Nearly-complete Decomposability and Stochastic Stability with an Application to Cournot Oligopoly*

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Abstract

This paper presents a general framework for analysing stochastic stability in models with evolution at two levels. Under certain conditions the theory of nearly-complete decomposability can be used to disentangle these two levels. They can then be studied separately and the equilibrium of one can be used to obtain the equilibrium of the other. This gives an approximation of the equilibrium of the combined dynamics. This approached is applied to a model of conjectural variation and imitation in Cournot oligopoly. If behavioural change takes place infrequently, the Walrasian equilibrium is the unique stochastically stable outcome. As a corollary, it is indicated that smaller industries are more competitive than larger ones.

Keywords: Stochastic stability, Near-complete decomposability, Oligopoly, Conjectural variations, Imitation

JEL classification: L13, C73

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1 Introduction

In recent years, many papers and books have used the concept of stochastic stability to explain equilibrium selection in games. Most of these models follow the same structure.[1] One starts by modelling a particular kind of dynamic behaviour in discrete time that evolves according to a Markov chain. This gives the *pure dynamics*. One can think, for example, of symmetric Cournot oligopoly where at each point in time firms imitate the output of the firm with highest profits in the previous period. Vega–Redondo (1997) shows that this dynamics has numerous equilibria (absorbing states), namely all those states where all firms produce the same quantity (the so-called *monomorphic states*). In order to select between all these equilibria, the pure dynamics is then perturbed by random noise, leading to the *perturbed dynamics*. In the oligopoly example, one assumes that with a certain small probability each firm chooses a random quantity. The stochastically stable states are those states that get positive probability mass in the limit distribution[2] as the random noise component vanishes. In the oligopoly example Vega–Redondo (1997) shows that the Walrasian equilibrium is the unique stochastically stable state, thus giving an evolutionary underpinning of Walrasian behaviour.

Stochastic stability has been analysed to study, for example, evolution in biology (e.g. Foster and Young (1990)), the evolution of conventions (e.g. Young (1993)), equilibrium selection in non-cooperative games (e.g. Kandori et al. (1993)), and a plethora of other fields. In oligopoly theory, stochastic stability has been applied in the seminal Vega–Redondo (1997) as outlined above. This model has been extended to study, for example, entry and exit (Alós-Ferrer et al. (1999)), Bertrand competition (Alós-Ferrer et al. (2000)) the comparison between Cournot and Walrasian equilibrium (Alós-Ferrer (2004)), and the interaction between different types of behaviour (e.g. Schipper (2003) and Kaarbøe and Tieman (1999)).

One crucial assumption in all these models is that agents may change their *decisions*, but never the *behaviour* that leads to these decisions. In Vega–Redondo (1997), for example, all firms are profit imitators. Even if there are multiple behavioural rules present in the population (as in e.g. Schipper (2003)), players cannot change their behaviour. This is a very restrictive assumption. One would like to be able to study models where agents can choose between different behavioural rules.[3] In a repeated non-cooperative game, this would lead to two levels of dynam-

[1] For a textbook exposition see, for example, Fudenberg and Levine (1998).

[2] The limit distribution is also called the "invariant probability measure", or "equilibrium distribution". The latter term can be confusing as the limit distribution need not correspond to an equilibrium of the underlying game.

[3] In a static context this is studied in the literature on indirect evolution. See, for example, Güth

ics. Given a configuration of behavioural rules for all players there is a dynamics of strategy choices, the *strategy dynamics*. At a higher level of aggregation, given the results of behavioural rules, players switch between different behavioural rules, the *behavioural dynamics*. The question then is how the two levels influence each other and what behaviour and strategy combinations result in stochastically stable states.

A major problem with such an analysis is that the resulting Markov chain describing the dynamics becomes very complicated. This paper discusses a very intuitive way of disentangling the two types of dynamics and, hence, obtaining a good approximation of the limit distribution of the original Markov chain. This approach is basically an application of the theory of nearly-complete decomposability as developed by Ando and Fisher (1963), Simon and Ando (1961), and Courtois (1977). Originally, this theory was developed to aggregate over large dynamic systems in the presence of limited computational power. The main idea is that under certain conditions one can study the two levels of dynamics separately. Intuitively, this means that one uses the limit distribution of the strategy dynamics to obtain the limit distribution of the behavioural dynamics. The theory of nearly-completely decomposable systems provides an upper bound on the interaction between the two dynamics below which the latter limit distribution is a good approximation of the limit distribution of the original Markov chain.

To illustrate the way this theory can be used, we study an extension to Vega–Redondo (1997). We consider a Cournot oligopoly with a finite number of identical firms. The strategy dynamics is driven by best responses given conjectural variations. For each configuration of conjectural variations this dynamics leads to a different equilibrium. In particular, there are configurations that lead to the Walrasian, Cournot-Nash, and Cartel equilibria. At the behavioural level it is assumed that firms imitate the behaviour (i.e. the conjectural variation) of the firm with the highest profit. It is shown that if behavioural change does not take place too frequently (a notion made precise below), Walrasian behaviour is (approximately) the unique stochastically stable state. This result reaffirms the strength of the Walrasian idea in competitive markets. A crucial assumption, however, is that behavioural change takes place at a sufficiently low rate. The upper bound on this rate is decreasing in the number of firms. If behavioural change takes place more frequently, the support of the limit distribution may consist of more elements than just the Walrasian equilibrium, indicating that larger industries may inherently be less competitive than smaller industries.

The remainder of the paper is organised as follows. In Section 2 the general

and Yaari (1992), or Possajennikov (2000) for an application to oligopoly theory.

framework is discussed. The theory of nearly-completely decomposable systems is described Section 3. In Section 4 we develop a model of Cournot oligopoly to illustrate the theory and Section 5 concludes.

2 A General Model of Multi-level Evolution

Let $\left(I_n, (S_i)_{i\in I_n}, (u_i)_{i\in I_n}\right)$ be a game in normal form. Let $S = \prod_{i\in I_n} S_i$. This game is infinitely repeated. At every time $t = 0, 1, 2, \ldots,$ the configuration of strategy choices is denoted by $s_t = (s_{it})_{i\in I_n}$, where $s_{it} \in S_i$ is the strategy chosen by player i at time t. It is assumed that every player $i \in I_n$ can choose from a finite set $\mathcal{B}_i$ of behavioural rules. A *behavioural rule* is a correspondence $B_i : S \to S_i$, with typical element $s_{it} = B_{i,t-1}(s_{t-1})$, where $B_{i,t-1}$ denotes the behavioural rule that player $i \in I_n$ chose at time $t-1$.[4] That is, $B_{i,t-1}(s_{t-1})$ describes the strategy choice of player i at time t, given the strategy choices of all players and player i's choice of behavioural rule at time $t-1$.[5] Furthermore, the fact that strategy choice at time t is influenced by behavioural choice at time $t-1$ suggests that strategy choices are made at the beginning of each period, whereas behavioural choices are made at the end.

It is assumed that players adapt their strategy choice with probability $p \in (0,1)$ every period. So, with probability p, $s_{it} \in B_{i,t-1}(s_{t-1})$, and with probability $1-p$, $s_{it} = s_{i,t-1}$. If $B_{i,t-1}(s_{t-1})$ contains more than one element, player i chooses an element at random according to a probability measure, η_i, with full support. The aforementioned dynamics describe the *pure strategy dynamics.*

The actual strategy choice can be influenced by several aspects. For example, a player can make a mistake and choose another strategy than her behavioural rule prescribes. Another possibility is that a player experiments and consciously chooses another strategy. Finally, a player may be replaced by another player who has the same behaviour, but chooses a different strategy at first. Since these effects are outside the model, they are treated as stochastic perturbations. To model these perturbations at the strategy level, it is assumed that with probability $\varepsilon > 0$ a player chooses an element from S randomly according to a probability distribution, ν_i, with full support.

[4]This formulation should not necessarily be interpreted as a player choosing a behavioural rule out of free will. The possibility that choice of behaviour is determined by, for example, genetics or evolutionary forces is not *a priori* excluded.

[5]This formulation does not explicitly include memory longer than one period. Models can, however, relatively easily be transformed to include finite memory. See, for example, Alós-Ferrer et al. (1999).

Let $s(k)$, $k = 1, \ldots, m$, and B^I, $I = 1, \ldots, N$, denote the k-th and I-th permutation of S and $\mathcal{B} = \prod_{i \in I_n} \mathcal{B}_i$, respectively. Then, pure strategy dynamics and perturbations, together, lead to a Markov chain on S with transition matrix $M_I^\varepsilon(k,l)$, a typical element of which is

$$M_I^\varepsilon(k,l) = \prod_{i \in I_n} \Big\{ (1-\varepsilon) \Big[p \mathbb{1}_{\big(s(l)_i \in B_i^I\big(s(k)_{-i}\big)} \eta_i\big(s(l)_i\big) + (1-p) \mathbb{1}_{\big(s(k)_i = s(l)_i\big)} \Big] + \varepsilon \nu_i\big(s(l)_i\big) \Big\},$$

where $\mathbb{1}_{(\cdot)}$ denotes the indicator function and the part between square brackets gives the transition probabilities for the pure strategy dynamics.

The behavioural dynamics takes place at the end of period t, when each player i gets the opportunity to revise its behaviour with probability $0 < \tilde{p} < 1$. Behavioural change can be thought of as a conscious or non-conscious change. In the case of conscious change one could think that once in a while a player analyses her past performance and assesses the payoffs her behaviour yield by comparing with the payoffs of the other players. The importance of *relative* payoffs has already been stressed by Alchian (1950). It is assumed that each player knows the model and can observe the choices of other players and can, therefore, deduce the behaviour of the other players as well. She can then change her behaviour accordingly. Since deriving the other players' behaviour requires more cognitive effort than simply following a behavioural rule in choosing a strategy, it seems reasonable to assume that players change their behaviour less often than their strategy choices which could be reflected in assuming that $\tilde{p} < p$. Non-conscious behavioural change can, for example, be thought of as genetic change, where "weaker genes" are replaced by "stronger" genes, leading to a Darwinian survival of the fittest. Again, it seems reasonable to assume that behavioural change takes place at a lower frequency than strategy change.

For each player $i \in I_n$, behavioural change is governed by a correspondence $\tilde{B}_i : \mathcal{B} \times S \to \mathcal{B}_i$. That is, given the strategy choices $(s_{1,t-1}, \ldots, s_{n,t-1}) \in S$ and the behavioural rules $(B_{1,t-1}, \ldots, B_{n,t-1}) \in \mathcal{B}$, player i chooses a behavioural rule at time t such that

$$B_{it} \in \tilde{B}_i\big(s_{1,t-1}, \ldots, s_{n,t-1}; B_{1,t-1}, \ldots, B_{n,t-1}\big).$$

If $\tilde{B}_i(\cdot)$ does not consist of a unique element, player i chooses any element from $\tilde{B}(\cdot)$ using a probability measure $\tilde{\eta}_i(\cdot)$ with full support.

This dynamic process constitutes the *pure behavioural dynamics*. Just as in the strategy dynamics, random perturbations are added.[6] So, each player chooses with probability $\tilde{\varepsilon} > 0$ any behavioural rule using a probability measure $\tilde{\nu}_i(\cdot)$ with full support. For each $k \in \{1, \ldots, m\}$ and corresponding strategy vector $s(k)$, the behavioural dynamics gives rise to a Markov chain on $\mathcal{B}$ with transition matrix $\lambda_k^{\tilde{\varepsilon}}$, a typical element of which equals

$$\lambda_k^{\tilde{\varepsilon}}(I,J) = \prod_{i \in I_n} \Big\{ (1-\tilde{\varepsilon}) \Big[\tilde{p} \mathbb{1}_{\left(B_i^J \in \tilde{B}_i(B^I, s(k))\right)} \tilde{\eta}_i(B_i^J) + (1-\tilde{p}) \mathbb{1}_{\left(B_i^J = B_i^I\right)} \Big] + \tilde{\varepsilon} \tilde{\nu}_i\big(B_i^J\big) \Big\}, \tag{1}$$

where the part between square brackets gives the transition probabilities for the *pure behavioural dynamics*.

The combined strategy and behavioural dynamics yield a Markov chain on $S \times \mathcal{B}$ with transition matrix $Q^{\varepsilon,\tilde{\varepsilon}}$. For future convenience, it is assumed that entries in this transition matrix are grouped according to the behavioural index. So, the k-th row in $Q^{\varepsilon,\tilde{\varepsilon}}$ consists of the transition probabilities from the state with behavioural rules B^1 and strategies $s(k)$. Similarly, the $m(I-1)+k$-th row contains the transition probabilities from the state with behavioural rules B^I and strategies $s(k)$. A typical element of $Q^{\varepsilon,\tilde{\varepsilon}}$ is given by

$$Q^{\varepsilon,\tilde{\varepsilon}}(k_I, l_J) = M_I^{\varepsilon}(k,l) \lambda_k^{\tilde{\varepsilon}}(I,J),$$

which should be read as the transition probability form the state with behavioural rules B^I and strategies $s(k)$ to the state with behavioural rules B^J and strategies $s(l)$. Note that, for every $\varepsilon > 0$ and every $\tilde{\varepsilon} > 0$, this Markov chain is ergodic (i.e. $Q^{\varepsilon,\tilde{\varepsilon}}$ is irreducible) and, hence, has a unique limit distribution.

Consider the Markov chain with transition matrix

$$Q = \lim_{\tilde{\varepsilon} \downarrow 0} \lim_{\varepsilon \downarrow 0} Q^{\varepsilon,\tilde{\varepsilon}}.$$

This is the transition matrix of the Markov chain that results after the stochastic perturbations at the strategy and behavioural level have converged to zero, respectively. The order of taking limits is essential to the analysis. The stochastically stable states are defined to be those states that have positive probability mass in the limit distribution, $\mu(\cdot)$, of Q. The dynamics in Q is complicated due to the interaction between the strategy and the behavioural levels. Therefore, the techniques developed in Freidlin and Wentzell (1984), which are usually used to determine the stochastically stable state will, in general, be hard to apply.

[6] In the case of non-conscious behavioural change, these perturbations can be thought of as *mutations*.

One way of proceeding is to decompose the Markov chain Q. Let for every $I = 1, \ldots, N$, $Q^*_I = \lim_{\varepsilon \downarrow 0} M^\varepsilon_I$. The standard techniques as applied in most of the literature (cf. Young (1998)) can be used to the unique limit distribution of Q^*_I, denoted by $\mu^I(\cdot)$. Then, construct a Markov chain on $\mathcal{B}$, $\tilde{Q}$, where $\mu^I(\cdot)$ is used to aggregate over the strategy dynamics. That is the transition probabilities in $\tilde{Q}$ are based on the assumption that the strategy dynamics has settled in equilibrium. Standard techniques can then be used to obtain the limit distribution, $\tilde{\mu}(\cdot)$ of $\tilde{Q}$. The theory of nearly-complete decomposability gives conditions on when $\tilde{\mu}(\cdot)$ is a good approximation of the measure of interest, $\mu(\cdot)$.

3 Nearly-complete Decomposability

Intuitively, a nearly-completely decomposable system is a Markov chain where the matrix of transition probabilities can be divided into blocks such that the interaction *between* blocks is small relative to interaction *within* blocks. This section presents the formal theory.[7] In the remainder, let Q be an $n \times n$ irreducible stochastic matrix, representing, for example, the transition matrix of an ergodic Markov chain. The dynamic process $(y_t)_{t \in \mathbb{N}}$, where $y_t \in \mathbb{R}^n$ for all $t \in \mathbb{N}$, is then given by

$$(y_{t+1})^\top = (y_t)^\top Q. \tag{2}$$

Note that Q can be written as follows:

$$Q = Q^* + \zeta C, \tag{3}$$

where Q^* is of order n and given by

$$Q^* = \begin{bmatrix} Q^*_1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & Q^*_N \end{bmatrix}. \tag{4}$$

The matrices Q^*_I, $I = 1, \ldots, N$, are irreducible stochastic matrices of order $n(I)$. Hence, $n = \sum_{I=1}^N n(I)$. Therefore the sums of the rows of C are zero. We choose ζ and C such that for all rows k_I, $I = 1, \ldots, N$, $k = 1, \ldots, n$, it holds that

$$\zeta \sum_{J \neq I} \sum_{l=1}^{n(J)} C_{k_I l_J} = \sum_{J \neq I} \sum_{l=1}^{n(J)} Q_{k_I l_J} \tag{5}$$

[7] The analysis closely follows Courtois (1977).

and

$$\zeta = \max_{k_I}\Big(\sum_{J\neq I}\sum_{l=1}^{n(J)} Q_{k_I l_J}\Big), \tag{6}$$

where the k_I denotes the k-th element in the I-th block. The parameter ζ is called the *maximum degree of coupling between subsystems* Q_I^*.

It is assumed that all eigenvalues of Q and Q^* are distinct. Then the spectral composition[8] of the t-step probabilities – Q^t – can be written as

$$Q^t = \sum_{I=1}^{N}\sum_{k=1}^{n(I)} \lambda^t(k_I)Z(k_I), \tag{7}$$

where

$$Z(k_I) = s(k_I)^{-1}v(k_I)v(k_I)^\top,$$

$\lambda(k_I)$ is the k_I-th maximal eigenvalue in absolute value of Q, $v(k_I)$ is the corresponding eigenvector normalised to one using the vector norm $\|\cdot\|_1$, and $s(k_I)$ is the condition number $s(k_I) = v(k_I)^\top v(k_I)$. Since Q is a stochastic matrix, the Perron-Frobenius theorem gives that the maximal eigenvalue of Q equals 1. Therefore, (7) can be rewritten as

$$Q^t = Z(1_1) + \sum_{I=2}^{N} \lambda^t(1_I)Z(1_I) + \sum_{I=1}^{N}\sum_{k=2}^{n(I)} \lambda^t(k_I)Z(k_I). \tag{8}$$

If one defines for each matrix Q_I^* in a similar way $Z^*(k_I)$, $s^*(k_I)$, $\lambda^*(k_I)$, and $v^*(k_I)$, e.g. $\lambda^*(k_I)$ is the k-th maximal eigenvalue in absolute value of Q_I^*, then one can find a similar spectral decomposition for Q^*, i.e.

$$(Q^*)^t = \sum_{I=1}^{N} Z^*(1_I) + \sum_{I=1}^{N}\sum_{k=2}^{n(I)} (\lambda^*)^t(k_I)Z^*(k_I), \tag{9}$$

using the fact that $v^*_{k_I}(1_I) = n(I)^{-1}$ for all k_I. The behaviour through time of y_t and y_t^*, where the dynamics of $(y_t)_{t\in\mathbf{N}}$ is described by (2) and the process $(y_t^*)_{t\in\mathbf{N}}$ is defined by

$$(y_{t+1}^*)^\top = (y_t^*)^\top Q^*,$$

are, therefore, also specified by (8) and (9). The behaviour of y_t can be seen as long-run behaviour whereas y_t^* describes short-run behaviour. The comparison between both processes follows from two theorems as stated by Simon and Ando (1961).

[8] See, for example, Lay (1994, Section 8.1).

Theorem 1 *For an arbitrary positive real number ξ, there exists a number ζ_ξ such that for $\zeta < \zeta_\xi$,*

$$\max_{p,q} |Z_{pq}(k_I) - Z^*_{pq}(k_I)| < \xi,$$

for any $2 \leq k \leq n(I)$, $1 \leq I \leq N$, where $1 \leq p, q \leq n$.

Theorem 2 *For an arbitrary positive real number ω, there exists a number ζ_ω such that for $\zeta < \zeta_\omega$,*

$$\max_{k,l} |Z_{k_I l_J}(k_I) - v^*_{l_J}(1_J)\alpha_{IJ}(1_K)| < \omega,$$

for any $1 \leq k \leq n(I)$, $1 \leq l \leq n(J)$, $1 \leq K, I, J \leq N$, and where $\alpha_{IJ}(1_K)$ is given by

$$\alpha_{IJ}(1_K) = \sum_{k=1}^{n(I)} \sum_{l=1}^{n(J)} v^*_{k_I} z_{k_I l_J}(1_K).$$

It can be shown that for all $I = 1, \ldots, N$, $\lambda(1_I)$ is close to unity. Therefore $\lambda^t(1_I)$ will also be close to unity for small t. Hence, the first two terms on the right-hand side of (8) will not vary much for $t < T_2$, for some $T_2 > 0$. The first term of the right-hand-side of (9) does not change at all. Hence, for $t < T_2$ the behaviour through time of y_t and y^*_t is determined by the last terms of Q^t and $(Q^*)^t$, respectively. Also, if $\zeta \to 0$ it can be shown that $\lambda(k_I) \to \lambda^*(k_I)$ and from Theorem 1 it follows that $Z(k_I) \to Z^*(k_I)$, for all $k = 2, \ldots, n(I)$ and $I = 1, \ldots, N$. This means that for ζ small and $t < T_2$ the paths of y_t and y^*_t are very close.

The eigenvalues $\lambda^*(k_I)$ are strictly less than unity in absolute value for all $k = 2, \ldots, n(I)$, and $I = 1, \ldots, N$. For any positive real number ξ_1 we can therefore define a smallest time T^*_1 such that

$$\max_{1 \leq p,q \leq n} \Big| \sum_{I=1}^{N} \sum_{k=2}^{n(I)} (\lambda^*)^t(k_I) Z^*_{pq}(k_I) \Big| < \xi_1 \qquad \text{for } t > T^*_1.$$

Similarly, we can find a T_1 such that

$$\max_{1 \leq p,q \leq n} \Big| \sum_{I=1}^{N} \sum_{k=2}^{n(I)} \lambda^t(k_I) Z_{pq}(k_I) \Big| < \xi_1 \qquad \text{for } t > T_1.$$

Theorem 1 plus convergence of the eigenvalues with ζ then ensures that $T_1 \to T^*_1$ as $\zeta \to 0$. We can always choose ζ such that $T_2 > T_1$. As long as ζ is not identical to zero it holds that $\lambda(1_I)$ is not identical to unity for $I = 2, \ldots, N$.[9] Therefore, there will be a time $T_3 > 0$ such that for sufficiently small ξ_3,

$$\max_{1 \leq p,q \leq n} \Big| \sum_{I=1}^{N} \sum_{k=2}^{n(I)} \lambda^t(1_I) Z_{pq}(1_I) \Big| < \xi_3 \qquad \text{for } t > T_3.$$

[9] If $\zeta = 0$, all blocks Q_I are irreducible and then we would have $\lambda(1_I) = \lambda^*(1_I) = 1$ for all I.

This implies that for $T_2 < t < T_3$, the last term of Q^t is negligible and the path of y_t is determined by the first two components of Q^t. According to Theorem 2 it holds that for any I and J the elements of $Z(1_K)$,

$$Z_{k_I 1_J}(1_K), \ldots, Z_{k_I l_J}(1_K), \ldots, Z_{k_I n(J)_J}(1_K),$$

depend essentially on I, J and l, and are almost independent of k. So, for any I and J they are proportional to the elements of the eigenvector of Q^*_J corresponding to the largest eigenvalue. Since Q^* is stochastic and irreducible, this eigenvector corresponds to the limit distribution μ^*_J of the Markov chain with transition matrix Q^*_J. Thus, for $T_2 < t < T_3$ the elements of the vector y_t, $(y_{l_J})_t$, will approximately have a constant ratio that is similar to that of the elements of μ^*_J. Finally, for $t > T_3$ the behaviour of y_t is almost completely determined by the first term of Q^t. So, y_t evolves towards $v(1_1)$, which corresponds to the limit distribution μ of the Markov chain with transition matrix Q. Summarising, the dynamics of y_t can be described as follows.

1. Short-run dynamics: $t < T_1$. The predominant terms in Q^t and $(Q^*)^t$ are the last ones. Hence, y_t and y^*_t evolve similarly.

2. Short-run equilibrium: $T_1 < t < T_2$. The last terms of Q^t and $(Q^*)^t$ have vanished while for all I, $\lambda^t(1_I)$ remains close to unity. A similar equilibrium is therefore reached within each subsystem of Q and Q^*.

3. Long-run dynamics: $T_2 < t < T_3$. The predominant term in Q^t is the second one. The whole system moves to equilibrium, while the short-run equilibria in the subsystems are approximately maintained.

4. Long-run equilibrium: $t > T_3$. The first term of Q^t dominates. Therefore, a global equilibrium is attained.

The above theory implies that one can estimate $\mu(\cdot)$ by calculating μ^*_I for $I = 1, \ldots, N$, and the invariant measure $\tilde{\mu}$ of the process

$$(\tilde{y}_{t+1})^\top = (\tilde{y}_t)^\top P, \tag{10}$$

where $(\tilde{y}_I)_t = \sum_{k=1}^{n(I)} (y_{k_I})_t$ for all $I = 1, \ldots, N$, and some transition matrix P. For $t > T_2$ we saw that $\frac{(y_{k_I})_t}{(\tilde{y}_I)_t} \approx \mu^*_{I,k}$. Hence, the probability of a transition from group I to group J is given by

$$(p_{IJ})_{t+1} = (\tilde{y}_I)_t^{-1} \sum_{k=1}^{n(I)} (y_{k_I})_t \sum_{l=1}^{n(J)} Q_{k_I l_J}.$$

For $t > T_2$ this can be approximated by

$$(p_{IJ})_{t+1} \approx \sum_{k=1}^{n(I)} \mu^*_{I,k} \sum_{l=1}^{n(J)} Q_{k_I l_J} \equiv p_{IJ}. \tag{11}$$

So, by taking $P = [p_{IJ}]$, the process in (10) gives a good approximation for $t > T_2$ of the entire process $(y_t)_{t\in\mathbb{N}}$.

Until now we have not been concerned by how large ζ can be. It was stated that for $T_1^* < t < T_2$, the original system Q is in a short-run equilibrium close to the equilibrium of the completely decomposable system Q^*. If this is to occur it must hold that $T_1^* < T_2$. Every matrix Q can be written in the form of (3), but not for all matrices it holds that $T_1^* < T_2$. Systems that satisfy the condition $T_1^* < T_2$ are called *nearly-completely decomposable* systems (cf. Ando and Fisher (1963)). Since T_1^* is independent of ζ and T_2 increases with $\zeta \to 0$, the condition is satisfied for ζ sufficiently small.

The main results concerning nearly-complete decomposability are given in the theorem below.

Theorem 3 (Courtois (1977)) *Let Q be an irreducible stochastic matrix, with a decomposition as given in (3)-(6). If*

$$\zeta < \tfrac{1}{2}\Big[1 - \max_{I=1,\dots,N} |\lambda^*(2_I)|\Big], \tag{12}$$

then Q is nearly-completely decomposable. Furthermore, the limit distribution of the Markov chain with transition matrix P, as defined in (11) gives an $O(\zeta)$ approximation of the limit distribution of Q.

4 An Application: Cournot Oligopoly with Conjectural Variations and Imitation

In this section, an application of the theory of nearly-completely decomposable systems to stochastic stability with multi-level evolution is discussed. Let be given a dynamic market for a homogeneous good with n firms, indexed by $I_n = \{1, 2, \dots, n\}$. At each point in time, $t \in \mathbb{N}$, competition takes place in a Cournot fashion, i.e. by means of quantity setting. Inverse demand is given by a smooth function $P : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $P'(\cdot) < 0$. The production technology is assumed to be the same for each firm and is reflected by a smooth cost function $C : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying $C'(\cdot) > 0$ and either $C''(\cdot) > 0$ or $C''(\cdot) < 0$. If at time $t \in \mathbb{N}$ the vector of quantities is given by $q \in \mathbb{R}^n_+$, the profit for firm $i \in I_n$ at time t is given by

$$\pi(q_i, q_{-i}) = P(q_i + Q_{-i})q_i - C(q_i),$$

where $q_{-i} = (q_j)_{j \neq i}$ and $Q_{-i} = \sum_{j \neq i} q_j$.

Each firm $i \in I_n$ chooses quantities from a finite grid Γ_i. Define $\Gamma = \prod_{i \in I_n} \Gamma_i$. For further reference let $q(k)$, $k = 1, \dots, m$, be the k-th permutation of Γ. It is assumed that in setting their quantities firms conjecture that their change in quantity results in an immediate change in the total quantity provided by their competitors. This can also be seen to reflect the firm's conjecture of the competitiveness of the market. Formally, firm $i \in I_n$ conjectures a value for the partial derivative of Q_{-i} with respect to q_i. Using this conjecture, the firm wants to maximise next period's profit. Hence, the firm is a myopic optimiser, which reflects its bounded rationality. The first-order condition for profit maximisation of firm i reads

$$P'(q_i + Q_{-i})\Big(1 + \frac{\partial Q_{-i}}{\partial q_i}\Big)q_i + P(q_i + Q_{-i}) - C'(q_i) = 0. \tag{13}$$

As can be seen from (13) we assume that there is only a first order conjecture effect. Furthermore, we assume that this effect is linear. These assumptions add to the firm's bounded rationality.[10]

To facilitate further analysis, the conjectures are parameterised by a vector $\alpha \in \mathbb{R}^n$ such that for all $i \in I_n$

$$(1 + \alpha_i)\frac{n}{2} = 1 + \frac{\partial Q_{-i}}{\partial q_i}.$$

Given a vector of conjectures an equilibrium for the market is given by $q \in \mathbb{R}^n_+$ such that for all $i \in I_n$ the first-order condition (13) is satisfied. Note that if all firms $i \in I_n$ have the conjecture $\alpha_i = -1$, the equilibrium coincides with the Walrasian equilibrium. Furthermore, if all firms have $\alpha_i = \frac{2-n}{n}$ or $\alpha_i = 1$, the equilibrium coincides with the Cournot-Nash equilibrium or the cartel equilibrium, respectively. Therefore, the conjectures $\alpha_i = -1$, $\alpha_i = \frac{2-n}{n}$, and $\alpha_i = 1$ will be called the Walrasian, Cournot-Nash, and cartel conjectures, respectively.

Each firm chooses its conjecture from a finite grid Λ on $[-1, 1]$, where it is assumed that $\Lambda \supset \{-1, \frac{2-n}{n}, 1\}$. The bounds of this finite grid represent the extreme cases of full competition ($\alpha = -1$) and cartel ($\alpha = 1$). For further reference, let $\alpha(I)$, $I = 1, \dots, N$, be the I-th permutation of $\Lambda^n = \prod_{i \in I_n} \Lambda$.

This model is a special case of the general framework presented in Section 2. The pure strategy dynamics is such that firm $i \in I_n$ seeks to find $q_i^t \in \Gamma_i$ so as to approximate as closely as possible the first-order condition (13). That is, $q_i^t \in$

[10]The first-order and linearity assumptions are also made throughout the static literature on conjectural variations. This seems incompatible with the assumption of fully rational firms in these models.

$B_i(q_{-i}^{t-1}; \alpha_i^{t-1})$, where[11] for $q_{-i} \in \prod_{j \neq i} \Gamma_j$ and $\alpha_i \in \Lambda_i$,

$$B(q_{-i}, \alpha_i) = \underset{q \in \Gamma_i}{\arg\min} \left\{ \left| P'(q + Q_{-i})(1 + \alpha_i)\frac{n}{2}q + P(q + Q_{-i}) - C'(q) \right| \right\}.$$

The pure behavioural dynamics consists of firms imitating at time t the conjecture of the firm(s) with the highest profit in period $t-1$.[12] Formally, firm i's choice α_i^t is such that $\alpha_i^t \in \tilde{B}(\alpha^{t-1}, q^t)$, where for given $\alpha \in \Lambda^n$ and $q \in \Gamma$,

$$\tilde{B}(\alpha, q) = \underset{\gamma \in \Lambda}{\arg\max} \Big\{ \exists_{j \in I_n} : \alpha_j = \gamma, \forall_{k \in I_n} : \pi(q_j, q_{-j}) \geq \pi(q_k, q_{-k}) \Big\}.$$

The combined strategy and behavioural dynamics yield a Markov chain on $\Gamma \times \Lambda^n$ with transition matrix $Q^{\varepsilon,\tilde{\varepsilon}}$. A typical element of $Q^{\varepsilon,\tilde{\varepsilon}}$ is given by

$$Q^{\varepsilon,\tilde{\varepsilon}}(k_I, l_J) = M_I^{\varepsilon}(k,l)\lambda_k^{\tilde{\varepsilon}}(I,J),$$

which should be read as the transition probability form the state with conjectures $\alpha(I)$ and quantities $q(k)$ to the state with conjectures $\alpha(J)$ and quantities $q(l)$. We want to determine the unique limit distribution, $\mu(\cdot)$, of the Markov chain with transition matrix

$$Q = \lim_{\tilde{\varepsilon} \downarrow 0} \lim_{\varepsilon \downarrow 0} Q^{\varepsilon,\tilde{\varepsilon}}.$$

First the strategy dynamics is studied. As before, for each $I = 1, \ldots, N$, let $Q_I^* = \lim_{\varepsilon \downarrow 0} M_I^{\varepsilon}$ be the limit Markov chain when the perturbations in the strategy dynamics vanish. Note that M_I has a unique limit distribution, $\mu^I(\cdot)$. To facilitate further analysis it is assumed that for any vector of conjectures there is a unique equilibrium, i.e. a unique vector of quantities that solves (13) for all firms. Furthermore, we assume that this equilibrium is an element of the quantity grid Γ.

Assumption 1 *For all $\alpha \in \Lambda^n$ there exists a unique $q^\alpha \in \Gamma$ such that for all $i \in I_n$,*

$$P'(q_i^\alpha + Q_{-i}^\alpha)(1 + \alpha_i)\frac{n}{2}q_i^\alpha + P(q_i^\alpha + Q_{-i}^\alpha) - C'(q_i^\alpha) = 0.$$

Let the permutation on Γ that corresponds to q^α be denoted by $k(I)$, i.e. $q(k(I)) = q^\alpha$. The following proposition states that for each vector of conjectures $\alpha(I)$ the unique stochastically stable state of the strategy dynamics is given by $q^{\alpha(I)}$.

Proposition 1 *Let $I \in \{1, \ldots, N\}$ be given. Under Assumption 1, the unique limit distribution $\mu^I(\cdot)$ of the Markov chain with transition matrix Q_I^* is such that*

$$\mu^I(q^{\alpha(I)}) = 1.$$

[11]From the definition of behavioural rules, α_i is not an argument of B_i. We include it for clarification.

[12]Here, imitation takes place at the behavioural level, not at the strategy level (as in, for example, Vega–Redondo (1997)).

Proof. The proposition is proved using the theory developed by Milgrom and Roberts (1991). First note that for all $i \in I_n$, Γ_i is a compact subset of $\mathbb{R}_+$. Define for all $i \in I_n$ the (continuous) function $\tilde{\pi}_i : \mathbb{R}_+ \times \mathbb{R}_+^{n-1} \to \mathbb{R}_+$, given by

$$\tilde{\pi}_i(q_i, q_{-i}) = -\left| P'(q_i + Q_{-i})(1 + \alpha_i(I))\frac{n}{2}q_i + P(q_i + Q_{-i}) - C'(q_i) \right|.$$

Consider the normal-form game $\left(I_n, (\Gamma_i)_{i \in I_n}, (\tilde{\pi}_i)_{i \in I_n}\right)$. Let $S \subset \Gamma$, denote by S_i the projection of S on Γ_i and define $S_{-i} = \prod_{j \neq i} S_j$. For all $i \in I_n$ the set of undominated strategies with respect to S is given by the set

$$U_i(S) = \left\{ q_i \in \Gamma_i \middle| \forall_{y \in S_i} \exists_{q_{-i} \in S_{-i}} : \tilde{\pi}_i(q_i, q_{-i}) \geq \tilde{\pi}_i(y, q_{-i}) \right\}.$$

Let $U(S) = \prod_{i \in I_n} U_i(S)$, the k-th iterate of which is given by $U^k(S) = U\left(U^{k-1}(S)\right)$, $k \geq 2$, where $U^1(S) = U(S)$. Note that since $q^{\alpha(I)}$ is unique we have

$$U^\infty(\Gamma) = \{q^{\alpha(I)}\}.$$

Following Milgrom and Roberts (1991) we say that $\{q^t\}_{t \in \mathbb{N}}$ is *consistent with adaptive learning* if

$$\forall_{\hat{t} \in \mathbb{N}} \exists_{\bar{t} > \hat{t}} \forall_{\tilde{t} \geq \bar{t}} : q^{\tilde{t}} \in U\big(\{q^s | \hat{t} \leq s < \tilde{t}\}\big).$$

Let $\hat{t} \in \mathbb{N}$, take $\bar{t} = \hat{t} + 1$ and let $\tilde{t} = \bar{t} + k$ for some $k \in \{0, 1, 2, \dots\}$. Then

$$\{q^s | \hat{t} \leq s < \tilde{t}\} = \{q^s | s = \hat{t}, \dots, \bar{t} + k - 1\}.$$

Let $\{q^t\}_{t \in \mathbb{N}}$ be generated by the pure strategy dynamics, i.e. the strategy dynamics without the random perturbations. Then we have, by definition, that

$$\forall_{y \in \Gamma_i} : \tilde{\pi}_i(q_i^{\tilde{t}}, q_{-i}^{\tilde{t}-1}) \geq \tilde{\pi}_i(y, q_{-i}^{\tilde{t}-1}).$$

Furthermore, it holds that $q^{\tilde{t}-1} \in \{q^s | \bar{t} \leq s < \tilde{t}\}$. Hence, we can conclude that $\{q^t\}_{t \in \mathbb{N}}$ is consistent with adaptive learning. From Milgrom and Roberts (1991, Theorem 7) one obtains that $\|q^t - q^{\alpha(I)}\| \to 0$ as $t \to \infty$. Since Γ is finite we have

$$\exists_{\bar{t} \in \mathbb{N}} \forall_{t \geq \bar{t}} : q^t = q^{\alpha(I)}.$$

So, $\{q^{\alpha(I)}\}$ is the only recurrent state of the pure strategy dynamics. From Young (1993) we know that the stochastically stable states are among the recurrent states of the mutation-free dynamics. Hence, $\mu^I(q^{\alpha(I)}) = 1$. □

Before we turn to Proposition 2, the following lemma is introduced, which plays a pivotal role in its proof. It compares the equilibrium profits for different conjectures. Suppose that the market is in a monomorphic state, i.e. all firms have the same conjecture. The question is what happens to equilibrium profits if k firms deviate to another conjecture. If $n-k$ firms have a conjecture equal to α and k firms have a conjecture equal to α', let the (unique) equilibrium quantities be denoted by q_k^{α} and $q_k^{\alpha'}$, respectively.

Lemma 1 *For all $k \in \{1, 2, \dots, n-1\}$ and $\alpha > \alpha' \geq -1$ it holds that*

$$P\big((n-k)q_k^{\alpha} + kq_k^{\alpha'}\big)q_k^{\alpha'} - C(q_k^{\alpha'}) > P\big((n-k)q_k^{\alpha} + kq_k^{\alpha'}\big)q_k^{\alpha} - C(q_k^{\alpha}).$$

The proof of this lemma can be found in Appendix A. Lemma 1 plays a similar role as the claim in Vega–Redondo (1997, p. 381). The main result in that paper is driven by the fact that if at least one firm plays the Walrasian quantity against the other firms playing another quantity, the firm with the Walrasian quantity has a strictly higher profit. In our model the dynamics is more elaborate. Suppose that all firms have the Walrasian conjecture and that the strategy dynamics is in equilibrium, i.e. the Walrasian equilibrium. If at least one player has another conjecture not only its own equilibrium quantity changes, but also the equilibrium *quantities* of the firms that still have the Walrasian *conjecture*. Lemma 1 states that the firms with the lower conjecture still have the highest *equilibrium* profit. This is intuitively clear form the first-order condition (13). The firms with the lower conjecture increase their production until the difference between the price and the marginal costs reaches a lower, but positive, level than the firms with the higher conjecture. Therefore, the total profit of having a lower conjecture is higher. This happens because the firms do not realise that in the future their behaviour will be imitated by other firms which puts downward pressure on industry profits.

Some additional notation and assumptions are needed in the following. For a matrix A let $\lambda_j(A)$ denote the j-th largest eigenvalue in absolute value of A. Furthermore, define $\lambda_k(I,J) = \lim_{\tilde{\varepsilon}\downarrow 0} \lambda_k^{\tilde{\varepsilon}}(I,J)$ and let $\zeta = \max_{k_I} \Big\{ \sum_{K \neq I} \sum_{l=1}^{m} Q_{k_I l_K} \Big\}$. The following assumptions are made.

Assumption 2 *All eigenvalues of Q are distinct.*

Assumption 3 $\zeta < \frac{1}{2}\Big[1 - \max_{I \in \{1,\dots,N\}} \lambda_2(Q_I^*)\Big]$.

Since the probability measures $\nu_i(\cdot)$ and $\tilde{\nu}_i(\cdot)$ have full support for all $i \in I_n$, all eigenvalues of Q will generically be distinct and, hence, Assumption 2 will generically be satisfied. Let $\alpha(1)$ be the monomorphic state where all firms have the Walrasian conjecture, i.e. $\alpha(1) = (-1, \dots, -1)$ We can now state the following proposition.

Proposition 2 *Suppose that Assumptions 1–3 hold. Then there exists an ergodic Markov chain on Λ^n with transition matrix $\tilde{Q}$ and unique limit distribution $\tilde{\mu}(\cdot)$. For $\tilde{\mu}(\cdot)$ it holds that $\tilde{\mu}(q^{\alpha(1)}) = 1$. Furthermore, $\tilde{\mu}(\cdot)$ is an approximation of $\mu(\cdot)$ of order $O(\zeta)$.*

Proof. First, decompose Q as in (3)-(6). So, the transition matrix Q is decomposed into a block diagonal matrix Q^*, where each diagonal block is the transition matrix for the strategy dynamics for a given vector of conjectures, and a matrix that reflects the behavioural dynamics. The constant ζ is the maximum degree of coupling between subsystems Q_I^*.

Given the result of Proposition 1 one can aggregate Q using $\mu^I(\cdot)$ in the following way. Define a Markov chain on Λ^n with transition matrix $\tilde{Q}$, which has typical element

$$\begin{aligned}\tilde{Q}(I,J) &= \sum_{k=1}^{m} \mu^I\big(q(k)\big) \sum_{l=1}^{m} Q_{k_I l_J} \\ &= \sum_{k=1}^{m} \mu^I\big(q(k)\big) \lambda_k(I,J) \sum_{l=1}^{m} M_I(k,l) \\ &= \sum_{k=1}^{m} \mu^I\big(q(k)\big) \lambda_k(I,J) = \lambda_{k(I)}(I,J).\end{aligned}$$

Note that the transition matrix $\tilde{Q}$ is the limit of a sequence of ergodic Markov chains with transition matrices $\tilde{Q}^{\tilde{\varepsilon}}$ with $\tilde{Q}^{\tilde{\varepsilon}}(I,J) = \lambda_{k(I)}^{\tilde{\varepsilon}}(I,J)$. So, $\tilde{Q}$ has a unique limit distribution $\tilde{\mu}(\cdot)$. Under Assumptions 2 and 3, Theorem 3 directly yields that $\tilde{\mu}(\cdot)$ is an $O(\zeta)$ approximation of $\mu(\cdot)$.

The result on $\tilde{\mu}(\cdot)$ is obtained by using the familiar techniques developed by Freidlin and Wentzell (1984). First we establish the set of recurrent states for the mutation-free dynamics of $\tilde{Q}^{\tilde{\varepsilon}}$. This is the dynamics without the experimentation part and is thus equal for all $\tilde{\varepsilon} > 0$. From (1) one can see that the transition probabilities for this dynamics are equal to the transition probabilities of going from one vector of conjectures $\alpha(I)$ to another vector $\alpha(J)$ given that the current quantity vector is the equilibrium $q^{\alpha(I)}$. So, the dynamics of $\tilde{Q}^{\tilde{\varepsilon}}$ is the pure conjecture dynamics if the quantity dynamics gets sufficient time to settle in equilibrium. Let the transition matrix for this aggregated pure conjecture dynamics be denoted by $\tilde{Q}_0$.

Lemma 2 *The set $\mathcal{A}$ of recurrent states for the aggregated mutation-free conjecture dynamics with transition matrix $\tilde{Q}_0$ is given by the set of monomorphic states, i.e.*

$$\mathcal{A} = \big\{\{(\alpha,\ldots,\alpha)\}\big|\alpha \in \Lambda\big\}.$$

The proof of this lemma can be found in Appendix B. Define the costs between $\alpha(I)$ and $\alpha(J)$ to be

$$c(\alpha(I),\alpha(J)) = \min_{K=1,\ldots,N}\big\{d(\alpha(I),\alpha(K))\big|\tilde{Q}_0(K,J) > 0\big\},$$

where $d(\alpha(I), \alpha(K)) = \sum_{i \in I_n} \mathbb{1}_{\left(\alpha_i(I) \neq \alpha_i(K)\right)}$. The cost between $\alpha(I)$ and $\alpha(J)$ is the minimum number of mutations from $\alpha(I)$ that is needed for the pure conjecture dynamics to have positive probability of reaching $\alpha(J)$. Let $\alpha \in \Lambda^n$. An α-tree H_α is a collection of ordered pairs (α', α'') such that:

1. every $\alpha' \in \Lambda^n \backslash \{\alpha\}$ is the first element of exactly one pair;
2. for all $\alpha' \in \Lambda^n \backslash \{\alpha\}$ there exists a path (α', α^1),(α^1, α^2),...,(α^{s-1}, α^s), (α^s, α) in H_α.

For each α-tree H_α the cost of tree H_α is defined by

$$c(H_\alpha) = \sum_{(\alpha', \alpha'') \in H_\alpha} c(\alpha', \alpha'').$$

First, we build an $\alpha(1)$-tree H^* with minimal costs. Then it is shown that for any state $\alpha \in \mathcal{A} \backslash \{\alpha(1)\}$ and any α-tree H_α the costs will be higher. From Freidlin and Wentzell (1984, Lemma 6.3.1) one can then conclude that $\alpha(1)$ is the unique element in the support of $\tilde{\mu}(\cdot)$. Young (1993) has shown that the minimum cost tree is among the α-trees where α is an element of a recurrent class of the mutation-free dynamics. Thus, from Lemma 2 we know that we only need to consider the monomorphic states in $\mathcal{A}$. This implies that for all α-trees H_α, $\alpha \in \mathcal{A}$, we have $c(H_\alpha) \geq |\mathcal{A}| - 1$, since one always needs at least one experiment to leave a monomorphic state.

Consider $\alpha(1)$ and the $\alpha(1)$-tree H^* that is constructed in the following way. Let $\alpha \in \mathcal{A} \backslash \{\alpha(1)\}$. For all $i \in I_n$ we have $\alpha_i > \alpha_i(1)$. Suppose that one firm i experiments to $\alpha_i(1) = -1$, while the other firms cannot revise their output. According to Lemma 1 with $k = 1$ this firm has a higher profit in quantity equilibrium than the other firms. If one period later all other firms $j \neq i$ get the opportunity to revise their conjectural variation (which happens with positive probability) they will all choose $\alpha_j(1) = -1$. Hence, one mutation suffices to reach $\alpha(1)$ and, therefore, $c(H^*) = |\mathcal{A}| - 1$.

Conversely, let H_α be an α-tree for some $\alpha \in \mathcal{A} \backslash \{\alpha(1)\}$. Then somewhere in this tree there is a path between $\alpha(1)$ and a monomorphic state α' with $\alpha'_i > -1$ for all $i \in I_n$. Suppose that starting from $\alpha(1)$ one firm i experiments to α'_i. From Lemma 1 with $k = n - 1$ it is obtained that firm i has a strictly lower profit than the other firms in quantity equilibrium. So, to drive the system away from $\alpha(1)$ to α' at least two mutations are needed. Hence, $c(H_\alpha) > c(H^*)$. □

Proposition 2 gives a result on the convergence of market interaction to the Walrasian equilibrium that is similar to the result of Vega–Redondo (1997). Apparently, profit imitation is such a strong force that it also drives this more elaborate behavioural model to the Walrasian equilibrium. Note, however, that the result in

Proposition 2 is an approximation. It might well be that the support of $\mu(\cdot)$ consists of more states than just the Walrasian equilibrium.

A crucial assumption is the one on the maximum degree of coupling between subsystems Q_I^*, ζ, as stated in Assumption 3. This parameter should not be too large. Intuitively, this condition requires that the interaction between subsystems Q_I^* is sufficiently low, i.e. that the conjecture dynamics does not happen too frequent. In Proposition 3 a sufficient condition on $\tilde{p}$ is given for Assumption 3 to hold.

Proposition 3 *If $\tilde{p} < 1 - \left(\frac{3}{4}\right)^{1/n}$, then Assumption 3 is satisfied.*

Proof. Let $I \in \{1, 2, \dots, N\}$. From Bauer et al. (1969) we obtain an upper bound for the second largest eigenvalue in absolute value of Q_I^*:

$$\begin{aligned} \lambda_2(Q_I^*) \leq \min\Big\{ &\max_{1\leq\theta,\rho\leq m} \frac{1}{2}\sum_{i=1}^{m} v_i^1(Q_I^*)\Big|\frac{Q_I^*(i,\theta)}{v_\theta^1(Q_I^*)} - \frac{Q_I^*(i,\rho)}{v_\rho^1(Q_I^*)}\Big|, \\ &\max_{1\leq\theta,\rho\leq m} \frac{1}{2}\sum_{i=1}^{m} |Q_I^*(\theta,i) - Q_I^*(\rho,i)|\Big\}, \end{aligned} \tag{14}$$

where $v^1(Q_I^*)$ is the eigenvector corresponding to the largest eigenvalue of Q_I^*. Since Q_I^* is a stochastic matrix we have that

$$v_i^1(Q_I^*) = \mu_i^I = \mathbb{1}_{(q=q^{\alpha(I)})}.$$

Consider the first term on the right-hand side of (14). For $\theta = k(I)$ and $\rho \neq k(I)$, we get

$$\frac{1}{2}\sum_{i=1}^{m} v_i^1(Q_I^*)\Big|\frac{Q_I^*(i,\theta)}{v_\theta^1(Q_I^*)} - \frac{Q_I^*(i,\rho)}{v_\rho^1(Q_I^*)}\Big| = \frac{1}{2}\Big|\frac{Q_I^*(k(I),\theta)}{\mu_{k(I)}^I} - \frac{Q_I^*(k(I),\rho)}{\mu_\rho^I}\Big|. \tag{15}$$

Take $\theta = k(I)$, and $\rho \neq k(I)$ such that $q(\rho)_i = q(k(I))_i$ for some $i \in I_n$. Then $Q^*(k(I),\rho) > 0$ and $\mu_\rho^I = 0$, so that (15) is unbounded.

The maximum of the second term on the right-hand side of (14) is attained for $\theta = k(I)$ and some $\rho \neq k(I)$, such that $q(k(I))$ is not a best response to $q(\rho)$. One obtains that

$$\frac{1}{2}\sum_{i=1}^{m} |Q_I^*(k(I),i) - Q_I^*(\rho,i)| \leq \frac{1}{2}|Q_I^*(k(I),k(I))| = \frac{1}{2},$$

since $q(k(I))$ is a best response to $q(k(I))$. Hence, we find that $\lambda_2(Q_I^*) \leq \frac{1}{2}$ for all $I = 1, \dots, N$. So, we have that

$$\frac{1}{2}[1 - \max_{I=1,\dots,N} \lambda_2(Q_I^*)] \geq \frac{1}{4}.$$

Note that it holds that

$$\begin{aligned}\zeta &= \max_{k_I}\Big\{\sum_{K\neq I}\sum_{l=1}^{m} Q_{k_I l_K}\Big\}\\ &= \max_{k_I}\Big\{\sum_{K\neq I}\lambda_k(I,K)\Big\}\\ &= \max_{k_I}\Big\{1-\lim_{\tilde{\varepsilon}\downarrow 0}\lambda_k^{\tilde{\varepsilon}}(I,I)\Big\}.\end{aligned}$$

Furthermore, by definition we have that

$$\lambda_k^{\tilde{\varepsilon}}(I,I) \geq \prod_{i\in I_n}\{(1-\tilde{\varepsilon})(1-\tilde{p})+\tilde{\varepsilon}\tilde{\nu}_i(\alpha_i(I))\}.$$

Therefore, we conclude that

$$\begin{aligned}\zeta &\leq 1-\lim_{\tilde{\varepsilon}\downarrow 0}\prod_{i\in I_n}\{(1-\tilde{\varepsilon})(1-\tilde{p})+\tilde{\varepsilon}\tilde{\nu}_i(\alpha_i(I))\}\\ &= 1-(1-\tilde{p})^n < \frac{1}{4}\\ \iff \tilde{p} &< 1-\left(\frac{3}{4}\right)^{1/n},\end{aligned}$$

which proves the proposition. □

Note that this upper bound is exponentially decreasing in the number of firms, from $\tilde{p} < 0.14$ for $n = 2$, to $\tilde{p} < 0.03$ for $n = 10$. This implies that, for the Walrasian equilibrium to be the only stochastically stable state in a large industry, the rate of behavioural change needs to be very low. In other words, leaving the rate of behavioural change constant, this analysis indicates that smaller industries are possibly more competitive than larger industries, since uniqueness of the Walrasian equilibrium as the only stochastically stable state cannot be guaranteed.

5 Discussion

This paper analysed a general framework for analysing stochastic stability in models with multi-level evolution, namely strategy and behavioural evolution. It was shown that under certain conditions, the theory of near-complete decomposability can be used to disentangle the two levels of evolution. They can then be studied separately and the equilibrium of the strategy dynamics can be used to obtain the equilibrium of the behavioural dynamics, which, in turn, is an approximation of the equilibrium of the combined dynamics. This approach is applied to an extension of the Vega–Redondo (1997) model of imitation in Cournot oligopoly.

In the application, we model strategy dynamics based on myopic optimisation by firms that includes the conjectured market response to the firm's own quantity-setting behaviour which is modelled by means of a conjecture parameter. At a second level, we allow firms to change or adapt their behaviour in the sense that they can change their conjecture. This decision is also modelled to be boundedly rational. Firms look at their competitors and imitate the behaviour of the most successful firm.

The main conclusion of Proposition 2 is that if behavioural adjustment takes place at a sufficiently low frequency, the market ends up in the Walrasian equilibrium in the long-run. An explicit upper bound for this frequency is provided and is shown to be exponentially decreasing in the number of firms. So, even with more elaborate behavioural dynamics than e.g. Vega–Redondo (1997), evolution still selects the Walrasian equilibrium. The appeal of this equilibrium lies in the fact that if behaviour is guided by profit imitation, i.e. relative payoffs, this leads to spitefulness in a firm's actions. This in turn leads to selection of the Walrasian equilibrium. The analysis indicates that smaller industries are possibly more competitive than larger industries.

Modelling explicit dynamic processes where players learn from the past is important, since in a "pure repeated game framework[...]history matters only because firms threaten it to matter" (Vives (1999)). With learning or evolution, history matters *per se*. Until recently, most models of learning are restricted to dynamics at one level. The analysis in this paper suggests ways in which to include several levels of evolution. This makes it possible to study both learning and evolution separately in a unified framework, i.e. the short-run and the long-run. In the example of the oligopolistic industry: how do (short-run) strategic choices based on conjectures and (long-run) competitive and Darwinian pressures interact? We restricted ourselves to two levels, but in principle it is straightforward to extend the theory of near-complete decomposability to more levels of learning or evolution.

Another application of the presented theory is for simulation analysis. Analysing large dynamic agent based systems can be computationally very intensive. If one is willing to assume relatively infrequent interactions between different levels of dynamics, the theory of near-complete decomposability can greatly reduce the computational burden of these models.

Appendix

A Proof of Lemma 1

Since all firms are identical and solutions to the first-order conditions are unique, firms with the same conjecture have the same equilibrium quantity. Therefore, the equilibrium quantities q_k^α and $q_k^{\alpha'}$ satisfy[13]

$$P'\big((n-k)q_k^\alpha + kq_k^{\alpha'}\big)(1+\alpha)\frac{n}{2}q_k^\alpha + P\big((n-k)q_k^\alpha + kq_k^{\alpha'}\big) - C'(q_k^\alpha) = 0$$
$$P'\big((n-k)q_k^\alpha + kq_k^{\alpha'}\big)(1+\alpha')\frac{n}{2}q_k^{\alpha'} + P\big((n-k)q_k^\alpha + kq_k^{\alpha'}\big) - C'(q_k^{\alpha'}) = 0.$$

These first-order conditions imply that

$$\begin{aligned} &P'\big((n-k)q_k^\alpha + kq_k^{\alpha'}\big)(1+\alpha)\frac{n}{2}q_k^\alpha - C'(q_k^\alpha) \\ &\qquad = P'\big((n-k)q_k^\alpha + kq_k^{\alpha'}\big)(1+\alpha')\frac{n}{2}q_k^{\alpha'} - C'(q_k^{\alpha'}). \end{aligned} \tag{A.1}$$

Suppose that $q_k^\alpha \geq q_k^{\alpha'}$. There are two possible cases:

1. if $C'(q_k^\alpha) \geq C'(q_k^{\alpha'})$, then (A.1) immediately gives a contradiction;

2. if $C'(q_k^\alpha) < C'(q_k^{\alpha'})$, then according to (A.1) it should hold that

$$-P'\big((n-k)q_k^\alpha + kq_k^{\alpha'}\big)(1+\alpha)\frac{n}{2}q_k^\alpha \leq P'\big((n-k)q_k^\alpha + kq_k^{\alpha'}\big)(1+\alpha')\frac{n}{2}q_k^{\alpha'}.$$

   This implies that $\frac{q_k^\alpha}{q_k^{\alpha'}} \leq \frac{1+\alpha'}{1+\alpha}$. However, since $\frac{q_k^\alpha}{q_k^{\alpha'}} \geq 1$ and $\frac{1+\alpha'}{1+\alpha} < 1$ this gives a contradiction.

According to the mean-value theorem there exists a $q \in (q_k^\alpha, q_k^{\alpha'})$ such that

$$C'(q) = \frac{C(q_k^{\alpha'}) - C(q_k^\alpha)}{q_k^{\alpha'} - q_k^\alpha},$$

since the cost function is continuous. Furthermore, it holds that

$$\begin{aligned} C'(q) &< \max\{C'(q_k^\alpha), C'(q_k^{\alpha'})\} \\ &\leq P\big((n-k)q_k^\alpha + kq_k^{\alpha'}\big) \\ \iff P\big((n-k)q_k^\alpha + kq_k^{\alpha'}\big)q_k^{\alpha'} - C(q_k^{\alpha'}) &> P\big((n-k)q_k^\alpha + kq_k^{\alpha'}\big)q_k^\alpha - C(q_k^\alpha), \end{aligned}$$

which proves the lemma. □

[13] Here the assumption that $\alpha' \geq -1$ is crucial. For if $\alpha > -1$ and $\alpha' < -1$ the system of first-order conditions has no solution.

B Proof of Lemma 2

Given a monomorphic state, the pure conjecture dynamics remains in the same monomorphic state with probability one. So $\mathcal{A} \supset \big\{\{(\alpha, \ldots, \alpha)\}\big|\alpha \in \Lambda\big\}$. Conversely, let $\alpha \in \Lambda^n \backslash \mathcal{A}$. With positive probability all firms may adjust their conjecture and with positive probability all choose the same conjecture, leading to a monomorphic state. Hence,

$$\mathcal{A} \subset \big\{\{(\alpha, \ldots, \alpha)\}\big|\alpha \in \Lambda\big\},$$

which proves the lemma. □

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