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POPULATION AND ENDOGENOUS GROWTH

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Using a general three sector growth model, this paper derives general conditions for positive growth in the economy along a balanced growth path under the alternative assumptions of a static population and a growing population. The framework is general enough to replicate endogenous and semi-endogenous R&D based growth models. This paper challenges the conventional wisdom that (non-)linearity is synonymous with (semi-)endogenous growth. CES technology is introduced to human capital accumulation to obtain positive balanced growth with or without population growth.

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Keywords: (Semi-)endogenous growth; linearity; population; R&D; CES

JEL classification: O30, O41

1 Introduction

Total population of the OECD today may be one and half times what it was in 1950, but it is expected to be static for the next fifty years (United Nations 2005). This projection takes into account immigration and moderate fertility assumptions. In fact, depopulation is anticipated if fertility remains constant. The prospect of zero population growth in the world's hub of research and development (R&D) has generated a flurry of R&D based growth models establishing feasibility of long run economic growth in the absence of population growth.

This new branch of the literature is largely derived from semi-endogenous growth models that assume strictly positive population growth. The precedent Romer (1990) type models treat population as an exogenous constant and would therefore seem pertinent to the current theoretical challenge. However, such models have been overlooked because they typically assume linearity in the accumulation of knowledge. *Linearity* implies that the

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output of new knowledge will double whenever we double the existing stock of knowledge. Semi-endogenous growth models feature the more palatable assumption of diminishing marginal returns to knowledge. This paper examines the tenability of this association between (non-)linearity and (semi-)endogenous growth.

The objective of this paper is two fold. First, to establish general conditions for positive growth in output per capita along a balanced growth path under the alternative assumptions of a static population and a growing population. Second, to construct a specific model that delivers long run growth in the economy with or without population growth, using the most realistic application of these general conditions.

Since Romer's (1990) seminal paper, models of R&D-based growth have become increasingly sophisticated. A recent paper (Strulik 2005) comprises 48 equations, not including those contained in the appendix. The new breed of semi-endogenous growth models typically comprise two aspects of R&D or endogenous fertility. In a decentralized setting, the assumptions that are critical to positive and balanced growth are obscured by the intricacy of these models. Often, simplifying assumptions that make these models tractable, for example, the absence of physical capital, are costly in terms of realism. We overcome such difficulties by abstracting from the microeconomic foundations of R&D and modelling the decision making of a central planner.

This paper introduces a general model comprising three sectors (final output, R&D and human capital accumulation) in order to prove the following assertions. First, in any model, positive growth along a balanced growth path requires restrictions in terms of a matrix of structural elasticities. Second, if strictly positive population growth is assumed, the notions of diminishing marginal returns and semi-endogenous growth are logically independent. Third, if zero population growth is assumed, linearity in the accumulation of knowledge is not necessary for endogenous growth.

Interested in contributing to the literature, rather than utilizing existing assumptions that deliver perpetual growth with a static population, the general model in this paper assumes one aspect of R&D, exogenous population growth and allows for all inputs to be productive in all sectors. The general model also allows for heterogenous labor, an assumption which has been absent from the literature since Romer (1990).

Previous generalized models (Eicher & Turnovsky (1999), Christiaans (2004) and Steger (2005)) comprise two sectors, final output and R&D. By including a third sector, human capital accumulation, we obtain richer results. More importantly, this is the first generalized model that allows for either a growing or static population. In doing so, this paper provides a simple, unified treatment of endogenous and semi-endogenous growth models. This paper further contributes by exploring asymptotic linearity in either R&D or human capital accumulation as a general condition for endogenous growth.

The benefits of this work are manifold. This paper contributes to the literature both by establishing conditions for positive economic growth with or without population growth in a general three sector growth model and by constructing a specific model where Constant Elasticity of Substitution (CES) technology describes the accumulation of human capital.

This paper also provides a useful framework for summarizing the two main strands of the literature and a methodology for obtaining central planner solutions for endogenous and semi-endogenous growth models alike.

2   Background

Let y denote income per capita, L the total population, H the stock of human capital and let g_x denote the long run growth rate of any variable x. We classify models of R&D-based growth into two broad types: those that assume population is an exogenous constant and those that allow for population to grow over time.

Models of the first type, exemplified by Romer (1990), Aghion & Howitt (1992) and Grossman & Helpman (1991), are widely condemned for their scale effect: long run per capita output growth is proportional to population size. The implication that the growth rate of the economy will rise exponentially over time should population grow at a constant rate is not supported by empirical evidence. It is, in fact, a slight misrepresentation of Romer (1990) and Aghion & Howitt (1992) to say that g_y is proportional to L. By allowing for heterogeneous labor, both predict:

$$g_y = a.H \tag{1a}$$

where all constant parameters are summarized by the term, $a > 0$. Thus, long run growth per capita growth is proportional to the skill employed in R&D. However, to the extent that the skill is embodied in the population, long run growth of the economy is still proportional to the size of the population.

Regardless, presenting the prediction in its original form shows how existing models are derivative of these seminal models of R&D-based growth. The stock of labor in these models can be homogenized into either the stock of human capital or total population. Most literature stems from the latter assumption. However, an example of a model that assumes the former is Funke & Strulik (2000). They retain the assumption that population is an exogenous constant and are therefore a first-type model. By endogenizing the accumulation of human capital, they remove the empirically inconsistent scale effect from the long run growth rate of the economy:

$$g_y = a \tag{1b}$$

All these first type models share the common feature of sectoral linearity in a knowledge accumulation equation, whether knowledge be non-rivalrous ideas or rivalrous human capital. And so, *sectoral linearity* has become synonymous with *endogenous growth* models that treat population as an exogenous constant. Like the scale effect, linearity in the accumulation of knowledge is widely condemned. Jones (2001) argues that, with the exception of the population equation, the assumption of linearity is ad hoc. This brings

us to R&D-based growth models of the second type, that introduce a linear population equation.

Early examples of second-type models are Jones (1995), Kortum (1997) and Segerstrom (1998). Their common feature is diminishing marginal returns to ideas (or knowledge spillovers of degree less than one) in the creation of new ideas. Diminishing marginal returns in the stock of ideas requires increasing effort to create an idea. This increasing effort can come from more researchers. Since the fraction of the labor force engaged in R&D is constant in steady state, strictly positive population growth satisfies the increasing efforts needed for strictly positive growth in technology and the overall economy. This is the intuition behind semi-endogenous growth. Jones (1995) coined the phrase, which basically means technological change is endogenously determined, but long run growth in the economy requires growth in a factor exogenous to the model, population. And so, *diminishing marginal returns* to knowledge has become synonymous with *semi-endogenous growth*.

Population growth is the engine of long run economic growth in these models:

$$g_y = cg_L \tag{1c}$$

where all constant parameters are summarized by the multiplicative term $c > 0$. On the flip side, long growth of the economy is inextricably dependent on population growth. To establish feasibility of long run economic growth in the absence of population growth, recent literature adapts second-type models. Two main branches have emerged.

The first new branch of second type models assume two aspects of R&D in the one model. Examples are Young (1998), Dinopolous & Thompson (1998), Peretto (1998) and Li (2000). In brief, R&D may involve either the creation of new products, so that technological improvement is measured by increased variety of intermediate goods (Romer 1990) or the improvement of existing products as in Aghion & Howitt's (1992) quality-ladder model. We refer to these two aspects as simply variety R&D and quality R&D. Li (2000) shows that if there are no knowledge spillovers in variety R&D and spillovers of degree one (or linearity) in quality R&D then the long run growth rate of the economy is an additively separable function of population growth and a constant term:

$$g_y = b + cg_L \tag{1d}$$

where all constant parameters are summarized in the terms b and c. The absence of knowledge spillovers in variety R&D implies a one-to-one correspondence between variety growth and population growth. This explains the second term of equation (1d). If population is static, the variety of intermediate goods stays constant. However, endogenous technological change is still possible through improving existing products, since linearity in quality R&D implies quality growth is proportional to the population size. This explains the first term of equation (1d). Consequently, the long run growth rate of the economy can be strictly positive without strictly positive population growth. To obtain this result,

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note that these models move away from diminishing marginal returns to knowledge in the form of quality improvement. Thus, the strong association between diminishing marginal returns and semi-endogenous growth prevails.

The second branch of second type R&D-based growth models endogenize either population (Jones 2001) or human capital (as in, Strulik (2005) and Dalgaard & Kreiner (2001)) or both (Galor & Weil 2000). Just as Funke & Strulik (2000) removes the "strong" scale effect from the early endogenous growth models, these models remove the "weak" scale effect from semi-endogenous growth models by endogenizing the culpable variable. They predict a long run rate of growth in the economy:

$$g_y = d \tag{1e}$$

where d is a constant term summarizing, for example, exogenous efficiency parameters. In a decentralized setting, these models are intricate. Simplifying assumptions prevent the models from being unwieldy. Examples of such assumptions are the absence of physical capital in final production in Dalgaard & Kreiner (2001) and a reduced form specification of R&D in Galor & Weil (2000). There is a trade-off between sophistication and realism.

Thus, to establish feasibility of long run growth in the economy in the absence of population growth, existing literature extends semi-endogenous theory by either modelling two aspects of R&D or endogenizing fertility. To explore the reasoning behind founding this development in semi-endogenous growth theory, this paper establishes conditions for perpetual growth in a generalized setting. To be inclusive of the early endogenous growth theory, we allow for population growth to be either zero or strictly positive. Interested in exploring new ways to establish long run growth in the economy without population growth, we assume one aspect of R&D and exogenize population growth.

3   A General Three Sector Growth Model

The model is general in four aspects: Firstly, two types of labor, skilled and unskilled, accumulate.[1] The allowance for both types of labor, albeit as exogenous constants, appears in Romer (1990), but has been absent from the R&D based growth literature since. Secondly, the economy consists of three sectors (final goods, the accumulation of ideas and the accumulation of human capital) enabling us to replicate the features of a wide variety of R&D-based growth models. Only the accumulation of physical labor alone is exogenized. Thirdly, each factor of production is allowed to be productive in each sector. Finally, non-parameterized general production functions are employed. Restrictions on parameters and functional forms are introduced only when necessary.

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[1]The assumption of heterogenous labor confers realism to the model. Explored in a follow on paper, a secondary motivation for this assumption is the possibility of rising research intensity along a balanced growth path.

We model the decision making of a central planner over our three generalized sectors. In doing so we abstract from issues related to the microfoundations of R&D-based growth models, such as household decision making, the patenting of ideas, monopoly power in the intermediate goods sector and perfect competition in final goods sector. In the words of Eicher & Turnovsky (1999) (p.397),

> We make these abstractions, not because we feel that such issues are unimportant, but to facilitate the identification of the characteristics common to alternative approaches.

All the models presented in this paper, whether original or central planner versions of existing models, can be given microfoundations, and in each case the equilibrium growth rates in the corresponding decentralized economy can be derived. It is worth noting that growth rates derived for a corresponding decentralized economy differ only by the absence of terms, such as a monopoly markup, that capture the negative spillovers that a central planner internalizes.

The economy produces two goods, final output (Y) and change in technology (the stock of which is denoted by A), and accumulates stocks of physical capital (K), human capital (H) and physical labor (L). H is measured by total, not average, years of education attained by a pool of workers, so that L is measured by a count of people in the labor force.[2] Alternatively, H could refer to the number of skilled workers and L to the number of unskilled workers. Under either interpretation, H can vary separately from L and replicating a given pool of workers requires that both H and L double.

Consider the following general three-sector production structure:

$$
\begin{aligned}
Y &= F\left(a_Y A, h_Y H, l_Y L, k_Y K\right) & \text{(2a)} \\
\dot{A} &= J\left(a_A A, h_A H, l_A L, k_A K\right) & \text{(2b)} \\
\dot{H} &= Q\left(a_H A, h_H H, l_H L, k_H K\right) & \text{(2c)}
\end{aligned}
$$

where the x_i ($x = a, h, l, k;\quad i = Y, A, H, L$) assume values to reflect general assumptions, that are both intuitive and standard in endogenous growth models. We start with the broad assumption that, with the exception of physical labor, which is used only in human reproduction and the production of final output, inputs may be productive in all sectors. If $\sum_i x_i = 1 \forall i$, the respective input is private. If $x_i = 1 \forall i$, the input is non-rivalrous in use. Thus, we distinguish rivalrous private knowledge (H) from non-rivalrous knowledge (A). Letting h denote the average skill level, we note that $H = hL$. Finally, since physical labor is non-rivalrous in its employment in final production and human reproduction, let l_i denote the portion of human capital (or equivalently, the portion of labor with a given average skill level) allocated to sector i.

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[2]Physical labor can be thought of as brawn and basic skills that do not need to be taught, such as, eye-hand co-ordination.

Assume $\dot{L} = nL$, where $n \geq 0$. By allowing for either a growing or static population, our generalized framework can be used to analyze the two main branches of existing R&D-based literature.

After imposing the general assumptions, the generalized production structure simplifies to:

$$Y = F\left(A, (1 - l_A - l_H)H, L, (1 - k_A - k_H)K\right) \tag{3a}$$

$$\dot{A} = J\left(A, l_A H, k_A K\right) \tag{3b}$$

$$\dot{H} = Q\left(A, l_H H, k_H K\right) \tag{3c}$$

The representative agent of the economy derives utility solely from the consumption of the final output good, so their preferences are used only to derive the Euler equation. As is standard in the existing literature, the representative agent of the economy has intertemporal utility of isoelastic form:

$$\int_0^\infty e^{-\rho t} \frac{c^{1-\theta}}{1-\theta} dt \quad \rho > 0; \theta > 0$$

where c denotes consumption per capita, to be replaced by aggregate consumption, C, when $n = 0$. In the absence of depreciation, physical capital accumulates as a residual after aggregate consumption needs have been met:

$$\dot{K} = Y - C \tag{3d}$$

The central planner chooses consumption, and the fractions of labor and capital employed in each sector so as to maximize intertemporal utility of the representative agent subject to the production and accumulation constraints, equations (3a) - (3d). For the purposes of this paper, we note that the following discussion is premised on sectoral allocations of factors that are strictly positive and constant, as required for balanced growth.[3]

4 Balanced Growth Equilibrium

Definition 1 *A balanced growth path is a path along which all real variables grow at constant, though not necessarily equal, rates.*

The balanced growth rates of the real variables (Y, K, A, H) are obtained by total differentiation of the production functions (3a) - (3c), noting that constant growth rates requires $g_Y = g_K$,[4] $g_{\dot{A}} = g_A$ and $g_{\dot{H}} = g_H$. The resulting system of equations can be

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[3]First optimality conditions are used to solve for sectoral allocations of factors in a follow on paper.

[4]The growth rate in physical capital, given by $g_K = \frac{\dot{K}}{K} = \frac{Y}{K} - \frac{C}{Y}\frac{Y}{K}$, is constant if Y, K and C grow at the same rate.

expressed in matrix form:

$$
\begin{bmatrix}
(1 - \sigma_K) & -\sigma_A & -\sigma_H \\
-\eta_K & (1 - \eta_A) & -\eta_H \\
-\omega_K & -\omega_A & (1 - \omega_H)
\end{bmatrix}
\begin{bmatrix}
g_K \\
g_A \\
g_H
\end{bmatrix}
=
\begin{bmatrix}
\sigma_L n \\
0 \\
0
\end{bmatrix}
\tag{4}
$$

where $\sigma_i \equiv F_i i / F \geqslant 0$, $\eta_i \equiv J_i i / J \geqslant 0$ and $\omega_i \equiv Q_i i / Q \geqslant 0$; $i = K, A, H$ denote the structural elasticities in the production, technology and human capital sectors, respectively.

The system of linear equations in (4) is non-homogeneous ($Ax = d$ in matrix form) or homogenous ($Ax = 0$) depending on whether the population is growing or static, respectively.

4.1 Conditions for Positive and Balanced Growth with a growing population

First, consider the case where population growth is strictly positive. The system of equations in (4) jointly determine the growth rates of real variables as functions of population growth and the structural elasticities:

$$
g_K = \frac{\sigma_L \left[(1 - \eta_A)(1 - \omega_H) - \omega_A \eta_H \right] n}{|A|}
\tag{5a}
$$

$$
g_A = \frac{\sigma_L \left[\eta_K (1 - \omega_H) + \omega_K \eta_H \right] n}{|A|}
\tag{5b}
$$

$$
g_H = \frac{\sigma_L \left[\eta_K \omega_A + \omega_K (1 - \eta_A) \right] n}{|A|}
\tag{5c}
$$

4.1.1 Strictly Positive Growth and Diminishing Marginal Returns

Proposition 1 (Conditions for Positive Growth)
For strictly positive population growth, $|A| > 0$ and $\sigma_K < 1$, $\eta_A < 1$ and $\omega_H < 1$, together with $\eta_K > 0$ and/or $\omega_K > 0$ are necessary and sufficient for strictly positive growth in output, capital, consumption, technology and human capital.

Consider each of the conditions for strictly positive growth, in turn.

It is a rudimentary result of linear algebra that a non-homogeneous system of linear equations has a unique non-trivial solution if $|A| \neq 0$. Further, stability of the underlying dynamic system requires $|A| > 0$.

The next three conditions, $\sigma_K < 1$, $\eta_A < 1$ and $\omega_H < 1$, imply diminishing marginal returns to capital, human capital and technology in the sector that produces each input, respectively. On first inspection of (5a) - (5c), diminishing returns to capital in the production of final output ($\sigma_K < 1$) does not seem a condition for positive growth, but it implies and is implied by $(1 - \eta_A)/\eta_H > \omega_A/(1 - \omega_H)$ in combination with $|A| > 0$

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and the other conditions.[5] The equivalent condition $(1 - \eta_A)/\eta_H > \omega_A/(1 - \omega_H)$, as required by (5a), relates the returns to scale to technology and human capital across the R&D and human capital accumulation sectors. If, for instance, we have decreasing returns to technology and human capital in the R&D sector, this condition implies we must also have decreasing returns to technology and human capital in the accumulation of human capital.

Examining (5a) - (5c), diminishing returns to technology and diminishing returns to human capital in their respective sectors ($\eta_A < 1$ and $\omega_H < 1$) is clearly sufficient for positive growth, but is it necessary? From (5a), we can rule out $\eta_A = 1$ or $\omega_H = 1$, although $\eta_A > 1$ and $\omega_H > 1$ is not inconsistent with strictly positive growth in output, capital and consumption. However, if $\eta_A > 1$ and $\omega_H > 1$, additional conditions would be required for strictly positive growth in technology and human capital.

Certainly, the three conditions $\sigma_K < 1$, $\eta_A < 1$ and $\omega_H < 1$ are provided by the Hawkins-Simon conditions: a necessary and sufficient condition that the stationary solutions to $Ax = d$ be all strictly positive is that all principal minors of the matrix A are strictly positive. Denoting $|D_i|$ as the ith principal minor of matrix A, $|D_1| > 0 \Leftrightarrow \sigma_K < 1$, together with $|D_2| > 0 \Rightarrow \eta_A < 1$ and $|D_3| = |A| > 0 \Rightarrow \omega_H < 1$. Hawkins & Simon (1949) assume all elements of the vector d are strictly positive, to simplify the statement of their theorem and its proof. They note, however, that elements of vector d may be weakly positive, as in the system (4), without any essential loss of generality.

The final condition $\eta_K > 0$ and/or $\omega_K > 0$ says that physical capital must be productive in either R&D or human capital, to obtain positive growth in either sector. This condition results from our allowance for heterogeneous labor and the assumption that physical labor is employed only in the production of final output. Exogenous growth in population or raw labor therefore drives growth in final output and physical capital, and, indirectly, R&D and human capital only if physical capital is employed in these sectors. Introducing a Lucas (1988) specification for human capital accumulation (with diminishing returns) and a Jones (1995) type R&D sector to our four sector growth model implies zero growth in both types of knowledge and therefore would be redundant. There is a trade-off between the realism of heterogeneous labor and the simplicity of single input knowledge accumulation equations. Existing literature opts for the latter and we introduce the restriction of homogeneous labor later in the paper to illustrate these models as special cases of our general model.

Corollary 1 (to Proposition 1) *A further sufficient condition for strictly positive growth in <u>per capita</u> output and capital is $\sigma_L > (1 - \sigma_K)$.*

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[5] $(1 - \sigma_K) [(1 - \eta_A)(1 - \omega_H) - \omega_A \eta_H] > \sigma_A [\eta_K (1 - \omega_H) + \omega_K \eta_H] + \sigma_H [\eta_K \omega_A + \omega_K (1 - \eta_A)]$
$\Leftrightarrow |A| > 0$

From equation (5a),[6] the economy grows at the per capita rate:

$$g_y = \{\sigma_L - (1 - \sigma_K)\} \frac{g_K}{\sigma_L} + \sigma_A \frac{g_A}{\sigma_L} + \sigma_H \frac{g_H}{\sigma_L} \tag{6}$$

where $\sigma_L > (1 - \sigma_K) \Rightarrow g_y > 0$, given the above conditions for positive growth in output, physical capital, technology and human capital are satisfied. Eicher & Turnovsky (1999) obtain the same condition for a general two sector growth model with homogeneous labor. Thus, the condition has generality. It implies increasing returns to scale in the final output sector, since $\sigma_L + \sigma_K > 1$, together with non-negativity of σ_i, implies $\sum \sigma_i > 1$. Note that this condition is sufficient but not necessary for strictly positive growth in per capita output. That is, growth in per capita output may still be strictly positive if $\sigma_L \leq (1 - \sigma_K)$. For instance, if $\sigma_L + \sigma_K = 1$, growth in per capita output is strictly positive, given strictly positive growth in technology and human capital.

4.2 Balanced Growth and Cobb-Douglas or Constant Returns to Scale Technology

While positive growth is provided by Proposition 1, *balanced* growth requires constancy of the growth rates in (5a)-(5c), which, in turn, requires constant population growth and constant multiplicative terms.

The multiplicative terms in (5a)-(5c) are constant if the structural elasticities are constant (as for Cobb-Douglas production functions). In this case, the constancy of the multiplicative terms is independent of the returns to scale, so that output, technology and human capital may grow at different rates. If the structural elasticities are not constant (as for Constant Elasticity of Substitution (CES) production functions), balanced growth requires that production in each sector exhibit constant returns to scale (i.e. $\sum \sigma_i = 1; \sum \eta_i = 1; \sum \omega_i = 1$), in which case the multiplicative terms reduce to unity and all sectors grow at the common rate: $g_Y = g_K = g_A = g_H = n$.[7]

Thus, for a constant rate of population growth, Cobb-Douglas production technology or constant returns to scale in all sectors are sufficient conditions for balanced growth. This result is known, albeit for a two sector generalized growth model (see Eicher & Turnovsky (1999), who also discuss a third condition, that of homogeneously separable forms). Accordingly, this paper does not state these conditions in a formal proposition. However, since these conditions are widely used, but obscured in sophisticated models, we note some of their implications.

Constant returns to scale, particularly in the production of final output, is unlikely. When non-rivalrous knowledge is employed, final production most likely exhibits increasing returns to scale (Romer 1990). Moreover, constant returns to scale in all sectors implies

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[6] $g_y = g_K - n = \frac{\{\sigma_L - (1 - \sigma_K)\}[(1 - \eta_A)(1 - \omega_H) - \omega_A \eta_H] + \sigma_A [\eta_K (1 - \omega_H) + \omega_K \eta_H] + \sigma_H [\eta_K \omega_A + \omega_K (1 - \eta_A)]}{|A|} n$

[7] To aid the reader in verifying this, if $\sum \sigma_i = 1; \sum \eta_i = 1; \sum \omega_i = 1$ then $|A| = \sigma_H \{\omega_K \eta_H + \eta_K (\omega_A + \omega_K)\}$.

zero growth in per capita output. This may explain why Cobb-Douglas technology most commonly appears the literature, but this is not without its limitations, since it assumes input shares are exogenous constants. An all or nothing approach is not required. For instance, we may assume Cobb-Douglas technology in one sector, CES technology in another or even CES technology nested within Cobb-Douglas production function.

Notwithstanding these caveats, the main problem with these sufficient conditions for balanced growth is the implication that (positive) balanced growth seems inextricably dependent on a (positive) constant population growth. We now establish that positive, balanced growth arises without positive population growth if growth in either technology or human capital asymptotes to a positive constant.

4.2.1 Does Strictly Positive and Balanced Growth require Strictly Positive Population Growth?

Differentiating (3a) with respect to time, and noting that constant g_K requires that Y and K grow at the same rate, yields:

$$g_Y = g_K = \frac{\sigma_A}{1 - \sigma_K} g_A + \frac{\sigma_H}{1 - \sigma_K} g_H + \frac{\sigma_L}{1 - \sigma_K} n \tag{7}$$

When both g_A and g_H depend on g_K, equation (7) reduces to equation (5a). However, if either g_A or g_H are independent of g_K, then g_Y is an additively separable function of two terms, only one of which is a function of population growth. Hence, strictly positive, balanced growth no longer requires strictly positive population growth.

Assuming that physical capital is not employed in one knowledge sector does not imply g_A or g_H are independent of g_K. To illustrate, if physical capital is used in human capital accumulation but not in R&D, under normal conditions, g_A is still a function of g_K because human capital is used in R&D. This raises the question, under what condition(s) is either g_A or g_H independent of g_K? The answer lies in the asymptotic nature of g_A or g_H. For instance, if either asymptotes to a positive constant that exceeds g_K, then g_Y is an additively separable function, as required.

Equations (3b) and (3c) describe the accumulation of non-rivalrous ideas and rivalrous human capital, respectively. The accumulation of each type of knowledge is a function of its own stock, the stock of the alternative knowledge and the stock of physical capital. As a result, there are several ways by which asymptotic limits may imply growth in one type of knowledge is independent of growth in physical capital. To simplify, we confine our analysis to the case where knowledge accumulation is a function of its existing stock of knowledge and one other input.[8] In order to generalize the following proposition, we

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[8]The following section introduces CES technology to illustrate the implications of positive asymptotic limits. Whilst it is neater to discuss the degree of substitutability between two inputs, we could apply CES technology to three inputs or nest a CES technology (with two inputs) within a Cobb-Douglas production function (with a third input).

could define two generic types of knowledge. However, this is a redundant exercise if we consider that certain input combinations are implausible. For instance, can innovation occur through the interaction of existing ideas and physical capital? Researchers are most likely an essential input in R&D. Accordingly, from equations (3b) and (3c), the two types of knowledge grow at the rates:

$$g_A = \frac{J(A, l_A H,)}{A} \equiv j(A, H; \beta_A) \tag{8a}$$

$$g_H = \frac{Q(l_H H, k_H K)}{H} \equiv q(H, K; \beta_H) \tag{8b}$$

where β_A and β_H are shift parameters encapsulating, respectively, l_A and l_H, k_H. The growth rates asymptote to:

$$\lim_{A \to \infty} j(A, H; \beta_A) = \bar{j}(H; \beta_A) \tag{9a}$$

$$\lim_{H \to \infty} q(H, K; \beta_H) = \bar{q}(K; \beta_H) \tag{9b}$$

where $\bar{j}$ and $\bar{q}$ are constants, that may depend on the shift parameters.

Proposition 2 (Condition for Positive, Balanced Growth and Population)
Strictly positive balanced growth may arise without population growth if $\bar{q}(K; \beta_H) > 0$.

Strictly positive balanced growth requires strictly positive, constant population growth if $\bar{q}(K; \beta_H) = 0$, unless $\bar{j}(H; \beta_A) > 0$.

The first part of Proposition 2, refers to the case where g_H asymptotes to a positive constant. If this positive constant exceeds g_K, then g_H does not depend, indirectly on n. Thus, the accumulation of human capital features an endogenous stock of knowledge. Curve 1 in Figure 1 illustrates such a case. Substituting $\bar{q}$ for g_H in equation (7) then, regardless of whether or not g_A is a function of g_H, g_Y is an additively separable function of two terms, one of which does not depend on population growth.

If g_H asymptotes to zero, as in the second part of Proposition 2, then g_H depends on g_K in steady state (see Curve 2 in Figure 1). Two possibilities arise. If g_A asymptotes to zero, implying g_A is a function of g_H, then growth in both types of knowledge requires growth in physical capital. Substituting for g_A and g_H in equation (7) gives unambiguous semi-endogenous growth: strictly positive, constant growth in output requires strictly positive, constant population growth. However, if g_A asymptotes to a positive constant that exceeds g_H, then g_A is independent of g_H. Substituting $\bar{j}$ for g_A in equation (7) yields an additively separable function for g_Y, so that strictly positive balanced growth does not require strictly positive, constant population growth.

This proposition provides a basis for distinguishing endogenous and semi-endogenous growth, often synonymous in the literature with scale and non-scale growth, respectively, which in turn are strongly associated with sectoral linearity and diminishing marginal
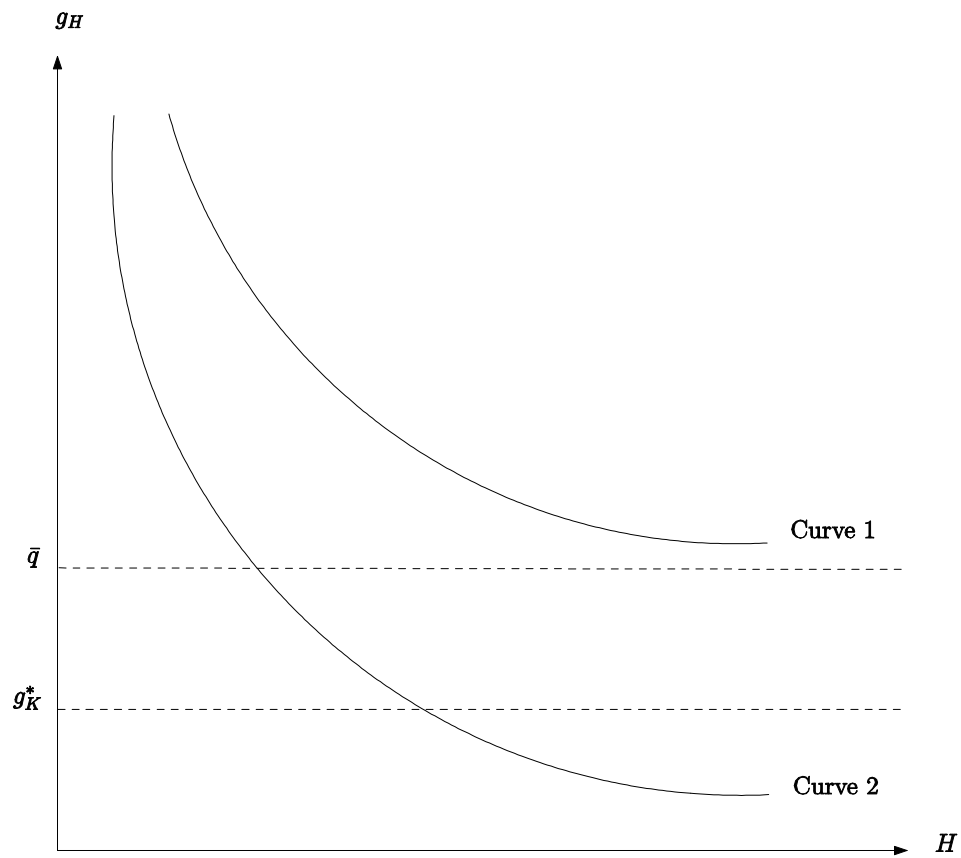
Figure 1: Asymptotic Growth in Human Capital

returns, respectively. Endogenous growth differs from semi-endogenous growth in that economic growth does not require strictly positive growth in an exogenous factor, such as population. Proposition 2 reminds us that endogenous growth requires only that growth in knowledge asymptote to a strictly positive constant. Both curves in Figure 1 are monotonically decreasing, reflecting diminishing marginal returns to knowledge in the accumulation of knowledge. However, Curve 1 implies endogenous growth, while Curve 2 implies semi-endogenous growth. Thus, the association between diminishing marginal returns and semi-endogenous growth is tenuous.

4.2.2 CES technology - which sector?

Pitchford (1960) and Barrro & Sala-i-Martin (1999) demonstrate the capacity for endogenous growth with CES technology in the standard neoclassical growth model.[9] More recently, this has been extended to a Romer (1990) type R&D-based growth model by Zuleta (2004), where the production function for final output is a CES combination of physical capital and labor. Whereas Zuleta (2004) introduce CES technology to the production of final output, Proposition 2 suggests introducing CES technology to the accumulation of knowledge: the growth in knowledge will asymptote to zero or a positive constant, depending on whether the elasticity of substitution between inputs is less than one or greater than one, respectively. To which knowledge accumulation sector do we introduce CES technology?

Taking inspiration from Dalgaard & Kreiner (2003), we could introduce CES technology to the accumulation of non-rivalrous knowledge, $J(A, l_A H,)$. Because they assume physical labor rather than human capital is employed in the production of new ideas, their application of CES technology addresses a *direct* relationship between g_A and n. Except for the fact that the relationship between g_A and n is indirect in our model, therefore, introducing CES technology to $J(A, l_A H,)$ would effectively replicate existing literature. This is not to suggest that simply by introducing CES technology to an alternative sector we contribute to the literature. In the following section, we introduce CES technology to the accumulation of human capital $Q(l_H H, k_H K)$ not only because it hasn't been done before, but also because it is the more plausible application of CES technology.

Applying l'Hopital's rule[10] to equation (8b),

$$\lim_{H \to \infty} q(H, K; \beta_H) = \lim_{H \to \infty} Q_H \tag{10}$$

where Q_H is the marginal product of human capital in the generation of new human capital. Thus, another interpretation of Proposition 2 is that endogenous growth arises if the marginal product of knowledge in producing new knowledge tends to a positive constant

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[9]Technically, endogenous growth may arise in the neoclassical growth model when the marginal product of capital is bounded below by a positive constant, as demonstrated by Jones & Manuelli (1990).

[10]$\lim_{x \to a}\left(\frac{m(x)}{n(x)}\right) = \lim_{x \to a}\left(\frac{m'(x)}{n'(x)}\right)$

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as the stock of knowledge tends to infinity. This suggests one criterion for introducing CES technology.

Dalgaard & Kreiner (2003) argue that while the marginal product of physical capital most likely tends to zero as the stock of physical capital becomes infinite, the marginal product of knowledge is bounded below by a positive constant. In their words,

> ...why would a new piece of information be completely unproductive in producing new ideas even if there did in fact exist infinitely many other pieces of information?

They make a good point. Ideas are boundless. On this criterion, our application of CES technology to human capital accumulation is closer to Barrro & Sala-i-Martin's (1999) application, since human capital is embodied knowledge. However, this is not the only criterion for introducing CES technology.

Positive asymptotic limits arise with CES technology only when the elasticity of substitution exceeds one. This suggests we should look to define CES technology over inputs that, a priori, we expect may be highly substitutable. In this sense, Barrro & Sala-i-Martin's (1999) application of CES technology has an intuitive appeal that Dalgaard & Kreiner (2003) lacks. There are several real world examples, such as the demise of the typist pool, where physical capital has replaced physical labor in the production of final output. Moreover, empirical evidence supports an elasticity of substitution between physical capital and labor higher than one (Duffy & Papageorgiou 2000).

In contrast, it is hard to think of examples where ideas have replaced researchers in the process of innovation. A scientist works with an existing idea, say $e = mc^2$, to create new ideas, suggesting a high degree complementarity between the two inputs in R&D. The notion that ideas may be increasingly substituted for physical capital in the process of R&D may be more palatable. Contrast the physical capital requirements of a modern researcher with those of, say, Thomas Edison. Or better still, consider the way the idea to use silicon in a microchip means researchers no longer require computers that take up a floor of a building. There are examples of physical capital complementing ideas in the process of R&D. The notion that knowledge can be substituted for physical capital is more plausible than the notion that knowledge can be substituted for researchers in R&D.

We could broaden Proposition 2 to the case of three inputs and introduce CES technology to the employment of ideas and physical capital in R&D. However, we need not do this if we consider that just as non-rivalrous knowledge (ideas) may replace physical capital in the process of R&D, private knowledge may replace physical capital in the process of learning. For instance, as an economy's stock of human capital accumulates, increased reliance on self-education may be consistent with rivalrous knowledge replacing physical infrastructure. The degree of complementarity between human capital and physical capital in the accumulation of human capital may be high or low, whereas, a priori, the degree of complementarity between human capital and ideas in R&D is high. This reasoning suggests introducing CES technology to $Q\left(l_H H, k_H K\right)$ rather than $J\left(A, l_A H\right)$.

4.3 A Model of Endogenous Growth with or without population growth

Consider an economy comprised of three sectors with the following production technologies:

$$Y = A^{\sigma_A} \left((1 - l_A) H \right)^{\sigma_H} L^{\sigma_L} \left((1 - k_H) K \right)^{\sigma_K} \tag{11a}$$

$$\dot{A} = A^{\eta_A} (l_A H)^{\eta_H} \tag{11b}$$

$$\dot{H} = \left[(\phi_1 H)^{\rho} + (\phi_2 k_H K)^{\rho} \right]^{1/\rho} \tag{11c}$$

where ρ (the substitution parameter) determines the constant elasticity of substitution between human capital and physical capital in the accumulation of human capital, given by $\epsilon = 1/(1 - \rho) : \epsilon > 0, \epsilon \neq 0$. Define $\phi_1 \equiv \alpha^{1/\rho}$ and $\phi_2 \equiv (1 - \alpha)^{1/\rho}$, where $\alpha \in (0, 1)$ is the distribution parameter.[11] The parameters satisfy $0 < n \leq g_K < \phi_1$.

We keep the model as close as possible to the generalized production structure in (3). For simplicity, physical capital and the stock of technology are dropped as inputs to R&D and human capital accumulation, respectively, but these assumption can be relaxed without loss of generality.[12] Also for simplicity, we assume $l_H = 1$, meaning human capital is as a private input allocated to the production of final output and R&D, while at the same time being used in the accumulation of human capital. This assumption is not critical. If we relax the assumption, steady state growth in human capital will be a function of l_H, but the presence of this term is innocuous since sectoral shares of labor and capital are constant along a balanced growth path.

Differentiating g_H with respect to time and recognizing that $\omega_H = (\phi_1/g_H)^{\rho}$ and $\omega_K = 1 - (\phi_1/g_H)^{\rho}$ (see Appendix for detail), we obtain:

$$\dot{g}_H = g_H (g_K - g_H) \left\{ 1 - \left(\frac{\phi_1}{g_H} \right)^{\rho} \right\} \tag{12}$$

This equation has three steady states: $g_H = 0$, $g_H = g_K$ and $g_H = \phi_1$. Referring to Figure 2, g_H converges to either g_K or ϕ_1, depending on whether ϵ is less than one or greater than one, respectively. In the case where $\epsilon > 1$, the growth in human capital is bounded below by ϕ_1.

Thus, when $\epsilon > 1$, human capital grows permanently, independent of and at a higher rate than physical capital. As suggested in the previous section, the intuition for this

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[11]When Dalgaard & Kreiner (2003) introduce CES technology to the interaction of A and $l_A L$ in R&D, they assume the equivalent parameter to ϕ_1 is not a function of ρ and α and impose the restriction $\phi_1 \in (0, 1)$. In a standard CES production function, such as equation (11c), $\phi_1 \equiv \alpha^{1/\rho} \gtrless 1$, depending on $\rho \in (-\infty, 1)$; $\rho \neq 1$. A standard CES production is consistent with a growth rate less than 1, since g_K only converges to ϕ_1 when $\rho > 0$, implying $g_K = \phi_1 < 1$.

[12]By dropping physical capital from R&D, our solution for g_K more closely resembles (7). Also, this aids derivation of research intensity, measured by l_A, in a follow up paper. Allowing for physical capital as an input to R&D, the long run growth rate of output is still independent of n when $\epsilon > 1$: $g_Y = \frac{1}{(1 - \sigma_K) - \sigma_A \eta_K} \left[\left\{ \frac{\sigma_A \eta_H}{(1 - \eta_A)} + \sigma_H \right\} \phi_1 + \sigma_L n \right]$.

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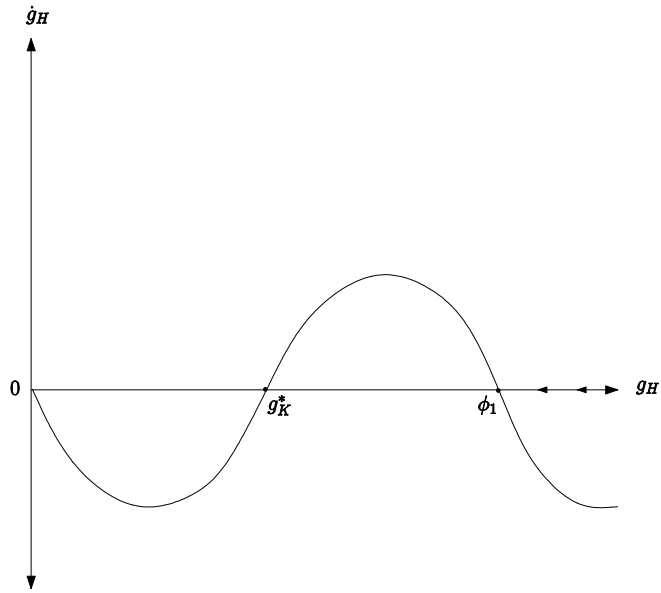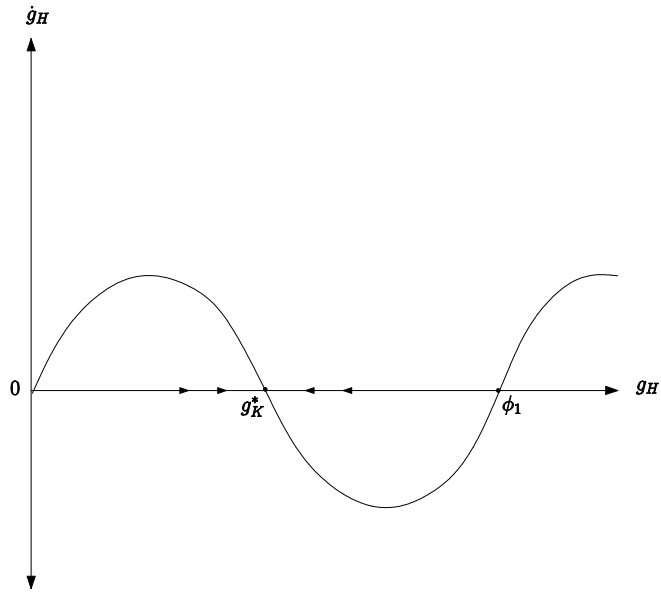
(a) Dynamics when $\epsilon > 1$

(b) Dynamics when $\epsilon < 1$

Figure 2: Dynamics of Growth in Human Capital

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result lies in the fact that human capital is increasingly substituted for physical capital as the accumulation of human capital proceeds.

To solve for the growth rates of the other real variables in the model, we note that when $\epsilon > 1$, we cannot use the solutions given by (5a) and (5b), obtained by Cramer's Rule, since $g_H = \phi_1$ implies $\omega_H = 1$ and $\omega_K = 0$, which in turn imply $|A| = 0$ for the matrix system (4). We therefore proceed with total differentiation of (11b) and substitution for g_A and g_H in (7). The balanced growth rates in physical capital, output and consumption on the one hand and technology on the other are:

$$\left.\begin{array}{l} g_K = \tau\phi_1 + \upsilon n \\[2ex] g_A = \frac{\eta_H}{(1-\eta_A)}\phi_1 \end{array}\right\} \quad \text{if } \epsilon \ > \ 1 \tag{13a}$$

$$\left.\begin{array}{l} g_K = \left(\frac{\upsilon}{1-\tau}\right)n \\[2ex] g_A = \frac{\eta_H}{(1-\eta_A)}g_K \end{array}\right\} \quad \text{if } \epsilon \ < \ 1 \tag{13b}$$

where $\tau = \frac{\sigma_A}{(1-\sigma_K)}\frac{\eta_H}{(1-\eta_A)} + \frac{\sigma_H}{(1-\sigma_K)}$ and $\upsilon = \frac{\sigma_L}{1-\sigma_K}$. When $\epsilon < 1$, strictly positive rates of growth requires $(1-\tau) > 0$, as implied by $|A| > 0$.[13] When $\epsilon > 1$, strictly positive growth in output no longer requires strictly positive population growth n. This is the key result. Note that the restriction $g_K < \phi_1$ requires that σ_i and η_i satisfy $\frac{\upsilon}{1-\tau} < \frac{\phi_1}{n}$. As an illustration, if we assume constant returns to scale in final production and R&D (i.e. $\sum \sigma_i = 1; \sum \eta_i = 1$) this restriction simplifies to $\phi_1 > n$.

The long run *per capita* growth rate of the economy is given by:

$$g_y = \begin{cases} \tau\phi_1 + (\upsilon - 1)n & \text{if } \epsilon > 1 \\[2ex] \left(\frac{\upsilon}{1-\tau} - 1\right)n & \text{if } \epsilon < 1 \end{cases} \tag{14}$$

where $\upsilon > 1$ is *sufficient* for positive per capita growth, as per Corollary 1 to Proposition 1.

Thus, growth in the economy does not require growth in the population, when knowledge is highly substitutable for physical capital in the accumulation of knowledge.

4.4 Conditions for Positive Growth with a static population

Consider now the case where population is static. If $n = 0$, vector d in matrix system (4) is a null vector.

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[13] $|A| = [(1-\sigma_K)(1-\eta_A) - \sigma_A\eta_H - \sigma_H(1-\eta_A)]\omega_K$

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Proposition 3 (Condition for Positive Growth)
For a static population, $|A| = 0$ is necessary for strictly positive growth in output, capital, consumption, technology and human capital.

It is a well known result of linear algebra that a homogeneous linear system of equations (in matrix form $Ax = 0$) has non trivial solutions iff $|A| = 0$. So the existence of a solution with positive growth rates implies $|A| = 0$. However, $|A| = 0$ does not imply positive growth since a non trivial solution may be one of negative growth. Thus, for a static population, $|A| = 0$ is necessary for strictly positive growth in output, capital, consumption, technology and human capital.

Corollary 2 (to Proposition 3) *Sectoral linearity is a sufficient but not necessary condition for $|A| = 0$.*

A sufficient condition for $|A| = 0$ is that each of the entries in one or more of the rows or columns in matrix A is zero. Existing models with a static population commonly assume that either (3b) or (3c) are single input linear equations (as in a Romer (1990) type R&D equation ($\eta_A = 1$) or a Lucas (1988) specification for human capital accumulation ($\omega_H = 1$)). This sectoral linearity assumption implies each of the entries in either the second or third rows of the coefficient matrix in the system (4) is zero. Thus, sectoral linearity is introduced to a knowledge accumulation equation in order to solve for strictly positive rates of growth in the real variables of the model. This assumption is widely criticized. To quote Jones (2001) (p.5),

> The linearity in existing models is assumed ad hoc, with no motivation other than that we must have linearity somewhere to generate endogenous growth.

It is therefore worth considering whether we can solve for strictly positive rates of growth under an alternative, more palatable assumption.

Another sufficient condition for $|A| = 0$ is that one row (column) is a linear combination of the other rows (columns) of the matrix. To explore this further, letting v_i denote the ith column vector of the coefficient matrix A, the system $Ax = 0$ can be written as the vector equation:

$$g_K v_1 + g_A v_2 + g_H v_3 = 0$$

where

$$v_1 = \begin{pmatrix} (1 - \sigma_K) \\ -\eta_K \\ -\omega_K \end{pmatrix}, \quad v_2 = \begin{pmatrix} -\sigma_A \\ (1 - \eta_A) \\ -\omega_A \end{pmatrix}, \quad v_3 = \begin{pmatrix} -\sigma_H \\ -\eta_H \\ (1 - \omega_H) \end{pmatrix} \tag{15}$$

$|A| = 0$, as required to obtain a strictly positive solution to the system, if the three vectors are linearly dependent:

$$v_1 = a v_2 + b v_3 \quad a < 0; b < 0 \tag{16}$$

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The easiest and most obvious case to consider is that of constant returns to scale to growing factors ($a = b = -1$). Note that, because we have physical labor employed in the production of final output, this case corresponds to increasing returns to scale in final production ($\sum \sigma_i > 1$).

However, we can move away from constant returns to scale to growing factors and still obtain $|A| = 0$. In the case where $a = b = -k$, we have increasing or decreasing returns to scale to growing factors in the final output sector, when k is less than or greater than one, respectively, with ambiguous returns to scale in the other sectors. We can solve for strictly positive growth rates in the physical capital, technology and human capital with varying degrees of returns of scale across sectors, so long as the vectors are linearly dependent.

Example 1 *Consider a Cobb Douglas economy where $(1 - \sigma_K) = \sigma_A = \sigma_H = 0.66$; $\eta_K = \eta_A = \eta_H = 0.25$; $\omega_K = \omega_A = \omega_H = 0.25$. These values suggest increasing returns to scale in final output and decreasing returns to scale in both knowledge accumulation sectors. The three column vectors are linearly dependent: $v_1 = -0.5v_2 - 0.5v_3$, as required to obtain a strictly positive solution to the system.*

To demonstrate that linear dependence and sectoral linearity are both sufficient for $|A| = 0$, as required for strictly positive growth, we can use the generalized setting to describe the decentralized two sector R&D-based growth models of Romer (1990) and Rivera-Batiz & Romer (1991). With no accumulation of human capital and population an exogenous constant, the system of equations which determine positive growth rates reduces to:

$$\begin{bmatrix} (1 - \sigma_K) & -\sigma_A \\ -\eta_K & (1 - \eta_A) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{17}$$

Both papers assume the same Cobb-Douglas specification for the production of final output, which in a centralized decision making model is given by:

$$Y = \eta^{(\sigma_H + \sigma_L) - 1} A^{\sigma_H + \sigma_L} \left((1 - l_A) H \right)^{\sigma_H} L^{\sigma_L} K^{1 - (\sigma_H + \sigma_L)} \tag{18a}$$

where η is a constant term that, in a decentralized setting measures the units of foregone consumption (or equivalently, physical capital) required to create one unit of any type of intermediate good.[14] The term l_A is absent in Rivera-Batiz & Romer (1991) but, since the allocation of inputs between sectors is constant along a balanced growth path, this is a harmless omission.

The modelling of the R&D sector is the major distinction between the two papers. Romer (1990) assumes neither physical capital nor physical labor are productive in R&D:

$$\dot{A} = \delta \left(l_A H \right) A \tag{18b}$$

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[14] In a decentralized setting, A determines the range of intermediate goods that can be produced.

where δ is a constant efficiency parameter. Rivera-Batiz & Romer (1991), on the other hand, allow for both physical capital and physical labor to be productive inputs in R&D. They propose the lab equipment model of R&D:

$$\dot{A} = \delta A^{\sigma_H + \sigma_L} H^{\sigma_H} L^{\sigma_L} K^{1-(\sigma_H + \sigma_L)} \tag{18c}$$

Substituting for the sectoral elasticities from (18b) in (17):

$$\begin{bmatrix} (\sigma_H + \sigma_L) & -(\sigma_H + \sigma_L) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} g_K \\ g_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

where Romer's assumption of sectoral linearity implies $|A| = 0$. We can therefore solve the system for strictly positive rates of growth in physical capital and technology. From (19), we have one equation with two unknowns: $g_K = g_A$. Because $|A| = 0$, we employ a different solution method to that used in Section 4.1. From (18b), $g_A = \delta (l_A H)$, where δ and H are exogenous constants and l_A, the portion of human capital allocated to R&D, is constant, as required for balanced growth.[15]

Substituting for the sectoral elasticities from (18c) in (17):

$$\begin{bmatrix} (\sigma_H + \sigma_L) & -(\sigma_H + \sigma_L) \\ (\sigma_H + \sigma_L) - 1 & 1 - (\sigma_H + \sigma_L) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{20}$$

where $v_1 = -v_2$, sufficient for $|A| = 0$. Thus, instead of introducing sectoral linearity to the accumulation of knowledge, Rivera-Batiz & Romer (1991) assume diminishing marginal returns to the stock of existing ideas in the creation of new ideas ($\eta_A < 1$). Constant returns to scale to physical capital and ideas, the growing factors, is sufficient for strictly positive rates of growth. From (20), we have one equation with two unknowns which again is simply $g_K = g_A$.

Thus, we have demonstrated that static population R&D-based growth models can be solved for strictly positive growth without introducing sectoral linearity to the accumulation of knowledge. Note that while Rivera-Batiz & Romer (1991) have assumed constant returns to scale to the growing factors, we know from our discussion of Corollary 2 and the numerical example that we could have varying returns to scale across sectors in a three sector model.

5 Homogeneous Labor

In order to evaluate existing R&D based growth models as special cases of our general three sector model, the restriction of homogeneous labor is introduced. We assume all labor is skilled, in order to analyze both endogenous and exogenous labor accumulation.

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[15] A follow on paper derives a generalized expression for l_A from the first order conditions.

5.1 Endogenous Labor

If the accumulation of skilled labor is endogenized, we obtain a homogeneous form (in matrix algebra, $Ax = 0$) of the system of equations in (4).

5.1.1 Illustration of Proposition 3

Funke & Strulik (2000) model the development to an innovative economy. We simplify their model only by removing an exogenous productivity parameter and a distinction between the stock intermediate goods and physical capital which, in an innovative economy, are one and the same. Their decentralized model is detailed, comprising forty six equations. We can use our generalized model to reveal the salient features of their innovative economy. The production structure is:

$$
\begin{align}
Y &= A^{1-\sigma} \left((1 - l_A - l_H) H \right)^{1-\sigma} K^{\sigma} \eta^{-\sigma} \tag{21a} \\
\dot{A} &= \delta (l_A H) \tag{21b} \\
\dot{H} &= \xi (l_H H) \tag{21c}
\end{align}
$$

where δ and ξ are constant efficiency parameters.

As per Proposition 3, Funke & Strulik's (2000) assumption of sectoral linearity in the human capital accumulation equation implies positive rates of growth in physical capital, (output, consumption) and technology and human capital, as jointly determined by the system:

$$
\begin{bmatrix}
(1 - \sigma) & -(1 - \sigma) & -(1 - \sigma) \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
g_K \\
g_A \\
g_H
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\tag{22}
$$

Because $|A| = 0$, we employ a different solution method to that used when there is an exogenously growing factor. Sectoral linearity in the human capital accumulation equation gives us two equations with three unknowns, which Funke & Strulik (2000) reduce to one equation with two unknowns by assuming no physical capital is employed in R&D. From (22), $g_K = 2 \, g_A = 2 \, g_H$. Referring to the Appendix, we use the first order optimality conditions to solve. Final output and the two types of knowledge in the economy grow at the rates:

$$
g_Y = \frac{\xi - \rho}{\theta}; \quad g_A = g_H = \frac{\xi - \rho}{2\theta}
\tag{23}
$$

5.2 Exogenous Labor

Since $H = hL$, where h is the average skill level, skilled labor accumulates over time due to growth in the average skill level (measured by growth in average educational attainment,

e) and/or growth in the labor force (measured by population growth, n). The system of equations which determine positive and balanced growth rates reduces to:

$$\begin{bmatrix} (1 - \sigma_K) & -\sigma_A \\ -\eta_K & (1 - \eta_A) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \end{bmatrix} = \begin{bmatrix} \sigma_H (e + n) \\ \eta_H (e + n) \end{bmatrix} \tag{24}$$

Despite empirical evidence that a significant portion of income growth (equivalent to g_K in (24)) is attributable to growth in educational attainment, e is absent from existing models of R&D-based growth. The reason for this is best articulated by Jones (2001):

> ...roughly 80 percent of post-war US growth is due to increases in human capital investment rates and research intensity and only 20 percent is due to the general increase in population (Jones 2002). However, ... neither educational attainment nor the share of labor force devoted to research can increase forever. So unless there is an ad-hoc Lucas-style linearity in human capital accumulation, population growth remains the only possible source of long run growth...

Introducing the restriction that skilled labor accumulates at the exogenous rate of population growth, n, yields the general two-sector growth model analyzed by Eicher & Turnovsky (1999). Note that the generalized growth rates of real variables, jointly determined by (24), are not obtained by setting $\omega_i = 0$ in (5a) and (5b). However, the rates of growth of physical capital, output and consumption, on the one hand, and knowledge, on the other, are readily obtained (see p. 400 of Eicher & Turnovsky (1999) for a full analysis).

Whilst the general model encompasses several well known non-scale R&D-based growth models as special cases, this paper focuses on the dependency of economic growth on population growth and we discuss an example from the literature that best reflects this focus, namely, a re-parameterization of Jones (1995), suggested by Proposition 1 (Corollary 1).

5.2.1 Illustration of Proposition 1 (especially Corollary 1)

The model of Jones (1995) is well known. He obtains non-scale growth by introducing diminishing marginal returns to the stock of ideas in R&D ($\eta_A < 1$) to a homogenized labor version of Romer (1990). The production structure is:

$$Y = (A (1 - l_A) H)^\sigma K^{1-\sigma} \tag{25a}$$
$$\dot{A} = \delta A^{\eta_A} (l_A H)^{\eta_H} \tag{25b}$$

where $\dot{H} = nH$, so that population growth is the only exogenous source of growth in the economy.

As per Proposition 1, diminishing marginal returns implies positive rates of growth of physical capital, output and consumption, on the one hand, and knowledge, on the other, as jointly determined by the system:

$$\begin{bmatrix} \sigma & -\sigma \\ 0 & (1 - \eta_A) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \end{bmatrix} = \begin{bmatrix} \sigma n \\ \eta_H n \end{bmatrix} \tag{26}$$

Technology and *per capita* output, physical capital and consumption grow at a common rate determined by population growth and the shares of labor and stock of knowledge in the R&D sector:

$$g_A = g_y = \psi n \tag{27}$$

where $\psi = \eta_H / (1 - \eta_A)$. Constant population growth and Cobb-Douglas technology imply balanced growth.

Of most interest, however, is the implication of Jones's (1995) final production parameter restrictions for the rate of growth in per capita output. In the general two sector growth model, *per capita* rates of growth of physical capital, output and consumption are given by:

$$g_y = \frac{[\{\sigma_H - (1 - \sigma_K)\}(1 - \eta_A) + (\eta_H + \eta_K)\sigma_A]}{[(1 - \sigma_K)(1 - \eta_A) - \eta_K \sigma_A]} n \tag{28}$$

By introducing the restriction $\sigma_H = (1 - \sigma_K) = \sigma_A = \sigma$, together with $\eta_K = 0$, Jones (1995) obtains a long run growth in per capita output that is determined by relative factor shares in the R&D sector, as encapsulated in the parameter ψ.

Jones (2002) provides estimates of ψ for the United States (U.S.) economy, ranging from a low value of 0.05 to a high value of 1/3. These estimates suggest a long run rate of growth of per capita output that is less than a third of the rate of population growth, from which Jones (2002) draws two inferences. Firstly, growth rates in the U.S. for the last century are not indicative of steady state. Secondly, we should anticipate a future slowdown as the economy transits to a long run rate of growth that is *lower* than the rate of population growth.

Would alternative parameter restrictions suggest a less pessimistic outlook for long run growth of the U.S. economy? The restriction $\sigma_H > (1 - \sigma_K) = \sigma_A = \sigma$, by Corollary 1 to Proposition 1, is sufficient for positive *per capita* growth in output:

$$g_y = (\chi + \psi - 1) n \tag{29}$$

where $\chi = \sigma_H / (1 - \sigma_K)$. Since $\chi > 1$, our re-parameterization of Jones (1995) yields a higher long run rate of growth in per capita output. Moreover, if $(\chi - 1) > (1 - \psi)$, a restriction which does not violate any of the conditions for positive and balanced growth[16], the economy will transit to a long run rate of growth that is *higher* than the rate of population growth.

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[16] $|A| > 0$ is the only condition that relates relative factor shares in the R&D sector to relative factor shares in the final output sector: $|A| > 0 \Rightarrow \frac{1 - \sigma_K}{\sigma_A} > \frac{\eta_K}{1 - \eta_A}$.

6 Seven Principles for Model Construction

We bring together the formal propositions of this paper in a set of principles for constructing a model of either endogenous or semi-endogenous growth. These principles were derived using a generalized model where physical capital and two types of knowledge grow endogenously and the optimization problem is that of a central planner. Because the principles are general, they can be applied to any R&D-based growth model with specific microfoundations and decentralized decision making. They are equally applicable to models with two or three endogenous factors; homogeneous labor or heterogeneous labor. Recognizing the two broad approaches in the literature, we tailor the principles to the treatment of population as a growing or static factor.

If you want to allow for the possibility that population can grow at exogenous rate, we propose five principles for constructing a model:

1. To obtain strictly positive growth in output, capital, consumption, technology and human capital, it is necessary and sufficient to assume diminishing marginal returns to each input in its productive sector and a relationship between the structural elasticities such that $|A| > 0$.

2. In terms of obtaining a *balanced* growth path, Cobb-Douglas production functions have the benefit of allowing for varying degrees of returns to scale. Constant returns to scale must be assumed if you use CES production functions.

3. For strictly positive *per capita* growth in the economy, it is sufficient to assume increasing returns to scale to physical capital and physical labor in the production of final output.

4. Strictly positive balanced growth may arise without population growth if growth in knowledge (either non-rivalrous ideas or human capital) asymptotes to a positive constant. CES technology is one such production technology for which this possible.

5. The notions of diminishing marginal returns and semi-endogenous growth are logically independent. Given our fourth principle, diminishing marginal returns to knowledge is consistent with both endogenous growth (i.e. growth in the economy without exogenous population growth) and semi-endogenous growth.

The first three principles are known, albeit for two sector R&D-based growth models, but are often hidden behind the complexity of the decentralized solutions to these models. With the introduction of additional sectors, such models become increasingly complex. This paper demonstrates the generality of these principles to multi-sector growth models, even when we allow for heterogenous labor. Even if these principles are known for two sector R&D-based growth models, they are not always fully utilized. This is demonstrated

in the previous section, by applying the third principle to a well-known model to achieve a more general result.

The last two principles are based on new results obtained in this paper. Introducing CES technology to a knowledge accumulation equation to obtain endogenous growth is more than a mathematical peculiarity. We recommend that CES technology be introduced where the inputs are, a priori, highly substitutable. A CES combination of human capital and physical capital in the accumulation of human capital, as analyzed in Section 4.3 is such a plausible application.

If you start out with the assumption that population is static, we propose two principles:

1. To obtain strictly positive growth in output, capital, consumption, technology and human capital, it is necessary to assume $|A| = 0$.

2. Diminishing marginal returns to existing knowledge in the accumulation of knowledge is consistent with strictly positive growth. Sectoral linearity is sufficient for $|A| = 0$. However, constant returns to scale to growing factors is also sufficient. Moreover, we can move away from constant returns to scale, so long as the degree of returns to scale vary across sectors such that the column or row vectors of matrix A are linearly dependent.

These two principles are also based on new results in this paper and shed new light on early R&D-based growth models. Being able to construct such models without resorting to sectoral linearity, theorists may rediscover early endogenous growth theory in their endeavour to establish strictly positive long run economic growth with a static population.

7 Conclusion

Positive and balanced growth in an economy cannot be obtained without knife edge conditions, whether growth is endogenous or semi-endogenous. By construct of a general three sector growth model, these conditions can be expressed in terms of a matrix of structural elasticities and tailored to the treatment of population as a growing or static factor.

If population grows at a positive, exogenous rate, as in semi-endogenous growth models, diminishing marginal returns and a positive determinant are necessary and sufficient for positive growth in real variables of the models. However, diminishing marginal returns is consistent with endogenous growth if growth in human capital asymptotes to a positive constant. If population is static, the necessary condition for positive growth is singularity of the matrix, which is achieved by imposing either sectoral linearity, constant returns to scale to growing factors or returns to scale that vary across sectors such that vectors of the elasticity matrix are linearly dependent.

Since our general three sector growth model allows for heterogeneity of knowledge and labor, the conditions are universal. That is, they apply to growth models with either

R&D or human capital accumulation or both and either physical labor or human capital augmented labor or both.

The key conditions challenge the convention in growth theory that diminishing marginal returns is synonymous with semi-endogenous growth and linearity is synonmymous with endogenous growth.

Semi-endogenous growth models are premised on the possibility that population grows at a positive rate. However, endogenous growth, or growth independent of the growth rate of population, may arise in such models if growth in human capital asymptotes to a positive constant. This is consistent with diminishing marginal returns to human capital as human capital accumulates. Admittedly, the tendency of growth in human capital to a positive constant implies linearity, albeit asymptotic.

Perhaps the more powerful result of this paper pertains to the early endogenous growth models which assume population is static. We establish that diminishing marginal returns to knowledge in the accumulation of knowledge is consistent with singularity of the matrix of structural elasticities, the only necessary condition for strictly positive rates of growth in real variables. Endogenous growth models, premised on the assumption of zero population growth, can be solved for a strictly positive growth rate in the economy without imposing the restriction to sectoral linearity.

Thus, whether population is growing or static, strictly positive long run economic growth, driven by knowledge accumulation, can be obtained under the more palatable assumption of diminishing marginal returns to knowledge in the accumulation of knowledge.

All this suggests that concerns of zero long run economic growth due to forward projections of zero population growth in the world's hub of R&D may be missplaced. Such fears generate pressure on public policy to boost fertility and immigration rates. To the extent that, under reasonable assumptions, the long run growth rates of economies engaging in R&D may be strictly positive without population growth, these policy reforms are, at best, innocuous.

Because the knife edge conditions for positive growth along a balanced growth path are in terms of structural elasticities, we provide a neat and concise framework for analysing the long run central planner solutions for endogenous and semi-endogenous growth models alike. CES production function are sometimes characterized as cumbersome and difficult to manipulate (Sato 1987). However, using our framework, we solve for the growth rates of all real variables along a balanced growth path in a three sector economy where CES technology desribes the accumulation of human capital in less than one page. Similarly, we solve a central planner version of Funke & Strulik (2000) from a single matrix system and four optimality conditons.

A simple, unified framework for analysing the solutions along a balanced growth path has several benefits. Firstly, we can apply our conditions to well-known models to achieve a more general result, such as, our reparameterisation of Jones (1995). Secondly, we can select the most realistic application of conditions to construct original models to address empirical

anomalies, such as, our introduction of CES technology to human capital accumulation to establish the result that growth in the economy does not require growth in population.

All the models presented in this paper, whether original or central planner versions of existing models, can be given microfoundations, and in each case the equilibrium growth rates in the corresponding decentralized economy can be derived. It is worth noting that growth rates derived for a corresponding decentralized economy differ only by the absence of terms, such as a monopoly markup, that capture the negative spillovers that a central planner internalizes.

We can improve the analysis of our general three sector growth model with a static population by formally defining the relationship between returns to scale across sectors, as implied by equation (16). Mulligan & Sala-i-Martin (1993) and Rebelo (1991) analyze a similar relationship for two sector endogenous growth. We may also flesh out the stability of the system by reference to the first order optimality conditions and analyze the transitional dynamics.

As it is straightforward to generalize conditions in terms of determinants to higher dimensions, the model is readily extended model along the lines suggested by Papageorgiou (2003), which allows for technological imitation in addition to innovation.

A   Appendix

A.1   Derivation of equation (12)

$$g_H = \frac{\dot{H}}{H} = \frac{Q\left(H, k_H K\right)}{H}$$

Differentiating g_H with respect to time,

$$\dot{g}_H = \frac{\left(Q_{k_H}\dot{k}_H + Q_H\dot{H} + Q_K\dot{K}\right)H - Q\left(H, k_H K\right)\dot{H}}{H^2}$$

Noting that $\dot{k}_H = 0$ and $\omega_H \equiv \frac{Q_H H}{Q}$ and $\omega_K \equiv \frac{Q_K K}{Q}$, we obtain:

$$\dot{g}_H = g_H\left\{\omega_H g_H + \omega_K g_K - g_H\right\}$$

Substituting for ω_H and ω_K:

$$\omega_H = \frac{(\phi_1 H)^\rho}{[(\phi_1 H)^\rho + (\phi_2 k_H K)^\rho]}$$

$$= \left(\frac{(\phi_1 H)}{[(\phi_1 H)^\rho + (\phi_2 k_H K)^\rho]^{1/\rho}}\right)^\rho$$

$$= \left(\frac{\phi_1}{g_H}\right)^\rho$$

$$\omega_K = \frac{(\phi_2 k_H K)^\rho}{[(\phi_1 H)^\rho + (\phi_2 k_H K)^\rho]}$$

$$= \frac{(\phi_1 H)^\rho + (\phi_2 k_H K)^\rho - (\phi_1 H)^\rho}{[(\phi_1 H)^\rho + (\phi_2 k_H K)^\rho]}$$

$$= 1 - \left(\frac{\phi_1}{g_H}\right)^\rho$$

gives us equation (12).

A.2   Solution to Section 5.1.1

For the production structure (21a)-(21c), the first order optimality conditions relevant to solving the model are:

$$C^{-\theta} = \lambda \qquad \text{(4a')}$$

$$\lambda(1-\sigma)\frac{Y}{(1-l_A-l_H)H} = \gamma\xi \qquad \text{(4b')}$$

$$\dot{\lambda} = \rho\lambda - \sigma\frac{Y}{K}\lambda \qquad \text{(4d')}$$

$$\dot{\gamma} = \rho\gamma - \xi\gamma \qquad \text{(4f')}$$

Total differentiation of (4a') with respect to time, after inserting (4d') gives Euler's equation: $g_C = \frac{1}{\theta}\left(\sigma\frac{Y}{K} - \rho\right)$. As shown in the beginning of Section 4, the growth rate in physical capital is constant when $g_K = g_Y = g_C$. Differentiating (4b') with respect to time, we obtain $\dot{\lambda}(1-\sigma)\frac{Y}{(1-l_A-l_H)H} = \dot{\gamma}\xi$, so that from (4d) and (4f), we get $\sigma\frac{Y}{K} = \xi$. Substituting for $\sigma\frac{Y}{K} = \xi$ in Euler's equation yields:

$$g_Y = g_K = g_C = \frac{\xi - \rho}{\theta}$$

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