Level shifts in a panel data based unit root test. An application to the rate of unemployment

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Abstract

Several unit root tests in panel data have recently been proposed. The test developed by Harris and Tzavalis (1999 JoE) performs particularly well when the time dimension is moderate in relation to the cross-section dimension. However, in common with the traditional tests designed for the unidimensional case, it was found to perform poorly when there is a structural break in the time series under the alternative. Here we derive the asymptotic distribution of the test allowing for a shift in the mean, and assess the small sample performance. We apply this new test to show how the hypothesis of (perfect) hysteresis in Spanish unemployment is rejected in favour of the alternative of the natural unemployment rate, when the possibility of a change in the latter is considered.

1 Introduction

Several recent papers have addressed the question of testing for a unit root in panel data. Levin and Lin (1993) can be considered the seminal paper in this field, stimulating further developments such as those reported in Im, Pesaran, and Shin (1997), Phillips and Moon (1999b, 1999b) and Breitung (2000). Among these proposals particular mention should be made of the test designed by Harris and Tzavalis (1999) -hereafter, HT- in which asymptotics in T are not required because results are derived considering $N \to \infty$ with T fixed. This makes their proposal particularly attractive when working with panels whose time dimension is moderated in relation to the cross-section dimen-

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sion.

Notwithstanding, this flourishing stream of literature has paid little attention to the deterministic component of these models. It is well known that any misspecification of the trend function can give misleading results when testing for unit roots in univariate time series. This was first demonstrated by Perron (1989) and extended by Banerjee, Dolado, and Galbraith (1990), Perron (1990, 1997), Banerjee, Lumsdaine, and Stock (1992), Perron and Vogelsang (1992), Zivot and Andrews (1992), Montañés and Reyes (1997), Vogelsang (1997) and Vogelsang and Perron (1998), among others.

In this paper we propose a procedure to test for a unit root in a panel setting that allows for the possibility of a structural break located at an unknown time. We extend the test proposed by HT to take into account the existence of a level shift in the deterministic part of the time series. Such a possibility has already been identified as being of interest in the analysis of certain economic time series -see, for instance, Perron and Vogelsang (1992) for the PPP hypothesis. Undoubtely, the consideration of just one structural break, with a breakpoint located at a common date for the whole panel is the main shortcoming of our proposal. However, it should be stressed that we allow for different reactions given that the magnitude of the shift can vary for each individual in the panel. Therefore, the procedure proposed here is particularly attractive when the individuals in the cross-section are affected by similar shocks, though respond differently. Thus, it can be applied to regional data sets since the set of regions that makes up a country are presumably subjected to the same shocks. Additionally, this could also be the case for samples of integrated economies such us the EU countries or even some sets of OECD economies -see Strauss (2000) for a recent application of a unit root test in panel data allowing for an exogenous structural break.

We illustrate the application of the test with an exercise that seeks to discriminate between the natural rate and the hysteresis hypothesis for Spanish unemployment. This is particularly interesting because the Spanish economy has been recognised as one of the most notable cases of extreme persistence in unemployment.

The rest of the paper is structured as follows. In section 2 we discuss the effects of a shift in the mean on the standard panel data unit root test proposed by HT. Section 3 generalises their test to allow for the presence of a structural break in the time dimension of the panel and derives the asymptotic distribution for this particular case. Section 4 deals with the estimation of the date of the break. Section 5 analyses the finite sample performance of our proposal while in section 6 the test is applied to the unemployment rates of the Spanish provinces. Finally, section 7 concludes.

2 The motivation

The earliest contributions from econometrics for testing the unit root hypothesis within a panel data framework have presented statistical tools that consider asymptotics in both dimensions of the panel ($T \to \infty$ and $N \to \infty$). Levin and Lin (1993), Quah (1994) and Im, Pesaran, and Shin (1997) have taken this approach and it is appropriate for testing for the unit root in panels with a sufficiently large number of cross-section individuals and time periods. However, they are frequently applied to panels of moderate time dimension compared to the number of individuals. Indeed, the use of tests developed for panels with both large sample size and a high number of individuals -double limiting distributions- for a panel in which one of the dimensions should be considered as fixed is questionable. HT address this concern and develop a unit root test that only considers asymptotics in N with T fixed. Our analysis here focuses on this latter approach.

Three models are specified by HT when testing for the unit root hypothesis, depending on the deterministic component specified under the alternative. These are the panel data counterparts of the usual models specified in the univariate unit root tests:

$$y_{i,t} = \varphi\, y_{i,t-1} + v_{i,t}; \tag{1}$$
$$y_{i,t} = \alpha_i + \varphi\, y_{i,t-1} + v_{i,t}; \tag{2}$$
$$y_{i,t} = \alpha_i + \beta_i\, t + \varphi\, y_{i,t-1} + v_{i,t}, \tag{3}$$

$i = 1, \dots, N; t = 1, \dots, T$, with $\{v_{i,t}\}$ satisfying the following set of assumptions.

Assumption 1

a. $\{v_{i,t}\}, i = 1, \dots, N; t = 1, \dots, T$, is a series of independent identically normally distributed random variables having $E(v_{i,t}) = 0$ and $Var(v_{i,t}) < \infty$, for all i and t;

b. the initial values, $y_{i,0}$, are fixed;

c. the individual effects, α_i, are fixed.

The first assumption indicates that each individual is independent of each other. HT claim that this can be guaranteed by removing the cross-section mean from the data. Normal distribution of $\{v_{i,t}\}$ is more a convenience than a requirement, given that it simplifies the derivation of the moments that define the asymptotic distribution of the test. This assumption can be relaxed provided that a consistent estimator for $E\left(v_{i,t}^4\right)$ is available. The second and third assumptions prevent the introduction of additional probability distributions in the model.

One of the special features of these models is that they consider a common value for the autoregressive parameter for all the individuals, that is, they are designed to test for the presence of a unit root in the whole set of time series. As for the deterministic component, (1) does not consider any deterministic element, (2) specifies individual fixed effects whereas (3) accounts for both individual and trend effects. Based on the following expression for the estimation of the autoregressive parameter:

$$\hat{\varphi} = \left[\sum_{i=1}^{N} y'_{i,-1} Q_T y_{i,-1}\right]^{-1} \left[\sum_{i=1}^{N} y'_{i,-1} Q_T y_i\right], \tag{4}$$

where $y_{i,-1} = (y_{i,0}, ..., y_{i,T-1})', y_i = (y_{i,1}, ..., y_{i,T})'$ with $Q_T = I_T - x\,(x'x)^{-1}\,x'$, being x the $(T \times k)$ matrix of non-stochastic regressors, HT derive the asymptotic distribution for the normalised bias test which is shown to converge to a normal distribution with parameters derived in Theorems 1 to 3 of their paper.

One of the obvious drawbacks of this test is that the specifications given by models (1) to (3) do not allow for structural breaks in the time dimension. This has already been shown to be a critical point when assessing the performance of a unit root test designed for the unidimensional case as it can lead to the drawing of incorrect inferences.

In the rest of this section we analyse the effect of a structural break in the time series mean on the normalised bias test of HT for model (2). We are conscious that this analysis can be extended to the other models and to other different effects of the structural break but this is left for future research. Our goal is to assess whether the test tends towards the non rejection of the null hypothesis of a unit root when the DGP is stationary around a level shift model as it does in the univariate framework -see Perron (1989, 1990) and Montañés and Reyes (1998). As a first step, we conducted a simple Monte Carlo experiment to analyse the effect of the structural break in the mean on the empirical power of the test. We consider the following DGP:

$$y_{i,t} = \alpha_i + \theta_i \, DU_t + \varphi \, y_{i,t-1} + v_{i,t}, \tag{5}$$

with $\alpha_i = 0 \, \forall i$, where $DU_t = 1$ for $t > T_b$ and 0 elsewhere, with T_b indicating the date of the break and $v_{i,t} \sim N(0,1)$, $i = 1, \ldots, N$; $t = 1, \ldots, T$. Different magnitudes of the structural break have were specified, $\theta_i = \{3, 5, 7\}$, $\forall i = 1, \ldots, N$, all positioned in the middle of the time period, $T_b = 0.5T$, with different values for the autoregressive parameter, $\varphi = \{0, 0.5, 0.8\}$. We selected the values that fit best with the proposal made by Harris and Tzavalis (1999) for the sample size and the number of individuals and so $T = \{25, 50\}$ and $N = \{10, 25, 50, 100\}$. We carried out 5,000 replications. The results of the simulation experiment are summarised in Table 1. The way in which the results are organized reveals some expected features: the power of the test decreases as the DGP tends to approach the null hypothesis ($\varphi = 1$), and increases with the number of individuals. More interestingly, the power decreases with the magnitude of the structural break (θ).

The evidence provided by the Monte Carlo experiment can be derived analytically. To do so we consider the DGP given by (5) with $|\varphi| < 1$. If a practitioner does not take the structural break into account he or she will proceed to estimate model (1b) of HT given by (2), for which it can be easily established that the estimator of φ is computed from (4) with $x = e_T = (1, \ldots, 1)'$ the $(T \times 1)$ vector of non-stochastic regressors. However, notice that the model assumed as the DGP, along with the independent term, considers the effect of a structural break through the DU_t dummy variable. Hence, the actual matrix of non-stochastic regressors is $z = [e_T, DU]$. Therefore, the numerator of (4) can be written as:

$$y'_{i,-1} Q_T y_i = y'_{i,-1} Q_T \left(z \, \zeta_i + \varphi \, y_{i,-1} + v_i \right),$$

with $\zeta_i = (\alpha_i, \theta_i)'$ and $v_i = (v_{i,1}, ..., v_{i,T})'$, so that (4) is equal to:

$$\hat{\varphi} = \varphi + \frac{\sum_{i=1}^{N} y'_{i,-1} Q_T v_i}{\sum_{i=1}^{N} y'_{i,-1} Q_T y_{i,-1}} + \frac{\sum_{i=1}^{N} y'_{i,-1} Q_T z \, \zeta_i}{\sum_{i=1}^{N} y'_{i,-1} Q_T y_{i,-1}}. \tag{6}$$

Table 1: Power of the Harris-Tzavalis test in the presence of a structural break in the mean

θ	$N\backslash T$	$\varphi=0$		$\varphi=0.5$		$\varphi=0.8$	
		25	50	25	50	25	50
3	10	1	1	1	1	0.86	1
	25	1	1	1	1	0.99	1
	50	1	1	1	1	1	1
	100	1	1	1	1	1	1
5	10	0.92	1	0.09	0.99	0.01	0.53
	25	1	1	0.45	1	0.02	0.97
	50	1	1	0.92	1	0.03	0.99
	100	1	1	0.99	1	0.05	1
7	10	0.01	0.99	0	0.13	0	0.01
	25	0.16	1	0	0.90	0	0.04
	50	0.75	1	0	1	0	0.17
	100	0.99	1	0	1	0	0.61

The DGP is given by $y_{i,t} = \alpha_i + \theta_i \; DU_t + \varphi \; y_{i,t-1} + v_{i,t}$ with $v_{i,t} \sim N(0,1)$, $\theta_i = \theta$, $\alpha_i = 0 \; \forall i = 1, \ldots, N$ and $T_b = 0.5T$.

Notice that if the DGP was not affected by a structural break, the third element of the right-hand side of the equality would not be present. Nickell (1981) derives the asymptotic bias of the estimation of the autoregressive coefficient in a first order stationary dynamic panel data model with fixed unobservable effects, such as (2). It is given by the second element of the right-hand side of (6). Thus, misspecification of the deterministic component of the process induces another element of bias in the estimation of φ, besides altering the denominator of Nickell's bias.

In the case of a stationary panel with $\varphi > 0$, the bias in the estimation of the autoregressive coefficient is given by the results in the following Theorem.

Theorem 1 *Let $\{y_{i,t}\}_0^T$ be a stochastic process defined by (5) with $\varphi > 0$, and $\{v_{i,t}\}_1^T$, $i = 1, \ldots, N$, satisfying Assumption 1. Then, as $N \to \infty$ the limit of the estimator given by (6) is:*

$$\hat{\varphi} \to \varphi + \frac{N_1\left(\varphi, T, \sigma_v^2\right)}{D\left(\varphi, T, \lambda, \sigma_\theta^2, \bar{\theta}, \sigma_v^2\right)} + \frac{N_2\left(\varphi, T, \lambda, \sigma_\theta^2, \bar{\theta}\right)}{D\left(\varphi, T, \lambda, \sigma_\theta^2, \bar{\theta}, \sigma_v^2\right)}, \tag{7}$$

where "$\to$" denotes convergence in probability, and $D\left(\varphi, T, \lambda, \sigma_\theta^2, \bar{\theta}, \sigma_v^2\right)$, $N_1\left(\varphi, T, \sigma_v^2\right)$ and $N_2\left(\varphi, T, \lambda, \sigma_\theta^2, \bar{\theta}\right)$ denote non zero complicated expressions of the autoregressive parameter, the sample size, the variance of the disturbance term $\left(\sigma_v^2\right)$, the date of the break -through the break fraction (λ)-, and of the mean $\left(\bar{\theta}\right)$ and the variance $\left(\sigma_\theta^2\right)$ of the distribution that characterizes the magnitude of the structural break that affects the individuals.

Theorem 1 is proved in the appendix, where expressions for $N_1(\bullet)$, $N_2(\bullet)$ and $D(\bullet)$ are given. It is also shown that the bias in the estimation of the autoregressive parameter collected in Theorem 1 equals the bias derived by Nickell (1981) in case that $\sigma_\theta^2 = \bar{\theta} = 0$, that is, for these situations where there is no structural break affecting the time series. The effect of the misspecification error can be assessed through the analysis of the contibution of $N_1(\bullet)$, $N_2(\bullet)$ and $D(\bullet)$ to the bias estimation. First of all, provided that the the denominator $D(\bullet)$ is a semi-definite positive quadratic form, its contribution to the two elements of the bias is always positive. As a consequence, the contribution to the direction of the bias of the two last terms in (7) depends on the sign of $N_1(\bullet)$ and $N_2(\bullet)$. Therefore, our interest is going to focus on the two numerators. From the expression of $N_1(\bullet)$ given in the appendix we can conclude that this term is negative for the admissible range of values of the parameter φ, that is, $0 < \varphi < 1$. The situation is more confused for the second numerator. The sign of $N_2(\bullet)$ depends on the combinations of T and λ. Thus, it can be shown that only for small values of T combined with high values of λ this numerator takes negative values. Graphical and numerical analyses reveal that for those situations in which $\lambda T = (T-1)$, $N_2(\bullet)$ is negative.[1] For the other situations the contribution of this numerator is always positive.

To sum up, the bias in the estimation of the autoregressive parameter can be affected by two opposite effects. The first effect is always negative and contributes to underestimate the autoregressive parameter. As discused above, the second one is positive for most of the cases. However, what is interesting is the net effect of both sources of bias. Once again, graphical and numerical simulations reveal that, excluding very small values of T, the net effect is positive as the structural break moves away from the extremes. Moreover, this positive effect increases with the ratio $k = \left(\sigma_\theta^2 + \bar{\theta}^2\right)/\sigma_v^2$. As a result the misspecification of the deterministic component of the model induces a positive bias in the estimation of the autoregressive coefficient. Therefore, a unit root test based on the estimation of such a coefficient would be biased towards the null hypothesis of non-stationarity when the DGP is stationary with a level shift, in accordance with the simulation results above.

This situation is even more striking when the DGP is a static panel -as already proved by Perron (1989) in the univariate case. The following theorem reflects the inconsistency of the estimator for the simplest situation in which the time series are white noise, $\varphi = 0$, around a breaking trend.

Theorem 2 *Let $\{y_{i,t}\}_0^T$ be a stochastic process defined by (5) with $\varphi = 0$, $y_{i,0} = \alpha_i$, and $\{v_{i,t}\}_1^T$, $i = 1, \ldots, N$, satisfying Assumption 1. Then, as $N \to \infty$ the limit of the estimator given by (6) is:*

$$\hat{\varphi} \to \frac{((1-\lambda)T - 1)\lambda\left(\sigma_\theta^2 + \bar{\theta}^2\right)}{(T-1)\sigma_v^2 + \left(\frac{(T-T\lambda-1)(T\lambda+1)}{T}\right)\left(\sigma_\theta^2 + \bar{\theta}^2\right)},$$

[1] These results are not provided to save space. However, they are available upon request.

where " $\rightarrow$ " denotes convergence in probability, with $\bar{\theta}$ and σ_{θ}^{2} being the mean and the variance, respectively, of distribution of the magnitude of the structural break that affects the deterministic component of the time series of each individual.

The proof of the theorem is outlined in the appendix. The bias identified by Nickell (1981) disappears when $\varphi = 0$, that is when the panel is static instead of dynamic. Notwithstanding, our analysis has indicated that the misspecification of the deterministic component introduces bias in the estimation of the autoregressive parameter. In accordance with this, our results from the simulation experiment shown in Table 1 indicate that even when $\varphi = 0$ there are certain combinations of T, N, λ, σ_{θ}^{2} and $\bar{\theta}$ that can bias the estimation of φ to the null hypothesis of a unit root in the HT test.

The marginal effect of the different parameters that involve the limiting expression collected in Theorem 2 is not straightforward. However, this analysis can be performed graphically after setting certain combinations of parameters. The main conclusion to bear in mind is that the misspecification error of not allowing for a structural break that has affected the mean of the time series means the estimation of the autoregressive parameter is inconsistent. And more important for our purpose here, there are combinations of the parameters that make the bias positive and large, thus pushing the estimated coefficient of the autoregressive parameter towards the region of non-stationarity, as already observed in the first column of Table 1.

Summing up, these results show that the presence of a level shift in the time dimension of the series alters the performance of the test proposed by HT. Hereafter, we develop a proposal to take into account such and eventuality.

3 The model

Based on the proposals of Perron (1989, 1990), we specify a model where the structural break shifts the intercept of the deterministic component of the time series:

$$y_{i,t} = \alpha_i + \theta_i DU_t + \delta_i D\left(T_b\right)_t + \varphi y_{i,t-1} + v_{i,t}, \tag{8}$$

where $DU_t = 1$ for $t > T_b$ and 0 elsewhere, and $D\left(T_b\right)_t = 1$ for $t = T_b$ and 0 elsewhere, $t = 1, ..., T$, being $T_b = \lambda T$ the date of the break, with λ the break fraction parameter, $\lambda \in \Lambda = (0, 1)$. Under the null hypothesis of unit root, (8) reduces to:

$$y_{i,t} = \delta_i D\left(T_b\right)_t + y_{i,t-1} + v_{i,t}, \tag{9}$$

whereas under the alternative hypothesis of stationarity over a breaking trend time series it evolves according to:

$$y_{i,t} = \alpha_i + \theta_i DU_t + \varphi y_{i,t-1} + v_{i,t}, \tag{10}$$

with $|\varphi| < 1$. The properties of $\{v_{i,t}\}$, $i = 1, \ldots, N;\ t = 1, \ldots, T$, are the same as those described in section 2 and given in HT. Notice that (8) nests equations (9) and

(10), the models under the null and alternative hypotheses, respectively. Under the null hypothesis, the estimator of φ in (8) satisfies:

$$\hat{\varphi} - 1 = \left[\sum_{i=1}^{N} y'_{i,-1} Q_T y_{i,-1} \right]^{-1} \left[\sum_{i=1}^{N} y'_{i,-1} Q_T v_i \right], \tag{11}$$

where $y_{i,-1} = (y_{i,0}, ..., y_{i,T-1})'$, $v_i = (v_{i,1}, ..., v_{i,T})'$ and $Q_T = I_T - x (x'x)^{-1} x'$ being x the matrix of deterministic regressors, $x = [e_T, DU, D(T_b)]$. However, from (11) it is proved in the appendix that as $N \to \infty$ with T fixed, $\hat{\varphi}$ is an inconsistent estimator of φ in (8). This result is summarized in the following theorem.

Theorem 3 *Under the null hypothesis that $\varphi = 1$ and provided that assumption 1 holds, the estimator given by (11) converges, as $N \to \infty$ with T fixed, to:*

$$\underset{N\to\infty}{plim} \frac{1}{N} (\hat{\varphi} - 1) = B_{An},$$

where $B_{An} = \frac{-3(T-3)}{(1+2\lambda^2-2\lambda)T^2+(2\lambda-2)T-1}$.

Theorem 3 shows the bias of the test defined by (11) as a function of the sample size (T) and of the break fraction parameter (λ). The bias decreases as the sample size grows. Although $B_{An} \to 0$ as $T \to \infty$, there is a bias for T fixed. However, we can still have a consistent estimator if a suitable modification of (11) is made. The limiting distribution of the test statistic resulting from this correction is given in the following theorem.

Theorem 4 *Under the null hypothesis that $\varphi = 1$ and provided that assumption 1 holds, it can be established that, as $N \to \infty$ with T fixed, the unbiased estimator converges to:*

$$\sqrt{N} (\hat{\varphi} - 1 - B_{An}) \xrightarrow{d} N(0, C_{An}),$$

where $\xrightarrow{d}$ *denotes convergence in distribution and*

$$\begin{aligned} C_{An} = \; & \frac{3}{5\left(T^2 + 2T^2\lambda^2 - 2T^2\lambda - 2T + 2T\lambda - 1\right)^4} \\ & \left(\left(40\lambda^6 - 78\lambda - 208\lambda^3 + 162\lambda^2 + 17 - 120\lambda^5 + 204\lambda^4\right) T^6 \right. \\ & + \left(-180 + 1056\lambda^3 - 1176\lambda^2 + 120\lambda^5 - 624\lambda^4 + 702\lambda\right) T^5 \\ & + \left(3144\lambda^2 - 1920\lambda^3 + 636\lambda^4 + 753 - 2400\lambda\right) T^4 \\ & + \left(-3408\lambda^2 + 1072\lambda^3 + 3768\lambda - 1552\right) T^3 \\ & \left. + \left(1158\lambda^2 - 2634\lambda + 1539\right) T^2 + (642\lambda - 420) T - 293 \right). \end{aligned}$$

The proof of Theorem 4 is given in the appendix, where the derivation of the variance for the unbiased test statistic is outlined. Notice that the test statistic depends on T and λ through both the bias correction term and the expression for the variance.

4 Break point estimation

Theorem 4 proves that the expanded test which allows for a break can be performed through the application of the standard normal distribution. That is, it is based on the same inference as the test that does not allow for a structural break, as proposed by HT. However, Theorem 4 shows that the asymptotic distribution of (11) depends on a nuisance parameter, the break fraction (λ). An important characteristic is given by the fact that λ is a parameter that is present both under the null and alternative hypotheses. Therefore, this case is close to the unit root test first considered in Banerjee, Lumsdaine, and Stock (1992), Perron and Vogelsang (1992) and Zivot and Andrews (1992), and subsequently applied, among others, in Bai, Lumsdaine, and Stock (1998), Hao (1996) and Perron (1997). From the proposals available in this literature, here we use the supremum functional to estimate the date of the break and, therefore, make the asymptotic distribution of the test free of the λ nuisance parameter. This allows us to present the following result.

Theorem 5 *If Assumption 1 holds, then under the null hypothesis that $\varphi = 1$, as $N \rightarrow \infty$ with T fixed, and being $W(\lambda) = \sqrt{N/C_{An}}\,(\hat{\varphi} - 1 - B_{An})$, it can be established that*

$$Sup\, W(\lambda) = Sup_{\lambda \in (\lambda^*, 1-\lambda^*)}\, W(\lambda) \xrightarrow{d} N(0,1),$$

where $\lambda^* = T_b^*/T$ denotes the amount of trimming.

The proof of the previous theorem follows from standard results since the supremum functional is a continuous function not involving N -for this proof see Hansen (1992) and Zivot and Andrews (1992). The definition of some trimming is needed when computing the test statistic and it means discarding some observations at the extremes of the period.

5 Finite sample performance

In this section we assess the finite sample properties of the test proposed in section 3. As usual we conduct this analysis by carrying out a set of simulation experiments. The first set of experiments deals with the empirical size of the test. Theorem 4 shows that the test converges in distribution to that of the standard normal as the number of individuals in the panel tends to grow. In order to check this, we generated different panel data sets using the DGP given by (9) for which the test outlined in section 3 has been computed.[2] Without loss of generality, we have assumed that there is no structural break under the null and thus have set $\delta_i = 0\ \forall i = 1, \ldots, N$. Moreover, the model is estimated assuming that the break point is positioned at the middle of the time period $(\lambda = 0.5)$. After conducting 5,000 replications we obtained the empirical distribution of the test from which the percentiles of interest were computed. These percentiles are shown in Table 2.

[2] The GAUSS code used to compute the test statistic is available upon request.

Table 2: Empirical distribution of the normalised bias test with a structural break

T	N	0.01	0.05	0.1	0.5	0.9	0.95	0.99	Empirical size
5	5	-3.61	-2.05	-1.48	0.01	1.48	1.93	3.43	0.08
5	10	-2.84	-1.84	-1.41	0.00	1.33	1.73	2.54	0.07
5	25	-2.54	-1.70	-1.30	0.03	1.35	1.77	2.53	0.05
5	50	-2.36	-1.68	-1.31	0.01	1.28	1.67	2.35	0.05
5	100	-2.23	-1.62	-1.29	0.01	1.34	1.71	2.33	0.05
10	5	-2.70	-1.87	-1.48	-0.11	1.09	1.42	1.98	0.08
10	10	-2.54	-1.84	-1.49	-0.07	1.12	1.46	2.03	0.07
10	25	-2.55	-1.78	-1.37	-0.04	1.21	1.55	2.19	0.06
10	50	-2.47	-1.70	-1.29	-0.02	1.21	1.57	2.24	0.06
10	100	-2.43	-1.66	-1.30	-0.01	1.23	1.59	2.32	0.05
25	5	-3.08	-2.10	-1.61	-0.11	1.05	1.34	1.88	0.10
25	10	-2.82	-1.88	-1.46	-0.06	1.14	1.44	2.03	0.07
25	25	-2.66	-1.88	-1.42	-0.05	1.15	1.45	2.07	0.07
25	50	-2.40	-1.74	-1.36	-0.05	1.21	1.54	2.17	0.06
25	100	-2.41	-1.72	-1.36	-0.02	1.23	1.59	2.15	0.06
50	5	-3.06	-2.11	-1.65	-0.14	1.04	1.35	1.82	0.10
50	10	-2.85	-1.98	-1.52	-0.09	1.11	1.43	1.94	0.08
50	25	-2.58	-1.80	-1.42	-0.08	1.17	1.50	2.05	0.07
50	50	-2.48	-1.76	-1.37	-0.02	1.21	1.54	2.22	0.06
50	100	-2.55	-1.77	-1.35	-0.01	1.22	1.56	2.17	0.06
	N(0,1)	-2.33	-1.64	-1.29	0	1.29	1.64	2.33	

The DGP is given by $y_{i,t} = \delta_i D(T_b)_t + y_{i,t-1} + v_{i,t}$ with $\delta_i = 0$ and $v_{i,t} \sim N(0,1)$ for all $i = 1, ..., N$. The unit root panel data test is given by results in Theorem 4.

To facilitate comparisons, we have included the percentiles of the standard normal distribution in the last row of Table 2. The first thing to notice is the symmetry that characterises the empirical distribution of the test. Moreover, as the number of individuals grows -remember that our derivations take $N \to \infty$- symmetry is around zero and the difference between the empirical distribution of our test and the normal distribution can be regarded as experimental errors, which as expected from theorems 3 and 4.

Also, notice that the empirical distribution of the normalised bias tends to resemble more closely the normal distribution not only as $N \to \infty$ but also for those situations for which T can be considered moderate relative to N. Hence, for the test proposed her to perform well it is not necessary to undertake a computation in which a large number of individuals is considered. Additionally, the last column of Table 2 reproduces the empirical size of the test when the nominal size has been set equal to 5%. The conclusions drawn from these results are in accordance with those above. Finally, the similarity between the results of the empirical distribution of the test proposed here and those reported in Table 1b of HT is not surprising, since the way in which the deterministic component of the model is specified is irrelevant provided that the suitable bias correction and variance is considered when defining the test statistic.

The empirical power of the test was analysed through the Monte Carlo simulation experiment defined by the following parameters. The DGP assumed for the time series was the one given by (10) where, without loss of generality, $\alpha_i = 0 \,\forall i = 1, \dots, N$. In the experiment we have considered three different values for the autoregressive parameter, $\varphi = \{0.8, 0.9, 0.95\}$, for three sample sizes, $T = \{10, 25, 50\}$, and three sets of individuals, $N = \{10, 25, 50\}$. Simulations for $N = 100$ have a large computational cost and were omitted from our experiment. We did however run some of them for this case and our results resembled those for $N = 50$. The power was computed for two magnitudes of the structural break, $\theta = \{3, 7\}$, positioned as given by the following values of the break fraction, $\lambda = \{0.25, 0.5, 0.75\}$. The combinations of all of these parameters lede to 162 simulation experiments. For each experiment 5,000 replications were carried out.

The results presented in Tables 3 and 4 show the empirical power of the test for $\theta = 3$ and $\theta = 7$, respectively. The main conclusion is that, in almost all situations, the power of the test is equal to 1. Exceptions were found for the DGPs that use the smallest values of T and N with $\lambda = 0.25$. However, we are not surprised by this result considering that in such cases the presence of a structural break splits the time period in two regimes which for $T = 10$ with $\lambda = 0.25$ means 2 observations for the first subperiod. In order to assess the precision of the estimation of the break point we computed the mean and the standard deviation of this estimated break point from the set of replications. It can be said that the mean of the break point estimations is equal to the true value defined in the Monte Carlo experiment. Besides, it should be noted that, in general, the standard deviation diminished as both the number of individuals and the magnitude of the structural break increased. In fact there are many cases in which the standard deviation was equal to zero which means that the date of the break was precisely estimated most of the times. Once again, the case in which the standard

Table 3: Empirical power of the normalised bias test with a structural break of magnitude $\theta = 3$

			$\lambda = 0.25$			$\lambda = 0.5$			$\lambda = 0.75$		
				T_b			T_b			T_b	
φ	T	N	Power	Mean	std	Power	Mean	std	Power	Mean	std
0.8	10	10	0.72	5.55	3.25	0.96	4.43	0.81	0.95	6.57	0.89
	10	25	0.97	5.75	3.30	1	4.18	0.58	1	6.33	0.74
	10	50	1	6.11	3.31	1	4.05	0.32	1	6.14	0.51
	25	10	1	5	0.53	1	11.01	0.38	1	17.02	0.59
	25	25	1	5	0	1	11	0	1	17	0
	25	50	1	5	0	1	11	0	1	17	0
	50	10	1	10.88	0.92	1	23.95	1.02	1	36.01	1.18
	50	25	1	11	0.05	1	24	0.01	1	36	0
	50	50	1	11	0	1	24	0	1	36	0
0.9	10	10	0.55	6.15	3.20	0.92	4.33	0.74	0.89	6.47	0.84
	10	25	0.85	6.91	3.04	1	4.11	0.45	1	6.18	0.58
	10	50	0.99	7.57	2.70	1	4.02	0.18	1	6.05	0.31
	25	10	0.99	5.26	2.23	0.98	11.04	1.21	0.99	16.95	1.21
	25	25	1	5.01	0.48	1	11	0	1	17	0
	25	50	1	5	0	1	11	0	1	17	0
	50	10	1	11.71	6.57	1	23.95	4.49	1	35.64	4.16
	50	25	1	10.97	1.65	1	23.99	0.72	1	36.01	0.58
	50	50	1	10.99	0.09	1	24.01	0.35	1	36	0
0.95	10	10	0.42	6.50	3.10	0.86	4.27	0.70	0.82	6.36	0.79
	10	25	0.71	7.43	2.76	1	4.07	0.38	1	6.10	0.44
	10	50	0.95	8.18	2.13	1	4.01	0.13	1	6.02	0.21
	25	10	0.92	5.61	3.18	0.92	11.11	1.86	0.93	16.85	1.74
	25	25	1	5.05	0.91	1	11	0.23	1	16.99	0.43
	25	50	1	5	0	1	11	0	1	17	0
	50	10	0.96	14.27	10.68	0.95	24.35	6.97	0.96	34.59	6.88
	50	25	1	11.60	5.27	1	24.08	2.72	1	35.89	2.12
	50	50	1	11.05	2.04	1	24	0.64	1	35.99	0.42

The DGP is given by $y_{i,t} = \alpha_i + \theta_i DU_t + \varphi y_{i,t-1} + v_{i,t}$ with $\alpha_i = 0$, $\theta_i = 3$ and $v_{i,t} \sim N(0,1)$ for all $i = 1, ..., N$. The unit root panel data test is given by results in Theorem 4.

Table 4: Empirical power of the normalised bias test with a structural break of magnitude $\theta = 7$

			$\lambda = 0.25$			$\lambda = 0.5$			$\lambda = 0.75$		
				T_b			T_b			T_b	
φ	T	N	Power	Mean	std	Power	Mean	std	Power	Mean	std
0.8	10	10	0.77	5	3.16	1	4.50	0.86	1	6.60	0.91
	10	25	1	5.36	3.22	1	4.26	0.67	1	6.37	0.78
	10	50	1	5.69	3.27	1	4.11	0.45	1	6.19	0.59
	25	10	1	5	0	1	11	0.05	1	17	0.06
	25	25	1	5	0	1	11	0	1	17	0
	25	50	1	5	0	1	11	0	1	17	0
	50	10	1	11	0	1	24	0	1	36	0
	50	25	1	11	0	1	24	0	1	36	0
	50	50	1	11	0	1	24	0	1	36	0
0.9	10	10	0.62	6.13	3.18	1	4.42	0.82	1	6.48	0.85
	10	25	0.99	6.96	2.97	1	4.18	0.57	1	6.25	0.66
	10	50	1	7.64	2.59	1	4.05	0.30	1	6.09	0.42
	25	10	1	5	0	1	11	0	1	17	0
	25	25	1	5	0	1	11	0	1	17	0
	25	50	1	5	0	1	11	0	1	17	0
	50	10	1	11	0	1	24	0	1	36	0
	50	25	1	11	0	1	24	0	1	36	0
	50	50	1	11	0	1	24	0	1	36	0
0.95	10	10	0.54	6.72	3.05	1	4.37	0.78	1	6.40	0.80
	10	25	0.98	7.73	2.52	1	4.15	0.53	1	6.18	0.57
	10	50	1	8.43	1.79	1	4.03	0.25	1	6.04	0.29
	25	10	1	5	0	1	11	0	1	17	0
	25	25	1	5	0	1	11	0	1	17	0
	25	50	1	5	0	1	11	0	1	17	0
	50	10	1	11	0	1	24	0	1	36	0
	50	25	1	11	0	1	24	0	1	36	0
	50	50	1	11	0	1	24	0	1	36	0

The DGP is given by $y_{i,t} = \alpha_i + \theta_i DU_t + \varphi y_{i,t-1} + v_{i,t}$ with $\alpha_i = 0$, $\theta_i = 7$ and $v_{i,t} \sim N(0,1)$ for all $i = 1, ..., N$. The unit root panel data test is given by results in Theorem 4.

deviation most frequently departs from zero is for the smallest sample size, $T = 10$, with $\lambda = 0.25$.

To sum up, the simulation experiments that have been carried out suggest that the test performs well in finite samples.

6 Empirical application

In this section we illustrate the application of the panel data unit root test developed above to unemployment rates in the Spanish provinces over the last few decades (1964-1997). These results are compared to those obtained when applying univariate tests and the panel data unit root test that does not allow for a change in the mean of the process under the alternative, as proposed by HT. Spain experienced a large increase in its unemployment rates up to the mid-eighties, having recorded low levels in the late sixties and early seventies. Additionally, it has been cited as the most striking case of persistence in Europe - see Blanchard and Jimeno (1995), Jimeno and Bentolila (1998) and Dolado and Jimeno (1997). As such it has been frequently used as an example that clearly contradicts the natural rate of unemployment theory.

The validity of the natural rate hypothesis is based on two assumptions -see Friedman (1968) and Phelps (1967, 1968). Firstly, the uniqueness of the equilibrium level of unemployment and its independence of monetary variables in the steady-state. Secondly, actual unemployment tends to return to the natural rate given that expectations tend to correct themselves sooner or later. The rejection of either of these two assumptions leads to the natural rate hypothesis being refuted.

While initially the absence of theories explaining the determination of the natural rate meant that in practice it was taken to be constant, subsequent developments have attempted to explain the reasons behind changes across economies and over time. Among the structural factors influencing the natural rate are productivity level and growth, energy prices, international trade, union power, and normative traditions -for a discussion of these issues see Bianchi and Zoega (1998).

The experience of Western economies since the mid-seventies cast serious doubts on the empirical validity of the natural rate theory. In such an environment it is hardly surprising that the hypothesis of hysteresis gained widespread support. In the early stages it was defined as the effect of past unemployment on the natural rate, Phelps (1972), whereas a slow adjustment toward a constant natural rate was described as persistence. Later on, a myriad of papers addressed hysteresis in unemployment and related it to physical and human capital, insider-outsider relations, the search effectiveness of the unemployed, the employees' perception of the unemployed -Lindbeck and Snower (1986), Blanchard and Summers (1987, 1988), Alogoskoufis and Manning (1988), Layard and Bean (1989), Lindbeck (1992) and Blanchard and Diamond (1994).

Empirically, most of the literature has analysed the sum of the coefficients in the autoregressive process representing the rate of unemployment. A value close to but lower than one for the sum was associated with partial hysteresis, that is, strong persistence. Meanwhile the case of perfect or pure hysteresis applies when the sum is equal to one.

It should be stressed that only in the latter case is the natural rate hypothesis violated, given that even in cases of strong persistence unemployment slowly converges to the natural rate. But it is sensible to state that in such a case the difference is negligible - Bianchi and Zoega (1998).

Tests of unit roots have been widely been used in series of unemployment rates -see Blanchard and Summers (1987), Decressin and Fatás (1995) and Bianchi and Zoega (1998)-, finding in favour of the hypothesis of hysteresis and against that of the natural rate when the null is not rejected. Results support hysteresis in the EU economies and the natural rate in the US and Nordic countries -see Papell, Murray, and Ghiblawi (2000).

However, those results are based on unit root tests that under the alternative assume a constant, unique, natural rate of unemployment. Recent contributions have sought to make the model under the alternative more flexible, allowing for changes over time in the natural rate. In this regard, Bianchi and Zoega (1997) estimate a Markov switching-regression model to test if medium to long run changes in unemployment for France, the UK and the US are more likely to due to (infrequent) changes in mean unemployment or to hysteresis. They conclude that unemployment for those economies is characterised by a stationary process around an infrequently changing mean. Bianchi and Zoega (1998) extend the analysis to a broader sample of OECD countries with a similar conclusion: unemployment in those countries is consistent with an endogenous (changing) natural rate.

Likewise, Arestis and Biefang-Frisancho Mariscal (1999) and Papell, Murray, and Ghiblawi (2000) apply unit root tests that allow for structural breaks in the unemployment rates of samples of OECD countries. In most of these countries the null of pure hysteresis (unit root) is rejected in favour of the alternative of stationarity around a changing equilibrium rate. Papell, Murray, and Ghiblawi (2000) conclude that such a finding seems to be more in keeping with the structuralist theories of unemployment (p. 315).

The empirical exercises discussed above are based on univariate unit root tests whose properties are based on $T \rightarrow \infty$. Thus, their implementation is sometimes restricted by the availability of a long enough number of time series observations for each one of the economies under analysis. In fact, some studies have used data with a higher than annual frequency to accommodate this requirement. However, as stressed in the previous sections, the test proposed by HT has been shown to perform reasonably well for a pool of moderate N and T. It thus represents a sensible alternative to unidimensional unit root tests when the time series is rather short. In addition, it is particularly attractive when applied to a sample of economies within a common economic context; e.g. regions within an integrated economy.

Our empirical exercise applies the HT test to the pool of unemployment rates in the 50 Spanish provinces (NUTS III EUROSTAT geographic classification) over the period 1964-1997. Unemployment figures come from the Labour Force Survey provided by the Spanish Statistical Office (http://www.ine.es). Figure 1 shows the evolution in unemployment rates. As stated above, a considerable increase in unemployment rates

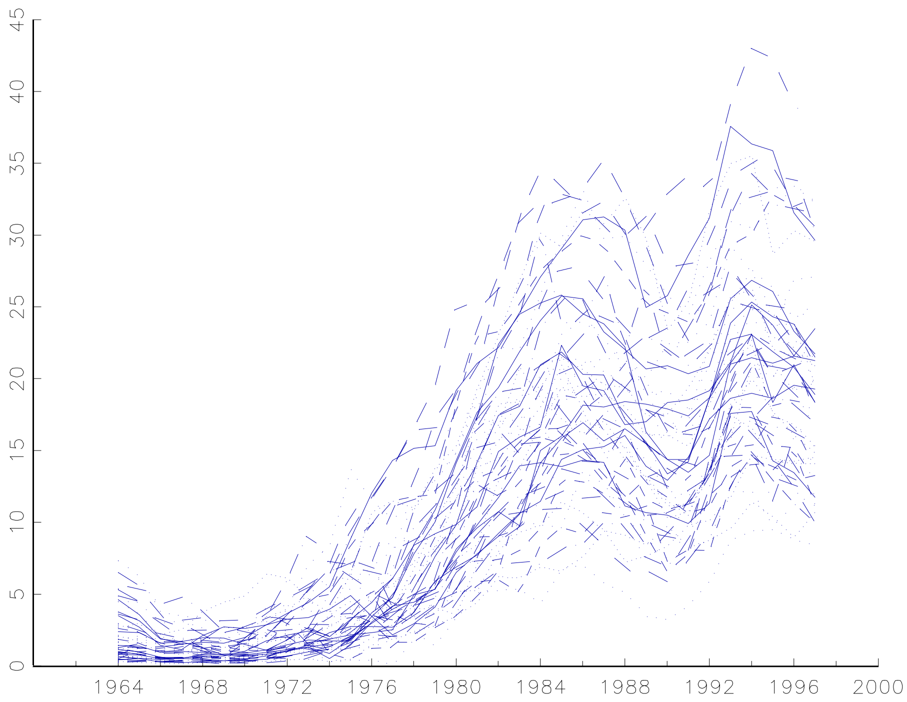

Figure 1: Unemployment rate of the Spanish provinces

Table 5: Results of the univariate ADF and KPSS tests

		Unemployment rates	Unemployment rates deviation to the yearly average
ADF	No RH_0 5%	50	48
	No RH_0 10%	50	46
KPSS	RH_0 5%	0	1
	RH_0 10%	49	23

Note: Figures correspond to the number of provinces in which the null of the test is rejected (KPSS) or not rejected (ADF). Total number of provinces is 50. All the ADF tests were computed using the parametric correction. The order of the autoregresive model was selected using the individual significance criteria of Ng and Perron (1995) with the specification of a maximum order of lag equal to 5. The bandwidth for the Quadratic spectral window used when computing the KPSS test was selected with the automatic procedure of Newey and West (1994) with an initial value for the bandwidth of 8 in all cases.

began in all the provinces in the mid-seventies. This culminated in a situation where unemployment rates fluctuated with the business cycle from the mid-eighties on but at a much higher level.

In order to compare the results of the panel data test, we start our analysis with the ADF test of Dickey and Fuller (1979) and the KPSS test of Kwiatkowski, Phillips, Schmidt, and Shin (1992) applied to each time series. Table 5 summarises these results. In both cases, the test is applied to the original series and the series in deviations to the average unemployment rate each year. In the first case, the ADF test does not reject the null of a unit root in any of the provinces, whereas for the series in deviations it rejects the null in 2 cases at 5% (4 at 10%). The KPSS rejects the null of stationarity at 10% in all but one case in the original series, although the results point to the absence of a unit root at the 5% level of probability. When applied to the series in deviations it only rejects the null in 23 cases at 10% and for only one case at 5%. Therefore, there is a discrepancy in the results provided by both tests. It has been shown that this might occur when the deterministic components of the process are misspecified -see Cheung and Chinn (1996).

Our results for the HT test for the model with a separate intercept for each province are shown in Table 6. We applied the test to the pool formed by all the provinces for the whole time period and also to the same pool for two sub-periods: one finishing in 1984, and the other from 1985 to the last year under consideration. The cross-section average in each time period was removed from the data to ensure independence across individuals, as required by Assumption 1a. The null of a unit root is clearly not rejected when using the whole period; the same is true of the first sub-period. The positive value of the test in these cases might be due to the misspecification of the deterministic components under the alternative. In contrast, in the last period the null is clearly rejected, in accordance with the fluctuation around a stable unemployment rate in that

Table 6: Results of the HT test for the unemployment rates in the Spanish provinces

	Without break in the mean		With a break in the mean		
	HT test	p-value	Test	p-value	T_b
1964-1997	0.222	0.587	-3.821	0.000	1974
1964-1984	6.618	1.000			
1985-1997	-4.280	0.000			

Note: The cross-section average in each time period was removed from the data to ensure independence across individuals.

period for all the provinces as depicted in Figure 1.

Contradictory results across time periods could point to the existence of a shift in the natural rate of unemployment in the Spanish provinces, thus influencing the properties of the HT test, as stressed in section 2. Therefore, we tested the null of unit root by model (8), in which a structural break shifts the intercept of the process under the alternative. The break point is endogeneously determined as described in section 4. There is a significant positive break in 1974, and the null is clearly rejected in this case for the whole period. Therefore, we can conclude that the Spanish unemployment rate is stationary around a natural rate that shifted from a low level in the sixties and early seventies to a large magnitude from the mid-eighties onwards.

7 Conclusions

This paper has shown that the specification of the deterministic component of the models used to test for the unit root hypothesis in a panel data framework, following the proposal of HT, is crucial, since a misspecification error can lead to the drawing of wrong conclusions. Thus, we have proposed a new panel data based unit root test that considers a structural break which shifts the mean of the model for those panels with moderate T compared to N. The simulation experiments conducted here have shown the performance of the test in finite samples to be adequate.

Although our approach might be understood as being somewhat limited in application, since it only allows for one structural break affecting all the time series at the same date, we believe that it is useful for those panels that might be subjected to similar shocks. Furthermore, our approach can easily be extended to time series with structural breaks at different dates for each individual in the cross-section. Our current research is addressing this concern.

The analysis carried out here on the unemployment rate for the Spanish regions indicates that any conclusions about the hysteresis hypothesis are dependent on the deterministic component specified in the model. Thus, after allowing for a structural shift in the mean, our results indicate that the hysteresis hypothesis is not supported by empirical evidence. The main conclusion to be drawn from our empirical application is that by using the modified HT test that allows for a structural break in the mean the Spanish unemployment rates are found to be stationary around a natural rate that shifted from a low level in the sixties and early seventies to large magnitudes from the mid-eighties onwards.

Appendix A. Mathematical appendix

In this appendix we proof the results given in the paper. Before proceeding to prove the different theorems we have collected some intermediate results in two lemmas. Some of the proofs of statements given in these lemmas are carried out in detail and some are referred to previous papers. Finally, we have indistinctive made use of two different ways to denote the date of the break (T_b and λT, since as defined in the paper $T_b = \lambda T$).

A.1 Useful intermediate results

Lemma 1 *Let* $\{y_{i,t}\}_0^T$ *be a stochastic process defined by (5) with* $\varphi \neq 0$, $|\varphi| < 1$, *and* $\{v_{i,t}\}_1^T$, $i = 1, \ldots, N$, *satisfying Assumption 1. Thus, it can be established that:*

$$\begin{aligned}
\sum_{t=1}^{T} y_{i,t-1} &= \frac{\alpha_i}{(1-\varphi)}T + \theta_i \sum_{t=1}^{T}\sum_{j=0}^{\infty} \varphi^j DU_{t-1-j} + \sum_{t=1}^{T}\sum_{j=0}^{\infty} \varphi^j v_{i,t-1-j};\\
\left(\sum_{t=1}^{T} y_{i,t-1}\right)^2 &= \left(\frac{\alpha_i}{(1-\varphi)}T\right)^2 + \left(\theta_i \sum_{t=1}^{T}\sum_{j=0}^{\infty} \varphi^j DU_{t-1-j}\right)^2 + \left(\sum_{t=1}^{T}\sum_{j=0}^{\infty} \varphi^j v_{i,t-1-j}\right)^2\\
&\quad +2\frac{\alpha_i}{(1-\varphi)}T\theta_i \sum_{t=1}^{T}\sum_{j=0}^{\infty} \varphi^j DU_{t-1-j}\\
&\quad +2\frac{\alpha_i}{(1-\varphi)}T \sum_{t=1}^{T}\sum_{j=0}^{\infty} \varphi^j v_{i,t-1-j}\\
&\quad +2\theta_i \sum_{t=1}^{T}\sum_{j=0}^{\infty} \varphi^j DU_{t-1-j} \sum_{t=1}^{T}\sum_{j=0}^{\infty} \varphi^j v_{i,t-1-j};\\
\left(\frac{1}{(1-\varphi L)} DU_{t-1}\right)^2 &= \frac{\left(\varphi^{t-\lambda T-1} - 1\right)^2}{(\varphi-1)^2};\\
\sum_{t=1}^{T}\left(\sum_{j=0}^{\infty} \varphi^j DU_{t-1-j}\right)^2 &= \left((\varphi-1)^2 \varphi^2 \left(\varphi^2-1\right)\right)^{-1} \left((T-\lambda T)\varphi^4 + 2\varphi^3 + \right.\\
&\quad \left.(-T+1+\lambda T)\varphi^2 - 2\varphi^{T-\lambda T+2} - 2\varphi^{-\lambda T+3+T} + \varphi^{-2\lambda T+2T+2}\right);\\
\sum_{t=1}^{T}\sum_{j=0}^{\infty} \varphi^j DU_{t-1-j} &= \frac{(-T+\lambda T)\varphi^2 + (T-1-\lambda T)\varphi + \varphi^{T-\lambda T+1}}{(\varphi-1)^2 \varphi}.
\end{aligned}$$

Proof: The statements collected in lemma 1 follow from direct calculations. Thus, the first statement is obtained after expressing the DGP as:

$$\begin{aligned}(1-\varphi L)\,y_{i,t} &= \alpha_i+\theta_i\,DU_t+v_{i,t},\\ y_{i,t} &= \frac{\alpha_i}{(1-\varphi)}+\frac{\theta_i}{(1-\varphi L)}\,DU_t+\frac{1}{(1-\varphi L)}v_{i,t},\\ y_{i,t} &= \frac{\alpha_i}{(1-\varphi)}+\theta_i\sum_{j=0}^{\infty}\varphi^j DU_{t-j}+\sum_{j=0}^{\infty}\varphi^j v_{i,t-j},\end{aligned}$$

and for the lagged variable:

$$y_{i,t-1}=\frac{\alpha_i}{(1-\varphi)}+\theta_i\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}+\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}. \tag{1}$$

Notice the equivalency between $\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}$ and $\sum_{j=0}^{t-1}\varphi^j DU_{t-1-j}$ since we know that $DU_t=0$ for $\forall t\le 0$, that is, we have out of sample information about the value of the dummy variable. The square of the lagged variable can be computed from (A-1) as:

$$\begin{aligned}y_{i,t-1}^2 &= \left(\frac{\alpha_i}{(1-\varphi)}+\theta_i\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}+\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right)^2 \qquad \text{(A-2)}\\ &= \left(\frac{\alpha_i}{(1-\varphi)}\right)^2+\left(\theta_i\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\right)^2+\left(\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right)^2\\ &\quad +2\frac{\alpha_i}{(1-\varphi)}\theta_i\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\\ &\quad +2\frac{\alpha_i}{(1-\varphi)}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\\ &\quad +2\theta_i\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}.\end{aligned}$$

From (A-1) it is easy to see that

$$\sum_{t=1}^{T}y_{i,t-1}=\frac{\alpha_i}{(1-\varphi)}T+\theta_i\sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}+\sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}.$$

Another intermediate result which refers to the second statement of 1 indicates that:

$$\left(\sum_{t=1}^{T} y_{i,t-1}\right)^2 = \left(\frac{\alpha_i}{(1-\varphi)}T\right)^2 + \left(\theta_i \sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\right)^2 + \left(\sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right)^2 \quad \text{(A-3)}$$
$$+2\frac{\alpha_i}{(1-\varphi)}T\theta_i \sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}$$
$$+2\frac{\alpha_i}{(1-\varphi)}T \sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}$$
$$+2\theta_i \sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j} \sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}.$$

For the third statement we have that:

$$\left(\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\right)^2 = \left(\frac{1}{(1-\varphi L)}DU_{t-1}\right)^2,$$

which for $t \leq T_b + 1$:

$$\left(\frac{1}{(1-\varphi L)}DU_{t-1}\right)^2 = 0$$

provided that $DU_t = 0$ for $t \leq T_b$. For $t > T_b + 1$ we have that:

$$\begin{aligned}\left(\frac{1}{(1-\varphi L)}DU_{t-1}\right)^2 &= \sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\sum_{k=0}^{\infty}\varphi^k DU_{t-1-k}, \\ &= \sum_{j=0}^{t-\lambda T-2}\sum_{k=0}^{t-\lambda T-2}\varphi^{j+k} DU_{t-1-j}DU_{t-1-k}, \\ &= \sum_{j=0}^{t-\lambda T-2}\sum_{k=0}^{t-\lambda T-2}\varphi^{j+k},\end{aligned}$$

since $DU_t = 1$ for $t > T_b$. Thus, after some calculations we get:

$$\sum_{j=0}^{t-\lambda T-2}\sum_{k=0}^{t-\lambda T-2}\varphi^{j+k} = \frac{\left(\varphi^{t-\lambda T-1}-1\right)^2}{(\varphi-1)^2}.$$

Hence, the summation over all the time period, which gives the fourth statement of lemma 1, produces:

$$\sum_{t=1}^{T}\left(\sum_{j=0}^{\infty}\varphi^{j}DU_{t-1-j}\right)^{2} = \sum_{t=\lambda T+2}^{T}\frac{\left(\varphi^{t-\lambda T-1}-1\right)^{2}}{(\varphi-1)^{2}}$$

$$= \frac{(T-\lambda T)\,\varphi^{4}+2\varphi^{3}+(-T+1+\lambda T)\,\varphi^{2}-2\varphi^{T-\lambda T+2}-2\varphi^{-\lambda T+3+T}+\varphi^{-2\lambda T+2T+2}}{(\varphi-1)^{2}\,\varphi^{2}\left(\varphi^{2}-1\right)}.$$

Other intermediate result indicates that for $t \leq T_b + 1$:

$$\sum_{j=0}^{\infty}\varphi^{j}DU_{t-1-j}=0,$$

whereas for $t > T_b + 1$:

$$\begin{aligned}\sum_{j=0}^{\infty}\varphi^{j}DU_{t-1-j} &= \sum_{j=0}^{t-\lambda T-2}\varphi^{j}DU_{t-1-j},\\ &= \sum_{j=0}^{t-\lambda T-2}\varphi^{j},\\ &= \frac{\varphi^{t-\lambda T-1}-1}{\varphi-1}.\end{aligned}$$

If we sum over the whole period, which gives the fifth statement of Lemma 1, is equal to:

$$\sum_{t=\lambda T+2}^{T}\frac{\varphi^{t-\lambda T-1}-1}{\varphi-1}=\frac{(-T+\lambda T)\,\varphi^{2}+(T-1-\lambda T)\,\varphi+\varphi^{T-\lambda T+1}}{(\varphi-1)^{2}\,\varphi}.$$

Thus, Lemma 1 has been proved. ■

Lemma 2 *Let $\{y_{i,t}\}_0^T$ be a stochastic process defined by (5) with $\varphi = 0$ and $y_{i,0} = \alpha_i$, and $\{v_{i,t}\}_1^T$, $i = 1, \ldots, N$, satisfying Assumption 1. Thus, it can be established that:*

$$\begin{aligned}\sum_{t=1}^{T}y_{i,t} &= \alpha_i T+\theta_i\,(1-\lambda)\,T+\sum_{t=1}^{T}u_{i,t};\\ \sum_{t=1}^{T}y_{i,t-1} &= \alpha_i T+\theta_i\,((1-\lambda)\,T-1)+\sum_{t=1}^{T}u_{i,t-1};\\ \sum_{t=1}^{T}y_{i,t-1}^{2} &= \alpha_i^{2}T+\theta_i^{2}\,((1-\lambda)\,T-1)+\sum_{t=1}^{T}u_{i,t-1}^{2}\\ &\quad +2\alpha_i\theta_i\,((1-\lambda)\,T-1)+2\alpha_i\sum_{t=1}^{T}u_{i,t-1}+2\theta_i\sum_{t=T_b+2}^{T}u_{i,t-1}.\end{aligned}$$

Proof: The statements collected in lemma 2 follow from direct calculations.

Lemma 3 *Let* $\varepsilon_{i,t-1} = \sum_{j=1}^{t-1} v_{i,j}$. *Then,*

a. $\sum_{t=1}^{T} \varepsilon_{i,t-1}^2 = \sum_{t=1}^{T} (T-t) v_{i,t}^2 + 2\sum_{t=1}^{T-1} \sum_{s=t+1}^{T} (T-s) v_{i,t} v_{i,s};$

b. $\left(\sum_{t=1}^{T} \varepsilon_{i,t-1}\right)^2 = \sum_{t=1}^{T} (T-t)^2 v_{i,t}^2 + 2\sum_{t=1}^{T-1} \sum_{s=t+1}^{T} (T-t)(T-s) v_{i,t} v_{i,s};$

c. $\sum_{t=T_b+1}^{T} \varepsilon_{i,t-1} = \sum_{t=1}^{T} (T-t) v_{i,t} - \sum_{t=1}^{T_b} (T_b - t) v_{i,t};$

d. $\left(\sum_{t=1}^{T-1} v_{i,t}\right)^2 = \sum_{t=1}^{T-1} v_{i,t}^2 + 2\sum_{t=1}^{T-2} \sum_{s=t+1}^{T-1} v_{i,t} v_{i,s};$

e. $\left(\sum_{t=T_b+2}^{T} \varepsilon_{i,t-1}\right)^2 = \sum_{t=1}^{T} (T-t)^2 v_{i,t}^2 + 2\sum_{t=1}^{T-1} \sum_{s=t+1}^{T} (T-t)(T-s) v_{i,t} v_{i,s}$
$+\sum_{t=1}^{T_b+1} \left((T_b+1-t)^2 - 2(T-t)(T_b+1-t)\right) v_{i,t}^2$
$+\sum_{t=1}^{T_b} \sum_{s=t+1}^{T_b+1} (2(T_b+1-t)(T_b+1-s) - 2(T-t)(T_b+1-s) -$
$2(T-s)(T_b+1-t)) v_{i,t} v_{i,s} - 2\sum_{t=1}^{T_b+1} \sum_{s=T_b+2}^{T} (T_b+1-t)(T-s) v_{i,t} v_{i,s};$

f. $\left(\sum_{t=1}^{T} (T-t) v_{i,t}\right)\left(\sum_{t=1}^{T-1} v_{i,t}\right) = \sum_{t=1}^{T-1} (T-t) v_{i,t}^2$
$+\sum_{t=1}^{T-2} \sum_{s=t+1}^{T-1} (2T-t-s) v_{i,t} v_{i,s}.$

g. $\left(\sum_{t=1}^{T-1} \sum_{s=t+1}^{T} f(t) g(s) v_{i,t} v_{i,s}\right)^2 = \sum_{t=1}^{T-1} \sum_{s=t+1}^{T} f(t)^2 g(s)^2 v_{i,t}^2 v_{i,s}^2$
$+2\sum_{t=1}^{T-2} \sum_{s=t+1}^{T-1} \sum_{j=s+1}^{T} f(t)^2 g(s) g(j) v_{i,t}^2 v_{i,s} v_{i,j}$
$+2\sum_{t=1}^{T-2} \sum_{i=t+1}^{T-1} \left(\sum_{s=t+1}^{T} f(t) g(s) v_{i,t} v_{i,s}\right)\left(\sum_{s=i+1}^{T} f(i) g(s) v_{i,i} v_{i,s}\right)$

h. $\sum_{t=1}^{T-1} v_{i,t} \sum_{t=1}^{T} v_{i,t} = \left(\sum_{t=1}^{T} v_{i,t}^2 + 2\sum_{t=1}^{T-1} \sum_{s=t+1}^{T} v_{i,t} v_{i,s}\right)$
$-v_{i,T}^2 - v_{i,T} \sum_{t=1}^{T-1} v_{i,t}$

i. $\sum_{t=T_b+2}^{T} \varepsilon_{i,t-1} (\varepsilon_{i,T} - \varepsilon_{i,T_b}) = \sum_{t=1}^{T_b} \sum_{s=T_b+1}^{T} (T - T_b - 1) v_{i,t} v_{i,s}$
$+\sum_{t=T_b+1}^{T} (T-t) v_{i,t}^2 + \sum_{t=T_b+1}^{T-1} \sum_{s=t+1}^{T} (2T-t-s) v_{i,t} v_{i,s};$

j. $\varepsilon_{i,T_b} (\varepsilon_{i,T} - \varepsilon_{i,T_b}) = \sum_{t=1}^{T_b} \sum_{s=T_b+1}^{T} v_{i,t} v_{i,s};$

k. $\sum_{t=T_b+1}^{T} \varepsilon_{i,t-1} \varepsilon_{i,T_b} = (T - T_b) \sum_{t=1}^{T_b} v_{i,t}^2 + 2(T - T_b) \sum_{t=1}^{T_b-1} \sum_{s=t+1}^{T_b} v_{i,t} v_{i,s}$
$+\sum_{t=1}^{T_b} \sum_{s=T_b+1}^{T} (T-s) v_{i,t} v_{i,s};$

l. $\sum_{t=T_b+1}^{T} \varepsilon_{i,t-1} \varepsilon_{i,T} = \sum_{t=1}^{T} (T-t) v_{i,t}^2 + \sum_{t=1}^{T-1} \sum_{s=t+1}^{T} (2T-t-s) v_{i,t} v_{i,s}$
$-\sum_{t=1}^{T_b} (T_b - t) v_{i,t}^2 - \sum_{t=1}^{T_b-1} \sum_{s=t+1}^{T_b} (2T_b - t - s) v_{i,t} v_{i,s}$
$-\sum_{t=1}^{T_b} \sum_{s=T_b+1}^{T} (T_b - t) v_{i,t} v_{i,s};$

m. $\sum_{t=1}^{T} \varepsilon_{i,t-1} \sum_{t=T_b+1}^{T} \varepsilon_{i,t-1} = \sum_{t=1}^{T} (T-t)^2 v_{i,t}^2 + 2\sum_{t=1}^{T-1} \sum_{s=t+1}^{T} (T-t)(T-s) v_{i,t} v_{i,s}$

$-\sum_{t=1}^{T_b}(T-t)(T_b-t)v_{i,t}^2 - \sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}((T-t)(T_b-s)+(T-s)(T_b-t))v_{i,t}v_{i,s}$
$-\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}(T_b-t)(T-s)v_{i,t}v_{i,s};$

n. $\left(\sum_{t=T_b+1}^{T}\varepsilon_{i,t-1}\right)^2 = \sum_{t=1}^{T}(T-t)^2 v_{i,t}^2 + 2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}(T-t)(T-s)v_{i,t}v_{i,s}$
$+\sum_{t=1}^{T_b}\left((T_b-t)^2 - 2(T-t)(T_b-t)\right)v_{i,t}^2$
$+\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(2T_b^2 - 2ts - 4TT_b + 2Ts + 2Tt\right)v_{i,t}v_{i,s}$
$-2\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}(T_b-t)(T-s)v_{i,t}v_{i,s}.$

Proof: The result in (a) is given in lemma A.1 of Harris and Tzavalis (1999), and (b) follows from the mentioned lemma. (c) and (d) are simple extensions of the previous ones. Thus, (c) is achieved through

$$\begin{aligned}\sum_{t=T_b+1}^{T}\varepsilon_{i,t-1} &= \sum_{t=1}^{T}\varepsilon_{i,t-1} - \sum_{t=1}^{T_b}\varepsilon_{i,t-1} \\ &= \sum_{t=1}^{T}(T-t)v_{i,t} - \sum_{t=1}^{T_b}(T_b-t)v_{i,t}.\end{aligned}$$

The proof for (d) is trivial and omitted. The result given by (e) is obtained from

$$\begin{aligned}\left(\sum_{t=T_b+2}^{T}\varepsilon_{i,t-1}\right)^2 &= \left(\sum_{t=1}^{T}(T-t)v_{i,t} - \sum_{t=1}^{T_b+1}(T_b+1-t)v_{i,t}\right)^2 \\ &= \left(\sum_{t=1}^{T}(T-t)v_{i,t}\right)^2 - 2\left[\left(\sum_{t=1}^{T}(T-t)v_{i,t}\right)\left(\sum_{t=1}^{T_b+1}(T_b+1-t)v_{i,t}\right)\right] \\ &\quad + \left(\sum_{t=1}^{T_b+1}(T_b+1-t)v_{i,t}\right)^2,\end{aligned}$$

that is,

$$\begin{aligned}\left(\sum_{t=T_b+2}^{T}\varepsilon_{i,t-1}\right)^2 &= \sum_{t=1}^{T}(T-t)^2 v_{i,t}^2 + 2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}(T-t)(T-s)v_{i,t}v_{i,s} \\ &\quad + \sum_{t=1}^{T_b+1}\left((T_b+1-t)^2 - 2(T-t)(T_b+1-t)\right)v_{i,t}^2 \\ &\quad + \sum_{t=1}^{T_b}\sum_{s=t+1}^{T_b+1}(2(T_b+1-t)(T_b+1-s) - 2(T-t)(T_b+1-s) - \\ &\quad 2(T-s)(T_b+1-t))v_{i,t}v_{i,s} \\ &\quad -2\sum_{t=1}^{T_b+1}\sum_{s=T_b+2}^{T}(T_b+1-t)(T-s)v_{i,t}v_{i,s}.\end{aligned}$$

The result given by (f) is

$$\begin{aligned}
\left(\sum_{t=1}^{T}(T-t)\,v_{i,t}\right)\left(\sum_{t=1}^{T-1}v_{i,t}\right) &\equiv \sum_{t=1}^{T-1}(T-t)\,v_{i,t}\sum_{t=1}^{T-1}v_{i,t} \\
&= \sum_{t=1}^{T-1}(T-t)\,v_{i,t}^{2}+\sum_{t=1}^{T-2}\sum_{s=t+1}^{T-1}(2T-t-s)\,v_{i,t}v_{i,s}.
\end{aligned}$$

To prove the statement in (g) we have followed an inductive process. Starting from the particular expression:

$$\left(\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}(1+\lambda T-t)\,(T-s)\right)^{2}=\frac{1}{576}T^{2}\,(T-1)^{2}\,(T-2)^{2}\,(4\lambda T-T+3)^{2}\,,$$

that can be suitable decomposed as:

$$\begin{aligned}
& \sum_{t=1}^{T-1}\left(\sum_{s=t+1}^{T}(1+\lambda T-t)\,(T-s)\right)^{2} \\
& +2\sum_{t=1}^{T-2}\sum_{i=t+1}^{T-1}\left(\sum_{s=t+1}^{T}(1+\lambda T-t)\,(T-s)\right)\left(\sum_{s=i+1}^{T}(1+\lambda T-i)\,(T-s)\right) \\
= \; & \frac{1}{576}T^{2}\,(T-1)^{2}\,(T-2)^{2}\,(4\lambda T-T+3)^{2}
\end{aligned}$$

or expressed as:

$$\begin{aligned}
& \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}(1+\lambda T-t)^{2}\,(T-s)^{2} \\
& +2\sum_{t=1}^{T-2}\sum_{s=t+1}^{T-1}\sum_{j=s+1}^{T}(1+\lambda T-t)\,(T-s)\,(1+\lambda T-t)\,(T-j) \\
& +2\sum_{t=1}^{T-2}\sum_{i=t+1}^{T-1}\left(\sum_{s=t+1}^{T}(1+\lambda T-t)\,(T-s)\right)\left(\sum_{s=i+1}^{T}(1+\lambda T-i)\,(T-s)\right) \\
= \; & \frac{1}{576}T^{2}\,(T-1)^{2}\,(T-2)^{2}\,(4\lambda T-T+3)^{2}\,.
\end{aligned}$$

If we introduce the cross-product between disturbances we can easily obtain that:

$$\left(\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}(1+\lambda T-t)(T-s)\,v_{i,t}v_{i,s}\right)^2=\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}(1+\lambda T-t)^2(T-s)^2\,v_{i,t}^2v_{i,s}^2$$

$$+2\sum_{t=1}^{T-2}\sum_{s=t+1}^{T-1}\sum_{j=s+1}^{T}(1+\lambda T-t)(T-s)(1+\lambda T-t)(T-j)\,v_{i,t}^2v_{i,s}v_{i,j}$$

$$+2\sum_{t=1}^{T-2}\sum_{i=t+1}^{T-1}\left(\sum_{s=t+1}^{T}(1+\lambda T-t)(T-s)\,v_{i,t}v_{i,s}\right)\left(\sum_{s=i+1}^{T}(1+\lambda T-i)(T-s)\,v_{i,i}v_{i,s}\right).$$

In terms of expectations:

$$\begin{aligned}E\left(\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}(1+\lambda T-t)(T-s)\,v_{i,t}v_{i,s}\right)^2 &= \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}(1+\lambda T-t)^2(T-s)^2\,\sigma_v^4;\\ &= \frac{1}{360}T(T-1)(T-2)\\ &\quad \big(30\lambda^2T^3-12T^3\lambda+2T^3+54\lambda T^2-\\ &\quad 30\lambda^2T^2-12T^2-54\lambda T+25T-21\big)\,\sigma_v^4.\end{aligned}$$

Another result like the previous one but more general is:

$$\left(\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}A(t,s)\,v_{i,t}v_{i,s}\right)\left(\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}B(t,s)\,v_{i,t}v_{i,s}\right)=$$

$$\begin{aligned}=\; & \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}A(t,s)\,B(t,s)\,v_{i,t}^2v_{i,s}^2\\ &+\sum_{t=1}^{T-2}\sum_{s=t+1}^{T-1}\sum_{j=s+1}^{T}\big(A(t,s)\,B(t,j)\,v_{i,t}^2v_{i,s}v_{i,j}+A(t,j)\,B(t,s)\,v_{i,t}^2v_{i,j}v_{i,s}\big)\\ &+\sum_{t=1}^{T-2}\sum_{s=t+1}^{T-1}\left(\sum_{j=t+1}^{T}A(t,j)\,v_{i,t}v_{i,j}\sum_{j=s+1}^{T}B(s,j)\,v_{i,s}v_{i,j}+\right.\\ &\left.\sum_{j=t+1}^{T}B(t,j)\,v_{i,t}v_{i,j}\sum_{j=s+1}^{T}A(s,j)\,v_{i,s}v_{i,j}\right)\end{aligned}$$

To prove (h) it only has to be seen that

$$\begin{aligned}\sum_{t=1}^{T-1}v_{i,t}\sum_{t=1}^{T}v_{i,t} &= \left(\sum_{t=1}^{T}v_{i,t}-v_{i,T}\right)\sum_{t=1}^{T}v_{i,t},\\ &= \left(\sum_{t=1}^{T}v_{i,t}\right)^2-v_{i,T}\sum_{t=1}^{T}v_{i,t},\end{aligned}$$

that can be expressed as:

$$\sum_{t=1}^{T-1} v_{i,t} \sum_{t=1}^{T} v_{i,t} = \left(\sum_{t=1}^{T} v_{i,t}^2 + 2 \sum_{t=1}^{T-1} \sum_{s=t+1}^{T} v_{i,t} v_{i,s} \right) - v_{i,T}^2 - v_{i,T} \sum_{t=1}^{T-1} v_{i,t}.$$

For (i)

$$\begin{aligned}
\sum_{t=T_b+2}^{T} \varepsilon_{i,t-1} \left(\varepsilon_{i,T} - \varepsilon_{i,T_b} \right) &= \left(\sum_{t=1}^{T} (T-t) v_{i,t} - \sum_{t=1}^{T_b+1} (T_b + 1 - t) v_{i,t} \right) \left(\sum_{t=T_b+1}^{T} v_{i,t} \right) \\
&= \left(\sum_{t=1}^{T_b} (T-t) v_{i,t} \right) \left(\sum_{t=T_b+1}^{T} v_{i,t} \right) \\
&\quad + \left(\sum_{t=T_b+1}^{T_b} (T-t) v_{i,t} \right) \left(\sum_{t=T_b+1}^{T} v_{i,t} \right) \\
&\quad - \left(\sum_{t=1}^{T_b+1} (T_b + 1 - t) v_{i,t} \right) \left(\sum_{t=T_b+1}^{T} v_{i,t} \right),
\end{aligned}$$

that can be simplified as:

$$\begin{aligned}
\sum_{t=T_b+2}^{T} \varepsilon_{i,t-1} \left(\varepsilon_{i,T} - \varepsilon_{i,T_b} \right) &= \sum_{t=1}^{T_b} \sum_{s=T_b+1}^{T} (T - T_b - 1) v_{i,t} v_{i,s} + \sum_{t=T_b+1}^{T} (T-t) v_{i,t}^2 \\
&\quad + \sum_{t=T_b+1}^{T-1} \sum_{s=t+1}^{T} (2T - t - s) v_{i,t} v_{i,s}.
\end{aligned}$$

Statement (j) can be obtained from

$$\varepsilon_{i,T_b} \left(\varepsilon_{i,T} - \varepsilon_{i,T_b} \right) = \sum_{t=1}^{T_b} v_{i,t} \sum_{t=1}^{T} v_{i,t} - \sum_{t=1}^{T_b} v_{i,t} \sum_{t=1}^{T_b} v_{i,t}.$$

We can see that:

$$\begin{aligned}
\sum_{t=1}^{T_b} v_{i,t} \sum_{t=1}^{T} v_{i,t} &= \left(v_{i,1} + \ldots + v_{i,T_b} \right) \left(v_{i,1} + \ldots + v_{i,T} \right) \\
&= \left(v_{i,1} + \ldots + v_{i,T_b} \right)^2 + \left(v_{i,1} + \ldots + v_{i,T_b} \right) \left(v_{i,T_b+1} + \ldots + v_{i,T} \right) \\
&= \left(\sum_{t=1}^{T_b} v_{i,t}^2 + 2 \sum_{t=1}^{T_b-1} \sum_{s=t+1}^{T_b} v_{i,t} v_{i,s} \right) + \sum_{t=1}^{T_b} \sum_{s=T_b+1}^{T} v_{i,t} v_{i,s}.
\end{aligned}$$

On the other hand,

$$\left(\sum_{t=1}^{T_b} v_{i,t} \right)^2 = \left(\sum_{t=1}^{T_b} v_{i,t}^2 + 2 \sum_{t=1}^{T_b-1} \sum_{s=t+1}^{T_b} v_{i,t} v_{i,s} \right).$$

Hence,

$$\begin{aligned}
\varepsilon_{i,T_b+1}\left(\varepsilon_{i,T}-\varepsilon_{i,T_b}\right) &= \left(\sum_{t=1}^{T_b} v_{i,t}^2 + 2\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b} v_{i,t}v_{i,s}\right) + \sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T} v_{i,t}v_{i,s} \\
&\quad - \left(\sum_{t=1}^{T_b} v_{i,t}^2 + 2\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b} v_{i,t}v_{i,s}\right) \\
&= \sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T} v_{i,t}v_{i,s}.
\end{aligned}$$

Finally, (k) is proved through

$$\begin{aligned}
\sum_{t=T_b+1}^{T} \varepsilon_{i,t-1}\varepsilon_{i,T_b} &= \sum_{t=1}^{T_b}\left(T-t\right)v_{i,t}\sum_{t=1}^{T_b} v_{i,t} + \sum_{t=T_b+1}^{T}\left(T-t\right)v_{i,t}\sum_{t=1}^{T_b} v_{i,t} - \sum_{t=1}^{T_b}\left(T_b-t\right)v_{i,t}\sum_{t=1}^{T_b} v_{i,t} \\
&= \sum_{t=1}^{T_b}\left(T-T_b\right)v_{i,t}\sum_{t=1}^{T_b} v_{i,t} + \sum_{t=T_b+1}^{T}\left(T-t\right)v_{i,t}\sum_{t=1}^{T_b} v_{i,t} \\
&= \left(T-T_b\right)\sum_{t=1}^{T_b} v_{i,t}^2 + 2\left(T-T_b\right)\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b} v_{i,t}v_{i,s} + \sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(T-s\right)v_{i,t}v_{i,s}.
\end{aligned}$$

Finally, to prove (l) we use the following development:

$$\begin{aligned}
\sum_{t=T_b+1}^{T} \varepsilon_{i,t-1}\varepsilon_{i,T} &= \left(\sum_{t=1}^{T}\left(T-t\right)v_{i,t}\right)\left(\sum_{t=1}^{T} v_{i,t}\right) - \left(\sum_{t=1}^{T_b}\left(T_b-t\right)v_{i,t}\right)\left(\sum_{t=1}^{T} v_{i,t}\right) \\
&= \sum_{t=1}^{T}\left(T-t\right)v_{i,t}^2 + \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(2T-t-s\right)v_{i,t}v_{i,s} \\
&\quad - \sum_{t=1}^{T_b}\left(T_b-t\right)v_{i,t}^2 - \sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(2T_b-t-s\right)v_{i,t}v_{i,s} \\
&\quad - \sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(T_b-t\right)v_{i,t}v_{i,s}.
\end{aligned}$$

For statement (m)

$$\begin{aligned}
\sum_{t=1}^{T}\varepsilon_{i,t-1}\sum_{t=T_b+1}^{T}\varepsilon_{i,t-1} &= \left(\sum_{t=1}^{T}\varepsilon_{i,t-1}\right)^2-\sum_{t=1}^{T}\varepsilon_{i,t-1}\sum_{t=1}^{T_b}\varepsilon_{i,t-1} \\
&= \sum_{t=1}^{T}(T-t)^2 v_{i,t}^2+2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}(T-t)(T-s)v_{i,t}v_{i,s} \\
&\quad -\sum_{t=1}^{T_b}(T-t)(T_b-t)v_{i,t}^2-\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}((T-t)(T_b-s)+ \\
&\quad (T-s)(T_b-t))v_{i,t}v_{i,s} \\
&\quad -\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}(T_b-t)(T-s)v_{i,t}v_{i,s}.
\end{aligned}$$

For the statement (n)

$$
\begin{aligned}
\left(\sum_{t=T_b+1}^{T}\varepsilon_{i,t-1}\right)^2 &= \left(\sum_{t=1}^{T}(T-t)\,v_{i,t}-\sum_{t=1}^{T_b}(T_b-t)\,v_{i,t}\right)^2\\
&= \left(\sum_{t=1}^{T}(T-t)\,v_{i,t}\right)^2-2\sum_{t=1}^{T}(T-t)\,v_{i,t}\sum_{t=1}^{T_b}(T_b-t)\,v_{i,t}\\
&\quad+\left(\sum_{t=1}^{T_b}(T_b-t)\,v_{i,t}\right)^2\\
&= \sum_{t=1}^{T}(T-t)^2\,v_{i,t}^2+2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}(T-t)\,(T-s)\,v_{i,t}v_{i,s}\\
&\quad-2\sum_{t=1}^{T_b}(T-t)\,(T_b-t)\,v_{i,t}^2-2\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}((T-t)\,(T_b-s)+\\
&\quad(T-s)\,(T_b-t))\,v_{i,t}v_{i,s}\\
&\quad-2\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}(T_b-t)\,(T-s)\,v_{i,t}v_{i,s}\\
&\quad+\sum_{t=1}^{T_b}(T_b-t)^2\,v_{i,t}^2+2\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}(T_b-t)\,(T_b-s)\,v_{i,t}v_{i,s}\\
&= \sum_{t=1}^{T}(T-t)^2\,v_{i,t}^2+2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}(T-t)\,(T-s)\,v_{i,t}v_{i,s}\\
&\quad+\sum_{t=1}^{T_b}\Big((T_b-t)^2-2\,(T-t)\,(T_b-t)\Big)\,v_{i,t}^2\\
&\quad+\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}(2\,(T_b-t)\,(T_b-s)-\\
&\quad 2\,((T-t)\,(T_b-s)+(T-s)\,(T_b-t)))\,v_{i,t}v_{i,s}\\
&\quad-2\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}(T_b-t)\,(T-s)\,v_{i,t}v_{i,s}\\
&= \sum_{t=1}^{T}(T-t)^2\,v_{i,t}^2+2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}(T-t)\,(T-s)\,v_{i,t}v_{i,s}\\
&\quad+\sum_{t=1}^{T_b}\Big((T_b-t)^2-2\,(T-t)\,(T_b-t)\Big)\,v_{i,t}^2\\
&\quad+\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(2T_b^2-2ts-4TT_b+2Ts+2Tt\right)v_{i,t}v_{i,s}\\
&\quad-2\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}(T_b-t)\,(T-s)\,v_{i,t}v_{i,s}.
\end{aligned}
$$

$(2(T_b - t)(T_b - s) - 2((T-t)(T_b - s) + (T-s)(T_b - t))) =$
Hence lemma 3 has been proved. ■

Lemma 4 *Some little algebra manipulations allow us to write the following expressions:*

- $\sum_{t=1}^{T}(T-t)^2 = \frac{1}{6}T(2T-1)(T-1)$;
- $\sum_{t=1}^{T-1}\sum_{s=t+1}^{T} 2(T-t)(T-s) = \frac{1}{12}T(T-1)(T-2)(3T-1)$;
- $\sum_{t=1}^{T-1}(T-t-T\lambda)^4 = \frac{1}{30}T(T-1)\left(-30T^3\lambda + 6T^3 + 60T^3\lambda^2 + 30T^3\lambda^4\right.$

$\left.-60T^3\lambda^3 - 9T^2 + 30T^2\lambda - 30T^2\lambda^2 + T + 1\right)$;

- $\sum_{t=1}^{T-2}\sum_{s=t+1}^{T-1} 2(T-t-T\lambda)^2(T-s-T\lambda)^2 = \frac{1}{180}T(T-1)(T-2)$

$\left(-120T^3\lambda - 360T^3\lambda^3 + 300T^3\lambda^2 + 180T^3\lambda^4 + 20T^3\right.$
$\left.-36T^2 + 120T^2\lambda - 120T^2\lambda^2 + 7T + 3\right)$;

- $\sum_{t=1}^{T_b+1}(-T+2t+T\lambda)^2 = \frac{1}{3}(T\lambda+1)$

$\left(13T^2\lambda^2 - 12T^2\lambda + 3T^2 + 26T\lambda - 12T + 12\right)$;

- $\sum_{t=1}^{T_b}\sum_{s=t+1}^{T_b+1} 2(-T+2t+T\lambda)(-T+2s+T\lambda) = \frac{1}{3}T\lambda(T\lambda+1)$

$\left(12T^2\lambda^2 - 12T^2\lambda + 3T^2 + 23T\lambda - 12T + 10\right)$;

- $\sum_{t=1}^{T-1}(T-t)(T-t-T\lambda)^2 = \frac{1}{12}T^2(T-1)\left(3T - 8T\lambda + 6T\lambda^2 + 4\lambda - 3\right)$;
- $\sum_{t=1}^{T-2}\sum_{s=t+1}^{T-1}\left[(T-t)(T-s-T\lambda)^2 + (T-s)(T-t-T\lambda)^2\right] = \frac{1}{6}T^2(T-1)(T-2)$

$\left(T - 3T\lambda + 3T\lambda^2 + \lambda - 1\right)$;

- $\sum_{t=1}^{T_b+1}(T-t)(-T+2t+T\lambda) = -\frac{1}{6}(T\lambda+1)$

$\left(7T^2\lambda^2 - 15T^2\lambda + 6T^2 + 20T\lambda - 18T + 12\right)$;

- $\sum_{t=1}^{T_b}\sum_{s=t+1}^{T_b+1}[(T-t)(-T+2s+T\lambda) + (T-s)(-T+2t+T\lambda)] = -\frac{1}{6}T\lambda(T\lambda+1)$

$\left(6T^2\lambda^2 - 15T^2\lambda + 6T^2 + 17T\lambda - 18T + 10\right)$;

- $\sum_{t=1}^{T_b+1}(T-t-T\lambda)^2(-T+2t+T\lambda) = \frac{1}{6}(T\lambda+1)$

$\left(31T^3\lambda^3 - 52T^3\lambda^2 + 30T^3\lambda - 6T^3 + 74T^2\lambda^2 - 83T^2\lambda + 24T^2 + 54T\lambda - 30T + 12\right)$;

- $\sum_{t=1}^{T_b}\sum_{s=t+1}^{T_b+1}\left[(T-t-T\lambda)^2(-T+2s+T\lambda) + (T-s-T\lambda)^2(-T+2t+T\lambda)\right]$

$= \frac{1}{6}T\lambda(T\lambda+1)\left(28T^3\lambda^3 - 50T^3\lambda^2 + 30T^3\lambda\right.$
$\left.-6T^3 + 63T^2\lambda^2 - 77T^2\lambda + 24T^2 + 42T\lambda - 26T + 8\right)$;

Proof: The results collected in lemma 4 are obtained from direct calculations. To save space all these calculations are omitted. ■

A.2 Proof of Theorem 1

The denominator that involves the bias terms in (6) is defined by:

$$y'_{i,-1}Q_T y_{i,-1} = \sum_{t=1}^{T} y_{i,t-1}^2 - \frac{1}{T}\left(\sum_{t=1}^{T} y_{i,t-1}\right)^2,$$

with limit in probability

$$\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N} y'_{i,-1}Q_T y_{i,-1} = \underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T} y_{i,t-1}^2 - \underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\frac{1}{T}\left(\sum_{t=1}^{T} y_{i,t-1}\right)^2.$$

We are going to compute this limit considering each part separately. Thus, for the first one we use the expression (A-2) and the statements collected in Lemma 1, from which it can be seen that:

$$\begin{aligned}
\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N} y_{i,t-1}^2 \quad = \quad & \underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\alpha_i}{(1-\varphi)}\right)^2 + \underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\theta_i\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\right)^2 \\
& + \underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right)^2 \\
& + 2\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\alpha_i}{(1-\varphi)}\theta_i\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\right) \\
& + 2\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\alpha_i}{(1-\varphi)}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right) \\
& + 2\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\theta_i\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right),
\end{aligned}$$

or equivalently,

$$\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}y_{i,t-1}^{2}=\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\alpha_i}{(1-\varphi)}\right)^2+\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\theta_i\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\right)^2$$

$$+\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right)^2$$

$$+2\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\alpha_i}{(1-\varphi)}\theta_i\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\right)$$

$$=\frac{1}{(1-\varphi)^2}\left(\sigma_\alpha^2+\bar{\alpha}^2\right)+\left(\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\right)^2\left(\sigma_\theta^2+\bar{\theta}^2\right)$$

$$+2\frac{1}{(1-\varphi)}\left(\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\right)\frac{1}{N}\sum_{i=1}^{N}(\alpha_i\theta_i)$$

$$+\frac{1}{(1-\varphi^2)}\sigma_v^2,$$

since $\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\alpha_i}{(1-\varphi)}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right)$ and $\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\theta_i\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right)$ are equal to zero provided the properties of the disturbance term. Hence, if we sum over T,

$$\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}y_{i,t-1}^{2}=\sum_{t=1}^{T}\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}y_{i,t-1}^{2}$$

$$=\frac{1}{(1-\varphi)^2}\left(\sigma_\alpha^2+\bar{\alpha}^2\right)T$$

$$+\frac{(T-\lambda T)\varphi^4+2\varphi^3+(-T+1+\lambda T)\varphi^2-2\varphi^{T-\lambda T+2}-2\varphi^{-\lambda T+3+T}+\varphi^{-2\lambda T+2T+2}}{(\varphi-1)^2\varphi^2(\varphi^2-1)}\left(\sigma_\theta^2+\bar{\theta}^2\right)$$

$$+2\frac{1}{(1-\varphi)}\frac{(-T+\lambda T)\varphi^2+(T-1-\lambda T)\varphi+\varphi^{T-\lambda T+1}}{(\varphi-1)^2\varphi}\frac{1}{N}\sum_{i=1}^{N}(\alpha_i\theta_i)$$

$$+\frac{T}{(1-\varphi^2)}\sigma_v^2.$$

Now we are going to analyse the second element that defines the denominator given by

$\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\frac{1}{T}\left(\sum_{t=1}^{T}y_{i,t-1}\right)^2$. From (A-3) we get that:

$$\begin{aligned}
&\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\frac{1}{T}\left(\sum_{t=1}^{T}y_{i,t-1}\right)^2 = \frac{1}{(1-\varphi)^2}T\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\alpha_i^2\\
&+\frac{1}{T}\left(\sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\right)^2\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\theta_i^2\\
&+\frac{1}{T}\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right)^2\\
&+\frac{2}{(1-\varphi)}\sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}(\alpha_i\theta_i)\\
&+\frac{2}{(1-\varphi)}\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\alpha_i\sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right)\\
&+\frac{2}{T}\sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\theta_i\sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right),
\end{aligned}$$

or

$$\begin{aligned}
&\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\frac{1}{T}\left(\sum_{t=1}^{T}y_{i,t-1}\right)^2 = \frac{T}{(1-\varphi)^2}\left(\sigma_\alpha^2+\bar{\alpha}^2\right)\\
&+\frac{1}{T}\left(\frac{(-T+\lambda T)\varphi^2+(T-1-\lambda T)\varphi+\varphi^{T-\lambda T+1}}{(\varphi-1)^2\varphi}\right)^2\left(\sigma_\theta^2+\bar{\theta}^2\right)\\
&+\frac{1}{T}\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right)^2\\
&+\frac{2}{(1-\varphi)}\frac{(-T+\lambda T)\varphi^2+(T-1-\lambda T)\varphi+\varphi^{T-\lambda T+1}}{(\varphi-1)^2\varphi}\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}(\alpha_i\theta_i).
\end{aligned}$$

Notice that

$$\left(\sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right)^2=\sum_{t=1}^{T}\left(\frac{1}{(1-\varphi L)}v_{i,t-1}\right)^2+2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(\frac{1}{(1-\varphi L)}v_{i,t-1}\right)\left(\frac{1}{(1-\varphi L)}v_{i,s-1}\right).$$

Taking the limit in N, we get that the covariances can be expressed as:

$$\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\sum_{j=0}^{\infty}\varphi^{2j+(s-t)}\sigma_v^2=\varphi\frac{(\varphi-1)T-\varphi^T+1}{(\varphi^2-1)(\varphi-1)^2}\sigma_v^2,$$

whereas the variance can be reduced to:

$$\sum_{t=1}^{T}\left(\frac{1}{(1-\varphi L)}v_{i,t-1}\right)^2=\frac{T}{1-\varphi^2}\sigma_v^2.$$

Thus,

$$\frac{1}{T}\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right)^2=\frac{1}{(1-\varphi^2)}\sigma_v^2+\frac{2}{T}\varphi\frac{(\varphi-1)T-\varphi^T+1}{(\varphi^2-1)(\varphi-1)^2}\sigma_v^2.$$

Therefore, the denominator converges in the limit to:

$$\begin{aligned}\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}y_{i,-1}'Q_Ty_{i,-1}=&\frac{1}{(\varphi-1)^2\varphi^2(\varphi^2-1)}\\&\big((T-\lambda T)\varphi^4+2\varphi^3+(-T+1+\lambda T)\varphi^2-\\&2\varphi^{T-\lambda T+2}-2\varphi^{-\lambda T+3+T}+\varphi^{-2\lambda T+2T+2}\big)\left(\sigma_\theta^2+\bar{\theta}^2\right)\\&+\frac{T}{(1-\varphi^2)}\sigma_v^2\\&-\frac{1}{T}\left(\frac{(-T+\lambda T)\varphi^2+(T-1-\lambda T)\varphi+\varphi^{T-\lambda T+1}}{(\varphi-1)^2\varphi}\right)^2\left(\sigma_\theta^2+\bar{\theta}^2\right)\\&-\frac{1}{(1-\varphi^2)}\sigma_v^2-\frac{2}{T}\varphi\frac{(\varphi-1)T-\varphi^T+1}{(\varphi^2-1)(\varphi-1)^2}\sigma_v^2,\end{aligned}$$

and after rearranging terms,

$$\begin{aligned}\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}y_{i,-1}'Q_Ty_{i,-1}=&\left(\frac{1}{(\varphi-1)^2\varphi^2(\varphi^2-1)}\big((T-\lambda T)\varphi^4+2\varphi^3+(-T+1+\lambda T)\varphi^2-\right.\\&2\varphi^{T-\lambda T+2}-2\varphi^{-\lambda T+3+T}+\varphi^{-2\lambda T+2T+2}\big)\\&\left.-\frac{1}{T}\left(\frac{(-T+\lambda T)\varphi^2+(T-1-\lambda T)\varphi+\varphi^{T-\lambda T+1}}{(\varphi-1)^2\varphi}\right)^2\right)\left(\sigma_\theta^2+\bar{\theta}^2\right)\\&-\frac{\left(\varphi^2-2\varphi+1\right)T^2+\left(\varphi^2-1\right)T-2\varphi^{T+1}+2\varphi}{T(\varphi^2-1)(\varphi-1)^2}\sigma_v^2;\\&=D\left(\varphi,T,\lambda,\sigma_\theta^2,\bar{\theta},\sigma_v^2\right).\end{aligned}$$

Notice that the denominator of the bias computed by Nickell (1981) -see the equation just before expression (14) and the equation just before (17) of his paper- is given by:

$$\begin{aligned}\sum_{t=1}^{T} B_t &= \sum_{t=1}^{T}\left(\frac{\sigma_v^2}{(1-\varphi^2)} - \frac{2\sigma_v^2}{T(1-\varphi^2)}\left(\frac{1-\varphi^t}{(1-\varphi)} + \varphi\frac{1-\varphi^{T-t}}{(1-\varphi)}\right) + \frac{\sigma_v^2}{T(1-\varphi)^2}\left(1 - \frac{2\varphi\left(1-\varphi^T\right)}{T(1-\varphi^2)}\right)\right) \\ &= -\frac{1}{T}\frac{T^2\varphi^2 + T\varphi^2 - 2T^2\varphi + T^2 - T + 2\varphi - 2\varphi^{T+1}}{(\varphi-1)^2(\varphi^2-1)}\sigma_v^2,\end{aligned}$$

that corresponds with the last term of the expression we have derived.
The numerator of the first element of the bias is given by $\sum_{i=1}^{N} y_{i,-1}' Q_T v_i$ which can be suitable decomposed as:

$$\begin{aligned}y_{i,-1}' Q_T v_i &= y_{i,-1}' v_i - y_{i,-1}' x\left(x'x\right)^{-1} x' v_i, \\ &= \sum_{t=1}^{T} y_{i,t-1} v_{i,t} - \frac{1}{T}\sum_{t=1}^{T} y_{i,t-1}\sum_{t=1}^{T} v_{i,t}.\end{aligned}$$

Using the definition given by (A-1), it can be seen that

$$\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T} y_{i,t-1} v_{i,t} = 0,$$

whereas

$$\begin{aligned}\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T} y_{i,t-1}\sum_{t=1}^{T} v_{i,t}\right) &= \underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}\sum_{j=0}^{\infty}\varphi^j v_{i,t-1-j}\right)\left(\sum_{t=1}^{T} v_{i,t}\right) \\ &= \sum_{t=2}^{T}\sum_{j=0}^{t-2}\varphi^j\sigma_v^2, \\ &= -\frac{-\varphi^{T+1} + \varphi^2 T - \varphi T + \varphi}{\varphi(\varphi-1)^2}\sigma_v^2.\end{aligned}$$

Thus, the limit of the numerator of the first element is given by:

$$\begin{aligned}\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N} y_{i,-1}' Q_T v_i &= \frac{-\varphi^{T+1} + \varphi^2 T - \varphi T + \varphi}{T\varphi(\varphi-1)^2}\sigma_v^2; \qquad \text{(A-4)} \\ &= N_1\left(\varphi, T, \sigma_v^2\right).\end{aligned}$$

The limit of the numerator of Nickell (1981) -see the equation (13) and the equation just before equation (17) of his paper- is equal to:

$$\begin{aligned}\sum_{t=1}^{T} A_t &= \sum_{t=1}^{T}\left(-\frac{\sigma_v^2}{T(1-\varphi)}\left(1 - \varphi^{t-1} - \varphi^{T-t} + \frac{1}{T}\frac{\left(1-\varphi^T\right)}{(1-\varphi)}\right)\right) \\ &= \frac{-\varphi^{T+1} + \varphi^2 T - \varphi T + \varphi}{T\varphi(\varphi-1)^2}\sigma_v^2,\end{aligned}$$

that, as can be seen, equals the expression given in (A-4).
Finally, the last term that has to be analysed is the numerator of the second element that defines the bias in the estimation of the autoregressive parameter. Now the numerator deals with $\sum_{i=1}^{N} y'_{i,-1} Q_T z\, \zeta_i$, that can be also expressed as:

$$\begin{aligned} y'_{i,-1} Q_T z\, \zeta_i &= \theta_i \sum_{t=1}^{T} \left(DU_t - (1-\lambda)\right) y_{i,t-1} \\ &= \theta_i \sum_{t=1}^{T} \left(\left(DU_t - (1-\lambda)\right) \left(\frac{\alpha_i}{(1-\varphi)} + \theta_i \sum_{j=0}^{\infty} \varphi^j DU_{t-1-j} + \sum_{j=0}^{\infty} \varphi^j v_{i,t-1-j} \right) \right). \end{aligned}$$

Taking the limit in N, we have that:

$$\underset{N\to\infty}{plim} \frac{1}{N} \sum_{i=1}^{N} \theta_i \sum_{t=1}^{T} \left(DU_t - (1-\lambda)\right) \frac{\alpha_i}{(1-\varphi)} = 0,$$

since $\sum_{t=1}^{T} \left(DU_t - (1-\lambda)\right) = 0$. On the other hand,

$$\begin{aligned} \sum_{t=1}^{T} \left(\left(DU_t - (1-\lambda)\right) \sum_{j=0}^{\infty} \varphi^j DU_{t-1-j} \right) &= \sum_{t=1}^{T} \left(DU_t \sum_{j=0}^{\infty} \varphi^j DU_{t-1-j} \right) \\ &\quad - (1-\lambda) \sum_{t=1}^{T} \sum_{j=0}^{\infty} \varphi^j DU_{t-1-j}. \end{aligned}$$

Notice that from the definition of the dummy variable:

$$\sum_{t=1}^{\lambda T+1} \left(DU_t \sum_{j=0}^{\infty} \varphi^j DU_{t-1-j} \right) = \sum_{t=1}^{\lambda T+1} \sum_{j=0}^{\infty} \varphi^j DU_{t-1-j} = 0,$$

and that for the rest of the time period it can be established that:

$$\begin{aligned} \sum_{t=\lambda T+2}^{T} \left(DU_t \sum_{j=0}^{\infty} \varphi^j DU_{t-1-j} \right) &= \sum_{t=\lambda T+2}^{T} \sum_{j=0}^{t-2} \varphi^j \\ &= \frac{\left(-\varphi^2 + \varphi + \varphi^2\lambda - \varphi\lambda\right) T + \varphi^{T+1} + \varphi^2 - \varphi - \varphi^{\lambda T+2}}{\varphi\left(\varphi - 1\right)^2}, \end{aligned}$$

and

$$\begin{aligned} -(1-\lambda) \sum_{t=\lambda T+2}^{T} \sum_{j=0}^{\infty} \varphi^j DU_{t-1-j} &= -(1-\lambda) \sum_{t=\lambda T+2}^{T} \sum_{j=0}^{t-2} \varphi^j \\ &= (-1+\lambda) \frac{\varphi^{T+1} - \varphi^2 T + \varphi^2 + \varphi T - \varphi - \varphi^{\lambda T+2} + \varphi^2 \lambda T - \varphi \lambda T}{\varphi\left(\varphi - 1\right)^2}. \end{aligned}$$

Therefore,

$$\sum_{t=1}^{T}\left((DU_t-(1-\lambda))\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\right)=\lambda\frac{\varphi^{T+1}-\varphi^2T+\varphi^2+\varphi T-\varphi-\varphi^{\lambda T+2}+\varphi^2\lambda T-\varphi\lambda T}{\varphi\left(\varphi-1\right)^2},$$

and the corresponding limit expression is:

$$\begin{aligned}
&\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}\theta_i^2\sum_{t=1}^{T}\left((DU_t-(1-\lambda))\sum_{j=0}^{\infty}\varphi^j DU_{t-1-j}\right)=\\
=\;&\lambda\frac{\varphi^{T+1}-\varphi^2T+\varphi^2+\varphi T-\varphi-\varphi^{\lambda T+2}+\varphi^2\lambda T-\varphi\lambda T}{\varphi\left(\varphi-1\right)^2}\left(\sigma_\theta^2+\bar{\theta}^2\right).
\end{aligned}$$

Finally, the limit involving the disturbance term is equal to zero. Thus, the limit of the numerator is given by:

$$\begin{aligned}
\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1i,-1}^{N\prime}yQ_Tz\,\zeta_i &= \lambda\frac{\varphi^{T+1}-\varphi^2T+\varphi^2+\varphi T-\varphi-\varphi^{\lambda T+2}+\varphi^2\lambda T-\varphi\lambda T}{\varphi\left(\varphi-1\right)^2}\left(\sigma_\theta^2+\bar{\theta}^2\right);\\
&= N_2\left(\varphi,T,\lambda,\sigma_\theta^2,\bar{\theta}\right),
\end{aligned}$$

which not only depends on the sample size, T, but also on the position of the date of the break -through the break fraction parameter, λ- and on the mean and variance of the effect of the structural change, $\bar{\theta}$ and σ_θ^2, respectively. Notice that this term would not be present in the bias estimation of the autoregressive parameter if there is no structural break affecting the mean of the time series, since in this situation $\bar{\theta}=\sigma_\theta^2=0$.

If we put together the previous results we obtain that the bias in the estimation of the autoregressive parameter is given by:

$$\hat{\varphi}\to\varphi+\frac{N_1\left(\varphi,T,\sigma_v^2\right)}{D\left(\varphi,T,\lambda,\sigma_\theta^2,\bar{\theta},\sigma_v^2\right)}+\frac{N_2\left(\varphi,T,\lambda,\sigma_\theta^2,\bar{\theta}\right)}{D\left(\varphi,T,\lambda,\sigma_\theta^2,\bar{\theta},\sigma_v^2\right)}.$$

Our previous analysis has revealed that the estimation of the autoregressive parameter of a dynamic panel data gets its bias increased by the misspecification error of not to take account for a structural break that shifts the mean. Thus, Theorem 1 has been proved. ■

A.3 Proof of Theorem 2

The inconsistency of (4) when the time series have been affected by a structural break and it is not specified in the model relies on the third element of (6). Let us begin with the quadratic form that defines the denominator of this element. Using lemma 2 the denominator can be expressed as:

$$
\begin{aligned}
y_{i,-1}'Q_T y_{i,-1} &= y_{i,-1}'y_{i,-1} - y_{i,-1}'x\left(x'x\right)^{-1}x'y_{i,-1} \\
&= \sum_{t=1}^{T} y_{i,t-1}^2 - \frac{1}{T}\left(\sum_{t=1}^{T} y_{i,t-1}\right)^2 \\
&= \alpha_i^2 T + \theta_i^2\left(\left(1-\lambda\right)T-1\right) + \sum_{t=1}^{T} u_{i,t-1}^2 + 2\alpha_i\theta_i\left(\left(1-\lambda\right)T-1\right) \\
&\quad +2\alpha_i\sum_{t=1}^{T} u_{i,t-1} + 2\theta_i\sum_{t=T_b+2}^{T} u_{i,t} - \frac{1}{T}\left(\alpha_i T + \theta_i\left(\left(1-\lambda\right)T-1\right) + \sum_{t=1}^{T} u_{i,t-1}\right)^2
\end{aligned}
$$

Simple algebra manipulations gives:

$$
\begin{aligned}
y_{i,-1}'Q_T y_{i,-1} &= \alpha_i^2 T + \theta_i^2\left(\left(1-\lambda\right)T-1\right) + \sum_{t=1}^{T} u_{i,t-1}^2 + 2\alpha_i\theta_i\left(\left(1-\lambda\right)T-1\right) \\
&\quad +2\alpha_i\sum_{t=1}^{T} u_{i,t-1} + 2\theta_i\sum_{t=T_b+2}^{T} u_{i,t} \\
&\quad -\frac{1}{T}\left(\alpha_i^2 T^2 + \theta_i^2\left(\left(1-\lambda\right)T-1\right)^2 + \sum_{t=1}^{T} u_{i,t-1}^2 + 2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T} u_{i,t-1}u_{i,s-1}\right. \\
&\quad \left. +2\alpha_i T\theta_i\left(\left(1-\lambda\right)T-1\right) + 2\alpha_i T\sum_{t=1}^{T} u_{i,t-1} + 2\theta_i\left(\left(1-\lambda\right)T-1\right)\sum_{t=1}^{T} u_{i,t-1}\right) \\
&= \theta_i^2\left(\left(1-\lambda\right)T-1\right) + T\sigma_v^2 + 2\theta_i\sum_{t=T_b+2}^{T} u_{i,t} - \frac{1}{T}\theta_i^2\left(\left(1-\lambda\right)T-1\right)^2 - \sigma_v^2 \\
&\quad -\frac{1}{T}2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T} u_{i,t-1}u_{i,s-1} - \frac{1}{T}2\theta_i\left(\left(1-\lambda\right)T-1\right)\sum_{t=1}^{T} u_{i,t-1} \\
&= \theta_i^2\left(\left(1-\lambda\right)T-1\right) + \left(T-1\right)\sigma_v^2 + 2\theta_i\sum_{t=T_b+2}^{T} u_{i,t} - \frac{1}{T}\theta_i^2\left(\left(1-\lambda\right)T-1\right)^2 \\
&\quad -\frac{1}{T}2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T} u_{i,t-1}u_{i,s-1} - \frac{1}{T}2\theta_i\left(\left(1-\lambda\right)T-1\right)\sum_{t=1}^{T} u_{i,t-1}.
\end{aligned}
$$

For the numerator given by $y_{i,-1}'Q_T z\,\zeta_i$ we have that $Q_T z = \left[0, DU - \left(1-\lambda\right)e_T\right]$ is a $(T \times 2)$ matrix, the first column being a vector of zeros. Hence,

$$
\begin{aligned}
y_{i,-1}' Q_T z\, \zeta_i &= \theta_i \sum_{t=1}^{T} \left(DU_t - (1-\lambda)\right) y_{i,t-1} \\
&= \theta_i \left(\sum_{t=1}^{T} DU_t y_{i,t-1} - (1-\lambda) \sum_{t=1}^{T} y_{i,t-1} \right) \\
&= \theta_i \left(\sum_{t=1}^{T} DU_t \left(\alpha_i + \theta_i DU_{t-1} + u_{i,t-1}\right) - (1-\lambda) \sum_{t=1}^{T} y_{i,t-1} \right) \\
&= \theta_i \left((1-\lambda) T \alpha_i + \theta_i \left((1-\lambda) T - 1\right) + \sum_{t=T_b+1}^{T} u_{i,t-1} - (1-\lambda) \sum_{t=1}^{T} y_{i,t-1} \right) \\
&= \theta_i \left((1-\lambda) T \alpha_i + \theta_i \left((1-\lambda) T - 1\right) \right. \\
&\quad \left. + \sum_{t=T_b+1}^{T} u_{i,t-1} - (1-\lambda) \left(\alpha_i T + \theta_i \left((1-\lambda) T - 1\right) + \sum_{t=1}^{T} u_{i,t-1} \right) \right) \\
&= \theta_i \left(\lambda \theta_i \left((1-\lambda) T - 1\right) + \sum_{t=T_b+1}^{T} u_{i,t-1} - (1-\lambda) \sum_{t=1}^{T} u_{i,t-1} \right).
\end{aligned}
$$

The limit in probability of the denominator is equal to:

$$
\begin{aligned}
\underset{N\to\infty}{plim} \frac{1}{N} \sum_{i=1}^{N} y_{i,-1}' Q_T y_{i,-1} &= (T-1)\, \sigma_v^2 + \left(\frac{(T - T\lambda - 1)(T\lambda + 1)}{T} \right) \underset{N\to\infty}{plim} \frac{1}{N} \sum_{i=1}^{N} \theta_i^2, \\
&= (T-1)\, \sigma_v^2 + \left(\frac{(T - T\lambda - 1)(T\lambda + 1)}{T} \right) \left(\sigma_\theta^2 + \bar{\theta}^2 \right),
\end{aligned}
$$

whereas for the numerator we have that:

$$
\begin{aligned}
\underset{N\to\infty}{plim} \frac{1}{N} \sum_{i=1}^{N} y_{i,-1}' Q_T z\, \zeta_i &= \left((1-\lambda) T - 1\right) \lambda \underset{N\to\infty}{plim} \frac{1}{N} \sum_{i=1}^{N} \theta_i^2, \\
&= \left((1-\lambda) T - 1\right) \lambda \left(\sigma_\theta^2 + \bar{\theta}^2 \right).
\end{aligned}
$$

Therefore and using the Slutzky's theorem, the probability limit of the third element of (6) is:

$$
\underset{N\to\infty}{plim} \frac{\frac{1}{N} \sum_{i=1}^{N} y_{i,-1}' Q_T z\, \zeta_i}{\frac{1}{N} \sum_{i=1}^{N} y_{i,-1}' Q_T y_{i,-1}} = \frac{\left((1-\lambda) T - 1\right) \lambda \left(\sigma_\theta^2 + \bar{\theta}^2 \right)}{(T-1)\, \sigma_v^2 + \left(\frac{(T - T\lambda - 1)(T\lambda + 1)}{T} \right) \left(\sigma_\theta^2 + \bar{\theta}^2 \right)}.
$$

As can be seen, this complicated expression is function of the date of the break, the sample size, the variance of the disturbance and of the variance and the mean of the magnitude of the break. Thus, Theorem 2 has been proved. ■

A.4 Proof of Theorems 3 and 4

In this subsection we conduct the proof of theorems 3 and 4. Under the null hypothesis of unit root $\alpha_i = \theta_i = 0$ and $\varphi = 1$, model A described by (8) can be expressed as:

$$y_{i,t} = \delta_i D\left(T_b\right)_t + y_{i,t-1} + v_{i,t}. \tag{5}$$

Recursive substitution allows to express (A-5) as:

$$\begin{aligned} y_{i,t} &= \delta_i \sum_{j=1}^{t} D\left(T_b\right)_j + y_{i,0} + \sum_{j=0}^{t-1} v_{i,t-j}, \\ &= \delta_i DU_t + y_{i,0} + \sum_{j=0}^{t-1} v_{i,t-j}. \end{aligned} \tag{A-6}$$

Using matrix notation, (A-6) can be written as follows:

$$y_i = \delta_i DU + e_T y_{i,0} + D_T v_i, \tag{7}$$

where $y_i = (y_{i,1}, ..., y_{i,T})'$, $DU = (DU_1, ..., DU_T)'$, $e_T = (1, ..., 1)'$, and

$$D_T = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 1 & 1 & 1 & 1 & \ddots & \\ 1 & 1 & 1 & 1 & \dots & 0 \end{bmatrix}_{T\times T}.$$

From (A-7), under the null hypothesis the vector $y_{i,-1}$ can be decomposed as:

$$\begin{aligned} y_{i,-1} &= \delta_i DU_{-1} + e_T y_{i,0} + C_T v_i, \\ &= e_T y_{i,0} + \delta_i DU - \delta_i D\left(T_b\right) + C_T v_i, \end{aligned} \tag{A-8}$$

provided that

$$DU_t = DU_{t-1} + D\left(T_b\right)_t,$$

where

$$C_T = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 1 & 1 & 1 & 1 & \ddots & \\ 1 & 1 & 1 & 1 & \dots & 0 \end{bmatrix}_{T\times T}.$$

Notice that since $Q_T Z = Q_T \left[e_T, DU, D\left(T_b\right)\right] = 0$,

$$Q_T y_{i,-1} = Q_T C_T v_i,$$

due to the fact that Z matrix is also present as a matrix of regressors in (A-8) which allows (11) be simplified to:

$$\varphi - 1 = \left[\sum_{i=1}^{N} v_i' C_T' Q_T C_T v_i\right]^{-1} \left[\sum_{i=1}^{N} v_i' C_T' Q_T v_i\right]. \tag{9}$$

First, let us analyse the denominator of (A-9). The quadratic form can be decomposed as follows:

$$v_i' C_T' Q_T C_T v_i = v_i' C_T' C_T v_i - v_i' C_T' Z \left(Z'Z\right)^{-1} Z' C_T v_i,$$

where

$$v_i' C_T' C_T v_i = \sum_{t=1}^{T-1} \left(\sum_{j=1}^{t} v_{i,j}\right)^2 = \sum_{t=1}^{T-1} \varepsilon_{i,t}^2 = \sum_{t=1}^{T} \varepsilon_{i,t-1}^2,$$

provided that $\varepsilon_{i,0} = 0$. Another intermediate result shows that:

$$\begin{aligned}
\left(Z'Z\right)^{-1} &= \begin{bmatrix} T & (1-\lambda)T & 1 \\ (1-\lambda)T & (1-\lambda)T & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \frac{1}{T\lambda} & -\frac{1}{T\lambda} & 0 \\ -\frac{1}{T\lambda} & -\frac{T-1}{T\lambda(-T+T\lambda+1)} & \frac{1}{-T+T\lambda+1} \\ 0 & \frac{1}{-T+T\lambda+1} & T\frac{-1+\lambda}{-T+T\lambda+1} \end{bmatrix}.
\end{aligned}$$

Since

$$\begin{aligned}
Z' C_T v_i &= \begin{bmatrix} 0 + \sum_{j=1}^{1} v_{i,j} + \ldots + \sum_{j=1}^{T-1} v_{i,j} \\ 0 + \ldots + \sum_{j=1}^{T_b+1} v_{i,j} + \ldots + \sum_{j=1}^{T-1} v_{i,j} \\ 0 + \ldots + \sum_{j=1}^{T_b+1} v_{i,j} + \ldots + 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{t=1}^{T} \varepsilon_{i,t-1} \\ \sum_{t=T_b+1}^{T} \varepsilon_{i,t-1} \\ \varepsilon_{i,T_b} \end{bmatrix},
\end{aligned}$$

therefore we can see that the quadratic form of the denominator of (A-9) is equal to:

$$\begin{aligned}
v_i' C_T' Q_T C_T v_i &= \sum_{t=1}^{T} \varepsilon_{i,t-1}^2 - \frac{1}{T\lambda}\left(\sum_{t=1}^{T} \varepsilon_{i,t-1}\right)^2 + \frac{2}{T\lambda} \sum_{t=1}^{T} \varepsilon_{i,t-1} \sum_{t=T_b+1}^{T} \varepsilon_{i,t-1} \\
&\quad + \frac{(T-1)}{T\lambda\left(-T+T\lambda+1\right)} \left(\sum_{t=T_b+1}^{T} \varepsilon_{i,t-1}\right)^2 \\
&\quad - \frac{2}{-T+T\lambda+1} \sum_{t=T_b+1}^{T} \varepsilon_{i,t-1}\varepsilon_{i,T_b} + \frac{T\left(1-\lambda\right)}{-T+T\lambda+1} \varepsilon_{i,T_b}^2.
\end{aligned}$$

Let us now write this equation in terms of the original disturbances:

$$
\begin{aligned}
&v_i' C_T' Q_T C_T v_i = \sum_{t=1}^{T} (T-t)\, v_{i,t}^2 + 2 \sum_{t=1}^{T-1} \sum_{s=t+1}^{T} (T-s)\, v_{i,t} v_{i,s} \\
&-\frac{1}{T\lambda} \left(\sum_{t=1}^{T} (T-t)^2 v_{i,t}^2 + 2 \sum_{t=1}^{T-1} \sum_{s=t+1}^{T} (T-t)(T-s)\, v_{i,t} v_{i,s} \right) \\
&+\frac{2}{T\lambda} \left(\sum_{t=1}^{T} (T-t)^2 v_{i,t}^2 + 2 \sum_{t=1}^{T-1} \sum_{s=t+1}^{T} (T-t)(T-s)\, v_{i,t} v_{i,s} \right. \\
&-\sum_{t=1}^{T_b} (T-t)(T_b-t)\, v_{i,t}^2 - \sum_{t=1}^{T_b-1} \sum_{s=t+1}^{T_b} ((T-t)(T_b-s) + (T-s)(T_b-t))\, v_{i,t} v_{i,s} \\
&\left. -\sum_{t=1}^{T_b} \sum_{s=T_b+1}^{T} (T_b-t)(T-s)\, v_{i,t} v_{i,s} \right) \\
&+\frac{(T-1)}{T\lambda(-T+T\lambda+1)} \left(\sum_{t=1}^{T} (T-t)^2 v_{i,t}^2 + 2 \sum_{t=1}^{T-1} \sum_{s=t+1}^{T} (T-t)(T-s)\, v_{i,t} v_{i,s} \right. \\
&+\sum_{t=1}^{T_b} \left((T_b-t)^2 - 2(T-t)(T_b-t) \right) v_{i,t}^2 \\
&+\sum_{t=1}^{T_b-1} \sum_{s=t+1}^{T_b} \left(2T_b^2 - 2ts - 4TT_b + 2tT + 2sT \right) v_{i,t} v_{i,s} \\
&\left. -2 \sum_{t=1}^{T_b} \sum_{s=T_b+1}^{T} (T_b-t)(T-s)\, v_{i,t} v_{i,s} \right) \\
&-\frac{2}{-T+T\lambda+1} \left((T-T_b) \sum_{t=1}^{T_b} v_{i,t}^2 + 2(T-T_b) \sum_{t=1}^{T_b-1} \sum_{s=t+1}^{T_b} v_{i,t} v_{i,s} + \sum_{t=1}^{T_b} \sum_{s=T_b+1}^{T} (T-s)\, v_{i,t} v_{i,s} \right) \\
&+\frac{T(1-\lambda)}{-T+T\lambda+1} \left(\sum_{t=1}^{T_b} v_{i,t}^2 + 2 \sum_{t=1}^{T_b-1} \sum_{s=t+1}^{T_b} v_{i,t} v_{i,s} \right).
\end{aligned}
$$

If we collect terms involving squared elements we define the $D\left(T,\lambda\right)_i$ function:

$$
\begin{aligned}
v_i'C_T'Q_TC_Tv_i &= \sum_{t=1}^{T}\left(T-t\right)v_{i,t}^2+\left(\frac{1}{T\lambda}+\frac{T-1}{T\lambda\left(-T+T\lambda+1\right)}\right)\sum_{t=1}^{T}\left(T-t\right)^2 v_{i,t}^2 \\
&\quad -\frac{2}{T\lambda}\sum_{t=1}^{T_b}\left(T-t\right)\left(T_b-t\right)v_{i,t}^2 \\
&\quad +\frac{T-1}{T\lambda\left(-T+T\lambda+1\right)}\sum_{t=1}^{T_b}\left(\left(T_b-t\right)^2-2\left(T-t\right)\left(T_b-t\right)\right)v_{i,t}^2 \\
&\quad -\frac{T\left(1-\lambda\right)}{-T+T\lambda+1}\sum_{t=1}^{T_b}v_{i,t}^2+f\left(v_{i,t}v_{i,s}\right),
\end{aligned}
$$

and obtain

$$
\begin{aligned}
v_i'C_T'Q_TC_Tv_i &= \sum_{t=1}^{T}\frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1}v_{i,t}^2 \\
&\quad +\sum_{t=1}^{T_b}\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^2+f\left(v_{i,t}v_{i,s}\right) \\
&= D\left(T,\lambda\right)_i, \qquad \text{(A-10)}
\end{aligned}
$$

where $f\left(v_{i,t}v_{i,s}\right)$ summarises all terms of cross product of disturbances for $\forall s,t\ =$

$\{1, \ldots, T\}$; $s \neq t$. Specifically,

$$f\left(v_{i,t}v_{i,s}\right) = 2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(T-s\right)v_{i,t}v_{i,s} - \frac{2}{T\lambda}\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(T-t\right)\left(T-s\right)v_{i,t}v_{i,s}$$
$$+\frac{2}{T\lambda}\left(2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(T-t\right)\left(T-s\right)v_{i,t}v_{i,s}\right.$$
$$-\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(\left(T-t\right)\left(T_b-s\right)+\left(T-s\right)\left(T_b-t\right)\right)v_{i,t}v_{i,s}$$
$$\left.-\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(T_b-t\right)\left(T-s\right)v_{i,t}v_{i,s}\right)$$
$$+\frac{\left(T-1\right)}{T\lambda\left(-T+T\lambda+1\right)}\left(2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(T-t\right)\left(T-s\right)v_{i,t}v_{i,s}\right.$$
$$+\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(2T_b^2-2ts-4TT_b+2tT+2sT\right)v_{i,t}v_{i,s}$$
$$\left.-2\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(T_b-t\right)\left(T-s\right)v_{i,t}v_{i,s}\right)$$
$$-\frac{2}{-T+T\lambda+1}\left(2\left(T-T_b\right)\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}v_{i,t}v_{i,s}+\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(T-s\right)v_{i,t}v_{i,s}\right)$$
$$+\frac{2T\left(1-\lambda\right)}{-T+T\lambda+1}\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}v_{i,t}v_{i,s}.$$

Some simplification allow to express this function as:

$$f\left(v_{i,t}v_{i,s}\right) = \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(2\left(-T\lambda-1+t\right)\frac{-T+s}{-T+T\lambda+1}\right)v_{i,t}v_{i,s} \tag{A-11}$$
$$+\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(-2\frac{\left(T^3\lambda^2+ts+T^2\lambda-sT^2\lambda^2-tsT-sT\lambda+2tsT\lambda-tT\lambda-tT^2\lambda^2\right)}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}v_{i,s}$$
$$+\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(-2\left(-T\lambda-1+t\right)\frac{-T+s}{-T+T\lambda+1}\right)v_{i,t}v_{i,s}.$$

Hence, from (A-10) we can see that the limit probability of the denominator is:

$$\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}v_i'C_T'Q_TC_Tv_i=\sum_{t=1}^{T}\frac{(T-t)(T\lambda+1-t)}{-T+T\lambda+1}\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}v_{i,t}^2$$
$$+\sum_{t=1}^{T_b}\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda(-T+T\lambda+1)}\right)\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}v_{i,t}^2$$

since $\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}f(v_{i,t}v_{i,s})=0$ provided that $v_{i,t}$ are iid across i and t. Noticing that $\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}v_{i,t}^2=\sigma_v^2$ the probability of the denominator reduces to:

$$\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}v_i'C_T'Q_TC_Tv_i=\sum_{t=1}^{T}\frac{(T-t)(T\lambda+1-t)}{-T+T\lambda+1}\sigma_v^2$$
$$+\sum_{t=1}^{T_b}\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda(-T+T\lambda+1)}\right)\sigma_v^2$$
$$=\left(\left(\frac{1}{3}\lambda^2-\frac{1}{3}\lambda+\frac{1}{6}\right)T^2+\left(-\frac{1}{3}+\frac{1}{3}\lambda\right)T-\frac{1}{6}\right)\sigma_v^2. \qquad \text{(A-12)}$$

Notice that for $\lambda=1$, that is to say, no break is present in the model, the probability of the denominator equals the corresponding expression for the model 2 of Harris and Tzavalis (1999).

Let us now analyse the numerator of the test. First of all, we decompose the matrix product in two parts:

$$v_i'C_T'Q_Tv_i=v_i'C_T'v_i-v_i'C_T'Z\left(Z'Z\right)^{-1}Z'v_i.$$

It is straightforward to see that the first part is equal to:

$$v_i'C_T'v_i=\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}v_{i,t}v_{i,s},$$

while the second one is:

$$
\begin{aligned}
v_i' C_T' Z \left(Z'Z\right)^{-1} Z' v_i &= \left[\begin{array}{c} \sum_{t=1}^{T} \varepsilon_{i,t-1} \\ \sum_{t=T_b+1}^{T} \varepsilon_{i,t-1} \\ \varepsilon_{i,T_b} \end{array}\right]' \left[\begin{array}{ccc} \frac{1}{T\lambda} & -\frac{1}{T\lambda} & 0 \\ & -\frac{T-1}{T\lambda(-T+T\lambda+1)} & \frac{1}{-T+T\lambda+1} \\ & & T\frac{-1+\lambda}{-T+T\lambda+1} \end{array}\right] \times \\
& \left[\begin{array}{c} \sum_{t=1}^{T} v_{i,t} \\ \sum_{t=T_b+1}^{T} v_{i,t} \\ v_{i,T_b+1} \end{array}\right] \\
&= \left(\frac{\sum_{t=1}^{T} \varepsilon_{i,t-1}}{T\lambda} - \frac{\sum_{t=T_b+1}^{T} \varepsilon_{i,t-1}}{T\lambda}\right) \sum_{t=1}^{T} v_{i,t} \\
& + \left(-\frac{\sum_{t=1}^{T} \varepsilon_{i,t-1}}{T\lambda} - \left(\sum_{t=T_b+1}^{T} \varepsilon_{i,t-1}\right) \frac{T-1}{T\lambda\left(-T+T\lambda+1\right)} + \right. \\
& \left. \frac{\varepsilon_{i,T_b}}{-T+T\lambda+1}\right) \sum_{t=T_b+1}^{T} v_{i,t} \\
& + \left(\frac{\sum_{t=T_b+1}^{T} \varepsilon_{i,t-1}}{-T+T\lambda+1} + \varepsilon_{i,T_b} T \frac{-1+\lambda}{-T+T\lambda+1}\right) v_{i,T_b+1} \\
&= \left(\frac{\sum_{t=1}^{T} \varepsilon_{i,t-1}}{T\lambda} - \frac{\sum_{t=T_b+1}^{T} \varepsilon_{i,t-1}}{T\lambda}\right) \varepsilon_{i,T} \\
& + \left(-\frac{\sum_{t=1}^{T} \varepsilon_{i,t-1}}{T\lambda} - \left(\sum_{t=T_b+1}^{T} \varepsilon_{i,t-1}\right) \frac{T-1}{T\lambda\left(-T+T\lambda+1\right)} + \right. \\
& \left. \frac{\varepsilon_{i,T_b}}{-T+T\lambda+1}\right) \left(\varepsilon_{i,T} - \varepsilon_{i,T_b}\right) \\
& + \left(\frac{\sum_{t=T_b+1}^{T} \varepsilon_{i,t-1}}{-T+T\lambda+1} + \varepsilon_{i,T_b} T \frac{-1+\lambda}{-T+T\lambda+1}\right) v_{i,T_b+1}
\end{aligned}
$$

Let us analyse each part of the previous expression. First term:

$$
\left(\frac{\sum_{t=1}^{T} \varepsilon_{i,t-1}}{T\lambda} - \frac{\sum_{t=T_b+1}^{T} \varepsilon_{i,t-1}}{T\lambda}\right) \varepsilon_{i,T} = \frac{1}{T\lambda}\left(\sum_{t=1}^{T} \varepsilon_{i,t-1}\varepsilon_{i,T} - \sum_{t=T_b+1}^{T} \varepsilon_{i,t-1}\varepsilon_{i,T}\right).
$$

From Harris and Tzavalis (1999), lemma A.1:

$$
\sum_{t=1}^{T} \varepsilon_{i,t-1}\varepsilon_{i,T} = \sum_{t=1}^{T} (T-t)\, v_{i,t}^2 + \sum_{t=1}^{T-1} \sum_{s=t+1}^{T} (2T-s-t)\, v_{i,t} v_{i,s},
$$

while

$$\begin{aligned}\sum_{t=T_b+1}^{T} \varepsilon_{i,t-1}\varepsilon_{i,T} &= \left(\sum_{t=1}^{T}(T-t)\,v_{i,t} - \sum_{t=1}^{T_b}(T_b-t)\,v_{i,t}\right)\sum_{t=1}^{T} v_{i,t} \\ &= \sum_{t=1}^{T}(T-t)\,v_{i,t}^2 + \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}(2T-t-s)\,v_{i,t}v_{i,s} \\ &\quad -\sum_{t=1}^{T_b}(T_b-t)\,v_{i,t}^2 - \sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}(2T_b-t-s)\,v_{i,t}v_{i,s} \\ &\quad -\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}(T_b-t)\,v_{i,t}v_{i,s}.\end{aligned}$$

Hence, the first term is equal to:

$$\left(\frac{\sum_{t=1}^{T}\varepsilon_{i,t-1}}{T\lambda} - \frac{\sum_{t=T_b+1}^{T}\varepsilon_{i,t-1}}{T\lambda}\right)\varepsilon_{i,T} = \left(\frac{1}{T\lambda}\right)\left(\sum_{t=1}^{T_b}(T_b-t)\,v_{i,t}^2 + \right.$$
$$\left.\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}(2T_b-t-s)\,v_{i,t}v_{i,s} + \sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}(T_b-t)\,v_{i,t}v_{i,s}\right).$$

Now we are going to analyse the second term given by:

$$\left(-\frac{\sum_{t=1}^{T}\varepsilon_{i,t-1}}{T\lambda} - \frac{T-1}{T\lambda\,(-T+T\lambda+1)}\sum_{t=T_b+1}^{T}\varepsilon_{i,t-1} + \frac{\varepsilon_{i,T_b}}{-T+T\lambda+1}\right)(\varepsilon_{i,T}-\varepsilon_{i,T_b})$$

in different steps. The first element is:

$$\begin{aligned}-\frac{\sum_{t=1}^{T}\varepsilon_{i,t-1}}{T\lambda}(\varepsilon_{i,T}-\varepsilon_{i,T_b}) &= -\frac{1}{T\lambda}\left[\left(\sum_{t=1}^{T}(T-t)\,v_{i,t}^2 + \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}(2T-s-t)\,v_{i,t}v_{i,s}\right)\right. \\ &\quad -\left(\sum_{t=1}^{T_b}(T-t)\,v_{i,t}^2 + \sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}(2T-s-t)\,v_{i,t}v_{i,s}\right) \\ &\quad \left.-\sum_{t=T_b+1}^{T}(T-t)\,v_{i,t}\sum_{t=1}^{T_b}v_{i,t}\right].\end{aligned}$$

The second element is:

$$
\begin{aligned}
& -\frac{T-1}{T\lambda\left(-T+T\lambda+1\right)}\sum_{t=T_b+1}^{T}\varepsilon_{i,t-1}\left(\varepsilon_{i,T}-\varepsilon_{i,T_b}\right) \\
= & -\frac{T-1}{T\lambda\left(-T+T\lambda+1\right)}\left(\left(\sum_{t=1}^{T}\left(T-t\right)v_{i,t}^{2}+\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(2T-t-s\right)v_{i,t}v_{i,s}\right.\right. \\
& -\sum_{t=1}^{T_b}\left(T_b-t\right)v_{i,t}^{2}-\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(2T_b-t-s\right)v_{i,t}v_{i,s} \\
& \left.-\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(T_b-t\right)v_{i,t}v_{i,s}\right) \\
& -\left(\left(T-T_b\right)\sum_{t=1}^{T_b}v_{i,t}^{2}+\right. \\
& \left.\left.2\left(T-T_b\right)\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}v_{i,t}v_{i,s}+\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(T-s\right)v_{i,t}v_{i,s}\right)\right),
\end{aligned}
$$

and after some rearrangements,

$$
\begin{aligned}
& -\frac{T-1}{T\lambda\left(-T+T\lambda+1\right)}\sum_{t=T_b+1}^{T}\varepsilon_{i,t-1}\left(\varepsilon_{i,T}-\varepsilon_{i,T_b}\right) \\
= & -\frac{T-1}{T\lambda\left(-T+T\lambda+1\right)}\left(\sum_{t=1}^{T}\left(T-t\right)v_{i,t}^{2}+\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(2T-t-s\right)v_{i,t}v_{i,s}\right. \\
& -\sum_{t=1}^{T_b}\left(T-t\right)v_{i,t}^{2}-\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(2T-t-s\right)v_{i,t}v_{i,s} \\
& \left.-\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(T+T_b-t-s\right)v_{i,t}v_{i,s}\right).
\end{aligned}
$$

The third element is equal to:

$$
\frac{\varepsilon_{i,T_b}}{-T+T\lambda+1}\left(\varepsilon_{i,T}-\varepsilon_{i,T_b}\right)=\frac{1}{-T+T\lambda+1}\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}v_{i,t}v_{i,s}.
$$

Thus, the second term is equal to:

$$
\begin{aligned}
&\left(-\frac{\sum_{t=1}^{T}\varepsilon_{i,t-1}}{T\lambda}-\frac{T-1}{T\lambda\left(-T+T\lambda+1\right)}\sum_{t=T_b+2}^{T}\varepsilon_{i,t-1}+\frac{\varepsilon_{i,T_b+1}}{-T+T\lambda+1}\right)\left(\varepsilon_{i,T}-\varepsilon_{i,T_b}\right)=\\
= \quad &-\frac{1}{T\lambda}\left[\left(\sum_{t=1}^{T}\left(T-t\right)v_{i,t}^2+\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(2T-s-t\right)v_{i,t}v_{i,s}\right)\right.\\
&-\left(\sum_{t=1}^{T_b}\left(T-t\right)v_{i,t}^2+\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(2T-s-t\right)v_{i,t}v_{i,s}\right)\\
&\left.-\sum_{t=T_b+1}^{T}\left(T-t\right)v_{i,t}\sum_{t=1}^{T_b}v_{i,t}\right]\\
&-\frac{T-1}{T\lambda\left(-T+T\lambda+1\right)}\left(\sum_{t=1}^{T}\left(T-t\right)v_{i,t}^2+\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(2T-t-s\right)v_{i,t}v_{i,s}\right.\\
&-\sum_{t=1}^{T_b}\left(T-t\right)v_{i,t}^2-\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(2T-t-s\right)v_{i,t}v_{i,s}\\
&\left.-\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(T+T_b-t-s\right)v_{i,t}v_{i,s}\right)\\
&+\frac{1}{-T+T\lambda+1}\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}v_{i,t}v_{i,s},
\end{aligned}
$$

and some simplification can reduce it to:

$$\begin{aligned}
&\left(-\frac{\sum_{t=1}^{T}\varepsilon_{i,t-1}}{T\lambda}-\frac{T-1}{T\lambda\left(-T+T\lambda+1\right)}\sum_{t=T_b+2}^{T}\varepsilon_{i,t-1}+\frac{\varepsilon_{i,T_b+1}}{-T+T\lambda+1}\right)\left(\varepsilon_{i,T}-\varepsilon_{i,T_b}\right)= \\
=\;& -\frac{1}{T\lambda}\sum_{t=1}^{T}\left(T-t\right)v_{i,t}^{2}-\frac{T-1}{T\lambda\left(-T+T\lambda+1\right)}\sum_{t=1}^{T}\left(T-t\right)v_{i,t}^{2} \\
& -\frac{1}{T\lambda}\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(2T-s-t\right)v_{i,t}v_{i,s}-\frac{T-1}{T\lambda\left(-T+T\lambda+1\right)}\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(2T-t-s\right)v_{i,t}v_{i,s} \\
& +\frac{1}{T\lambda}\sum_{t=1}^{T_b}\left(T-t\right)v_{i,t}^{2}+\frac{T-1}{T\lambda\left(-T+T\lambda+1\right)}\sum_{t=1}^{T_b}\left(T-t\right)v_{i,t}^{2} \\
& +\frac{1}{T\lambda}\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(2T-s-t\right)v_{i,t}v_{i,s}+\frac{T-1}{T\lambda\left(-T+T\lambda+1\right)}\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(2T-t-s\right)v_{i,t}v_{i,s} \\
& +\frac{1}{T\lambda}\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(T-s\right)v_{i,t}v_{i,s}+\frac{1}{-T+T\lambda+1}\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}v_{i,t}v_{i,s} \\
& +\frac{T-1}{T\lambda\left(-T+T\lambda+1\right)}\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(T+T_b-t-s\right)v_{i,t}v_{i,s} \\
=\;& \sum_{t=1}^{T}\left(-\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^{2}+\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(-\frac{2T-s-t}{-T+T\lambda+1}\right)v_{i,t}v_{i,s}+\sum_{t=1}^{T_b}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^{2} \\
& +\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(\frac{2T-s-t}{-T+T\lambda+1}\right)v_{i,t}v_{i,s}+\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(\frac{2T^{2}\lambda-sT\lambda-Tt+t}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}v_{i,s}.
\end{aligned}$$

Finally, the last term:

$$\begin{aligned}
\left(\frac{\sum_{t=T_b+1}^{T}\varepsilon_{i,t-1}}{-T+T\lambda+1}+\varepsilon_{i,T_b}T\frac{-1+\lambda}{-T+T\lambda+1}\right)v_{i,T_b+1} &= \frac{1}{-T+T\lambda+1}\left(\sum_{t=1}^{T}\left(T-t\right)v_{i,t}v_{i,T_b+1}-\right. \\
&\qquad \left.\sum_{t=1}^{T_b}\left(T-t\right)v_{i,t}v_{i,T_b+1}\right) \\
&= \frac{1}{-T+T\lambda+1}\sum_{t=T_b+1}^{T}\left(T-t\right)v_{i,t}v_{i,T_b+1}.
\end{aligned}$$

Therefore, we can express the second element that defines the numerator of the test as follows:

$$
\begin{aligned}
&v_i' C_T' Z\left(Z'Z\right)^{-1} Z' v_i = \frac{1}{T\lambda}\sum_{t=1}^{T_b}\left(T_b - t\right) v_{i,t}^2 + \\
&+\frac{1}{T\lambda}\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(2T_b - t - s\right) v_{i,t}v_{i,s} + \frac{1}{T\lambda}\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(T_b - t\right) v_{i,t}v_{i,s} \\
&+\sum_{t=1}^{T}\left(-\frac{T-t}{-T+T\lambda+1}\right) v_{i,t}^2 + \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(-\frac{2T-s-t}{-T+T\lambda+1}\right) v_{i,t}v_{i,s} + \sum_{t=1}^{T_b}\left(\frac{T-t}{-T+T\lambda+1}\right) v_{i,t}^2 \\
&+\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(\frac{2T-s-t}{-T+T\lambda+1}\right) v_{i,t}v_{i,s} + \sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(\frac{2T^2\lambda - sT\lambda - Tt + t}{T\lambda\left(-T+T\lambda+1\right)}\right) v_{i,t}v_{i,s} \\
&+\frac{1}{-T+T\lambda+1}\sum_{t=T_b+1}^{T}\left(T-t\right) v_{i,t}v_{i,T_b+1},
\end{aligned}
$$

which some simplification allow us to express the previous equation as:

$$
\begin{aligned}
v_i' C_T' Z\left(Z'Z\right)^{-1} Z' v_i \quad = \quad & \sum_{t=1}^{T_b}\left(-\frac{-T^2\lambda^2 - T\lambda - tT + 2tT\lambda + t}{T\lambda\left(-T+T\lambda+1\right)}\right) v_{i,t}^2 \\
&+\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(-\frac{-2T^2\lambda^2 - 2T\lambda - tT + 2tT\lambda + t - sT + 2sT\lambda + s}{T\lambda\left(-T+T\lambda+1\right)}\right) v_{i,t}v_{i,s} \\
&+\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(-\frac{-T - T\lambda - 1 + t + s}{-T+T\lambda+1}\right) v_{i,t}v_{i,s} \\
&+\sum_{t=1}^{T}\left(-\frac{T-t}{-T+T\lambda+1}\right) v_{i,t}^2 + \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(-\frac{2T-s-t}{-T+T\lambda+1}\right) v_{i,t}v_{i,s} \\
&+\frac{1}{-T+T\lambda+1}\sum_{t=T_b+1}^{T}\left(T-t\right) v_{i,t}v_{i,T_b+1}.
\end{aligned}
$$

Hence, the numerator of the test is equal to:

$$\begin{aligned}
v_i' C_T' Q_T v_i &= \sum_{t=1}^{T-1}\sum_{s=t+1}^{T} v_{i,t}v_{i,s} + \sum_{t=1}^{T_b}\left(\frac{-T^2\lambda^2 - T\lambda - tT + 2tT\lambda + t}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^2 \\
&+ \sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(\frac{-2T^2\lambda^2 - 2T\lambda - tT + 2tT\lambda + t - sT + 2sT\lambda + s}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}v_{i,s} \\
&+ \sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(\frac{-T - T\lambda - 1 + t + s}{-T+T\lambda+1}\right)v_{i,t}v_{i,s} \\
&+ \sum_{t=1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^2 + \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(\frac{2T-s-t}{-T+T\lambda+1}\right)v_{i,t}v_{i,s} \\
&- \frac{1}{-T+T\lambda+1}\sum_{t=T_b+1}^{T}\left(T-t\right)v_{i,t}v_{i,T_b+1}.
\end{aligned}$$

or in a summarised way

$$\begin{aligned}
v_i' C_T' Q_T v_i &= \sum_{t=1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^2 + \sum_{t=1}^{T_b}\left(\frac{-T^2\lambda^2 - T\lambda - tT + 2tT\lambda + t}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^2 \\
&+ \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(-\frac{-T - T\lambda - 1 + t + s}{-T+T\lambda+1}\right)v_{i,t}v_{i,s} \\
&+ \sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(\frac{-2T^2\lambda^2 - 2T\lambda - tT + 2tT\lambda + t - sT + 2sT\lambda + s}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}v_{i,s} \\
&+ \sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(\frac{-T - T\lambda - 1 + t + s}{-T+T\lambda+1}\right)v_{i,t}v_{i,s} \\
&- \frac{1}{-T+T\lambda+1}\sum_{t=T_b+1}^{T}\left(T-t\right)v_{i,t}v_{i,T_b+1} \\
&= \sum_{t=1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^2 \\
&+ \sum_{t=1}^{T_b}\left(\frac{-T^2\lambda^2 - T\lambda - tT + 2tT\lambda + t}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^2 \\
&+ v_{i,T_b+1}^2 + g\left(v_{i,t}, v_{i,s}\right) \\
&= w\left(T,\lambda\right)_i, \qquad \text{(A-13)}
\end{aligned}$$

where $g\left(v_{i,t}, v_{i,s}\right)$ is a function of cross-products of disturbances between different moments of time. In concrete,

$$g\left(v_{i,t},v_{i,s}\right) = \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(-\frac{-T-T\lambda-1+t+s}{-T+T\lambda+1}\right)v_{i,t}v_{i,s} \tag{A-14}$$
$$+\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(\frac{-2T^2\lambda^2-2T\lambda-tT+2tT\lambda+t-sT+2sT\lambda+s}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}v_{i,s}$$
$$+\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(\frac{-T-T\lambda-1+t+s}{-T+T\lambda+1}\right)v_{i,t}v_{i,s}$$
$$+\sum_{t=T_b+2}^{T}\left(-\frac{(T-t)}{-T+T\lambda+1}\right)v_{i,t}v_{i,T_b+1}.$$

Once the numerator has been expressed in terms of the original disturbance we can compute its limiting probability. Thus,

$$\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}v_i'C_T'Q_Tv_i = \sum_{t=1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)\sigma_v^2$$
$$+\sum_{t=1}^{T_b}\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)\sigma_v^2+\sigma_v^2$$
$$= -\frac{1}{2}\left(T-3\right)\sigma_v^2.$$

Applying the Slutzky's theorem we can deduce the limit probability of the test:

$$B_{An} = \underset{N\to\infty}{plim}\frac{1}{N}\left(\varphi-1\right)=\frac{\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}w\left(T,\lambda\right)_i}{\underset{N\to\infty}{plim}\frac{1}{N}\sum_{i=1}^{N}D\left(T,\lambda\right)_i}$$
$$= \frac{-\frac{1}{2}\left(T-3\right)}{\left(\frac{1}{3}\lambda^2-\frac{1}{3}\lambda+\frac{1}{6}\right)T^2+\left(-\frac{1}{3}+\frac{1}{3}\lambda\right)T-\frac{1}{6}}$$
$$= \frac{-3\left(T-3\right)}{\left(1+2\lambda^2-2\lambda\right)T^2+\left(2\lambda-2\right)T-1}$$

B_{An} is the term that generates the inconsistency of the test statistic. Consequently, if we correct the test by this term we get a test that is consistent:

$$\left(\varphi-1-B_{An}\right)=\frac{\sum_{i=1}^{N}w\left(T,\lambda\right)_i}{\sum_{i=1}^{N}D\left(T,\lambda\right)_i}-B_{An}=\frac{\sum_{i=1}^{N}x\left(T,\lambda\right)_i}{\sum_{i=1}^{N}D\left(T,\lambda\right)_i},$$

where $x\left(T,\lambda\right)_i = w\left(T,\lambda\right)_i - B_{An}D\left(T,\lambda\right)_i$. Hence, $x\left(T,\lambda\right)_i$ is a iid process with zero mean and variance given by:

$$Var\left(x\left(T,\lambda\right)_i\right) = N_1\left(T,\lambda\right)k+N_2\left(T,\lambda\right)\sigma_v^4$$
$$= \sigma_x^2,$$

where

$$N_1(T,\lambda) = -\frac{(T-1)}{30\left(2T^2\lambda^2-2T^2\lambda+2T\lambda+T^2-2T-1\right)^2 T\lambda(-T+T\lambda+1)}$$
$$\Big(\left(20\lambda^6-60\lambda^5-6\lambda^3+53\lambda^4-11\lambda^2+4\lambda\right)T^6$$
$$+\left(2\lambda^4+188\lambda^2-198\lambda^3+60\lambda^5-56\lambda\right)T^5$$
$$+\left(670\lambda^3-183\lambda^4-662\lambda^2+220\lambda+5\right)T^4$$
$$+\left(804\lambda^2-20-466\lambda^3-364\lambda\right)T^3$$
$$+\left(-339\lambda^2+292\lambda+1\right)T^2+(74-96\lambda)T-76\Big),$$

and

$$N_2(T,\lambda) = \frac{1}{60(-T+T\lambda+1)^2 T\lambda\left(2T^2\lambda^2-2T^2\lambda+2T\lambda+T^2-2T-1\right)^2}$$
$$\Big(\left(-320\lambda^8-112\lambda^2+964\lambda^7-1456\lambda^6+395\lambda^3+1382\lambda^5+40\lambda^9-880\lambda^4+17\lambda\right)T^9$$
$$+\left(1342\lambda^2-1704\lambda^7+5666\lambda^4-6058\lambda^5-238\lambda+240\lambda^8+4272\lambda^6-3590\lambda^3\right)T^8$$
$$+\left(-3720\lambda^6+13\,765\lambda^3+9318\lambda^5-6738\lambda^2+1394\lambda-14\,788\lambda^4+716\lambda^7\right)T^7$$
$$+\left(18\,560\lambda^2+904\lambda^6+21\,392\lambda^4-29\,420\lambda^3-4484\lambda-6838\lambda^5-30\right)T^6$$
$$+\left(-30\,932\lambda^2-18\,124\lambda^4+37\,781\lambda^3+8986\lambda+180+2556\lambda^5\right)T^5$$
$$+\left(-11\,764\lambda-27\,006\lambda^3-276+31\,166\lambda^2+6974\lambda^4\right)T^4$$
$$+\left(9428\lambda-16\,482\lambda^2-312+7699\lambda^3\right)T^3$$
$$+\left(3036\lambda^2-3938\lambda+1338\right)T^2+(599\lambda-1356)T+456\Big).$$

This result has been achieved after some algebraic manipulations, derivation that is outlined in the following subsections.
Since we have assumed that $\{v_{i,t}\}$ is an iid stochastic process across $i=1,\ldots,N$ and $t=1,\ldots,T$, the application of the Central Limit Theorem (CLT) for $N\to\infty$ drives to the following result:

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{x(T,\lambda)_i}{\sigma_x}\xrightarrow{d}N(0,1),$$

where $\xrightarrow{d}$ denotes convergence in distribution. Previous result holds since $x(T,\lambda)_i$ is composed by summations of independent variables. Hence, provided that the probability limit of the denominator of (A-9) is given by (A-12) Cramer-Wold device -see, for instance, White (1984)- implies:

$$\sqrt{N}\left(\varphi-1-B_{An}\right)=\frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}x(T,\lambda)_i}{\frac{1}{N}\sum_{i=1}^{N}D(T,\lambda)_i}\xrightarrow{d}N\left(0,C_{An}\right)$$

with

$$
\begin{aligned}
C_{An} &= \frac{3}{5\left(T^2+2T^2\lambda^2-2T^2\lambda-2T+2T\lambda-1\right)^4} \\
&\left(\left(40\lambda^6-78\lambda-208\lambda^3+162\lambda^2+17-120\lambda^5+204\lambda^4\right)T^6\right. \\
&+\left(-180+1056\lambda^3-1176\lambda^2+120\lambda^5-624\lambda^4+702\lambda\right)T^5 \\
&+\left(3144\lambda^2-1920\lambda^3+636\lambda^4+753-2400\lambda\right)T^4 \\
&+\left(-3408\lambda^2+1072\lambda^3+3768\lambda-1552\right)T^3 \\
&\left.+\left(1158\lambda^2-2634\lambda+1539\right)T^2+\left(642\lambda-420\right)T-293\right),
\end{aligned}
$$

where the variance of the test is computed assuming that the original disturbances $\{v_{i,t}\}$ are normally distributed and that $k = 3\sigma_v^4$.

Variance of the numerator To compute the variance of the denominator we have to define the squared of $x(T,\lambda)_i$. Notice that:

$$
E\left[x(T,\lambda)_i\right]^2 = E\left[w(T,\lambda)_i\right]^2 + B_{An}^2 E\left[D(T,\lambda)_i\right]^2 - 2B_{An}E\left[w(T,\lambda)_i D(T,\lambda)_i\right].
$$

The square of w function First, we compute $w(T,\lambda)_i^2$ from the specification given by (A-13). Notice that the following general result can help us to simplify the steps followed in the proof:

$$
\left(\sum_{t=1}^{T} h(t)\, v_{i,t}^2\right)^2 = \sum_{t=1}^{T} h(t)^2\, v_{i,t}^4 + 2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T} h(t)\, h(s)\, v_{i,t}^2 v_{i,s}^2,
$$

with expectation given by:

$$
E\left(\sum_{t=1}^{T} h(t)\, v_{i,t}^2\right)^2 = \sum_{t=1}^{T} h(t)^2\, k + 2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T} h(t)\, h(s)\, \sigma_v^4,
$$

with $h(t)$ any polynomial on t and $k = E\left(v_i^4\right)$. Hence,

$$
\begin{aligned}
\left(\sum_{t=1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^2\right)^2 &= \sum_{t=1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)^2 v_{i,t}^4 \\
+2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)&\left(\frac{T-s}{-T+T\lambda+1}\right)v_{i,t}^2 v_{i,s}^2
\end{aligned}
$$

with expectation equals to:

$$
\begin{aligned}
E\left(\sum_{t=1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^2\right)^2 &= \frac{T(2T-1)(T-1)}{6(-T+T\lambda+1)^2}k \\
&+\frac{T(T-1)(T-2)(3T-1)}{12(-T+T\lambda+1)^2}\sigma_v^4.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& E\left(\sum_{t=1}^{T_b}\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^2\right)^2=\sum_{t=1}^{T_b}\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)^2 k \\
& +2\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)\left(\frac{-T^2\lambda^2-T\lambda-sT+2sT\lambda+s}{T\lambda\left(-T+T\lambda+1\right)}\right)\sigma_v^4 \\
= \quad & \frac{\left(T\lambda+1\right)\left(2T^3\lambda^3-2T^3\lambda^2+2T^3\lambda-2T^2\lambda+T^2-2T+1\right)}{6T\lambda\left(-T+T\lambda+1\right)^2}k \\
& +\left(\frac{\left(T\lambda+1\right)\left(T\lambda-1\right)\left(3\lambda T^3-4T^2\lambda^2-2T^2\lambda+2T^2-T\lambda-4T+2\right)}{12T\lambda\left(-T+T\lambda+1\right)^2}\right)\sigma_v^4.
\end{aligned}
$$

Finally,

$$
E\left(v_{i,T_b+1}^2\right)^2=k.
$$

The square of the $g\left(v_{i,t},v_{i,s}\right)$ function is cumbersome. It is derived in several steps. The first step is the partition of (A-14) in different components:

$$
g\left(v_{i,t},v_{i,s}\right)^2=\left(g1+g2+g3+g4\right)^2=\sum_{i=1}^{4}gi^2+2\sum_{i=1}^{3}\sum_{j=i+1}^{4}gi\,gj.
$$

Now it is necessary to analyse each element separately. In general by statement (*g)* in lemma 3, we can establish that:

$$
\begin{aligned}
\left(\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}h\left(t,s\right)v_{i,t}v_{i,s}\right)^2 &= \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}h\left(t,s\right)^2 v_{i,t}^2 v_{i,s}^2 \\
+2\sum_{t=1}^{T-2}\sum_{s=t+1}^{T-1}\sum_{j=s+1}^{T}h\left(t,s\right)h\left(t,j\right)v_{i,t}^2 v_{i,s}v_{i,j} \\
+2\sum_{t=1}^{T-2}\sum_{j=t+1}^{T-1}\left(\sum_{s=t+1}^{T}h\left(t,s\right)v_{i,t}v_{i,s}\sum_{s=j+1}^{T}h\left(s,j\right)v_{i,s}v_{i,j}\right)
\end{aligned}
$$

where, translated in terms of expectations, only matters the first element which involves the product between $v_{i,t}^2$ and $v_{i,s}^2$. Hence,

$$
E\left(\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}h\left(t,s\right)v_{i,t}v_{i,s}\right)^2=\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}h\left(t,s\right)^2\sigma_v^4.
$$

This fact will considerably reduce the algebra manipulations. Thus,

$$\begin{aligned} E\left(g1\right)^2 &= \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(-\frac{-T-T\lambda-1+t+s}{-T+T\lambda+1}\right)^2\sigma_v^4 \\ &= \frac{T\left(T-1\right)\left(T^2+6T^2\lambda^2-T-2\right)}{12\left(-T+T\lambda+1\right)^2}\sigma_v^4 \end{aligned}$$

$$E\left(g2\right)^2 = \sum_{t=1}^{T\lambda-1}\sum_{s=t+1}^{T\lambda}\left(\frac{-2T^2\lambda^2-2T\lambda-tT+2tT\lambda+t-sT+2sT\lambda+s}{T\lambda\left(-T+T\lambda+1\right)}\right)^2\sigma_v^4$$

$$= \frac{\left(T\lambda+1\right)\left(T\lambda-1\right)\left(4T^3\lambda^3-4T^3\lambda^2+7T^3\lambda-4T^2\lambda^2-6T^2\lambda+4T^2-T\lambda-8T+4\right)}{12T\lambda\left(-T+T\lambda+1\right)^2}\sigma_v^4;$$

$$\begin{aligned} E\left(g3\right)^2 &= \sum_{t=1}^{T\lambda}\sum_{s=T\lambda+1}^{T}\left(\frac{-T-T\lambda-1+t+s}{-T+T\lambda+1}\right)^2\sigma_v^4 \\ &= \frac{-T^2\lambda\left(-1+\lambda\right)\left(T^2\lambda^2-T^2\lambda+2T^2-1\right)}{6\left(-T+T\lambda+1\right)^2}\sigma_v^4; \end{aligned}$$

$$\begin{aligned} E\left(g4\right)^2 &= \sum_{t=T\lambda+2}^{T}\left(-\frac{\left(T-t\right)}{-T+T\lambda+1}\right)^2\sigma_v^4 \\ &= \frac{-\left(T\lambda-T+2\right)\left(2T\lambda-2T+3\right)}{6\left(-T+T\lambda+1\right)}\sigma_v^4. \end{aligned}$$

Now we look at cross-products of gi's.

$$E\left(g1\;g2\right) = E\left(\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(-\frac{-T-T\lambda-1+t+s}{-T+T\lambda+1}\right)v_{i,t}v_{i,s}\right.$$
$$\left.\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(\frac{-2T^2\lambda^2-2T\lambda-tT+2tT\lambda+t-sT+2sT\lambda+s}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}v_{i,s}\right)$$
$$\equiv E\left(\sum_{t=1}^{T\lambda-1}\sum_{s=t+1}^{T\lambda}\left(-\frac{-T-T\lambda-1+t+s}{-T+T\lambda+1}\right)\right.$$
$$\left.\left(\frac{-2T^2\lambda^2-2T\lambda-tT+2tT\lambda+t-sT+2sT\lambda+s}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^2v_{i,s}^2\right)$$
$$= \left(\frac{-\left(T\lambda+1\right)\left(T\lambda-1\right)\left(2T^2\lambda^2-T^2\lambda+6T^2-4T-3T\lambda-2\right)}{12\left(-T+T\lambda+1\right)^2}\right)\sigma_v^4$$

$$
\begin{aligned}
E\left(g1\, g3\right) &= E\left(\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(-\frac{-T-T\lambda-1+t+s}{-T+T\lambda+1}\right)v_{i,t}v_{i,s}\right.\\
&\quad\left.\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(\frac{-T-T\lambda-1+t+s}{-T+T\lambda+1}\right)v_{i,t}v_{i,s}\right)\\
&\equiv E\left(\sum_{t=1}^{T\lambda}\sum_{s=T\lambda+1}^{T}\left(-\frac{-T-T\lambda-1+t+s}{-T+T\lambda+1}\right)\left(\frac{-T-T\lambda-1+t+s}{-T+T\lambda+1}\right)v_{i,t}^{2}v_{i,s}^{2}\right)\\
&= \frac{T^{2}\lambda\left(-1+\lambda\right)\left(T^{2}\lambda^{2}-T^{2}\lambda+2T^{2}-1\right)}{6\left(-T+T\lambda+1\right)^{2}}\sigma_{v}^{4}
\end{aligned}
$$

$$
\begin{aligned}
E\left(g1\, g4\right) &= E\left(\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(-\frac{-T-T\lambda-1+t+s}{-T+T\lambda+1}\right)v_{i,t}v_{i,s}\right.\\
&\quad\left.\sum_{t=T_b+2}^{T}\left(-\frac{(T-t)}{-T+T\lambda+1}\right)v_{i,t}v_{i,T_b+1}\right)\\
&\equiv E\left(\sum_{t=T\lambda+2}^{T}\left(-\frac{-T-T\lambda-1+(T\lambda+1)+t}{-T+T\lambda+1}\right)\left(-\frac{(T-t)}{-T+T\lambda+1}\right)v_{i,t}^{2}v_{i,T_b+1}^{2}\right)\\
&= \frac{\left(T\lambda-T+2\right)\left(2T\lambda-2T+3\right)}{6\left(-T+T\lambda+1\right)}\sigma_{v}^{4}.
\end{aligned}
$$

Some little algebra will give us the cross-products involving the second element of (A-14). Let's go:

$$
\begin{aligned}
E\left(g2\, g3\right) &= E\left(\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(\frac{-2T^{2}\lambda^{2}-2T\lambda-tT+2tT\lambda+t-sT+2sT\lambda+s}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}v_{i,s}\right.\\
&\quad\left.\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(\frac{-T-T\lambda-1+t+s}{-T+T\lambda+1}\right)v_{i,t}v_{i,s}\right)\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
E\left(g2\, g4\right) &= E\left(\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(\frac{-2T^{2}\lambda^{2}-2T\lambda-tT+2tT\lambda+t-sT+2sT\lambda+s}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}v_{i,s}\right.\\
&\quad\left.\sum_{t=T_b+2}^{T}\left(-\frac{(T-t)}{-T+T\lambda+1}\right)v_{i,t}v_{i,T_b+1}\right)\\
&= 0
\end{aligned}
$$

Another set of cross-products is given by:

$$
\begin{aligned}
& E\left(g3\, g4\right)=E\left(\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(\frac{-T-T\lambda-1+t+s}{-T+T\lambda+1}\right)v_{i,t}v_{i,s}\right.\\
& \left.\sum_{t=T_b+2}^{T}\left(-\frac{(T-t)}{-T+T\lambda+1}\right)v_{i,t}v_{i,T_b+1}\right)\\
= \;& 0.
\end{aligned}
$$

Now we can compute the expression of the expectation of $g\left(v_{i,t}\; v_{i,s}\right)^2$. Thus, cumbersome manipulations produce:

$$
\begin{aligned}
E\left(g\left(v_{i,t}v_{i,s}\right)^2\right) \;=\; & \frac{1}{12\left(-T+T\lambda+1\right)T\lambda}\\
& \left(2T^4\lambda^4-4T^4\lambda^3+4T^3\lambda^3+3T^4\lambda^2-8T^3\lambda^2+15T^2\lambda^2\right.\\
& \left.-T^4\lambda+5T^3\lambda-17T^2\lambda+13T\lambda+4T-4\right)\sigma_v^4.
\end{aligned}
$$

Once the square of the elements that are involved in the numerator has been computed, now it is time to derive its cross-product. This is the final step before getting the expectation of $w\left(T,\lambda\right)^2$ function.

The first product is:

$$
\begin{aligned}
& \sum_{t=1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^2\sum_{t=1}^{T_b}\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^2=\\
= \;& \sum_{t=1}^{T_b}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^2\sum_{t=1}^{T_b}\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^2\\
& +\sum_{t=T_b+1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^2\sum_{t=1}^{T_b}\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^2
\end{aligned}
$$

with expectation:

$$
\begin{aligned}
& E\left(\sum_{t=1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^{2}\sum_{t=1}^{T_b}\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^{2}\right)= \\
= & \sum_{t=1}^{T\lambda}\left(\frac{T-t}{-T+T\lambda+1}\right)\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)k \\
& +\sum_{t=1}^{T\lambda-1}\sum_{s=t+1}^{T\lambda}\left(\left(\frac{T-t}{-T+T\lambda+1}\right)\left(\frac{-T^2\lambda^2-T\lambda-sT+2sT\lambda+s}{T\lambda\left(-T+T\lambda+1\right)}\right)+\right. \\
& \left.\left(\frac{T-s}{-T+T\lambda+1}\right)\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)\right)\sigma_v^4 \\
& +\sum_{t=T\lambda+1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)\sum_{t=1}^{T\lambda}\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)\sigma_v^4 \\
= & \left(\frac{-\left(T\lambda+1\right)\left(T^2\lambda^2-2T^2\lambda+3T^2+T\lambda-4T+1\right)}{6\left(-T+T\lambda+1\right)^2}\right)k \\
& +\left(-\frac{\left(T\lambda+1\right)\left(3T^3-2T^2\lambda^2+4T^2\lambda-12T^2-2T\lambda+11T-2\right)}{12\left(-T+T\lambda+1\right)^2}\right)\sigma_v^4.
\end{aligned}
$$

The expectation of the second product is given by:

$$
\begin{aligned}
E\left(\sum_{t=1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^{2}v_{i,T_b+1}^{2}\right) &= E\left(\left(\sum_{t=1}^{T_b}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^{2}-v_{i,T_b+1}^{2}+\right.\right. \\
&\quad \left.\left.\sum_{t=T_b+2}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^{2}\right)v_{i,T_b+1}^{2}\right) \\
&= \sum_{t=1}^{T_b}\left(\frac{T-t}{-T+T\lambda+1}\right)\sigma_v^4-k \\
&\quad +\sum_{t=T_b+2}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)\sigma_v^4 \\
&= \frac{-3T+2T\lambda+2+T^2}{2\left(-T+T\lambda+1\right)}\sigma_v^4-k.
\end{aligned}
$$

The expectation of the third product is:

$$
E\left(\sum_{t=1}^{T\lambda}\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^{2}v_{i,T_b+1}^{2}\right)=\frac{\left(1-T\right)\left(T\lambda+1\right)}{2\left(-T+T\lambda+1\right)}\sigma_v^4.
$$

It is straightforward to see that the expectation of the products involving $g\left(v_{i,t}\,v_{i,s}\right)$ function is equal to zero.

Therefore, after all these algebra calculations, we can write the expectation of $w\left(T,\lambda\right)_i^2$ as:

$$E\left(w\left(T,\lambda\right)_i^2\right) = A_1\left(T,\lambda\right)k + A_2\left(T,\lambda\right)\sigma_v^4,$$

where

$$A_1\left(T,\lambda\right) = \frac{2T^3\lambda^2 - 2T^3\lambda - 8T^2\lambda^2 + 10T^2\lambda - 8T\lambda - T + 1}{6\left(-T + T\lambda + 1\right)T\lambda}$$

and

$$\begin{aligned} A_2\left(T,\lambda\right) &= \frac{1}{6\left(-T+T\lambda+1\right)T\lambda}\big(T^4\lambda^4 - 2T^4\lambda^3 + 3T^4\lambda^2 - 2T^4\lambda + 2T^3\lambda^3 - \\ &\quad 15T^3\lambda^2 + 15T^3\lambda + 23T^2\lambda^2 - 35T^2\lambda + 16T\lambda + 3T - 3\big) \end{aligned}$$

The square of d function To compute the expectation of the square of $D\left(T,\lambda\right)_i$ function we start off (A-10) and proceed through the analysis of its three components in the similar way as the previous subsection. Thus,

$$D\left(T,\lambda\right)_i^2 = \left(d1 + d2 + d3\right)^2 = \sum_{i=1}^{3} di^2 + 2\sum_{i=1}^{2}\sum_{j=i+1}^{3} di\; dj,$$

where di, $i = \{1,2,3\}$, denotes the elements that are shown in **??**. Let us first analyse the square of each element. Following similar developments as for the numerator we can see that:

$$\begin{aligned} E\left(d1^2\right) &= E\left(\sum_{t=1}^{T}\frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1}v_{i,t}^2\right)^2 \\ &= \sum_{t=1}^{T}\left(\frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1}\right)^2 k \\ &\quad +2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1}\frac{\left(T-s\right)\left(T\lambda+1-s\right)}{-T+T\lambda+1}\sigma_v^4 \end{aligned}$$

The expectation of the square of the second element is:

$$E\left(d2^2\right)=E\left(\sum_{t=1}^{T_b}\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^2\right)^2$$

$$\begin{aligned}= & \sum_{t=1}^{T_b}\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)^2 k \\ & +2\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)\right. \\ & \left.\left(-\frac{T^3\lambda^2+T^2\lambda-2sT^2\lambda^2-2sT\lambda-s^2T+2s^2T\lambda+s^2}{T\lambda\left(-T+T\lambda+1\right)}\right)\right)\sigma_v^4\end{aligned}$$

For the third element we have to work a little hard. This element involves the square of the $f\left(v_{i,t}v_{i,s}\right)$ function that in its turn is function of cross-products of disturbances. From (A-11) we know that this functions can be decomposed in four elements. Hence, $f\left(v_{i,t}v_{i,s}\right)^2=\sum_{i=1}^{3} fi^2+2\sum_{i=1}^{2}\sum_{j=i+1}^{3} fi\ fj$. First, let us analyse the square of each element. Thus,

$$E\left(f1\right)^2=\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(2\left(-T\lambda-1+t\right)\frac{-T+s}{-T+T\lambda+1}\right)^2\sigma_v^4$$

$$E\left(f2\right)^2=\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(-2\frac{\left(T^3\lambda^2+ts+T^2\lambda-sT^2\lambda^2-tsT-sT\lambda+2tsT\lambda-tT\lambda-tT^2\lambda^2\right)}{T\lambda\left(-T+T\lambda+1\right)}\right)^2\sigma_v^4$$

$$E\left(f3\right)^2=\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(-2\left(-T\lambda-1+t\right)\frac{-T+s}{-T+T\lambda+1}\right)^2\sigma_v^4$$

The expectation of cross-products involving the first element is equal to:

$$\begin{aligned}E\left(f1\ f2\right) = & \sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(2\left(-T\lambda-1+t\right)\frac{-T+s}{-T+T\lambda+1}\right) \\ & \left(-2\frac{\left(T^3\lambda^2+ts+T^2\lambda-sT^2\lambda^2-tsT-sT\lambda+2tsT\lambda-tT\lambda-tT^2\lambda^2\right)}{T\lambda\left(-T+T\lambda+1\right)}\right)\sigma_v^4\end{aligned}$$

and

$$E\left(f1\ f3\right)=\sum_{t=1}^{T_b}\sum_{s=T_b+1}^{T}\left(2\left(-T\lambda-1+t\right)\frac{-T+s}{-T+T\lambda+1}\right)\left(-2\left(-T\lambda-1+t\right)\frac{-T+s}{-T+T\lambda+1}\right)\sigma_v^4.$$

The cross-product involving the second element is:

$$E\left(f2\ f3\right)=0.$$

Now we can express the expectation of the square of the $f\left(v_{i,t}v_{i,s}\right)$ function as:

$$\begin{aligned}E\left(f\left(v_{i,t}v_{i,s}\right)^2\right) &= \frac{1}{90\lambda T\left(-T+T\lambda+1\right)}\left(4T^6\lambda^6-12T^6\lambda^5+20T^6\lambda^4-20T^6\lambda^3\right.\\ &\quad +10T^6\lambda^2-2T^6\lambda+12T^5\lambda^5-58T^5\lambda^4+96T^5\lambda^3\\ &\quad -64T^5\lambda^2+16T^5\lambda+48T^4\lambda^4-152T^4\lambda^3+147T^4\lambda^2\\ &\quad -49T^4\lambda+76T^3\lambda^3-142T^3\lambda^2+71T^3\lambda+35T^2\lambda^2\\ &\quad \left.-35T^2\lambda-T\lambda-6T+6\right)\sigma_v^4.\end{aligned}$$

The last thing to do before defining the square of $D\left(T,\lambda\right)_i$ is to compute the expectation of the cross-products involving the different elements of this function. It is easy to see that:

$$
\begin{aligned}
E\left(d1\, d2\right) = E & \left(\sum_{t=1}^{T} \frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1} v_{i,t}^{2}\right. \\
& \left.\sum_{t=1}^{T_b} \left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right) v_{i,t}^{2}\right) \\
= & E\left(\sum_{t=1}^{T_b} \frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1} v_{i,t}^{2} \sum_{t=1}^{T_b} \left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right) v_{i,t}^{2}\right. \\
& \left.+\sum_{t=T_b+1}^{T} \frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1} v_{i,t}^{2} \sum_{t=1}^{T_b} \left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right) v_{i,t}^{2}\right) \\
= & E\left(\sum_{t=1}^{T_b} \left(\frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1}\right) \left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right) v_{i,t}^{4}\right. \\
& +\sum_{t=1}^{T_b-1} \sum_{s=t+1}^{T_b} \left(\frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1} \left(-\frac{T^3\lambda^2+T^2\lambda-2sT^2\lambda^2-2sT\lambda-s^2T+2s^2T\lambda+s^2}{T\lambda\left(-T+T\lambda+1\right)}\right)+\right. \\
& \left.\frac{\left(T-s\right)\left(T\lambda+1-s\right)}{-T+T\lambda+1} \left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)\right) v_{i,t}^{2} v_{i,s}^{2} \\
& \left.+\sum_{t=T_b+1}^{T} \frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1} v_{i,t}^{2} \sum_{t=1}^{T_b} \left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right) v_{i,t}^{2}\right) \\
= & \sum_{t=1}^{T_b} \left(\frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1}\right) \left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right) k \\
& +\sum_{t=1}^{T_b-1} \sum_{s=t+1}^{T_b} \left(\frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1} \left(-\frac{T^3\lambda^2+T^2\lambda-2sT^2\lambda^2-2sT\lambda-s^2T+2s^2T\lambda+s^2}{T\lambda\left(-T+T\lambda+1\right)}\right)+\right. \\
& \left.\frac{\left(T-s\right)\left(T\lambda+1-s\right)}{-T+T\lambda+1} \left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)\right) \sigma_v^4 \\
& +\sum_{t=T_b+1}^{T} \frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1} \sum_{t=1}^{T_b} \left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right) \sigma_v^4.
\end{aligned}
$$

Following the same reasoning as for the numerator, it is shown that:

$$E\left(d1\, d3\right) = E\left(d2\, d3\right) = 0,$$

that is to say, the products involving the $f\left(v_{i,t}v_{i,s}\right)$ function are equal to zero.
Therefore, we can conclude that:

$$E\left(D\left(T,\lambda\right)_i^2\right) = B_1\left(T,\lambda\right) k + B_2\left(T,\lambda\right) \sigma_v^4,$$

where

$$B_1(T,\lambda) = \frac{(T-1)}{30T\lambda(-T+T\lambda+1)}\left(\left(-\lambda+3\lambda^4-6\lambda^3+4\lambda^2\right)T^4 + \left(6\lambda^3-8\lambda^2+3\lambda\right)T^3+\left(-3\lambda+4\lambda^2\right)T^2+T\lambda+1\right)$$

and

$$\begin{aligned} B_2(T,\lambda) = {} & \frac{1}{180T\lambda(-T+T\lambda+1)}\left(\left(28\lambda^6-84\lambda^5+120\lambda^4-100\lambda^3+45\lambda^2-9\lambda\right)T^6\right. \\ & +\left(63\lambda+408\lambda^3-252\lambda^2+84\lambda^5-294\lambda^4\right)T^5 \\ & +\left(456\lambda^2-152\lambda+154\lambda^4-476\lambda^3\right)T^4 \\ & +\left(168\lambda+168\lambda^3-336\lambda^2\right)T^3 \\ & \left.+\left(-79\lambda+79\lambda^2\right)T^2+(-18+9\lambda)T+18\right). \end{aligned}$$

As a way to confirm this result it has to be stated that if we replace $\lambda = 1$ in the expressions above we will obtain the same expression as in Harris and Tzavalis (1999) -this is not shown in this appendix but the proof is straightforward.

The cross-product between the functions w and d The final step to compute the variance of the numerator of the test consists on the derivation of the cross product between functions $w(T,\lambda)_i$ and $D(T,\lambda)_i$. It has to be noticed that this covariance involves lots of cross-products since we have to multiply the functions $f(v_{i,t}v_{i,s})$ and $g(v_{i,t}v_{i,s})$. Before going on it has to be mentioned that the whole derivations starts from the decomposition given by (A-13) and (A-10). Hence, from (A-13) we see that $w(T,\lambda)_i = w1+w2+w3+w4$. On the other hand, from (A-10) we write $D(T,\lambda)_i = d1+d2+d3$. We analyse element by element. Thus,

$$\begin{aligned} E(w1\, d1) &= E\left(\sum_{t=1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^2\sum_{t=1}^{T}\frac{(T-t)(T\lambda+1-t)}{-T+T\lambda+1}v_{i,t}^2\right) \\ &= \sum_{t=1}^{T}\frac{(T-t)}{-T+T\lambda+1}\frac{(T-t)(T\lambda+1-t)}{-T+T\lambda+1}k \\ &\quad+\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(\frac{(T-t)}{-T+T\lambda+1}\frac{(T-s)(T\lambda+1-s)}{-T+T\lambda+1}+\right. \\ &\quad\left.\frac{(T-s)}{-T+T\lambda+1}\frac{(T-t)(T\lambda+1-t)}{-T+T\lambda+1}\right)\sigma_v^4. \end{aligned}$$

Similar developments has been followed for the rest of the cross-products.

$$\begin{aligned}
E\left(w1\ d2\right) &= E\left(\sum_{t=1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^{2}\right.\\
&\left.\sum_{t=1}^{T_b}\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^{2}\right)\\
&= E\left(\sum_{t=1}^{T_b}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^{2}\right.\\
&\sum_{t=1}^{T_b}\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^{2}\\
&+\sum_{t=T_b+1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)v_{i,t}^{2}\\
&\left.\sum_{t=1}^{T_b}\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^{2}\right)\\
&= \sum_{t=1}^{T_b}\left(\frac{T-t}{-T+T\lambda+1}\right)\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)k\\
&+\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(\left(\frac{T-t}{-T+T\lambda+1}\right)\left(-\frac{T^3\lambda^2+T^2\lambda-2sT^2\lambda^2-2sT\lambda-s^2T+2s^2T\lambda+s^2}{T\lambda\left(-T+T\lambda+1\right)}\right)+\right.\\
&\left.\left(\frac{T-s}{-T+T\lambda+1}\right)\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)\right)\sigma_v^4\\
&+\sum_{t=T_b+1}^{T}\left(\frac{T-t}{-T+T\lambda+1}\right)\sum_{t=1}^{T_b}\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)\sigma_v^4.
\end{aligned}$$

To compute the expectation of $E\left(w1\ d3\right)$ we use the decomposition of $f\left(v_{i,t}v_{i,s}\right)$ given by (A-11) from which we can see that $E\left(w1\ d3\right)=0$.

The second element of the numerator involves the following expectations:

$$
\begin{aligned}
E\left(w2\, d1\right) &= E\left(\sum_{t=1}^{T_b}\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^2\sum_{t=1}^{T}\frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1}v_{i,t}^2\right)\\
&= \sum_{t=1}^{T_b}\frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1}\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}k\\
&\quad+\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(\frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1}\frac{-T^2\lambda^2-T\lambda-sT+2sT\lambda+s}{T\lambda\left(-T+T\lambda+1\right)}+\right.\\
&\quad\left.\frac{\left(T-s\right)\left(T\lambda+1-s\right)}{-T+T\lambda+1}\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)\sigma_v^4\\
&\quad+\sum_{t=1}^{T_b}\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)\sum_{t=T_b+1}^{T}\frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1}\sigma_v^4,
\end{aligned}
$$

$$
\begin{aligned}
E\left(w2\, d2\right) &= E\left(\sum_{t=1}^{T_b}\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^2\right.\\
&\quad\left.\sum_{t=1}^{T_b}\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^2\right)\\
&= \sum_{t=1}^{T_b}\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)\\
&\quad\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)k\\
&\quad+\sum_{t=1}^{T_b-1}\sum_{s=t+1}^{T_b}\left(\left(\frac{-T^2\lambda^2-T\lambda-tT+2tT\lambda+t}{T\lambda\left(-T+T\lambda+1\right)}\right)\right.\\
&\quad\left(-\frac{T^3\lambda^2+T^2\lambda-2sT^2\lambda^2-2sT\lambda-s^2T+2s^2T\lambda+s^2}{T\lambda\left(-T+T\lambda+1\right)}\right)+\\
&\quad\left(\frac{-T^2\lambda^2-T\lambda-sT+2sT\lambda+s}{T\lambda\left(-T+T\lambda+1\right)}\right)\\
&\quad\left.\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)\right)\sigma_v^4.
\end{aligned}
$$

Finally, it is easy to see that $E\left(w2\, d3\right)=0$.

The expectation for the third element of the product is:

$$\begin{aligned}
E\left(w3\;d1\right) &= E\left(v_{i,T_b+1}^2\sum_{t=1}^{T}\frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1}v_{i,t}^2\right)\\
&= \left(\sum_{t=1}^{T_b}\frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1}+\sum_{t=T_b+2}^{T}\frac{\left(T-t\right)\left(T\lambda+1-t\right)}{-T+T\lambda+1}\right)\sigma_v^4,
\end{aligned}$$

$$\begin{aligned}
E\left(w3\;d2\right) &= E\left(v_{i,T_b+1}^2\sum_{t=1}^{T_b}\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)v_{i,t}^2\right)\\
&= \sum_{t=1}^{T_b}\left(-\frac{T^3\lambda^2+T^2\lambda-2tT^2\lambda^2-2tT\lambda-t^2T+2t^2T\lambda+t^2}{T\lambda\left(-T+T\lambda+1\right)}\right)\sigma_v^4,
\end{aligned}$$

$$E\left(w3\;d3\right)=0.$$

Finally, to analyse the cross-product between $f\left(v_{i,t}v_{i,s}\right)$ and $g\left(v_{i,t}v_{i,s}\right)$ functions we have to outperform the product element-by-element. Thus, after some tedious algebra (not shown in this appendix) we can conclude that this product is equal to zero, $E\left(f\left(v_{i,t}v_{i,s}\right)\;g\left(v_{i,t}v_{i,s}\right)\right)=0$.
Now we can see that:

$$E\left[w\left(T,\lambda\right)_i D\left(T,\lambda\right)_i\right]=C_1\left(T,\lambda\right)k+C_2\left(T,\lambda\right)\sigma_v^4,$$

where

$$C_1\left(T,\lambda\right)=\left(-\frac{1}{12}-\frac{1}{6}\lambda^2+\frac{1}{6}\lambda\right)T^2+\left(\frac{1}{6}-\frac{1}{6}\lambda\right)T+\frac{1}{12}$$

and

$$C_2\left(T,\lambda\right)=\left(-\frac{1}{12}-\frac{1}{6}\lambda^2+\frac{1}{6}\lambda\right)T^3+\left(\frac{1}{2}-\frac{5}{6}\lambda+\frac{2}{3}\lambda^2\right)T^2+\left(-\frac{7}{12}+\frac{2}{3}\lambda\right)T-\frac{1}{3}$$

Hence, after all these computations, we can express the variance of the numerator as:

$$Var\left(x\left(T,\lambda\right)_i\right)=N_1\left(T,\lambda\right)k+N_2\left(T,\lambda\right)\sigma_v^4,$$

where

$$\begin{aligned}
N_1\left(T,\lambda\right) &= -\frac{\left(T-1\right)}{30\left(2T^2\lambda^2-2T^2\lambda+2T\lambda+T^2-2T-1\right)^2T\lambda\left(-T+T\lambda+1\right)}\\
&\quad\left(\left(20\lambda^6-60\lambda^5-6\lambda^3+53\lambda^4-11\lambda^2+4\lambda\right)T^6\right.\\
&\quad+\left(2\lambda^4+188\lambda^2-198\lambda^3+60\lambda^5-56\lambda\right)T^5\\
&\quad+\left(670\lambda^3-183\lambda^4-662\lambda^2+220\lambda+5\right)T^4\\
&\quad+\left(804\lambda^2-20-466\lambda^3-364\lambda\right)T^3\\
&\quad\left.+\left(-339\lambda^2+292\lambda+1\right)T^2+\left(74-96\lambda\right)T-76\right)
\end{aligned}$$

and

$$
\begin{aligned}
N_2\left(T,\lambda\right) &= \frac{1}{60\left(T^2+2T^2\lambda^2-2T^2\lambda-2T+2T\lambda-1\right)^2\left(-T+T\lambda+1\right)T\lambda} \\
&\left(40\lambda^8-412\lambda^5-240\lambda^3+324\lambda^6+370\lambda^4-17\lambda-160\lambda^7+95\lambda^2\right)T^8 \\
&+\left(-2122\lambda^4+160\lambda^7+1524\lambda^5-1026\lambda^2+2004\lambda^3-744\lambda^6+221\lambda\right)T^7 \\
&+\left(-1740\lambda^5+4854\lambda^4-7152\lambda^3+396\lambda^6-1233\lambda+4749\lambda^2\right)T^6 \\
&+\left(30+12\,888\lambda^3-5110\lambda^4+388\lambda^5-11\,380\lambda^2+3601\lambda\right)T^5 \\
&+\left(-150+2368\lambda^4+14\,937\lambda^2-5995\lambda-11\,856\lambda^3\right)T^4 \\
&+\left(126+4836\lambda^3+5895\lambda-10\,554\lambda^2\right)T^3 \\
&+\left(438+2803\lambda^2-2875\lambda\right)T^2+\left(163\lambda-900\right)T+456
\end{aligned}
$$

References

ALOGOSKOUFIS, G., AND A. MANNING (1988): "Wage Setting and Unemployment Persistence in Europe, Japan and the USA," *European Economic Review*, 32, 698–706.

ARESTIS, P., AND I. BIEFANG-FRISANCHO MARISCAL (1999): "Unit Roots and Structural Breaks in OECD Unemployment," *Economics Letters*, 65, 149–156.

BAI, J., R. L. LUMSDAINE, AND J. H. STOCK (1998): "Testing for and Dating Common Breaks in Multivariate Time Series," *Review of Economic Studies*, 65, 395–432.

BANERJEE, A., J. J. DOLADO, AND J. W. GALBRAITH (1990): "Recursive and Sequential Tests for Unit Roots and Structural Breaks in Long Annual GNP Series," Discussion Paper 9010, Banco de España. Servicio de Estudios.

BANERJEE, A., R. L. LUMSDAINE, AND J. H. STOCK (1992): "Recursive and Sequential Tests of the Unit-Root and Trend-Break Hypotheses: Theory and International Evidence," *Journal of Business & Economic Statistics*, (13, 3), 271–287.

BIANCHI, M., AND G. ZOEGA (1997): "Assessing the Natural Rate Hypothesis. Challenges Facing Natural Rate Theory," *European Economic Review*, 41, 535–547.

——— (1998): "Unemployment Persistence: Does the Size of the Shock Matter?," *Journal of Applied Econometrics*, (13), 283–304.

BLANCHARD, O., AND P. DIAMOND (1994): "Ranking, Unemployment Duration and Wages," *Review of Economic Studies*, 61, 417–434.

BLANCHARD, O., AND J. F. JIMENO (1995): "Structural Unemployment: Spain versus Portugal," *American Economic Review*, 85(2), 212–218.

BLANCHARD, O., AND L. SUMMERS (1987): "Hysteresis in Unemployment," *European Economic Review*, 31, 288–295.

——— (1988): "Beyond the Natural Rate Hypothesis," *American Economic Review*, 78, 182–187.

BREITUNG, J. (2000): "The Local Power of some Unit Root Tests for Panel Data," Discussion paper, Institute of Statistics and Econometrics. Humbold University Berlin.

CHEUNG, Y.-W., AND M. D. CHINN (1996): "Further Investigation of the Uncertain Unit Root in GNP," Discussion Paper 206, National Bureau of Economic Research.

DECRESSIN, J., AND A. FATÁS (1995): "Regional Labor Market Dynamics in Europe," *European Economic Review*, 39, 1627–1655.

DICKEY, D. A., AND W. A. FULLER (1979): "Distribution of the Estimators for Autoregressive Time Series with a Unit Root," *Journal of the American Statistical Association*, pp. 423–431.

DOLADO, J. J., AND J. F. JIMENO (1997): "The Causes of Spanish Unemployment: A

Structural VAR Approach," *European Economic Review*, 41, 1281–1307.

FRIEDMAN, M. (1968): "The Role of Monetary Policy," *American Economic Review*, 58, 1–17.

HANSEN, B. E. (1992): "Tests for Parameter Instability in Regressions with I(1) Processes," *Journal of Business & Economic Statistics*, (10), 321–335.

HAO, K. (1996): "Testing for Structural Change in Cointegrated Regression Models: some Comparisons and Generalisations," *Econometric Reviews*, (15, 4), 401–429.

HARRIS, R. D. F., AND E. TZAVALIS (1999): "Inference for Unit Roots in Dynamic Panels Where the Time Dimension is Fixed," *Journal of Econometrics*, (90), 201–226.

IM, K. S., M. H. PESARAN, AND Y. SHIN (1997): "Testing for Unit Roots in Heterogeneous Panels," Discussion paper, Department of Applied Economics. University of Cambridge.

JIMENO, J. F., AND S. BENTOLILA (1998): "Regional Unemployment Persistence (Spain, 1976-1994)," *Labour Economics*, 5, 25–51.

KWIATKOWSKI, D., P. C. B. PHILLIPS, P. J. SCHMIDT, AND Y. SHIN (1992): "Testing the Null Hypothesis of Stationarity Against the Alternative of a Unit Root: How Sure are We that Economic Time Series Have a Unit Root," *Journal of Econometrics*, (54), 159–178.

LAYARD, R., AND C. BEAN (1989): "Why Does Unemployment Persist," *Scandinavian*, XCI, 371–396.

LEVIN, A., AND C. F. LIN (1993): "Unit Root Tests in Panel Data: Asymptotic and Finite-Sample Properties," Discussion paper, University of California. San Diego.

LINDBECK, A. (1992): "Macroeconomic Theory and the Labor Market," *European Economic Review*.

LINDBECK, A., AND D. J. SNOWER (1986): "Wage Setting, Unemployment and Insider-Outsider Relations," *American Economic Review*, 76, 235–239.

MONTAÑÉS, A., AND M. REYES (1997): "Testing for Unit Roots in Variable with a Change in the Mean: a Pseudo F-ratio Test," Discussion paper, Department of Economic Analysis. University of Zaragoza.

——— (1998): "Effect of a Shift in the Trend Function on Dickey-Fuller Unit Root Tests," Discussion paper, Departament of Economic Analysis. University of Zaragoza.

NEWEY, W. K., AND K. D. WEST (1994): "Automatic lag Selection in Covariance Matrix Estimation," *Review of Economic Studies*, (61), 631–653.

NG, S., AND P. PERRON (1995): "Unit Root Test in ARMA models with Data-Dependent Methods for the Selection of the Truncation Lag," *Journal of the American Statistical Association*, (90), 268–281.

NICKELL, S. (1981): "Biases in Dynamic Models with Fixed Effects," *Econometrica*, 49(6), 1417–1426.

PAPELL, D., C. MURRAY, AND H. GHIBLAWI (2000): "The Structure of Unemployment," *The Review of Economics and Statistics*, 82, 309–315.

PERRON, P. (1989): "The Great Crash, the Oil Price Shock and the Unit Root Hypothesis," *Econometrica*, (57, 6), 1361–1401.

——— (1990): "Testing for a Unit Root in a Time Series with a Changing Mean," *Journal of Business & Economic Statistics*, (8, 2), 153–162.

——— (1997): "Further Evidence on Breaking Trend Functions in Macroeconomic Variables," *Journal of Econometrics*, (80), 355–385.

PERRON, P., AND T. VOGELSANG (1992): "Nonstationarity and Level Shifts with an Application to Purchasing Power Parity," *Journal of Business & Economic Statistics*, (10, 3), 301–320.

PHELPS, E. S. (1967): "Phillips Curves," *Economica*, 34, 254–281.

——— (1968): "Money Wage Dynamics and Labor Market Equilibrium," *Journal of Political Economy*, 76, 678–711.

——— (1972): *Inflation Policy and Unemployment Theory*. MacMillan. London.

PHILLIPS, P. C. B., AND H. R. MOON (1999a): "Linear Regression Limit Theory for Nonstationary Panel Data," *Econometrica*, 67(5), 1057–1111.

——— (1999b): "Nonstationary Panel Data Analysis: An Overview of some Recent Developments," Discussion paper, Yale University.

QUAH, D. (1994): "Exploting Cross-Section Variations for Unit Root Inference in Dynamic Data," *Economics Letters*, 9-19(44).

STRAUSS, J. (2000): "Is There a Permanent Component in US Real GDP," *Economics Letters*, 66, 2, 137–142.

VOGELSANG, T. (1997): "Wald-Type Tests for Detecting Breaks in the Trend Function of a Dynamic Time Series," *Econometric Theory*, 13, 818–849.

VOGELSANG, T., AND P. PERRON (1998): "Additional Tests for a Unit Root Allowing for a Break in the Trend Function at an Unknown Time," *International Economic Review*, 39(4), 1073–1100.

WHITE, H. (1984): *Asymptotic Theory for Econometricians*. Academic Press.

ZIVOT, E., AND D. W. K. ANDREWS (1992): "Further Evidence on the Great Crash, the Oil Price Shock, and the Unit-Root Hypothesis," *Journal of Business & Economic Statistics*, (10, 3), 251–270.