Spatial Patterns of Segregation in a Monocentric City – A Model with a Special Production Function for Housing Services

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Abstract

Usually, in monocentric models of the city the spatial patterns of segregated ethnic groups are assumed to be ring-shaped, while early in the 1930ies Hoyt showed that wedge-shaped areas empirically predominate. After Rose-Ackerman's discussion of the influence of aversion one group of households has against another group within a ring-shaped segregated pattern, Yinger showd that a wedge-shaped pattern may arise, depending on the population mix, as long as border length is responsible for the spatial pattern. In this contribution, a monocentric model with different household groups, a specific production function for housing and other goods and a specific utility function of households is presented. At first, border length is founded as a criterion of optimality. Secondly, it is shown that mixed patterns of concentric and wedge-shaped areas represent multiple equilibria if more than two groups of households are being considered. The welfare optimal segregated pattern depends on the relative production coefficient of households of different groups in the production of goods.

1 Introduction

Empirical observations of current city structures show an increasing amount of segregation by ethnic or lifestyle groups (Sassen 1996; Harth, Herlyn, Scheller 1998; Schneider, Spellerberg 1999; Wagner 2001). Ethnic or other non-economic segregation takes place if there exist different household groups and if there are either negative externalities between households of different groups or positive externalities between households of the same group. An example of the former is racism while an example of the latter is the existence of social networks. Shelling (1978) shows that such externalities lead to a dynamic process of segregation because households choose their location so that either the number of households of the other group in the neighborhood is minimized or the number of households of the same group is maximized. This process is called *tipping-process*.

The analysis of urban segregation brought about two different spatial patterns of areas of different household groups. On the one hand, there is the well discussed ring-shaped pattern according to Alonso's (1960, 1964) description of households' location choice. On the other hand, in the 1930ies, Hoyt (1939) empirically discovered that the dominant spatial segregation pattern in American cities was more ore less wedge-shaped. The basic difference lies in the direction of borderlines which can be either concentric, leading to a ring-shaped pattern, or radial, with wedge-shaped patterns.

In the discussion of segregation caused by ethnic or other non-economic characteristics, focusing on density and pricing structure in space, the spatial pattern usually is given. The arising density structure depends on different assumed causes for segregation. In *border models,* such cause is the border itself. Its influence on density and price structure at any given location decreases with distance. In *amenity models*, density and price structure are affected by the composition of the population in a certain neighborhood while the effects are also decreasing by distance to the respective location.

Rose-Ackerman (1975) describes the effects of racism on the basis of a ring-shaped segregation pattern within a border model. She assumes a ring-shape as the pattern with the shortest border length and thus the least connection between households of different groups. Yinger (1976) shows that, depending on population mix, a wedge-shaped segregation pattern may lead to a minimal border length as well as the lowest number of households on a border.

In this article a special monocentric model of the Alonso-Mills-Muth-type is used to discuss the spatial segregation pattern of two, three and four household groups. Thus, there is a given city center which influences the location decision of households with regard to commuting between any location within the city and the city center. As in the model of Muth (1969), the amount of commuting is an argument of the utility function, based on the idea that a local public good is available in the city center which can be consumed as often as a household commutes. In addition to that, it is an argument of the budget constraint due to transportation costs.

Furthermore, externalities between different types of households are assumed, which affect the evaluation of a neighborhood by households according to their preferences. As a consequence the evaluation of a neighborhood varies with the household type. In this model, only concentric and radial positions of borders are examined. As in the amenity models, the externalities occur in the direct neighborhood of households while their effects disappear as soon as households are not located directly next to each other. This implies a very special distance function.

As a result of the model, the allocation efficiency and stability of different patterns are discussed. It is shown that urban space is divided into segregated areas of household groups according to the relation of their income. A second relation arises if, instead of exogenous income, income is endogenous, derived from a production function for private consumption goods. Then, urban space is divided according to the ratio of productivities, represented by production coefficients of different household types. With this ratio, the spatial segregation pattern can be examined also for more than two household groups.

This paper is divided into three parts. At first, a model containing a local public good, a special production function and externalities between households of different types is presented. The second section contains an extension with endogenous income. In a third part, this version of the model is examined numerically for two, three and four household groups.

2 The model

2.1 Assumptions

Assumption 1 *The population of the city is divided into different groups* $j = 1, ..., i, i', ..., J$ *of households* H_j*. The share of households of one group to the city population is* b_j*.*

Assumption 2 *The households maximize their utility which is represented by the Cobb-Douglas function:*

$$U = z^{\alpha_z} \cdot S^{\alpha_s} \cdot x^{\alpha_x}, \tag{1}$$

in which z represents a local public good, s the consumption of housing services and x the consumption of all other goods. The exponents a_z, a_s, a_x are exogenous and represent the preferences for the different goods.

This utility is a homogenous function of the degree $a_z + a_s + a_x$. It can be expected that $a_z, a_s, a_x < 1$ and $a_z + a_s + a_x \leq 1$. Therefore, there is decreasing marginal utility for every good and for all goods, i.e. for income, according to the usual neoclassical framework.

Assumption 3 *The local public good can be obtained by commuting between the location of housing and the city center. The transportation cost for a unit of the local public good z is t per distance r inside the city and t_{max} outside.*

This public good can either be the typical public service, like administration, infrastructure etc., or it can be interpreted as an immaterial good of the city itself, such as information, lifestyle etc. The important aspect is that it must be obtained by transport or commuting paid by the households.

Assumption 4 *The budget Y varies among the different household types.*

Assumption 5 *The housing service S is produced by the Leotief production function:*

$$S = Min(s, Q), \tag{2}$$

in which s represents land and Q characteristics of the lot's quality. Housing service is standardized to:

$$s = Min\,(1, q)\,, \tag{3}$$

the "qualified land". Then the characteristics of quality are a linear function of the neighborhood n and an amount a of producable characteristics of the lot itself.

$$q = a + n \tag{4}$$

Assumption 6 *The supply of space qm per land is inelastic.*

Assumption 7 *There is an alternative use of land which yields p_b per unit of land.*

The price for alternative land use may either be determined by rural land use or other alternative land uses. It is also the price for housing outside the city.

2.2 Household's behavior

Households maximize their utility subject to the budget constraint:

$$Y_i = t \cdot r \cdot z + p_s \cdot s + p_x \cdot x, \tag{5}$$

where p_x is the price for the consumption bundle x, and p_s is the price for qualified land. Thus optimal demand is:

$$z^* = \frac{\alpha_z \cdot Y_i}{t \cdot r}, \tag{6}$$

$$s^* = \frac{\alpha_s \cdot Y_i}{p_s} \text{ and} \tag{7}$$

$$x^* = \frac{\alpha_x \cdot Y_i}{p_x}. \tag{8}$$

It follows that the indirect utility is:

$$U_j = \left(\frac{\alpha_z \cdot Y_j}{t \cdot r}\right)^{\alpha_z} \cdot \left(\frac{\alpha_s \cdot Y_j}{p_s}\right)^{\alpha_s} \cdot \left(\frac{\alpha_x \cdot Y_j}{p_x}\right)^{\alpha_x}. \tag{9}$$

Solving for p_x, this leads to the well-known bidprice function:

$$\psi_j(r) = \alpha_s \cdot \left(\frac{\alpha_z}{t \cdot r}\right)^{\frac{\alpha_z}{\alpha_s}} \cdot \left(\frac{\alpha_x}{p_x}\right)^{\frac{\alpha_x}{\alpha_s}} \cdot \left(\frac{Y_j^{(\alpha_s+\alpha_x+\alpha_z)}}{U_j}\right)^{\frac{1}{\alpha_s}}. \tag{10}$$

The characteristics are discussed by Alonso (1965) and also by Wheaton (1974) who demonstrates the comparative statics.

2.3 Production of housing services

While the usual bidprice function is related to qualified land, the households are indifferent between locations and neighborhoods as long as the bidprice corresponds. The bidprice function then determines the factor demand in the production of qualified housing. The profit per unit of housing ρ is obtained as:

$$\begin{aligned} \rho_j(r) &= \psi_j(r)\, s - p_a a \\ &= \psi_j(r) \cdot Min\,(1, a+n) - p_a a. \end{aligned} \tag{11}$$

Thus, the profit maximizing level of produced characteristics of quality a follows as:

$$\begin{aligned} a &= 1-n \text{ for } \psi_j(r) > p_a \cdot (1-n) \\ &= 0 \text{ otherwise.} \end{aligned} \tag{12}$$

While for any unit of housing one unit of land is used, the amount of other quality characteristics depends on the quality of the neighborhood. In the bidprice function this leads to:

$$\rho_j(r) = \psi_j(r) - p_a(1-n). \tag{13}$$

The profit, thus, is a linear function of the neighborhood quality. This means that neighborhood quality perfectly substitutes other quality characteristics and reduces their costs. As a consequence the neighborhood quality is responsible for the profit.

2.4 The meaning of neighborhood quality for segregation

If there are externalities so that for households the value of housing is influenced by the population mix within a certain distance, this leads to the below following mechanism of segregation. While the quality level per unit of qualified housing and thus of land is fixed, the externalities n through quality q effect the amount of produced characteristics of quality a. Since the producer of housing has no influence on the quality of the neighborhood, he will choose a level of produced quality which leads to an optimal quality level per land, as long as the expenditure for the produced quality per unit of land is less than the price of this land. Thus the quality level of housing is equal even if there are differences in the quality of the neighborhood.

While the neighborhood's quality may be evaluated differently by households of different groups, the profit of the producer also varies with these households. Consequently a producer has a strong incentive to select a household which will obtain the qualified land. This selection process could occur in a way that the landlord offers housing with a certain level of produced quality characteristics, and only members of a certain household group accept this supply while members of other groups refuse it for neighborhood characteristics.

If these neighborhood characteristics only consist of positive externalities between households of the same group, then $n > 0$ if there are households of the same group in the neighborhood. Under this condition, the landlord will have an incentive to give the housing service to a household belonging to the group which dominates this neighborhood. If he did not discriminate in this way, he would have to add more produced quality characteristics to the same amount of land without getting a higher price.

If, as assumed, the neighborhood externalities disappear with distance, only the direct neighbors will influence the household choice of the landlord.

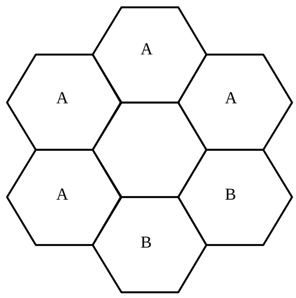

Figure 1: Possible neighborhood of a lot

This leads to a segregation process similar to the tipping-process described by Schelling (1978). If there is a vacant lot, the landlord will choose a household of, for instance, group A (see figure 1) which already dominates this neighborhood. As a result the borderline is minimized. For other locations within this neighborhood occupied by a household of group B, it is possible that households of group A value these locations higher. The landlord can lower the level of produced quality and give the lot to a household of group A when the household of group B finally moves out. By this reallocation, the neighborhood quality in other locations is effected with a similar process in suit. This process reaches an equilibrium when either there is no further demand by group A and the borderline and thus the expenditure of the landlord for produced quality is minimized, or when the whole city is occupied by group A.

In case of a mixed population an equilibrium is stable if any situation with a smaller sum of border lengths cannot be reached without accepting longer borderlines during the adjustment process. Thus, various equilibria may emerge which are equally stable but not equally efficient. The most efficient, however, is the one in which the border length is minimal in comparison to other equilibria. In case of an open city where the utility of all households is determined by the utility level obtainable outside the city, this effiency is the only measure for welfare.

In the following sections only radial and concentric shapes of borderlines will be discussed, assuming that the described segregation process leads to such spatial patterns in an equilibrium situation.

2.5 Equilibrium in the closed and in the open city model

While the supply of land is totally inelastic within the city, there is an equilibrium when every landlord achieves the highest bidprice for his supply of land and when no household can get a higher utility by changing its bidprice or location. A border between the areas of household groups is established when the bidprices of members of these different groups (i and l) are equal:

$$\psi_i(r) = \psi_l(r). \tag{14}$$

Thus, according to equation (10) for all households of different types, there is a constant ratio of income to utility:

$$\frac{Y_i^{(\alpha_s+\alpha_x+\alpha_z)}}{U_i} = \frac{Y_l^{(\alpha_s+\alpha_x+\alpha_z)}}{U_l}. \tag{15}$$

Accordingly, as a special result of this specific model, the bidprice functions of households of different groups at any given location are of equal value. Equation (10) also implies that the bidprice function decreases with distance to the city center:

$$\frac{\partial\psi(r)}{\partial r} < 0. \tag{16}$$

At the radius

$$R = \left(\frac{\alpha_s}{p_b}\right)^{\frac{\alpha_s}{\alpha_z}} \cdot \left(\frac{\alpha_z}{t}\right) \cdot \left(\frac{\alpha_x}{p_x}\right)^{\frac{\alpha_x}{\alpha_z}} \cdot \left(\frac{Y_j^{(\alpha_s+\alpha_x+\alpha_z)}}{U_j}\right)^{\frac{1}{\alpha_z}} \tag{17}$$

it reaches the level of an alternative price for land use p_b. While further away from the city center this alternative price is higher, the bidprice maximizing land use changes at R, at the border of the city. According to equation (15), R has the same value for any household type in the city and thus the shape of the city is circular.

Using of the bidprice function in the demand function for qualified housing, it follows that:

$$s_j(r) = \left(\frac{U_j}{Y_j^{(\alpha_s+\alpha_x+a_z)}}\right)^{\frac{1}{\alpha_s}} \cdot Y_j \cdot \left(\frac{t\cdot r}{\alpha_z}\right)^{\frac{\alpha_z}{\alpha_s}} \cdot \left(\frac{p_x}{\alpha_x}\right)^{\frac{\alpha_x}{\alpha_s}}. \tag{18}$$

The density of households h_i can be calculated by dividing the space at any given location by the demand for qualified housing measured in (qualified) units of land per household:

$$h_i\left(r\right) = 1/s_i = \left(\frac{Y_i^{(\alpha_s+\alpha_x+a_z)}}{U_i}\right)^{\frac{1}{\alpha_s}} \cdot \frac{1}{Y_i} \cdot \left(\frac{\alpha_z}{t \cdot r}\right)^{\frac{\alpha_z}{\alpha_s}} \cdot \left(\frac{\alpha_x}{p_x}\right)^{\frac{\alpha_x}{\alpha_s}}. \quad (19)$$

Thus, apart from the distance to the city center, the density of households depends on their income and the relation $Y_j^{(\alpha_s+\alpha_x+a_z)}/U_j$, which is constant for all households of different groups.

Due to this density function the total population can be calculated by

$$H = \sum_j \int_0^R h_j\left(Y_j, U_j, ..., r\right) dr \text{ for } j = 1, ..., i, ...J. \quad (20)$$

Hence, a relationship between total population and household's utility is established, leading to two different concepts of equilibrium. While either total population or utility has to be given, the other is endogenous. If the utility level is given, say, by a level outside the city, this is referred to as *open city*. If, instead, the total population is given, the city is a *closed city*, with endogenous utility. One reason for this may be prohibitive migration costs.

In case of a closed city the density function, equation (19), is the basis of the equilibrium city structure for concentric as well as for radial spatial patterns of segregated areas.

In the case of an open city there is migration between the city and locations outside as long as there are differences in utility. When these disappear, an equilibrium is reached because no household can be better off by migrating. Therefore, the utility inside and outside the city must be equal. While in this model it is assumed that the utility outside the city can be calculated by a fixed expense for transport of the public good t_{max} and perfect competition between the landlords. The indirect utility outside the city is:

$$\overline{U}_j = \left(\frac{\alpha_z \cdot Y_j}{t_{\max}}\right)^{\alpha_z} \cdot \left(\frac{\alpha_s \cdot Y_j}{p_b}\right)^{\alpha_s} \cdot \left(\frac{\alpha_x \cdot Y_j}{p_x}\right)^{\alpha_x}, \quad (21)$$

and thus the bidprice can be written as:

$$\psi_j\left(r\right) = p_b \cdot \left(\frac{t_{\max}}{t \cdot r}\right)^{\frac{\alpha_z}{\alpha_s}}. \quad (22)$$

Following equation 22 it is obvious that the bidprice function decreases with distance.

$$\frac{\partial \psi(r)}{\partial r} < 0.$$

The range R of the city can be simplified to:

$$R = t_{\max}/t.$$

In the open city the bidprice for a single unit of land (equation 22) is independent of the income of the different household groups, whereas the demand for qualified land:

$$s_j = \frac{\alpha_s \cdot Y_j}{p_b} \cdot \left(\frac{t \cdot r}{t_{\max}}\right)^{\frac{\alpha_z}{\alpha_s}} \tag{23}$$

does follow income. As in the case of the closed city this leads to the density function of any household group j due to the relationship between demand and density

$h_j = 1/s_j$:

$$h_j\left(r\right) = \frac{p_b}{\alpha_s \cdot Y_j} \cdot \left(\frac{t_{\max}}{t \cdot r}\right)^{\frac{\alpha_z}{\alpha_s}}. \tag{24}$$

As expected, the density decreases with distance. Furthermore it is dependent on income. A higher income corresponds with a lower density due to a higher demand for qualified land.

On the basis of these density functions the lengths of borders can be calculated if population mix b_j and income Y_j are known.

2.6 Wedge-shaped segregation patterns

If in a circular city of a range R household groups j segregate in a wedge-shaped pattern, then c_i is the share of such an area of the circular city in which share $\widetilde{b}_i$ of households lives. The number of households belonging to share $\widetilde{b}_i$ can be obtained by

$$\widetilde{H}_i = \int_0^R c_i \cdot 2\pi \cdot r \cdot h_i\left(r\right) \tag{25}$$

while a city's population is

$$\widetilde{H} = \sum_{j=1}^{J} \int_0^R c_j \cdot 2\pi \cdot r \cdot h_j\left(r\right), \tag{26}$$

with $j = 1, .., i, .., J$ being all household groups. Considering equations (18) and (23) in

$$\widetilde{H}_i = \widetilde{b}_i \cdot \widetilde{H} \tag{27}$$

for closed as well as for open cities leads to:

$$\widetilde{b}_i = \frac{c_i Y_i^{-1}}{\sum\limits_{j=1}^{J} c_j Y_j^{-1}}. \tag{28}$$

Solved for c_i this yields:

$$c_i = \frac{Y_i \widetilde{b}_i}{\sum\limits_{j=1}^{J} Y_j \widetilde{b}_j}. \tag{29}$$

Therefore the share of the area of a household group within the whole city corresponds with the share of its purchasing power. Same applies if only a part of the circular city - i.e. a certain ring with an inner border at r_{f-1} and an outer one at r_f - is taken into account. Then the amount of households of a group i is

$$\widetilde{H}_{f,i} = \int\limits_{r_{f-1}}^{r_f} c_i \cdot 2\pi \cdot r \cdot h_i(r) \tag{30}$$

while the amount of all households of this ring is

$$\widetilde{H}_f = \sum_{j=1}^{J} \int\limits_{r_{f-1}}^{r_f} c_j \cdot 2\pi \cdot r \cdot h_i(r). \tag{31}$$

The solution for c_i remains unchanged.

2.7 Ring-shaped segregation patterns

If in a circular city of a range R households j segregate in a ring-shaped pattern, population can be calculated as

$$\widehat{H} = \sum_{j=1}^{J} \int\limits_{r_{j-1}}^{r_j} 2\pi \cdot r \cdot h_j(r) dr. \tag{32}$$

Given radius r_i as the outer and r_{i-1} as the inner borderline of the segregated area of any group i, the number of households of this group follows as

$$\widehat{H}_i = \int_{r_{i-1}}^{r_i} 2\pi \cdot r \cdot h_i(r) dr. \tag{33}$$

Its share $\widehat{b}_i$ of the city's population, containing $j = 1, .., i, .., J$ groups, for closed as well as for open cities, can be written as:

$$\widehat{b}_i = \frac{\left(r_i^{\frac{2 \cdot \alpha_s - \alpha_z}{\alpha_s}} - r_{i-1}^{\frac{2 \cdot \alpha_s - \alpha_z}{\alpha_s}} \right) \cdot Y_i^{-1}}{\sum_{j=1}^{J} \left(r_j^{\frac{2 \cdot \alpha_s - \alpha_z}{\alpha_s}} - r_{j-1}^{\frac{2 \cdot \alpha_s - \alpha_z}{\alpha_s}} \right) \cdot Y_j^{-1}}, \tag{34}$$

while for households of different groups the ratio $Y_j^{(\alpha_s + \alpha_x + a_z)}/U_j$ is the same. The outer radius of any area follows as:

$$r_i = \left(r_{i-1}^{\frac{2 \cdot \alpha_s - \alpha_z}{\alpha_s}} + \frac{Y_i \cdot \widehat{b}_i}{\sum_{j=1}^{J} Y_j \cdot \widehat{b}_j} \cdot \left(r_{i+1}^{\frac{2 \cdot \alpha_s - \alpha_z}{\alpha_s}} - r_{i-1}^{\frac{2 \cdot \alpha_s - \alpha_z}{\alpha_s}} \right) \right)^{\frac{\alpha_s}{2 \cdot \alpha_s - \alpha_z}}. \tag{35}$$

For the area closest to city center $r_{i-1} = 0$, and for the area farthest from the city center $r_{i+1} = R$. Thus for any area the radius of the outer border is:

$$r_i = \left(\frac{\sum_{j=1}^{i} Y_j \cdot \widehat{b}_j}{\sum_{j=1}^{J} Y_j \cdot \widehat{b}_j} \right)^{\frac{\alpha_s}{2 \cdot \alpha_s - \alpha_z}} \cdot R \text{ with } i < J. \tag{36}$$

This shows that the ratio of the ranges and therefore, as in a wedge-shaped pattern, the share of areas are based on the shares of purchasing power.

In addition to that, equation (36) reveals that the radius and thus the area increase with rising b_i if $\alpha_s / (2\alpha_s - \alpha_z) > 0$. This is the case for a limited value domain with $\alpha_z < 2\alpha_s$ only. Since $a_z > 0$, the expression $\alpha_s / (2\alpha_s - \alpha_z)$ is between $1/2$ and infinity and the value domain for α_s is

$$0 < \alpha_z < 2\alpha_s. \tag{37}$$

If the households $\widehat{H}$ are not located throughout the whole city but only in a wedge-shaped part of it, the radius is calculated in nearly the same way.

When a wedge-shaped part with the share of c_f is regarded separately, the number of households of a ring-shaped part of this wedge-shaped area is

$$\widehat{H}_{f,i} = \int_{r_{i-1}}^{r_i} c_f \cdot 2\pi \cdot r \cdot h_j(r)dr \tag{38}$$

while the number of all households in this wedge-shaped area is

$$\widehat{H}_f = \sum_{j=1}^{J} \int_{r_{j-1}}^{r_j} c_f \cdot 2\pi \cdot r \cdot h_j(r)dr. \tag{39}$$

As in the case discussed above, the solution for the radius r_i remains unchanged.

Ring and wedge-shaped segregation patterns are connected by the following relationship. The total population is equal if the population mix is equal:

$$\widetilde{H} = \widehat{H} = H \tag{40}$$

if

$$\widetilde{b}_i = \widehat{b}_i = b_i. \tag{41}$$

In summary, in the wedge-shaped as well as in the ring-shaped segregation pattern the share of purchasing power is responsible for the location of borderlines.

2.8 Segregation patterns for two household groups with exogenous population mix

In this subsection the spatial pattern for two household groups will be examined, therefore set $J = 2$. The groups' shares of the population can be simplified to $b_1 = b$ and $b_2 = 1 - b$. For a ring-shaped segregation pattern border length G_R follows as:

$$G_R = 2\pi \left(\frac{Y_1 b}{Y_1 b + Y_2(1-b)} \right)^{\frac{\alpha_s}{2\cdot\alpha_s - \alpha_z}} \cdot R. \tag{42}$$

While the radius of the city border R is independent of total population, population mix etc., for a wedge-shaped segregation pattern the border length between two groups is constant:

$$G_S = 2 \cdot R. \tag{43}$$

In order to obtain the most efficient segregation pattern it is necessary to compare the border lengths of both cases. It follows:

$$G_R \gtreqless G_S, \tag{44}$$

if

$$\frac{Y_1 b}{Y_1 b + Y_2 (1 - b)} \gtreqless \frac{1}{\pi^{\frac{2 \cdot \alpha_s - \alpha_z}{\alpha_s}}}. \tag{45}$$

The critical value for b follows as:

$$b^* = \frac{Y_2}{\left(\pi^{\frac{2 \cdot \alpha_s - \alpha_z}{\alpha_s}} - 1\right) Y_1 + Y_2}. \tag{46}$$

If the share of a group is larger than this critical value, the length of the border in a ring-shaped pattern is longer and the pattern less efficient than in a wedge-shaped pattern and vice versa. Note that the critical value of the share is a linear function of the ratio of income of the household groups $\theta = Y_2/Y_1$:

$$b^* = \frac{\theta}{\left(\pi^{\frac{2 \cdot \alpha_s - \alpha_z}{\alpha_s}} - 1\right) \cdot \theta + 1}. \tag{47}$$

Thus, with increasing differences in income distribution, the critical value of the share of the household group with the lower income rises. But if this ratio of income approaches infinity, the critical share of the group will remain at a certain level:

$$\lim_{\theta \to \infty} b^* = \frac{1}{\pi^{\frac{2 \cdot \alpha_s - \alpha_z}{\alpha_s}} - 1}. \tag{48}$$

If the income is equal, the critical value is:

$$b^*_{\theta=1} = \frac{1}{\pi^{\frac{2 \cdot \alpha_s - \alpha_z}{\alpha_s}}}, \tag{49}$$

which for $\alpha_s = 0,3$ and $\alpha_z = 0,1$ amounts to about $0,148$. This finding is close to the results of Yinger (1976) who discussed this case with a different specification of the monocentric model.

While in this subsection the share of population and the income of household groups are exogenous, there is no endogenous reason for a mixed city. Without further assumptions the most efficient would be a city with only one household group and thus without any border if the population mix was not exogenously given. In order to discuss a motivation for a mixed city, in the next subsection the production of consumption goods and endogenous income of the household groups will be introduced.

2.9 Introduction of endogenous income

In order to introduce endogenous income it is nessescary to make a few additional assumptions:

Assumption 8 *The city is open.*

Assumption 9 *The households are employed in the production of consumption goods x.*

Assumption 10 *The production function is:*

$$x = \prod_j H_j^{\beta_j} \tag{50}$$

with H_j representing the number of households and j the household groups. β_j is a production coefficient varying accross household groups.

Assumption 11 *Each household supplies one unit of labour. While the supply of labour of a single household is inelastic, total labour supply is elastic due to migration.*

Assumption 12 *Outside the city, households obtain an alternative income of $\overline{Y}_j$.*

According to the production function assumed, income follows the marginal product of labour:

$$\begin{aligned} Y_i &= p_x \cdot \frac{\partial \prod_{j=1}^{J} H_j^{\beta_j}}{\partial x} \\ &= \beta_i \cdot H_i^{-1} \cdot \prod_{j=1}^{J} H_j^{\beta_j} \cdot p_x. \end{aligned} \tag{51}$$

The number of households per group can be substituted by their share of population and total population H:

$$Y_i = \beta_i \cdot \frac{\prod_{j=1}^{J} b_j^{\beta_j}}{b_i} \cdot H^{\sum_{j=1}^{J} \beta_j - 1} \cdot p_x. \tag{52}$$

Therefore, the population mix, besides its negative impact on land rent, has a positive effect on production and thus on the income of every household.

Thereby it is possible to derive the total population and the population mix from the exogenously given production coefficients and an alternative income $\overline{Y}_i$ obtainable outside the city. If the income of any household group i differs between inside and outside the city, due to migration the equilibrium condition for the labour market yields the following bidprice function:

$$\Psi_i(r) = p_b \cdot \left(\frac{t_{\max}}{t \cdot r}\right)^{\frac{\alpha_z}{\alpha_s}} \cdot \left(\frac{Y_i}{\overline{Y}_i}\right)^{\frac{\alpha_x+\alpha_s+\alpha_z}{\alpha_s}} \tag{53}$$

where Y_i is endogenous according to equation (52). Obviously, the higher the income inside the city relative to income outside the city, the higher the bidprices are:

$$\frac{\partial\Psi(r)}{\partial\frac{Y_i}{\overline{Y}_i}} > 0. \tag{54}$$

As a further equilibrium condition it has to be taken into account that for any household group the bidprice anywhere in the city must be the highest one of all groups. Otherwise they could not get any location at all. While income is independent of location, there is coexistence of different household groups, for example i und l, only if:

$$\frac{Y_i}{\overline{Y}_i} = \frac{Y_l}{\overline{Y}_l}. \tag{55}$$

In this case the bidprices of those two household groups are equal for every location in the city. This leads to the result that in an equilibrium, there is

$$\frac{Y_j}{\overline{Y}_j} = const. \tag{56}$$

for any household group j located in the city. In addition, the ratios of income of different household groups from outside the city will also be established inside the city:

$$\frac{Y_i}{Y_l} = \frac{\overline{Y}_i}{\overline{Y}_l}. \tag{57}$$

With these equilibrium conditions population mix and total population may be derived as follows. The demand for qualified land is:

$$s_i^* = \frac{\alpha_s}{p_b} \cdot Y_i \cdot \left(\frac{t \cdot r}{t_{\max}}\right)^{\frac{\alpha_z}{\alpha_s}} \cdot \left(\frac{\overline{Y}_j}{Y_j}\right)^{\frac{\alpha_x+\alpha_s+\alpha_z}{\alpha_s}}. \tag{58}$$

The density function may be reformulated as:

$$h_i^* = \frac{p_b}{\alpha_s} \cdot \frac{1}{Y_i} \cdot \left(\frac{t_{\max}}{t \cdot r}\right)^{\frac{\alpha_z}{\alpha_s}} \cdot \left(\frac{Y_j}{\overline{Y}_j}\right)^{\frac{\alpha_x+\alpha_s+\alpha_z}{\alpha_s}}. \tag{59}$$

Thus, the higher the income in the city is compared to outside the city, the higher the density inside the city. If this density is used for calculating the number of households of one group and total population, after a few manipulations the population mix follows as:

$$b_i = \frac{\beta_i \cdot \overline{Y}_i^{-1}}{\sum_{j=1}^{J} \left(\beta_j \cdot \overline{Y}_j^{-1} \right)}. \tag{60}$$

The total population is:

$$\begin{aligned} H \quad = \quad & \left[\left(2\pi \cdot \frac{\alpha_s}{2\alpha_s - \alpha_z} \cdot \left(\frac{t_{\max}}{t} \right)^2 \cdot p_b \cdot \sum_{j=1}^{J} \left(\beta_j \cdot \frac{1}{\overline{\overline{Y}}_j}^{\frac{\alpha_x + \alpha_s + \alpha_z}{\alpha_s}} \right) \cdot \right. \right. \\ & \left. \left. \left(\sum_{j=1}^{J} \beta_j \right)^{-1} \right)^{\alpha_s} \left(\prod_{j=1}^{J} \left(\frac{\beta_j}{\sum_{j=1}^{J} \beta_j} \right)^{\beta_j} \cdot p_x \right) \right]^{\left(\frac{1}{\alpha_x + \alpha_s + \alpha_z - (\alpha_x + \alpha_z) \sum_{j=1}^{J} \beta_j} \right)} \end{aligned} \tag{61}$$

Thus, population mix and total population are determined by production coefficients and alternative income of the household groups.

Using the income from equation (52) in (35), it follows that:

$$\frac{b_i Y_i}{\sum_{j=1}^{J} b_j Y_j} = \frac{\beta_i}{\sum_{j=1}^{J} \beta_j}. \tag{62}$$

The share of one group's purchasing power, which is responsible for the location of borders, corresponds with their share of the sum of production coefficients. Thus, the spatial pattern is neither dependent on an alternative income obtainable outside the city nor dependent on population mix or total population. The shares of areas and thus the size of segregated areas and borders are dependent on the production coefficients only.

2.10 Solutions: Spatial patterns of segregation

Taking, according to equation (62), the ratio of production coefficients instead of the ratio of purchasing power, the share of a wedge-shaped area at a ring follows as:

$$c_i = \frac{\beta_i}{\sum\limits_{j=1}^{J} \beta_j}. \tag{63}$$

The radius of an outer border in a ring-shaped pattern follows as:

$$r_i = \left(\frac{\sum\limits_{j=1}^{i} \beta_j}{\sum\limits_{j=1}^{J} \beta_j} \right)^{\frac{\alpha_s}{2 \cdot \alpha_s - \alpha_z}} \cdot R. \tag{64}$$

Comparing the border lengths of the different patterns for two household groups,

$$GL_R \gtreqless GL_S, \tag{65}$$

if

$$\frac{\beta_1}{\beta_1 + \beta_2} \gtreqless \frac{1}{\pi^{\frac{\alpha_s}{2 \cdot \alpha_s - \alpha_z}}}. \tag{66}$$

Thus, the ratio of production coefficients is responsible for the ratio of purchasing power and for the border length. Consequently, it also determines the most efficient segregation pattern:

$$\frac{\beta_1}{\beta_2} \gtreqless \frac{1}{\pi^{\frac{\alpha_s}{2 \cdot \alpha_s - \alpha_z}} + 1}. \tag{67}$$

In order to obtain the most efficient ring-shaped pattern, it is necessary that the group with the lower production coefficient lives closer to the city center than the other group. If α_z is restricted to values between 0 and $2\alpha_s$, as of equation (37), the expression on the right hand side of equation (67) is of a value smaller than or equal to $0,361$. Thus, depending on preferences for qualified land and the local public good, the ring-shaped pattern is more efficient only if there are strong differences in the productivity of the different household groups.

With three different household groups ($J = 3$ in equation (29) and (35)), there are five possible spatial segregation patterns (figure 2) with 22 possible distributions of household groups. The sum of border lengths of a certain pattern can be calculated by summing up all single border lines. The total length is dependent on the spatial order of groups. In any ring-shaped pattern, even only in parts of the city, to obtain the minimum border length the respective groups have to be located the closer to the city center the lower their production coefficients are. For wedge-shaped areas within a ring, the distribution of household groups is unimportant. Thus for every pattern one single obvious distribution of different household groups arises as the most efficient. In the fifth pattern the wedge-shaped area is occupied by the group with the highest production coefficient.

Table 1 presents the total border lengths for different numeric situations. While $R = 10$ and $a_s = 0,3$, different values for β_j and a_z are examined.The values of the production coefficients are set as $\sum_{j=1}^{J} \beta_j = 1$. As result, the ranking of border lengths varies with the cases. The shortest border lengths are printed bold.

Case	α_z	β_1	β_2	β_3	G_{301}	G_{302}	G_{303}	G_{304}	G_{305}
1	0.1	0.01	0.09	0.90	30.00	**19.75**	20.81	22.70	21.58
2		0.02	0.18	0.80	30.00	29.93	31.54	24.10	**23.16**
3		0.03	0.22	0.75	30.00	35.01	36.06	25.22	**24.40**
4		0.04	0.26	0.70	30.00	39.62	40.22	26.21	**25.63**
5		0.05	0.30	0.65	30.00	43.88	44.12	27.10	**26.84**
6		0.10	0.30	0.60	**30.00**	52.04	47.80	30.76	30.94
7		0.20	0.30	0.50	**30.00**	65.38	54.65	36.31	38.13
8		0.30	0.30	0.40	**30.00**	76.76	60.97	40.80	44.87
9	0.3	0.01	0.09	0.90	30.00	**6.91**	8.28	20.43	20.63
10		0.02	0.18	0.80	30.00	**13.82**	16.57	20.86	21.26
11		0.03	0.22	0.75	30.00	**17.59**	20.71	21.29	21.89
12		0.04	0.26	0.70	30.00	**21.36**	24.85	21.71	22.51
13		0.05	0.30	0.65	30.00	25.13	28.99	**22.14**	23.14
14		0.10	0.30	0.60	30.00	31.42	33.13	**24.28**	26.28
15		0.20	0.30	0.50	30.00	43.98	41.42	**28.57**	32.57
16		0.30	0.30	0.40	**30.00**	56.55	49.70	32.85	38.85

Table 1: Total border lengths of different patterns for three household groups

Therefore, the most efficient segregation patterns also vary from case to case. Especially in cases with strong differences in production coefficients, patterns with a mixture of ring and wedge-shaped areas are most efficient. For production coefficients converging to the same level, pure wedge-shaped patterns are superior.

In case of four household groups ($J = 4$) the number of possible patterns rises to 12 (figure 3). This, in turn, leads to $12 \cdot 4^4$ different possible distributions of household groups to segregated areas. Again, there obviously

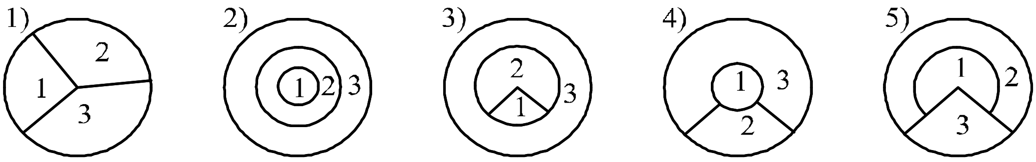

Figure 2: Possible patterns of segregation for three household groups

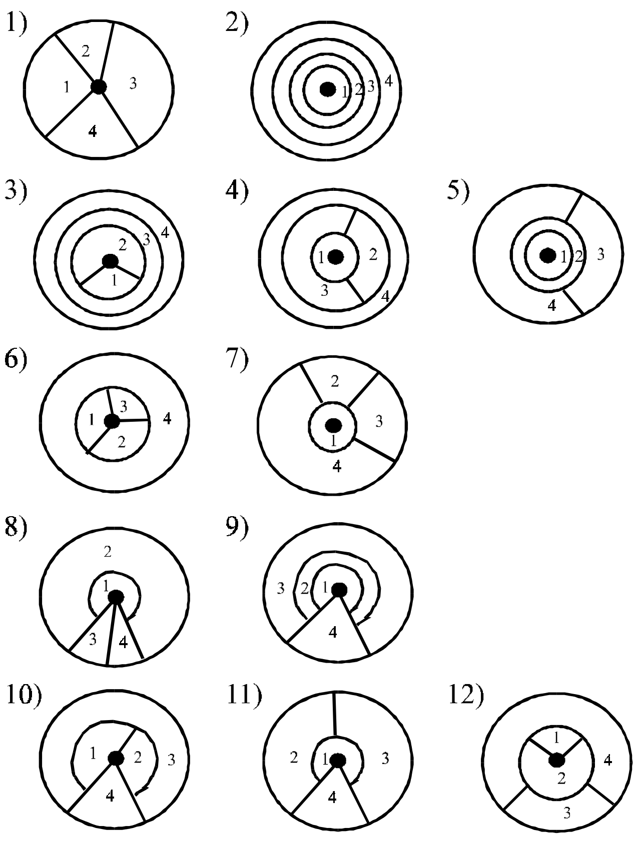

Figure 3: Possible segregation patterns for 4 household groups

is only one most efficient distribution of households in each pattern. The border lengths of these may be examined, as above, by adding up the border lengths of the different areas for each pattern:

As before, different values for β_j with $\sum_{j=1}^{J} \beta_j = 1$ and a_z are examined for $R = 10$ and $a_s = 0,3$. Again the most efficient pattern depends on numeric values of the variable (see tables 2a and 2b).

As with three household groups, for different values of production coefficients patterns with a mix of ring and wedge-shaped areas are more efficient whereas in cases with similar values the pure wedge-shaped pattern is the most efficient one.

Case	α_z	β_1	β_2	β_3	β_4	G_{401}	G_{402}	G_{403}	G_{404}
1	0.1	0.01	0.02	0.03	0.94	40.00	23,25	39.64	18.65
2		0.01	0.02	0.12	0.85	40.00	31,76	48.15	29.87
3		0.01	0.02	0.25	0.72	40.00	40,90	57.30	41.93
4		0.01	0.02	0.30	0.67	40.00	43,94	60.33	45.92
5		0.01	0.12	0.25	0.59	40.00	58,66	76.54	50.60
6		0.01	0.28	0.28	0.43	40.00	78,71	98.04	62.45
7		0.12	0.25	0.25	0.38	**40.00**	99,37	105.93	76.98
8		0.25	0.25	0.25	0.25	**40.00**	121,70	119.83	92.70
9	0.3	0.01	0.02	0.03	0.94	40.00	6.28	26.26	**5.50**
10		0.01	0.02	0.12	0.85	40.00	**11.94**	31.91	22.95
11		0.01	0.02	0.25	0.72	40.00	**20.11**	40.08	23.72
12		0.01	0.02	0.30	0.67	40.00	23.25	43.22	27.86
13		0.01	0.12	0.25	0.59	40.00	33.68	56.33	32.99
14		0.01	0.28	0.28	0.43	40.00	54.66	79.84	47.74
15		0.12	0.25	0.25	0.38	40.00	69.74	89.60	57.70
16		0.25	0.25	0.25	0.25	40.00	94.25	108.54	75.33

Table 2 a: Total border lengths of different patterns for four household groups (Part 1)

Case	$\mathbf{G_{405}}$	$\mathbf{G_{406}}$	$\mathbf{G_{407}}$	$\mathbf{G_{408}}$	$\mathbf{G_{409}}$	$\mathbf{G_{410}}$	$\mathbf{G_{411}}$	$\mathbf{G_{412}}$
1	30.37	**17.16**	32.07	30.98	23.77	29.09	31.19	27.66
2	30.37	29.74	32.07	30.98	**25.44**	27.40	31.76	27.66
3	30.37	43.25	32.07	30.98	**26.99**	27.22	32.28	27.66
4	30.37	47.73	32.07	30.98	27.46	**27.29**	32.44	27.66
5	41.57	52.91	32.11	**31.81**	35.71	38.19	32.67	38.81
6	52.60	66.26	32.07	**32.42**	47.04	50.54	33.07	49.90
7	66.61	69.68	39.20	41.83	63.12	55.92	43.34	54.60
8	80.10	78.12	44.29	50.73	81.32	64.79	51.88	61.45
9	22.31	5.57	30.33	30.63	22.51	26.89	30.53	21.89
10	22.31	13.93	30.33	30.63	22.51	23.89	30.53	21.89
11	22.31	25.99	30.33	30.63	22.51	22.96	30.53	21.89
12	22.31	30.64	30.33	30.63	22.51	22.79	30.53	**21.89**
13	28.86	36.37	30.34	30.65	29.07	31.84	30.55	**28.42**
14	38.65	52.91	**30.33**	30.63	38.85	43.31	30.53	38.22
15	48.39	57.56	**33.94**	37.54	50.79	49.22	36.34	43.25
16	62.12	69.62	**38.21**	45.71	67.12	58.08	43.21	51.42

Table 2 b: Total border lengths of different patterns for four household groups (Part 2)

3 Concluding remarks

In this contribution, within a specific model of the Alonso-Mills-Muth-type, spatial segregation caused by positive externalities between households of a same group, such as a social network, is discussed. When such externalities are introduced by a special production function for qualified land as housing service, segregation arises as a result of a selection by profit maximizing suppliers.

Due to the admission of radial as well as concentric borderlines, different possible spatial patterns of segregation emerge according to the number of household groups. The resulting efficiency of the city depends on border length. While in the case of an open city the utility of households is exogenously given, border length is the single criteria of economic efficiency and thus of welfare of the city.

The total border lengths of any pattern, besides household's preferences for housing service and for a local public good, depends on the relative purchasing power of different household groups. The most efficient segregation pattern hinges on the numeric value of the variables.

A mixed city only emerges when, apart from externalities in consumption, income is positively related to the population mix. The population mix can be endogenized by introducing a production function for consumption goods containig households of different groups as different production factors. Then, instead of purchasing power, the ratios of production coefficients are the driving parameter of the spatial structure. As a result of the special functions, spatial structure is independent of population mix and total population.

Examining the cases of three and four household groups, various possible spatial patterns containing a mix of concentric and radial borders between segregated areas emerge. The more similar households of different groups are with regard to their production coefficients, the more likely a pure wedge-shaped segregation pattern is the most efficient one. For less similar production coefficients mixed patterns with edge and ring-shaped areas are most efficient. This closes the gap between the comparative static analysis and the empirical results of Homer Hoyt (1939).

Last but not least, this model shows that with only a few additional assumptions the traditional monocentric model leads to other than ring-shaped patterns of residential land use.

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