_i^*$ for all i.*

[6]This is similar to the result obtained by Snyder (1989) at page 643.

2.2 Some Examples

2.2.1 Example: A Two Player, Two Item Symmetric Contest

Consider the most basic example of a multi-item contest in which there are two players and only two items, and suppose that the players have symmetric contest success functions, so that $\alpha_i = 1 = r_i$ for all i. Then, the expression we derived above indicate that each player's effort on contest i is proportional to their resource endowment and the size of the prize in contest i, relative to the aggregate prize $V \equiv V_1 + V_2$. More precisely, we have:

$$x_i = \mathcal{R}_x \frac{V_i}{V_1 + V_2} \quad i = 1, 2 \tag{16}$$

and

$$y_i = \mathcal{R}_y \frac{V_i}{V_1 + V_2} \quad i = 1, 2 \tag{17}$$

Using these equilibrium expenditures, we can find the equilibrium expected payoffs:

$$EU_x = \frac{\mathcal{R}_x}{\mathcal{R}_x + \mathcal{R}_y} (V_1 + V_2) \tag{18}$$

and

$$EU_y = \frac{\mathcal{R}_y}{\mathcal{R}_x + \mathcal{R}_y} (V_1 + V_2) \tag{19}$$

This example illustrates that in a two player, two item, symmetric contest, the player with the most resources has a higher expected payoff (this is not true in the more general asymmetric case). It is also straightforward to show that in the two player, two item symmetric contest, a player's equilibrium effort in contest i is increasing in the value of the prize in that contest, and decreasing in the value of prizes in other contests.

2.2.2 Example: A Two Player n-Item Symmetric Contest

Using the expressions for the general case, it is also easy to see that multi-item contests (just as we do with single item contests), players efforts in each contest will be proportional to the prize in that contest, as well as depending on each player's resources endowment and the *aggregate* prize $V \equiv \sum_{i=1}^{n} V_i$. More precisely, if $\alpha_i = r_i = 1$ for all i, we have:[7]

$$x_i^* = \mathcal{R}_x \frac{V_i}{V} \text{ and } y_i^* = \mathcal{R}_y \frac{V_i}{V}$$

with:

$$EU_x = \frac{\mathcal{R}_x}{\mathcal{R}_x + \mathcal{R}_y} V$$

and:

$$EU_y = \frac{\mathcal{R}_y}{\mathcal{R}_x + \mathcal{R}_y} V$$

[7] This result is obtained by Friedman (1958), page 702.

2.2.3 Example: A Simple Two Player n-item Asymmetric Contest

Consider the case where player x has an advantage (or disadvantage) in only one jurisdiction i. That is, we allow player i to have an advantage or disadvantage in contest i by altering the contest success function he faces in that contest, while specifying that the contest success functions in all other contests $j \neq i$ are identical and involve no advantages or disadvantages. In other words, we have:

$$p_i = \frac{\alpha x_i}{\alpha x_i + y_i} \text{ and } p_j = \frac{x_j}{x_j + y_j} \quad \text{for} \quad j \neq i \tag{20}$$

where $\alpha > 1$ indicates that player 1 has an advantage in contest i relative to contests $j \neq i$, and $\alpha < 1$ indicates that player 1 has a disadvantage in contest i. We again assume identical resource endowments $\mathcal{R}_x = \mathcal{R}_y = \mathcal{R}$ and risk neutrality. In this case, we have:

$$x_i^* = y_i^* = \frac{\alpha V_i \mathcal{R}}{(\alpha+1)^2 \left(\frac{V - V_i}{4} + \frac{\alpha V_i}{(\alpha+1)^2} \right)} = \frac{4\alpha V_i \mathcal{R}}{(\alpha+1)^2 V - V_i(\alpha-1)^2}$$

We also have:

$$x_j^* = y_j^* = \frac{(\alpha+1)^2 V_j \mathcal{R}}{(\alpha+1)^2 V - V^i(\alpha-1)^2} \; \forall \; j \neq i$$

with:

$$EU_x = \frac{\alpha}{\alpha+1} V_i + \frac{1}{2} \sum_{j \neq i} V_j \text{ and } EU_y = \frac{1}{\alpha+1} V_i + \frac{1}{2} \sum_{j \neq i} V_j$$

3 Comparative Statics

In this section we investigate how changes in the parameters V_i, α_i and r_i alter equilibrium efforts and payoffs.

3.1 Changes in the Values of Each Prize

Let us first examine how equilibrium effort levels vary with V_i, the value of the prize in contest i. Intuition suggests that an increase in the value of the prize in contest i increases the effort devoted to contest i, at the expense of efforts devoted to other contests $j \neq i$. This is straightforward to confirm.

An additional interesting property of the response of equilibrium efforts to changes in the values of other contests is that, in terms of elasticities, the ratio of effect of a change in V_j on x_i^* divided by effect of a change in V_i on x_j^* is simply equal to the inverse of the ratio of the original equilibrium efforts.

Proposition 4 *Suppose that for every i, p_i takes the functional form in equation (1). Then: (i) Equilibrium efforts in contest i are increasing in V_i and decreasing in V_j. In other words, ceteris paribus, both players devote greater effort to more valuable contests, and lower effort*

to less valuable contests.
(ii) Let $\varepsilon_{ij} \equiv \frac{\partial x_i^*}{\partial V_j} \cdot \frac{V_j}{x_i^*}$. *Then equilibrium efforts obey the following symmetry property:*

$$\frac{\varepsilon_{ij}}{\varepsilon_{ji}} = \frac{x_j^*}{x_i^*} \text{ for } i \neq j$$

(iii) For any i*, an increase in* V_i *always increases the expected payoff of both players.*

3.2 Changes in α_i

In the rent seeking literature, α_i is regarded as a measure of the advantage or disadvantage that a player has in a contest, in the following sense: if both players put in equal effort, the equilibrium win probability for player x is $\dfrac{\alpha_i}{\alpha_i + 1}$ and we say that if $\alpha_i > 1$, player x has a natural advantage in that contest or that it is "biased" towards player x. A common result in single item contests is that $\dfrac{\partial x_i}{\partial \alpha_i} > 0$ iff $\alpha_i < 1$. However, in multi-item contests this result only holds if the players have identical resource endowments. Otherwise, the sign of $\dfrac{\partial x_i}{\partial \alpha_i}$ depends on the ratio of the players' resource endowments and on r_i. On the other hand, even though in some instances the equilibrium effort can decrease in a reponse to a marginal increase in α_i, it is clear from the payoffs in (15) that such a change *always* increases the equilibrium payoff of player x and decreases the equilibrium payoff of player y. In other words:

$$\frac{\partial EU_x}{\partial \alpha_i} = \frac{(\mathcal{R}_x \mathcal{R}_y)^{r_i}}{\left[\alpha_i (\mathcal{R}_x)^{r_i} + (\mathcal{R}_y)^{r_i}\right]^2} V_i > 0 \text{ and } \frac{\partial EU_y}{\partial \alpha_i} = -\frac{(\mathcal{R}_x \mathcal{R}_y)^{r_i}}{\left[\alpha_i (\mathcal{R}_x)^{r_i} + (\mathcal{R}_y)^{r_i}\right]^2} V_i < 0 \;\; \forall \;\; i$$

Summarizing, we have:

Proposition 5 *Suppose that for every* i*,* p_i *takes the functional form in equation (1). Then:*
(i) Equilibrium efforts in contest i *are increasing in* α_i *for both players iff* α_i *is sufficiently small. That is :*

$$\frac{\partial x_i}{\partial \alpha_i} > 0 \;\; \text{and} \;\; \frac{\partial y_i}{\partial \alpha_i} > 0 \text{ iff } \alpha_i < \overline{\alpha}_i$$

where $\overline{\alpha}_i = \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}$. *Conversely, equilibrium efforts for are decreasing in* α_i *for both players iff* α_i *is sufficiently large. The critical value for changes in* α_i *to have these effects depends positively on* $\mathcal{R}_y$*, negatively on* $\mathcal{R}_x$*, and ambiguously on* r_i*.*
(ii) Equilibrium efforts in contest i *are increasing in* α_j *for both players iff* α_j *is sufficiently large. That is,*

$$\frac{\partial x_i}{\partial \alpha_j} > 0 \text{ and } \frac{\partial y_i}{\partial \alpha_j} > 0 \;\; \text{iff } \alpha_j > \overline{\alpha}_j$$

where where $\overline{\alpha}_j = \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_j}$.
(iii) For any contest i*, an increase in* α_i *always helps player* x *and hurts player* y*.*

3.3 Changes in r_i

Changes in r_i can also alter the equilibrium efforts and payoffs, although the effect is unclear unless we make further assumptions.

Proposition 6 *Suppose that for every i, p_i takes the functional form in equation (1). Then (i) In general, the signs of both $\frac{\partial x_i^*}{\partial r_i}$ and $\frac{\partial x_i^*}{\partial r_j}$ are indeterminate, and depend on the values of $\mathcal{R}_y, \mathcal{R}_x$, α_i and r_i.*
(ii)However, if the players have identical resource endowments, we always have:

$$\frac{\partial x_i}{\partial r_i} = \frac{\partial y_i}{\partial r_i} = \alpha_i + 1 > 0$$

and:

$$\frac{\partial x_i}{\partial r_j} = \frac{\partial y_i}{\partial r_j} = -\alpha_j - 1 < 0$$

(iii) For any contest i, a marginal increase in r_i helps player x and hurts player y if and only if $\mathcal{R}_x > \mathcal{R}_y$. If players have identical resource endowments, changes in r_i alter the equilibrium efforts in that and other contests, but have no effect on either player's overall expected payoff.

It is interesting to compare and contrast our results to those obtained by Snyder (1989), who models the behavior of two parties engaged in competition over many legislative seats in fashion slightly similar to ours. In Snyder's model, players are assumed to maximize the expected number of seats won, or maximize the probability of winning a majority of seats. Under the first behavioral assumption, Snyder obtains the result that players will put identical effort into each contest. This result is a function of his assumptions about resource constraints (he does not explicitly allow for resource constraints), and about candidate behavior (players only care about the *number* of seats won, so that the value of all seats can be regarded as being identical).

Snyder's results are therefore a special case of our more general results, where we have allowed for the possibility that different contests may be regarded by the players as having different values. This seems to be an attractive assumption in most multi-item contest settings, but is also especially appealing in political contests, where we often observe contestants placing more resources into electorates with greater returns in terms of political wealth, or to issues which are regarded as more important in terms of aggregate outcomes, such as purely economic issues, defense and foreign policy, or the environment. The result also implies the obvious conclusion that participants in political contests will devote more campaign resources to seats which are more politically prestigious, and which promise greater political and financial wealth to the winner.

4 Non-Participation in Multi-Item Contests

The results of the previous sections were derived for the case where the players put strictly positive effort into every contest; the primitives of our model and the functional form of the contest success function imposed this on our solution. In richer settings, strictly positive effort may not be a satisfactory prediction. Therefore, in this section we examine possible circumstances under which, in equilibrium, some players might not put any effort into a particular contest.[8]

4.1 Modifying the CSF

We consider a simple modification of the contest success functions. Note that with the functional form used in the previous section, the marginal benefit of effort for each player becomes unbounded as x_i goes to zero. That is, for every y_i,

$$\lim_{x_i \to 0} \frac{\partial p^i}{\partial x_i} = \lim_{x_i \to 0} \frac{\alpha_i r_i x_i^{r_i - 1} y_i^{r_i} V_i}{[\alpha_i x_i^{r_i} + y^{r_i}]^2} = \lim_{x_i \to 0} \alpha_i r_i V_i y_i^{r_i} \frac{x_i^{r_i - 3}}{\left[\alpha_i + \frac{y^{r_i}}{x_i^{r_i}}\right]^2} = \infty$$

since $r_i \leq 1$. Therefore, $x_i = 0$ can never be an equilibrium strategy in this model, since, for any fixed value of the opponent's effort in a particular contest, the marginal return from putting in the smallest effort is always positive and is unboundedly large for very small efforts. Thus, to get equilibria with $x_i = 0$, we need to make $\dfrac{\partial p^i}{\partial x_i}$ bounded as $x_i \longrightarrow 0$. One functional form where this property holds is:

$$p^i(x_i, y_i) \equiv \frac{(\beta_i + \alpha_i x_i)^{r_i}}{[(\beta_i + \alpha_i x_i)^{r_i} + (\delta_i + y_i)^{r_i}]} \tag{21}$$

where $\beta_i > 0, \delta_i > 0, r_i \in (0, 1]$ and $\alpha_i > 0$ for all i. This functional form has been used, for example, by Skaperdas and Syropoulos (1998), in their study of complementarity on contests.[9]

In what follows, we will restrict attention to *symmetric non-participation equilibria,* which occur when both players either expend no effort in a contest, or they both put in strictly positive effort. Depending on parameter values, other configurations may be possible, but we focus on this class of equilibria to illustrate the main ideas.

Note that with the functional form in equation (21), we have:

$$\frac{\partial p^i}{\partial x_i} = \frac{\alpha_i r_i (\beta_i + \alpha_i x_i)^{r_i - 1} (\delta_i + y_i)^{r_i}}{[(\beta_i + \alpha_i x_i)^{r_i} + (\delta_i + y_i)^{r_i}]^2}$$

and:

$$\begin{aligned} \left.\frac{\partial p^i}{\partial x_i}\right|_{x_i=0} &= \left.\frac{\alpha_i r_i (\beta_i + \alpha_i x_i)^{r_i - 1} (\delta_i + y_i)^{r_i}}{[(\beta_i + \alpha_i x_i)^{r_i} + (\delta_i + y_i)^{r_i}]^2}\right|_{x_i=0} \\ &= \frac{\alpha_i r_i \beta_i^{r_i - 1} (\delta_i + y_i)^{r_i}}{[\beta_i^{r_i} + (\delta_i + y_i)^{r_i}]^2} \end{aligned}$$

[8] Neither Snyder (1989) nor Monahan (1987) allow for the possibility that, in equilibrium, both players might put zero effort into a particular contest or subset of contests.

[9] See Skaperdas and Syropoulos (1998), page 670.

and:

$$\left.\frac{\partial p^i}{\partial x_i}\right|_{x_i=y_i=0} = \frac{\alpha_i r_i \beta_i^{r_i-1}\delta_i^{r_i}}{\left[\beta_i^{r_i}+\delta_i{}^{r_i}\right]^2} < \infty \tag{22}$$

Players again maximize their payoffs, subject to the resource constraints holding with equality. Let us adopt the following notation:

$$\mathcal{I}^+ \equiv \{i \in \mathcal{I} : x_i^* > 0 \text{ and } y_i^* > 0\}$$

That is, $\mathcal{I}^+$ is the set of contests in which both players put in strictly positive effort in equilibrium. A symmetric non-participation equilibrium is one where $\mathcal{I}^+ \neq \mathcal{I}$.

To find conditions under which a symmetric non-participation equilibrium might exist, note that the expression $\frac{\alpha_i r_i \beta_i^{r_i-1}\delta_i^{r_i}}{\left[\beta_i^{r_i}+\delta_i{}^{r_i}\right]^2}V_i$ is the value marginal product of player x's effort in contest i when both his effort and y's effort are zero. Similarly, for player y we have:

$$\frac{\alpha_i r_i \beta_i^{r_i}\delta_i^{r_i-1}}{\left[\beta_i^{r_i}+\delta_i{}^{r_i}\right]^2}V_i < \infty \tag{23}$$

which is the value marginal product of player y's effort in contest i when both his effort and x's effort are zero. Intuitively, if these are *both* sufficiently small, then neither player will be able to profitably put positive effort into these contests, even when the other player is not participating. In other words, both players putting no resources into these contests may be an equilibrium.

This argument is incomplete, however, because interdependence among contests dictates that we need to consider relative returns in other contests $\mathcal{I}^+$, where positive effort is expended in equilibrium. Suppose that there is a symmetric non-participation equilibrium, and suppose we were to endow an additional marginal unit of resources on player x. Then, player x will use these additional resources in $\mathcal{I}^+$ only if the marginal returns to contests in $\mathcal{I}^+$ are larger than marginal returns to contests not in $\mathcal{I}^+$. But we know that for player x, the marginal returns to contests in $\mathcal{I}^+$ are given by the Lagrange multipliers on the resource constraints, while the marginal returns to contests not in $\mathcal{I}^+$ are given by equations (22) and (23). The search for a sufficient condition for the existence of a symmetric non-participation equilibrium boils down to imposing conditions on these marginal returns:

Proposition 7 *Suppose that, in a two player contest with contest success function given by equation (21), there exists a subset of contests $\mathcal{I}^+ \subset \mathcal{I}$ such that:*

$$\frac{1}{\mathcal{R}_x + \mathcal{B}_\alpha^+}\sum_{i\in\mathcal{I}^+}\frac{\alpha_i^{r_i} r_i \mathcal{R}^{r_i} V_i}{\left(\alpha_i^{r_i}\mathcal{R}^{r_i}+1\right)^2} \geq \max_{j\notin\mathcal{I}^+}\frac{\alpha_j r_j \beta_j^{r_j-1}\delta_j^{r_j}}{\left[\beta_j^{r_j}+\delta_j^{r_j}\right]^2}V_j \tag{24}$$

and

$$\frac{1}{\mathcal{R}_y + \mathcal{D}^+}\sum_{i\in\mathcal{I}^+}\frac{\alpha_i^{r_i} r_i \mathcal{R}^{r_i} V_i}{\left(\alpha_i^{r_i}\mathcal{R}^{r_i}+1\right)^2} \geq \max_{j\notin\mathcal{I}^+}\frac{r_j \beta_j^{r_j}\delta_j^{r_j-1}}{\left[\beta_j^{r_j}+\delta_j^{r_j}\right]^2}V_j \tag{25}$$

where $\mathcal{D}^+ \equiv \sum_{i\in\mathcal{I}^+}\delta_i$, $\mathcal{B}^+_\alpha \equiv \sum_{i\in\mathcal{I}^+}\dfrac{\beta_i}{\alpha_i}$ *and* $\mathcal{R} \equiv \dfrac{\mathcal{R}_x+\mathcal{B}^+_\alpha}{\mathcal{R}_y+\mathcal{D}^+}$.. *Then, for all contests* $j \notin \mathcal{I}^+$ *the Nash equilibrium effort levels are* $x^*_j = y^*_j = 0$, *and for all* $i \in \mathcal{I}^+$, *the Nash equilibrium effort levels are strictly positive, with:*

$$x^*_i = \left(\mathcal{R}_x + \mathcal{B}^+_\alpha\right)\frac{\dfrac{\alpha_i^{r_i} r_i \mathcal{R}^{r_i} V_i}{\left(\alpha_i^{r_i}\mathcal{R}^{r_i}+1\right)^2}}{\sum_{i\in\mathcal{I}^+}\dfrac{\alpha_i^{r_i} r_i \mathcal{R}^{r_i} V_i}{\left(\alpha_i^{r_i}\mathcal{R}^{r_i}+1\right)^2}} - \frac{\beta_i}{\alpha_i} > 0 \tag{26}$$

and

$$y^*_i = \left(\mathcal{R}_y + \mathcal{D}^+\right)\frac{\dfrac{\alpha_i^{r_i} r_i \mathcal{R}^{r_i} V_i}{\left(\alpha_i^{r_i}\mathcal{R}^{r_i}+1\right)^2}}{\sum_{i\in\mathcal{I}^+}\dfrac{\alpha_i^{r_i} r_i \mathcal{R}^{r_i} V_i}{\left(\alpha_i^{r_i}\mathcal{R}^{r_i}+1\right)^2}} - \delta_i > 0 \tag{27}$$

This result immediately gives us:

Proposition 8 *Assume the hypothesis of Proposition 7 holds. Then the equilibrium expected payoffs are:*

$$U_x = \sum_{i\in\mathcal{I}^+}\frac{\alpha_i^{r_i}\left(\mathcal{R}_x+\mathcal{B}^+_\alpha\right)^{r_i}}{\alpha_i^{r_i}\left(\mathcal{R}_x+\mathcal{B}^+_\alpha\right)^{r_i}+\left(\mathcal{R}_y+\mathcal{D}^+\right)^{r_i}}V_i + \sum_{j\notin\mathcal{I}^+}\frac{\beta_j^{r_j}}{\beta_j^{r_j}+\delta_j^{r_j}}V_j \tag{28}$$

and

$$U_y = \sum_{i\in\mathcal{I}^+}\frac{\left(\mathcal{R}_y+\mathcal{D}^+\right)^{r_i}}{\alpha_i^{r_i}\left(\mathcal{R}_x+\mathcal{B}^+_\alpha\right)^{r_i}+\left(\mathcal{R}_y+\mathcal{D}^+\right)^{r_i}}V_i + \sum_{j\notin\mathcal{I}^+}\frac{\delta_j^{r_j}}{\beta_j^{r_j}+\delta_j^{r_j}}V_j \tag{29}$$

Examination of the efforts and payoffs in a non-participation equilibrium reveals that marginal changes in the parameters in the contests $j \notin \mathcal{I}^+$ will have no effect on the efforts of either player, although such parameter changes alter the equilibrium payoffs in a way that does not depend on either player's resource endowment. For example, if both players are putting zero effort into contest i in equilibrium and there is a small increase in the value of winning contest i, then both players expected payoff rises, even though both players continue to devote no resources to that contest. On the other hand, if V_i takes a large enough discrete "jump", then both players will immediately devote resources to contest i.

4.1.1 Example: A Two Player, Two Item Symmetric Contest

Let us illustrate the above results with a straightforward example. Suppose that $r_i = 1$, $\alpha_i = 1$, and that $\beta_i = \delta_i$. Then the contest success functions take the form:

$$p_i(x_i, y_i) \equiv \frac{\beta_i + x_i}{\beta_i + x_i + \beta_i + y_i} = \frac{\beta_i + x_i}{2\beta_i + x_i + y_i} \tag{30}$$

where $\beta_i > 0 \; \forall \; i$. We will call β_i the *upfront benefit* to player x or y in contest i, the idea being that if β_i is large enough, then the players may not find it optimal to put in any effort in that contest; in other words, β_i is the automatic contribution that players make to a contest i, without having to spend any resources. Consider first a two item contest, so that $\mathcal{I} = \{1, 2\}$. Suppose further that the resources of both players are identical, so that $\mathcal{R}_x = \mathcal{R}_y = \mathcal{R}$, and consider a candidate Nash equilibrium profile that has the players efforts in contest 1 of $(0, 0)$. Thus, in the terminology of the previous section, $\mathcal{I}^+ = \{2\}$. Then, the resource constraints dictate that the effort in contest 2 must be $x_2 = y_2 = \mathcal{R}$. The question is: Is this candidate profile a Nash equilibrium, and under which conditions? We note that the expected payoffs in this candidate equilibrium are:

$$\frac{1}{2}V_1 + \frac{1}{2}V_2 \tag{31}$$

Let us assume that the collection of profiles $\{(0, 0); (\mathcal{R}, \mathcal{R})\}$ is an equilibrium, and consider a deviation by player y of $2dy$ resources from contest 2 into contest 1, where dy is small relative to $\mathcal{R}$. Then, under the assumption that player x sticks to his equilibrium strategy, player y's expected payoffs are now:

$$\frac{2dy + \beta_1}{2dy + 2\beta_1}V_1 + \frac{\mathcal{R} - 2dy + \beta_2}{\mathcal{R} - 2dy + \mathcal{R} + 2\beta_2}V_2$$

Note that this deviation improves player y's expected payoff iff:

$$\frac{2dy + \beta_1}{2dy + 2\beta_1}V_1 + \frac{\mathcal{R} - 2dy + \beta_2}{\mathcal{R} - 2dy + \mathcal{R} + 2\beta_2}V_2 > \frac{1}{2}V_1 + \frac{1}{2}V_2$$

This inequality holds iff:

$$\frac{V_1}{2}\left(\frac{2dy + \beta_1}{dy + \beta_1} - 1\right) + \frac{V_2}{2}\left(\frac{\mathcal{R} - 2dy + \beta_2}{\mathcal{R} - dy + \beta_2} - 1\right) > 0$$

Alternatively, we can write this inequality as:

$$V_1\left(\frac{dy}{dy + \beta_1}\right) + V_2\left(\frac{-dy}{\mathcal{R} - dy + \beta_2}\right) > 0 \Leftrightarrow V_1\left(\frac{dy}{dy + \beta_1}\right) > V_2\left(\frac{dy}{\mathcal{R} - dy + \beta_2}\right)$$

Since this expression must hold for all $dy > 0$ we have, sending $dy \to 0$, a condition on β_1 for this deviation to be profitable:

$$\frac{V_2}{\mathcal{R} + \beta_2} < \frac{V_1}{\beta_1} \tag{32}$$

Similarly, for player x, $(0, \mathcal{R})$ is a best response to y playing $(0, \mathcal{R})$ if, for some deviation $2dx$,

$$\frac{2dx + \beta_1}{2dx + 2\beta_1}V_1 + \frac{\mathcal{R} - 2dx + \beta_2}{\mathcal{R} - 2dx + \mathcal{R} + 2\beta_2}V_2 > \frac{1}{2}V_1 + \frac{1}{2}V_2 \tag{33}$$

which gives us the same condition as equation (24). Therefore, we have proved the following result, which is a special case of the more general result in Proposition 7:

Proposition 9 *In a two player, two item contest, with CSFs given by equation (30), identical resource constraints, and symmetric up front benefits, the effort profile* $\{(0,0);(\mathcal{R},\mathcal{R})\}$ *is a Nash equilibrium if and only if:*

$$\frac{V_2}{\mathcal{R}+\beta_2} \geq \frac{V_1}{\beta_1} \tag{34}$$

More generally, in a two player, $n \geq 2$ *item contest with identical resource constraints and symmetric up front benefits, the degenerate effort profile*

$$\{(0,0);(0,0);...;(\mathcal{R},\mathcal{R});...;(0,0)\}$$

where $(\mathcal{R},\mathcal{R})$ *is the effort in the* i*th contest, is an equilibrium iff:*

$$\frac{V_i}{\mathcal{R}+\beta_i} \geq \max_{j\neq i} \left\{\frac{V_j}{\beta_j}\right\} \tag{35}$$

The intuition behind this result and Proposition 8 is straightforward: If the upfront benefits in some collection of contests $\mathcal{I}^0$ are sufficiently large, or if the payoffs are sufficiently small, then players do not participate in those contests, preferring to put effort into contests with relatively larger prizes, or with lower upfront benefits, where effort is needed most.

5 Conclusion

This paper examined a class of games – multi-item contests – in which the players competed for many prizes by exerting effort to increase their probability of winning, in contrast to the usual applications and examples of contests players only seek to obtain a single valuable prize. The key result in these games is that, even when outcomes in each contest are assumed to be mutually statistically independent, equilibrium efforts can exhibit strong interdependencies. In general, changes in the contest success function or value of the prize in one contest alters the amount of resources devoted to other all other contests by both players.

In multi-item contests, intrinsic advantages and disadvantages can be altered if relative resource endowments change. Changes in these intrinsic advantages in turn affect the equilibrium allocations of resources in multi-item contests in ways that cannot be captured by analysing each contest in isolation.

We derived further interesting results by altering the CSF, so that each player's marginal benefit of additional effort becomes bounded as effort approaches zero. Altering the CSF in this way can lead to situations in which both candidates put in zero effort in some contests in equilibrium, and where small changes in parameter values do not change this outcome in terms of efforts, even though the expected payoffs change. These results have implications for many situations in economics and political science that involve instances in which players are faced with a decision about how much resources to devote to a collection or sequence of different contests.

Appendix: Proofs of Propositions

Proof of Proposition 1:. The assumption of risk neutrality and mutual statistical independence plays a critical part in allowing us to express the payoffs in such a simple fashion. For any non-empty disjoint subsets S and T, define:

$$p_{S,T} \equiv \Pr\left(\text{player } x \text{ wins all contests in } S \text{ and wins all contests some other disjoint subset } T\ \right)$$

By mutual independence, we have $p_{S,T} = p_S.p_T$. Also, for any subset S of contests, define $V_S \equiv \sum_{i \in S} V_i$. To prove the result, we proceed by induction. First, note that the statement is trivially true for $n = 1$. It is also true for $n = 2$, since in that case the expected payoffs are:

$$EU_x = p_1 p_2 \left(V_1 + V_2\right) + p_1 \left(1 - p_2\right) V_1 + p_2 \left(1 - p_1\right) V_2 + \left(1 - p_1\right)\left(1 - p_2\right).0 = p_1 V_1 + p_2 V_2$$

and:

$$EU_y = p_1 p_2 .0 + p_1 \left(1 - p_2\right) V_2 + p_2 \left(1 - p_1\right) V_1 + \left(1 - p_1\right)\left(1 - p_2\right)\left(V_1 + V_2\right) = \left(1 - p_1\right) V_1 + \left(1 - p_2\right) V_2$$

Now assume that the statement is true for any n. We show that if this holds, it is also true for $n+1$. Let $\mathcal{I}$ be a set of n contests, and, by the induction hypothesis, suppose that $\sum_{S \subset \mathcal{I}} p_S V_S = \sum_{i=1}^{n} p_i V_i$. Now consider an additional contest, labelled $n+1$. Let p_{n+1} be the marginal probability that player x wins in contest $n+1$. The expected payoff for player x is:

$$\begin{aligned}
EU_x &= \sum_{S \subset \mathcal{I}} p_{n+1}.p_S \left[V_{n+1} + V_S\right] + \sum_{S \subset \mathcal{I}} \left(1 - p_{n+1}\right).p_S V_S \\
&\quad + \sum_{S \subset \mathcal{I}} p_{n+1}.\left(1 - p_S\right).V_{n+1} + \sum_{S \subset \mathcal{I}} \left(1 - p_{n+1}\right).\left(1 - p_S\right).0 \\
&= p_{n+1}.V_{n+1}.\sum_{S \subset \mathcal{I}} p_S + p_{n+1}.\sum_{S \subset \mathcal{I}} p_S V_S + \sum_{S \subset \mathcal{I}} p_S V_S \\
&\quad - p_{n+1}.\sum_{S \subset \mathcal{I}} p_S V_S + \sum_{S \subset \mathcal{I}} p_{n+1}.\left(1 - p_S\right).V_{n+1} \\
&= p_{n+1}.V_{n+1}.\sum_{S \subset \mathcal{I}} p_S + \sum_{S \subset \mathcal{I}} p_S V_S + \sum_{S \subset \mathcal{I}} p_{n+1}.\left(1 - p_S\right).V_{n+1} \\
&= p_{n+1}.V_{n+1} + \sum_{S \subset \mathcal{I}} p_S V_S = p_{n+1}.V_{n+1} + \sum_{i=1}^{n} p_i V_i
\end{aligned}$$

where the last equality follows from the induction hypothesis. Therefore $EU_x = \sum_{i=1}^{n} p_i V_i$ for any n. The expression for EU_y can be derived in a similar fashion. ■

Proof of Proposition 4:. For this and all subsequent proofs, let us define the notation:

$$\mathbf{I} = \sum_{i=1}^{n} \left\{ \frac{V_i \alpha_i r_i \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i + \left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2} \right\} > 0$$

and, for any i:

$$\mathbf{J} = \sum_{j \neq i} \left\{ \frac{V_j \alpha_j r_j \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_j}}{\left[\alpha_j + \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_j}\right]^2} \right\} > 0$$

Note that, for any i:

$$\mathbf{I} - \frac{V_i \alpha_i r_i \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i + \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2} = \mathbf{J}$$

Note that:

$$\begin{aligned} \operatorname{sgn} \frac{\partial x_i^*}{\partial V_i} &= \operatorname{sgn} \left[\sum_{j=1}^{n} \left\{ \frac{V_j \alpha_j r_j \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_j}}{\left[\alpha_j + \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_j}\right]^2} \right\} - \frac{V_i \alpha_i r_i \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i + \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2} \right] \\ &= \operatorname{sgn}[\mathbf{J}] > 0 \qquad \forall \;\; i \end{aligned}$$

Also, we have:

$$\operatorname{sgn} \frac{\partial x_i^*}{\partial V_j} = -\frac{V_i \alpha_i r_i \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i + \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2} . \frac{\alpha_j r_j \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_j}}{\left[\alpha_j + \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_j}\right]^2} < 0$$

To prove the third part of the proposition, note that

$$\frac{\partial x_i^*}{\partial V_j} \frac{1}{V_i} = \frac{\partial x_j^*}{\partial V_i} \frac{1}{V_j}$$

so therefore:

$$\varepsilon_{ij} x_i^* = \varepsilon_{ji} x_j^*$$

■

Proof of Proposition 5:. First, note that:

$$\begin{aligned} & \frac{\partial}{\partial \alpha_i} \frac{V_i \alpha_i r_i \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i + \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2} \\ &= V_i r_i \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i} \frac{\left[\alpha_i + \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2 - 2\alpha_i \left[\alpha_i + \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]}{\left[\alpha_i + \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^4} \end{aligned}$$

Then:

$$\frac{\partial x_i}{\partial \alpha_i} = \frac{\dfrac{\partial}{\partial \alpha_i}\left\{\dfrac{V_i\alpha_i r_i\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i+\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}\right\}\mathbf{I} - \dfrac{V_i\alpha_i r_i\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i+\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}\dfrac{\partial}{\partial \alpha_i}\left\{\dfrac{V_i\alpha_i r_i\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i+\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}\right\}}{\mathbf{I}^2}$$

so that

$$\begin{aligned}
\operatorname{sgn}\frac{\partial x_i}{\partial \alpha_i} &= \operatorname{sgn}\frac{\partial}{\partial \alpha_i}\left\{\frac{V_i\alpha_i r_i\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}\right\}\left\{\mathbf{I} - \frac{V_i\alpha_i r_i\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}\right\} \\
&= \operatorname{sgn}\frac{\partial}{\partial \alpha_i}\left\{\frac{V_i\alpha_i r_i\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}\right\}\times\mathbf{J} \\
&= \operatorname{sgn}\frac{\partial}{\partial \alpha_i}\left\{\frac{V_i\alpha_i r_i\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}\right\} \\
&= \operatorname{sgn}\left[\alpha_i+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2 - 2\alpha_i\left[\alpha_i+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right] \\
&= \operatorname{sgn}\left[\alpha_i+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right] - 2\alpha_i = \operatorname{sgn}\left[\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i} - \alpha_i\right]
\end{aligned}$$

The proof of the sign of $\frac{\partial x_i}{\partial \alpha_i}$ follows immediately from the derivatives found above. Clearly the critical value $\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}$ depends positively on $\mathcal{R}_y$ and negatively on $\mathcal{R}_x$. Finally, note that:

$$\frac{\partial}{\partial r_i}\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i} = \log\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}$$

which is positive if $\mathcal{R}_y > \mathcal{R}_x$, and negative otherwise.
For part (ii) of the proposition, note that:

$$\begin{aligned}
\frac{\partial x_i}{\partial \alpha_j} &= \operatorname{sgn} - \left\{\left[\alpha_j+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_j}\right]^2 - 2\alpha_j\left[\alpha_j+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_j}\right]\right\} \\
&= \operatorname{sgn}\left\{2\alpha_j - \left[\alpha_j+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_j}\right]\right\} \\
&= \operatorname{sgn}\left\{\alpha_j - \left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_j}\right\}
\end{aligned}$$

which gives us the required result. ■

Proof of Proposition 6:. We have:

$$
\begin{aligned}
\frac{\partial x_i}{\partial r_i} &= \mathcal{R}_x \frac{\dfrac{\partial}{\partial r_i}\dfrac{V_i\alpha_i r_i\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i+\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}\times \mathbf{I} - \dfrac{V_i\alpha_i r_i\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i+\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}\dfrac{\partial}{\partial r_i}\dfrac{V_i\alpha_i r_i\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i+\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}}{\mathbf{I}^2} \\
&= \mathcal{R}_x \frac{\dfrac{\partial}{\partial r_i}\dfrac{V_i\alpha_i r_i\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i+\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}\times \mathbf{J}}{\mathbf{I}^2}
\end{aligned}
$$

Then:

$$
\begin{aligned}
\operatorname{sgn}\frac{\partial x_i}{\partial r_i} &= \operatorname{sgn}\frac{\partial}{\partial r_i}\frac{V_i\alpha_i r_i\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i+\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2} \\
&= \operatorname{sgn}\frac{\partial}{\partial r_i}\frac{r_i\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i+\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2}
\end{aligned}
$$

To find the sign of this derivative, note that:

$$
\begin{aligned}
&\frac{\partial}{\partial r_i}\frac{r_i\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}}{\left[\alpha_i+\left(\dfrac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2} \\
= \;& \left[\alpha_i+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^{-4}\times\left[\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}+r_i\log\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]\left[\alpha_i+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2 \\
& -\left[\alpha_i+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^{-4}\times 2r_i\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\left[\alpha_i+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]\log\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\operatorname{sgn}\frac{\partial x_i}{\partial r_i} &= \operatorname{sgn}\left[\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}+r_i\log\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]\left[\alpha_i+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]^2\\
&\quad -2r_i\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\left[\alpha_i+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]\log\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\\
&= \operatorname{sgn}\left\{\left[1+r_i\log\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)\right]\left[\alpha_i+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right]-2r_i\log\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\right\}\\
&= \operatorname{sgn}\left\{\alpha_i\left[1+r_i\log\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)\right]+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_i}\left[1-r_i\log\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)\right]\right\}
\end{aligned}$$

and:

$$\begin{aligned}
\frac{\partial x_i^*}{\partial r_j} &= -\operatorname{sgn}\frac{\partial}{\partial r_j}\frac{r_j\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_j}}{\left[\alpha_j+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_j}\right]^2}\\
&= -\operatorname{sgn}\left\{\alpha_j\left[1+r_j\log\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)\right]+\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)^{r_j}\left[1-r_j\log\left(\frac{\mathcal{R}_y}{\mathcal{R}_x}\right)\right]\right\}
\end{aligned}$$

The second part of the proposition can be proved by substituting the condition $\mathcal{R}_x = \mathcal{R}_y$ into these two equations. To prove the third part of the proposition, note that we have:

$$\frac{\partial EU_x}{\partial r_i} = \alpha_i\left(\mathcal{R}_x\mathcal{R}_y\right)^{r_i}\frac{\log\mathcal{R}_x-\log\mathcal{R}_y}{\left[\alpha_i\left(\mathcal{R}_x\right)^{r_i}+\left(\mathcal{R}_y\right)^{r_i}\right]^2}V_i$$

which is positive if and only if $\mathcal{R}_x > \mathcal{R}_y$ and is zero if $\mathcal{R}_x = \mathcal{R}_y$. ■

Proof of Proposition 7:. In the subset $\mathcal{I}^+$ the necessary first order conditions are:

$$\frac{\alpha_i r_i\left(\beta_i+\alpha_i x_i^*\right)^{r_i-1}\left(\delta_i+y_i^*\right)^{r_i}}{\left[\left(\beta_i+\alpha_i x_i^*\right)^{r_i}+\left(\delta_i+y_i^*\right)^{r_i}\right]^2}V_i = \lambda_x \;\forall\; i\in\mathcal{I}^+ \tag{36}$$

and

$$\frac{r_i\left(\beta_i+\alpha_i x_i^*\right)^{r_i}\left(\delta_i+y_i^*\right)^{r_i-1}}{\left[\left(\beta_i+\alpha_i x_i^*\right)^{r_i}+\left(\delta_i+y_i^*\right)^{r_i}\right]^2}V_i = \lambda_y \;\forall\; i\in\mathcal{I}^+ \tag{37}$$

Dividing these two conditions yields:

$$\frac{\alpha_i\left(\delta_i+y_i^*\right)}{\beta_i+\alpha_i x_i^*} = \frac{\lambda_x}{\lambda_y}$$

or

$$\alpha_i x_i^*+\beta_i = \alpha_i\left(\delta_i+y_i^*\right)\frac{\lambda_y}{\lambda_x} \tag{38}$$

or

$$x_i^* = \frac{\lambda_y}{\lambda_x}y_i^*+\frac{\lambda_y}{\lambda_x}\delta_i-\frac{\beta_i}{\alpha_i}$$

Then, adding up this condition over all contests in $\mathcal{I}^+$, we have:

$$\sum_{i\in\mathcal{I}^+} x_i^* = \mathcal{R}_x = \frac{\lambda_y}{\lambda_x}\mathcal{R}_y + \frac{\lambda_y}{\lambda_x}\sum_{i\in\mathcal{I}^+}\delta_i - \sum_{i\in\mathcal{I}^+}\frac{\beta_i}{\alpha_i} \tag{39}$$

Define:

$$\mathcal{D}^+ \equiv \sum_{i\in\mathcal{I}^+}\delta_i \text{ and } \mathcal{B}_\alpha^+ \equiv \sum_{i\in\mathcal{I}^+}\frac{\beta_i}{\alpha_i} \tag{40}$$

Then:

$$\mathcal{R}_x = \frac{\lambda_y}{\lambda_x}\mathcal{D}^+ + \frac{\lambda_y}{\lambda_x}\mathcal{R}_y - \mathcal{B}_\alpha^+ \tag{41}$$

or:

$$\frac{\lambda_y}{\lambda_x} = \frac{\mathcal{R}_x + \mathcal{B}_\alpha^+}{\mathcal{R}_y + \mathcal{D}^+} \tag{42}$$

so then:

$$\alpha_i x_i + \beta_i = \alpha_i\left(\delta_i + y_i\right)\frac{\mathcal{R}_x + \mathcal{B}_\alpha^+}{\mathcal{R}_y + \mathcal{D}^+} \tag{43}$$

Define $\mathcal{R} \equiv \dfrac{\mathcal{R}_x + \mathcal{B}_\alpha^+}{\mathcal{R}_y + \mathcal{D}^+}$. Then substituting (3.45) back into the first order condition (3.37) yields:

$$\frac{\alpha_i r_i\left[\alpha_i\mathcal{R}\left(\delta_i + y_i^*\right)\right]^{r_i-1}\left(\delta_i + y_i^*\right)^{r_i}}{\left\{\left[\alpha_i\mathcal{R}\left(\delta_i + y_i^*\right)\right]^{r_i} + \left(\delta_i + y_i^*\right)^{r_i}\right\}^2}V_i = \lambda_x \tag{44}$$

or:

$$\alpha_i r_i\left(\alpha_i\mathcal{R}\right)^{r_i-1}\left(\delta_i + y_i^*\right)^{2r_i-1}V_i = \lambda_x\left(\alpha_i^{r_i}\mathcal{R}^{r_i} + 1\right)^2\left(\delta_i + y_i^*\right)^{2r_i} \tag{45}$$

so that:

$$y_i^* = \frac{\alpha_i r_i\left(\alpha_i\mathcal{R}\right)^{r_i-1}V_i}{\lambda_x\left(\alpha_i^{r_i}\mathcal{R}^{r_i} + 1\right)^2} - \delta_i = \frac{\alpha_i^{r_i} r_i\mathcal{R}^{r_i}V_i}{\mathcal{R}\lambda_x\left(\alpha_i^{r_i}\mathcal{R}^{r_i} + 1\right)^2} - \delta_i \tag{46}$$

Again, summing over all $i \in I^+$ yields:

$$\mathcal{R}_y + \mathcal{D}^+ = \frac{1}{\mathcal{R}\lambda_x}\sum_{i\in\mathcal{I}^+}\frac{\alpha_i^{r_i} r_i\mathcal{R}^{r_i}V_i}{\left(\alpha_i^{r_i}\mathcal{R}^{r_i} + 1\right)^2} \tag{47}$$

Therefore, the multipliers can be calculated as:

$$\begin{aligned}\lambda_x &= \frac{1}{\mathcal{R}}\frac{1}{\mathcal{R}_y + \mathcal{D}^+}\sum_{i\in\mathcal{I}^+}\frac{\alpha_i^{r_i} r_i\mathcal{R}^{r_i}V_i}{\left(\alpha_i^{r_i}\mathcal{R}^{r_i} + 1\right)^2} \\ &= \frac{1}{\mathcal{R}_x + \mathcal{B}_\alpha^+}\sum_{i\in\mathcal{I}^+}\frac{\alpha_i^{r_i} r_i\mathcal{R}^{r_i}V_i}{\left(\alpha_i^{r_i}\mathcal{R}^{r_i} + 1\right)^2}\end{aligned} \tag{48}$$

and:

$$\lambda_y = \frac{1}{\mathcal{R}_y + \mathcal{D}^+}\sum_{i\in\mathcal{I}^+}\frac{\alpha_i^{r_i} r_i\mathcal{R}^{r_i}V_i}{\left(\alpha_i^{r_i}\mathcal{R}^{r_i} + 1\right)^2} \tag{49}$$

Now, consider contests for which $x_j^* = y_j^* = 0$, with $j \notin \mathcal{I}^+$. In these contests, the necessary first-order Kuhn-Tucker conditions are:

$$\frac{\alpha_j r_j \left(\beta_j + \alpha_j x_j^*\right)^{r_j - 1} \left(\delta_j + y_j^*\right)^{r_j}}{\left[\left(\beta_j + \alpha_j x_j^*\right)^{r_j} + \left(\delta_j + y_j^*\right)^{r_j}\right]^2} V_j \leq \lambda_x \quad , \quad x_j^* = 0 \tag{50}$$

and

$$\frac{r_j \left(\beta_j + \alpha_j x_j^*\right)^{r_j} \left(\delta_j + y_j^*\right)^{r_j - 1}}{\left[\left(\beta_j + \alpha_j x_j^*\right)^{r_j} + \left(\delta_j + y_j^*\right)^{r_j}\right]^2} V_j \leq \lambda_y \quad , \quad y_j^* = 0 \tag{51}$$

where the multipliers λ_x and λ_y are given in equations (3.50) and (3.51). When these first order conditions are evaluated at $x_i = y_i = 0$, they hold if and only if:

$$\frac{\alpha_j r_j \beta_j^{r_j - 1} \delta_j^{r_j}}{\left[\beta_j^{r_j} + \delta_j^{r_j}\right]^2} V_j \leq \lambda_x = \frac{1}{\mathcal{R}_x + \mathcal{B}_\alpha^+} \sum_{i \in \mathcal{I}^+} \frac{\alpha_i^{r_i} r_i \mathcal{R}^{r_i} V_i}{\left(\alpha_i^{r_i} \mathcal{R}^{r_i} + 1\right)^2} \quad \forall \quad j \notin \mathcal{I}^+ \tag{52}$$

and:

$$\frac{r_j \beta_j^{r_j} \delta_j^{r_j - 1}}{\left[\beta_j^{r_j} + \delta_j^{r_j}\right]^2} V_j \leq \lambda_y = \frac{1}{\mathcal{R}_y + \mathcal{D}^+} \sum_{i \in \mathcal{I}^+} \frac{\alpha_i^{r_i} r_i \mathcal{R}^{r_i} V_i}{\left(\alpha_i^{r_i} \mathcal{R}^{r_i} + 1\right)^2} \quad \forall \quad j \notin \mathcal{I}^+ \tag{53}$$

so that:

$$\frac{1}{\mathcal{R}_x + \mathcal{B}_\alpha^+} \sum_{i \in \mathcal{I}^+} \frac{\alpha_i^{r_i} r_i \mathcal{R}^{r_i} V_i}{\left(\alpha_i^{r_i} \mathcal{R}^{r_i} + 1\right)^2} \geq \max_{j \notin \mathcal{I}^+} \frac{\alpha_j r_j \beta_j^{r_j - 1} \delta_j^{r_j}}{\left[\beta_j^{r_j} + \delta_j^{r_j}\right]^2} V_j \tag{54}$$

and:

$$\frac{1}{\mathcal{R}_y + \mathcal{D}^+} \sum_{i \in \mathcal{I}^+} \frac{\alpha_i^{r_i} r_i \mathcal{R}^{r_i} V_i}{\left(\alpha_i^{r_i} \mathcal{R}^{r_i} + 1\right)^2} \geq \max_{j \notin \mathcal{I}^+} \frac{r_j \beta_j^{r_j} \delta_j^{r_j - 1}}{\left[\beta_j^{r_j} + \delta_j^{r_j}\right]^2} V_j \tag{55}$$

■

Proof of Proposition 8:. Note that the expected payoffs can be calculated as follows:

$$\begin{aligned}
U_x &= \sum_{i\in\mathcal{I}^+} p_i(x_i,y_i)V_i + \sum_{j\notin\mathcal{I}^+} p_j\,(0,0)\,V_j \\
&\equiv \sum_{i\in\mathcal{I}^+} \frac{(\beta_i+\alpha_i x_i)^{r_i}}{(\beta_i+\alpha_i x_i)^{r_i}+(\delta_i+y_i)^{r_i}}V_i + \sum_{j\notin\mathcal{I}^+}\frac{\beta_j^{r_j}}{\left[\beta_j^{r_j}+\delta_j^{r_j}\right]}V_j \\
&= \sum_{i\in\mathcal{I}^+} \frac{(\alpha_i\,(\delta_i+y_i)\,\mathcal{R})^{r_i}}{[\alpha_i\,(\delta_i+y_i)\,\mathcal{R}]^{r_i}+(\delta_i+y_i)^{r_i}}V_i + \sum_{j\notin\mathcal{I}^+}\frac{\beta_j^{r_j}}{\left[\beta_j^{r_j}+\delta_j^{r_j}\right]}V_j \\
&= \sum_{i\in\mathcal{I}^+} \frac{(\alpha_i\mathcal{R})^{r_i}}{(\alpha_i\mathcal{R})^{r_i}+1}V_i + \sum_{j\notin\mathcal{I}^+}\frac{\beta_j^{r_j}}{\left[\beta_j^{r_j}+\delta_j^{r_j}\right]}V_j \\
&= \sum_{i\in\mathcal{I}^+} \frac{\alpha_i^{r_i}\left(\dfrac{\mathcal{R}_x+\mathcal{B}_\alpha^+}{\mathcal{R}_y+\mathcal{D}^+}\right)^{r_i}}{\alpha_i^{r_i}\left(\dfrac{\mathcal{R}_x+\mathcal{B}_\alpha^+}{\mathcal{R}_y+\mathcal{D}^+}\right)^{r_i}+1}V_i + \sum_{j\notin\mathcal{I}^+}\frac{\beta_j^{r_j}}{\left[\beta_j^{r_j}+\delta_j^{r_j}\right]}V_j \\
&= \sum_{i\in\mathcal{I}^+} \frac{\alpha_i^{r_i}\,(\mathcal{R}_x+\mathcal{B}_\alpha^+)^{r_i}}{\alpha_i^{r_i}\,(\mathcal{R}_x+\mathcal{B}_\alpha^+)^{r_i}+(\mathcal{R}_y+\mathcal{D}^+)^{r_i}}V_i + \sum_{j\notin\mathcal{I}^+}\frac{\beta_j^{r_j}}{\left[\beta_j^{r_j}+\delta_j^{r_j}\right]}V_j \qquad (56)
\end{aligned}$$

Similar calculations show that :

$$U_y = \sum_{i\in\mathcal{I}^+} \frac{(\mathcal{R}_y+\mathcal{D}^+)^{r_i}}{\alpha_i^{r_i}\,(\mathcal{R}_x+\mathcal{B}_\alpha^+)^{r_i}+(\mathcal{R}_y+\mathcal{D}^+)^{r_i}}V_i + \sum_{j\notin\mathcal{I}^+}\frac{\delta_j^{r_j}}{\beta_j^{r_j}+\delta_j^{r_j}}V_j \qquad (57)$$

■

6 References

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