# Value at Risk (VaR) and the α-stable distribution

John C. Frain

TEP Working Paper No. 0308

May 2008

**Trinity Economics Papers**

**Department of Economics**

**Trinity College Dublin**

# Value at Risk (VaR) and the $\alpha$-stable distribution

John C. Frain.*

May 29, 2008

**Abstract**

Volatility in financial markets is a matter of considerable concern to financial institutions and their supervisors. Already it is clear that this volatility has had an adverse effect on the real economy. Many measures of risk that are used today do not take full account of the kind of extreme changes in asset prices that have been observed. This paper finds that the Value at Risk measure of risk can be improved by the use of an $\alpha$-stable distribution in place of more conventional measures. The paper describes the use of this measure and implements it for six total returns equity portfolios. We find that $\alpha$-stable based measures are feasible and are better than conventional measures. They are a useful tool for the risk manager and the financial regulator.

## Contents



*Comments are welcome. My email address is frainj@tcd.ie. This document is work in progress. Please consult me before quoting. Thanks are due to Prof. Antoin Murphy and to Michael Harrison for help and suggestions and to participants at seminars in TCD , UL and at the IEA Annual Conference, Westport, 2008 for comments received. Any remaining errors in the paper are my responsibility. I would also like to thank my wife, Helen, for her great support and encouragement.

# 1 Introduction

Today Value at Risk (VaR) is the most common measures of risk used in many financial institutions. VaR at a $p\%$ level is estimated as the loss that might be exceeded $p\%$ of the time. Like many other models in finance it is often based on an assumption that losses follow a normal distribution. It is now well known that extreme losses are greater than, and occur much more often than, a normal distribution would predict. To allow for this, VaR measures are sometimes based on a t-distribution or an ARCH/GARCH systems with innovations having a normal or t-distribution. Several other distributions or mixtures of distributions have been proposed but none have received universal acceptance and it is probable that none ever will.

The $\alpha$-stable distribution, which is examined here, may be thought of as a generalisation of the normal distribution. A normal distribution of losses is often justified by an appeal the central limit theorem. Similar arguments can also be used to justify the use of an $\alpha$-stable distribution. The purpose of this exercise is to calculate VaR at various levels assuming that losses follow either a static $\alpha$-Stable distribution or a TS-GARCH type distribution with $\alpha$-stable innovations. The resulting estimates are compared with estimates obtained from static normal and t-distributions, and GARCH(1,1) systems with normal and t-innovations. The portfolios examined are six total returns[1] equity indices (ISEQ, CAC40, DAX30, FTSE100, S&P500, Dow Jones Composite (DJAC)). VaR is estimated at 10%, 5%, 1%, 0.5% and 0.1% levels.

Section 2 of the paper gives a brief outline of the development and definition of VaR. Section 3 introduces the $\alpha$-stable distribution and explains why it is a good candidate for the distribution of losses. The main results of the analysis are in section 4. All parameter estimates are maximum likelihood estimates. Technical details and results of the estimations along with descriptions of the data and software used are in the appendices. The results may be summarised as follows.

The main finding is that the $\alpha$-stable GARCH(1,1) model for losses provides the best measure of VaR. It gives good estimates at all VaR levels for all the indices considered. The theoretical justification for the good results is given in section 3. I have shown that the estimates of VaR derived from an $\alpha$-stable distribution are feasible and are a useful addition to the toolbox of a risk manager or a financial regulator.

The normal distribution performs very badly even at the conventional 5% and 1% levels. It tends to over estimate VaR at the higher probability levels and under estimate at the lower. This is what one would expect given the exponential decay in the tails of the normal distribution. A VaR at 1%, based on the normal distribution underestimates risk. It is misleading to management to the extent that they may agree to some investments that would not be accepted if a more accurate assessment of risk was used.

The t-distribution appears to perform very well, particularly in the tails of the distribution. Empirically it is marginally (but not statistically) better than the $\alpha$-stable. The simplicity of the t-distribution makes it an attractive alternative. While

1. In calculating these indices it is assumed that dividends are reinvested in the portfolio

it appears to work well empirically there is no good economic reason for its good performance. As is well known, empirical results with no theory usually lead to problems.

The GARCH distributions with normal innovations performs somewhat better than the static normal distribution. Curiously the GARCH distribution with t-innovations does not perform as well as the static t-distribution but is better than the GARCH with normal innovations.

The static $\alpha$-stable distribution performance is about equivalent to the t-distribution but is excellent at the conventional levels. Extreme VaR at levels less than 1% tends to be conservative.

Section 5 summarises the analysis and sets out the conclusions that may be drawn from the analysis.

# 2 Value at Risk (VaR)

The world wide equity crisis in 1987, the fall in Japanese equity market in 1990, the Mexican peso crisis in 1994/95 and the severe losses suffered in various derivative transactions in the 1990s were a strong incentive to both market participants and regulators to measure and monitor market and other exposures. Jorian (2007) (page 32) estimates losses in the 1990s publicly attributed to derivatives at over $ 30 billion. Given the overall volume of derivative trading this is not an enormous sum it is extremely problematic to the individual companies that incurred the losses. Financial regulators would also fear that losses such as these might have knock on effects that would effect the efficient functioning of markets. Jorian (2007) lists five firms that each had losses of more than $ 1 billion attributed to derivative trading.

- Orange County, California, December 1994, Reverse repos, loss $1810 billion
- Showa Shell Sekiyu, Japan, February 1993, Currency Forwards, loss $1580 billion
- Kashima Oil, Japan, April 1994, Currency Forwards, loss $1450 billion

- Metallgesellschaft, January 1994, Germany, Oil Futures, loss \$1340 billion
- Barings, U.K., February 1995, Stock Index futures, loss \$1330 billion

One lesson to be learned from these and similar events was the need to introduce better methods of risk assessment and monitoring. At that time often simple rules based on guidelines like "high liquidity", "low" interest rate risk, hedging, "highly" correlated, limits on amount invested, sectors etc. were often used. Such rules were often ambiguous or could easily be circumvented by "resourceful" traders. Many losses of the type outlined above were due to inadequate and/or circumvented supervisory controls. In the the US the Sarbanes-Oxley Act of 2002 creates a more rigourous legal environment for the board, the management committee, internal and external auditors, and the chief risk officer. These regulations apply to all companies with a quotation on a US exchange and thus apply to several large Irish companies. Management and directors of such institution are now required to have risk measurement, audit and control systems in place and to report regularly on these. The financial regulatory authorities have now adopted the Basel II Capital Adequacy Directive which allow institutions to use, subject to approval, their internal risk measurement systems to determine capital adequacy for regulatory purposes.

Value-at-Risk (VaR) is a commonly used measure of the risk of an investment or a portfolio. A $p$% VaR is the lower limit on the proportion of a portfolio can be lost $p$% of the time. Thus a $p$% VaR is the $(100 - p)$% quantile of the loss distribution This is illustrated in figure 1 where the value at the left boundary of the shaded area represents the 5% VaR.

$$Prob[\text{ loss} \geq V_p] = p \tag{1}$$

Thus if the daily loss on a portfolio is normally distributed with an expected value of 0.005% and a standard deviation of 0.010 one would expect to lose

- more than 0.0114% 5% of the time

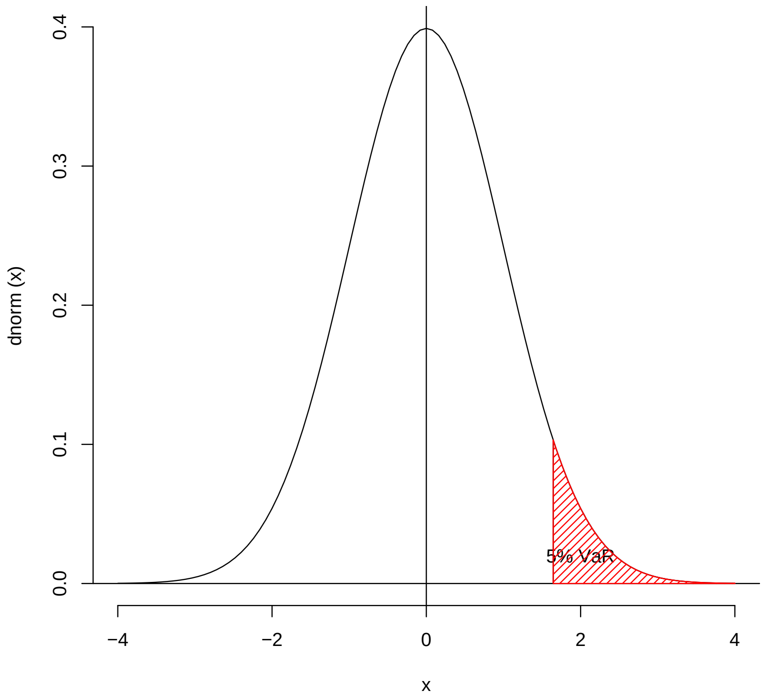


Figure 1: 5% Value at Risk

- more than 0.0183% 1% of the time

The daily VaR of the portfolio is then 0.0114% and 0.0183% at the 5% and 1% levels respectively. Here I consider VaR corresponding to a one day holding period. The period covered by the VaR calculated by a financial institution would depend on the nature of its business. A pension fund would calculate VaR over a long holding period whereas a bank would be interested in a shorter holding period (typically 10 days). We shall not consider here how the daily VaR might be aggregated to a longer holding period

A properly implemented VaR includes all sources of risk and should encompass market, operational, credit, liquidity and model risk. VaR may be calculated at enterprize level, at various sector levels within the organisation and at individual trader level - the VaR at lower levels being aggregated to estimate VaR at the higher levels. Operational VaR levels may be set for individual traders. VaR limits for individual traders should also facilitate control of operations as a dealer operating outside his limits[2] will be detected if his dealings are properly recorded by the system. It should be added that the risk management function in an organisation should not depend solely on a VaR system but should have a range of tools available to them. If one looks at many of the derivative disasters a proper VaR implementation might have saved a lot of embarrassment

Risk is a very complex subject which I am not going to examine in detail here. In brief it is the uncertainty in forecasted future returns. As such, like utility, it is an ordinal concept. Any one-one (strictly) monotonic transformation of a risk measure is an equivalent risk measure. The statement that one investment is 10% more risky than another simply does not make sense.

Artzner et al. (1999) set out a set of desirable properties that a measure of risk should have. Let $X$ and $Y$ be two assets. A risk measure $\rho()$ is coherent if it has the following four properties

**Subadditivity** $\rho(X+Y) \leq \rho(X) + \rho(Y)$ (diversification reduces risk)

**Homogeneity** For any number $\alpha > 0$, $\rho(\alpha X) = \alpha \rho(X)$

**Monotonicity** $\rho(X) \geq \rho(Y)$ if $X \leq Y$

**Risk Free Condition** $\rho(X+k) = \rho(X) - k$ for any constant $k$

VaR satisfies three of these conditions but may fail on subadditivity. To cope with this shortcoming alternative measures of risk have been proposed. Expected Shortfall (ES) is one such measure. Expected Shortfall is defined as the expected loss given that the VaR threshold has been exceeded. Daníelsson et al. (2005) has shown that subadditivity holds for VaR in all the distributions considered here. Subadditivity of VaR fails for assets which have super-fat tails (e.g. $\alpha$-stable distributions where $\alpha \leq 1$, return/loss distributions which show very little variation apart from occasional jumps (e.g. "fixed" exchange rates) and some transactions involving derivatives). In all the cases considered here VaR is a monotonic transformation of ES and thus an equivalent measure of risk. The difference is in the

2. A dealer making large profits but operating outside his limits should of course be subject to the same disciplinary action as his colleague who loses money in such circumstances

explanation given to each measure. In practice where there is doubt both measures might be calculated.

The main advantage of VaR is that it is a simple idea and may be relatively easy[3] to calculate and is easily explained to non-technical persons in management. In 1994, at 4,15 pm each evening, J.P Morgan started to take a snapshot of their global trading positions to estimate, for management, their Daily-Earnings-at-Risk . This system was based on estimated correlation matrices, IGARCH systems and innovations with a normal distribution. In 1996 they made the relevant data and programmes (Riskmetrics) available to all other users. This move allowed many smaller users to implement VaR systems without the required investment in data and programmes. The current version of the Riskmetrics package allows innovations to follow a t-distribution.

One problem with VaR is the apparent precision of the measurements which may lead management to underestimate the true risk or to miss some aspect of risk. Even in the simple cases considered here one can see that the estimates are subject to considerable margins of error. Risk managers must be aware of the limitations of VaR and avoid creating false impressions.

A second criticism of VaR is that it takes no account of the shape of the distribution beyond the VaR point. Strictly speaking VaR estimates of two portfolios may be comparable only if the distributions of losses arising from the two portfolios are similar. A dealer may be able to increase returns by selling derivatives which might hedge the purchaser against some extreme risk. If the probability of the extreme event was small this would have very little effect of his calculated VaR. He has however changed the distribution of his losses. This is a serious problem with VaR systems and demonstrates the need to keep watch on the entire loss distribution. Risk management is a dynamic process and not simply a black box. Risk managers need to be extremely competent and be aware of the ability of traders to adapt to various constrains imposed on them. The risk manager needs to oversee the entire loss profile and not depend solely on an individual measure such as VaR. The combined use of VaR and ES might prove useful in such circumstances.

---

3. For a large financial institution dealing with a large number of exotic option the calculation of VaR is not easy but it is difficult to think of an simpler alternative

Frain and Meegan (1996) contains an account of the concepts and analytics of Value-at Risk. For more details see see also Dowd (1998, 2002), Jorian (2007) and Crouhy et al. (2006).

## 3 The $\alpha$-Stable Distribution

It is well known that the unconditional distribution of losses[4] (returns) on equities and many other assets displays, relative to the normal distribution,

- fat tails,
- a high peak
- and may be skewed

The stylised facts regarding loss distributions are well set out in Chapter 4 of Taylor (2005). The normal distribution does not accommodate these stylised facts and many alternatives have been proposed. To date no distribution has been universally accepted and probably none ever will. The use of the $\alpha$-stable distribution was first advocated in the 60's by Mandelbrot (Mandelbrot (1962, 1964, 1967, 1997), Mandelbrot and Hudson (2004)) and Fama (1964, 1965, 1976). Mandelbrot examined the variation of prices of cotton (1816-1940), wheat (1883-1936), railroad stock (1857-1936) and interest and exchange rates (similar periods) and found a larger number of extreme values than could be justified by the assumption of a normal distribution. Fama examined the distribution of daily returns for the 30 stock in the Dow Jones Industrial Average in a period from about the end of 1957 to September 26 1962. There was considerable interest in the $\alpha$-stable distribution throughout the 60's and the early 70's but interest then declined. This decline

4. Throughout this paper returns are defined as 100 times the log difference of the asset price (including dividends). Thus if $P_{t-1}$ and $P_t$ are the prices of the asset in periods $t-1$ and $t$ respectively, and $D_t$ the dividend paid in period $t$ the return $R_t$ on the asset in period $t$ is given by

$$R_t = 100 \log \left( \frac{P_t + D_t}{P_{t-1}} \right) \approx 100 \left( \frac{P_t + D_t}{P_{t-1}} - 1 \right).$$

The loss on asset is then simply the negative of the return.

can be attributed to two causes. First the enormous success of the Merton-Black-Scholes theories which form the basis of much of the theory on modern finance. The implementation of much of their analyses are based on assumptions of normality in the distribution of returns. Secondly the mathematics and implementation of routines involving $\alpha$-stable distributions are not easy. For a more recent review of the application of $\alpha$-stable distributions in finance see Rachev and Mittnik (2000). Frain (2006) examines the fit of the total returns equity indices, considered here, to $\alpha$-stable distributions. Further details of the mathematical theory of $\alpha$-stable distributions may be found in Feller (1971), Janicki and Weron (1994), Samorodnitsky and Taqqu (1994), Uchaikin and Zolotarev (1999) or Zolotarev (1986).

An $\alpha$-stable distribution may be thought of as a generalisation of the normal distribution where the generalization allows greater concentration close to the mean, more extreme values and possible skewness. The distribution depends on four parameters $\alpha$, $\beta$, $\gamma$ and $\delta$. These parameters [5] can be interpreted as follows

- $\alpha$, $(0 < \alpha \leq 2)$, is the basic stability parameter. It determines the weight in the tails. The smaller the value of $\alpha$ the greater the frequency and size of extreme events.

- $\beta$ is a skewness parameter and $-1 \leq \beta \leq 1$. A zero beta implies that the distribution is symmetric. Negative or positive $\beta$ imply that the distribution is skewed to the left or right respectively

- The parameter $\gamma$ is positive and measures dispersion. It is similar to the variance of a normal distribution

- The parameter $\delta$ is a real number and may be thought of as a location measure. It is similar to the mean of a normal distribution

The $\alpha$-stable distribution may be thought of as a family of distributions indexed by the parameter $\alpha$. When $\alpha = 2$ the $\alpha$-stable distribution is a normal distribution. (In this case the $\beta$ parameter becomes redundant and may be taken as

5. There are several parameterisations of the $\alpha$-stable distribution. Here I follow the 1-parametrisation of Nolan (2007)

zero. The normal distribution has a variance of $2\gamma^2$) When $\alpha = 1$ and $\beta = 0$ the distribution becomes a Cauchy distribution. Apart from the normal and Cauchy distributions (and one other of less interest here) these are the only instances of $\alpha$-stable distributions whose probability densities can be expressed in terms of elementary functions. In general the evaluation of the $\alpha$-stable density function requires either the numerical inversion of a characteristic function or the possible compilation and interpolation of tabulated values. This process has been made feasible by recent advances in the power of micro-computers.

The origin of the attribute "stable" in the $\alpha$-stable distribution is derived from the property that the form of the density function of a sum of independent identically distributed $\alpha$-stable random variables is, up to a scale and location parameter the same (ie "stable") as the distribution of the original variables. Let $X_1$, $X_2$, $X_3$, $\ldots$, $X_n$ be mutually independent variables with a common distribution $R$ and let $S_n = X_1 + \cdots + X_n$. The distribution R is stable if for each $n$ there exists constants $c_n$ and $\gamma_n$ such that[6]

$$S_n \stackrel{d}{=} c_n X + \gamma_n \tag{2}$$

This implies that time aggregation of a variable with independent $\alpha$-stable increments leads, apart from a location and scale factors to the same distribution as before aggregation. $\alpha$-stable distributions are the **only** distributions with this property. (Note that this "stable" property is not to be confused with the concept of "infinite divisibility". $\alpha$-stable distributions are also infinitely divisible but this property is different and is shared with many other distributions)

$\alpha$-stable distributions also have a second unique property. One may recall the central limit theorem which in one of its simpler forms says that if $X_1$, $X_2$, $X_3$, $\ldots$, $X_n$ are independent identically distributed random variables with a finite variance and $S_n = X_1 + \cdots + X_n$ then the asymptotic distribution of $S_n$ is normal. These assumptions may be weaken considerably. Heuristically, if we drop the identically distributed and finite variance assumptions keep independence and specify that no individual variable has a significant effect on the mean the central limit theorem

6. The notation $U \stackrel{d}{=} V$ means that $U$ and $V$ have the same distribution

continues to hold. We might consider if this last condition holds in economics or finance. Ask yourself how many times you have inserted a dummy variable because you felt that the value was an outlier. If we keep the independence requirement but now allow for cases in which individual measurements may have an effect on the mean then if the sum converges it converges to an $\alpha$-stable distribution. Like the normal distribution, the $\alpha$-stable distribution will be a reasonable approximation to a family of distributions.

Each member of the $\alpha$-stable distribution (including the normal) is the asymptotic limit for some set of independent identically distributed random variables. That $\alpha$-stable distribution is said to be an attractor for that set of distributions and the set of distributions is the domain of attraction for the specific $\alpha$-stable distribution. It can be shown that $\alpha$-stable distributions are the only non-degenerate distributions that have domains of attraction.

Thus the $\alpha$-stable distribution can account for many of the typical properties of asset returns/losses

# 4 Empirical Results

In this section I calculate and evaluate static and dynamic estimates of VaR. The four static estimates are based on

1. a normal distribution,
2. a t-distribution, or
3. an $\alpha$-stable distribution and
4. a non-parametric quantile estimation procedure.

My initial evaluation of the parametric estimates is based on a comparison of the parametric and non-parametric estimators.

The dynamic VaR estimates are based on Garch(1,1) processes with normal, t, and $\alpha$-stable innovations. If an estimate of VaR at $p\%$ is good then it should be exceeded in the sample close to $p$% of the time. For each of the VaR estimates I calculate the exceedances and test the difference between the observed and predicted exceedances. This same test is also applied to the static estimates.

All distribution parameters are estimated by maximum likelihood. The Tables in appendices A.1, A.2 and A.3 give details of these estimates. Data sources and software used are described in Appendix B

## 4.1 VaR Estimates

Tables 1 to 5 set out static estimates of the VaR at 10%, 5%, 1%, 0.5% and 0.1% levels for an investment in each of the six total returns equity indices

- ISEQ (daily from 4 January 1988 to 31 January 2008)
- CAC40 (daily from 31 December 1987 to 31 January 2008)
- DAX30 (daily from 28 September 1959 to 31 January 2008)
- FTSE100 (daily from 31 December 1985 to 31 January 2008)
- Dow Jones Composite (DJC) (daily from 30 September 1987 to 31 January 2008)
- S&P500 (daily from 29 December 1989 to 31 January 2008)

The quantiles are calculated on the basis of returns following

- an $\alpha$-stable distribution with parameters estimated by maximum likelihood
- a normal distribution with parameters estimated by maximum likelihood
- a t-distribution[7] with nonzero mean, nonzero scale and degrees of freedom to be estimated by maximum likelihood

7. The probability density function of a t-distribution with mean $\mu$, scale $\sigma$ and degrees of freedom $\nu$ is given by

$$f(x|\mu,\sigma,\nu) = \frac{\Gamma[(\nu+1)/2]}{(\pi\nu)^{1/2}\Gamma(\nu/2)} \frac{\sigma^{-1}}{[1+(x-\mu)^2/(\sigma^2\nu)]^{(\nu+1)/2}}, \quad -\infty < x < \infty.$$

Table 1: 10% VaR for each equity index assuming the specified distribution

| Index | Distribution | | | (1) Sample Quantile | Quantile s.e. (2) |
|---|---|---|---|---|---|
| | Stable | Normal | t | | |
| ISEQ | **1.04** | 1.23 | 1.13 | 1.03 | 0.03 |
| CAC40 | **1.48** | 1.61 | 1.52 | 1.43 | 0.03 |
| DAX30 | **1.27** | 1.49 | 1.31 | 1.23 | 0.02 |
| FTSE100 | **1.15** | 1.30 | 1.22 | 1.13 | 0.02 |
| DJC | **1.05** | 1.26 | 1.11 | 1.04 | 0.03 |
| S&P500 | **1.10** | 1.20 | **1.14** | 1.09 | 0.03 |
| (1) Harrell and Davis (1982) | | | | | |
| (2) Bootstrap estimate | | | | | |

Table 2: 5% VaR for each equity index assuming the specified distribution

| Index | Distribution | | | Sample Quantile (1) | Quantile s.e. (2) |
|---|---|---|---|---|---|
| | Stable | Normal | t | | |
| ISEQ | **1.48** | 1.60 | **1.57** | 1.50 | 0.05 |
| CAC40 | **2.02** | **2.08** | **2.07** | 2.04 | 0.05 |
| DAX30 | **1.75** | 1.92 | **1.81** | 1.76 | 0.03 |
| FTSE100 | **1.59** | 1.68 | 1.66 | 1.55 | 0.03 |
| DJC | **1.46** | 1.63 | 1.53 | 1.47 | 0.05 |
| S&P500 | **1.54** | **1.55** | **1.56** | 1.55 | 0.04 |
| (1) Harrell and Davis (1982) | | | | | |
| (2) Bootstrap estimate | | | | | |

Table 3: 1% VaR for each equity index assuming the specified distribution

| Index | Distribution | | | Sample Quantile (1) | Quantile s.e. (2) |
|---|---|---|---|---|---|
| | Stable | Normal | t | | |
| ISEQ | **3.19** | 2.28 | **2.86** | 2.99 | 0.14 |
| CAC40 | **3.89** | 2.95 | **3.49** | 3.59 | 0.17 |
| DAX30 | **3.46** | 2.73 | **3.18** | 3.19 | 0.11 |
| FTSE100 | **3.09** | 2.40 | **2.81** | 2.92 | 0.12 |
| DJC | 2.97 | 2.32 | **2.69** | 2.59 | 0.10 |
| S&P500 | 3.25 | 2.21 | **2.75** | 2.73 | 0.10 |
| (1) Harrell and Davis (1982) | | | | | |
| (2) Bootstrap estimate | | | | | |

Table 4: 0.5% VaR for each equity index assuming the specified distribution

| Index | Distribution | | | Sample Quantile (1) | Quantile s.e. (2) |
|---|---|---|---|---|---|
| | Stable | Normal | t | | |
| ISEQ | 4.67 | 2.53 | **3.58** | 3.66 | 0.17 |
| CAC40 | 5.44 | 3.27 | **4.22** | 4.34 | 0.16 |
| DAX30 | 4.91 | 3.02 | **3.91** | 4.12 | 0.21 |
| FTSE100 | 4.33 | 2.66 | **3.40** | 3.48 | 0.23 |
| DJC | 4.25 | 2.57 | **3.31** | 3.25 | 0.20 |
| S&P500 | 4.70 | 3.45 | 3.40 | 3.10 | 0.09 |
| (1) Harrell and Davis (1982)) | | | | | |
| (2) Bootstrap estimate | | | | | |

Table 5: 0.1% VaR for each equity index assuming the specified distribution

| Index | Distribution | | | Sample Quantile (1) | Quantile s.e. (2) |
|---|---|---|---|---|---|
| | Stable | Normal | t | | |
| ISEQ | 12.00 | 3.04 | **5.91** | 5.55 | 0.55 |
| CAC40 | 12.98 | 3.94 | **6.33** | 6.15 | 0.48 |
| DAX30 | 11.98 | 3.63 | **6.12** | 6.44 | 0.42 |
| FTSE100 | 10.37 | 3.20 | **5.12** | 5.61 | 0.71 |
| DJC | 10.52 | 3.10 | **5.22** | 6.27 | 1.45 |
| S&P500 | 11.83 | 2.95 | **5.40** | 4.68 | 0.68 |
| (1) Harrell -Davis (1982) | | | | | |
| (2) Bootstrap estimate | | | | | |

- A distribution free estimate of each quantile based on Harrell and Davis (1982). A bootstrapped standard error of each non-parametric quantile estimate was also calculated.

The estimates for the parametric distributions,in bold case, in the first three columns are within two standard deviations of the non-parametric estimates. If we regard the nonparametric estimates and their bootstrapped standard errors as accurate such estimates are then, at least, consistent with the non-parametric estimates and may be regarded as "good".

On this criterion the estimates based on a normal distribution are of little value. They over-estimate VaR at 10% are a little high at 5% and underestimate risk at the lower levels.

The estimates for the $\alpha$ stable distribution are very good at the 10%, 5% and not that bad at the 1% levels. At the 0.5% and 0.1% levels they appear to overestimate the quantiles.

The t-distribution appears to perform well at the 1%, 0.5% and 0.1% levels and not that bad at the 5% level.

## 4.2 Exceedances of VaR Estimates

If a $p$% VaR estimate is reasonable I would expect that losses should exceed it approximately $p$% of the time. In these circumstances, the distribution or number of times that the $p$% VaR is exceeded (the exceedances) can be approximated by a Poisson [8] distribution with parameter given by $p$% of the sample size. Tables 6 to 10 present details of such counts of exceedances and an estimate of the probability of a higher value than that found based on the assumption of this Poisson distribution. Exceedances which accept the null at 95% level are set in bold font.

---

where $\Gamma(\cdot)$ is the gamma function. Note that the standard deviation of $x$ is $\sigma\sqrt{\frac{\nu}{\nu-2}}$. If $\mu = 0$ and $\sigma = 1$ this reduces to the standard Student's t-distribution with $\nu$ degrees of freedom. The heavier tails of the t-distribution are often used in economics and finance to model the fat tails that are often observed. Often the justification is empirical. A Bayesian justification involves a mixture of normal distributions with known mean and a prior inverse gamma distribution for the variance. For more details and references see Weitzman (2007).

8. The poisson approximation to the binomial is sufficient here.

Table 6: % Exceedances for 10% VaR given various distributional assumptions

| | Total Returns Index | | | | | |
|---|---|---|---|---|---|---|
| | ISEQ | CAC40 | DAX30 | FTSE100 | DJAC | S&P500 |
| Observations | 5037 | 5056 | 12098 | 5578 | 5158 | 4559 |
| Normal | 7.35 | 8.13 | 7.20 | 7.48 | 6.79 | 8.64 |
| | (1.00) | (1.00) | (1.00) | (1.00) | (1.00) | (1.00) |
| Garch(1,1) with | 9.18 | 10.88 | 9.18 | **9.77** | **9.36** | **10.13** |
| Normal Errors | (0.98) | (0.00) | (1.00) | (0.70) | (0.92) | (0.38) |
| t | 8.11 | 8.48 | 8.57 | 7.76 | 7.83 | 8.62 |
| | (1.00) | (0.99) | (1.00) | (1.00) | (1.00) | (1.00) |
| d.f. | 3.4 | 4.5 | 3.9 | 4.4 | 3.8 | 3.7 |
| Garch(1,1) with | 6.19 | **9.14** | 8.12 | 8.78 | 7.15 | 8.36 |
| t Errors | (1.00) | (0.02) | (1.00) | (1.00) | (1.00) | (1.00) |
| $\alpha$-Stable | **9.77** | **9.39** | **9.52** | **9.56** | **9.87** | **9.87** |
| | (0.69) | (0.91) | (0.95) | (0.84) | 0.61 | (0.60) |
| $\alpha$-Stable | **10.18** | **10.48** | **10.18** | **10.44** | **10.61** | **10.91** |
| GARCH(1,1) | (0.32) | (0.13) | (0.26) | (0.15) | (0.08) | (0.03) |

Figures in brackets are the estimated probability of a greater % than found based on a Poisson distribution for the number of exceedances.

Table 7: % Exceedances for 5% VaR given various distributional assumptions

| | Total Returns Index | | | | | |
|---|---|---|---|---|---|---|
| | ISEQ | CAC40 | DAX30 | FTSE100 | DJC | S&P500 |
| Observations | 5037 | 5056 | 12098 | 5578 | 5158 | 4559 |
| Normal | 4.40 | **4.79** | 4.07 | 4.10 | 4.11 | **4.98** |
| | (1.00) | (0.74) | (1.00) | (1.00) | (1.00) | (0.51) |
| Garch(1,1) with Normal Errors | 4.39 | **4.47** | 4.40 | 3.87 | 4.32 | **4.47** |
| | (1.00) | (0.95) | (1.00) | (1.00) | (0.99) | (0.94) |
| t | 4.17 | **4.27** | 4.40 | 3.87 | **4.65** | **4.47** |
| | (1.00) | (0.95) | (1.00) | (1.00) | (0.86) | (0.94) |
| d.f. | 3.4 | 4.5 | 3.9 | 4.4 | 3.8 | 3.7 |
| Garch(1,1) with t Errors | 3.87 | 4.27 | 3.87 | 4.12 | 3.28 | 3.90 |
| | (1.00) | (0.99) | (1.00) | (1.00) | (1.00) | (1.00) |
| $\alpha$-Stable | **5.18** | **4.98** | **5.00** | **4.66** | **5.02** | **5.13** |
| | (0.27) | (0.50) | (0.47) | (0.87) | (0.46) | (0.33) |
| $\alpha$-Stable GARCH(1,1) | **5.30** | **5.58** | **5.24** | **5.45** | **5.20** | **5.51** |
| | (0.16) | (0.03) | (0.11) | (0.06) | (0.25) | (0.06) |
| Figures in brackets are the estimated probability of a greater % than found based on a Poisson distribution for the number of exceedances. | | | | | | |

Table 8: % Exceedances for 1% VaR given various distributional assumptions

| | Total Returns Index | | | | | |
|---|---|---|---|---|---|---|
| | ISEQ | CAC40 | DAX30 | FTSE100 | DJC | S&P500 |
| Observations | 5037 | 5056 | 12098 | 5578 | 5158 | 4559 |
| Normal | 2.12 | 1.76 | 1.61 | 1.70 | 1.47 | 1.97 |
| | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) |
| Garch(1,1) with | 1.32 | 1.52 | 1.32 | 1.47 | 1.82 | 1.78 |
| Normal Errors | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) |
| t | **1.03** | **1.01** | **1.00** | **1.08** | **0.83** | **0.92** |
| | (0.37) | (0.44) | (0.51) | (0.26) | (0.87) | (.67) |
| d.f. | 3.4 | 4.5 | 3.9 | 4.4 | 3.8 | 3.7 |
| Garch(1,1) with | 0.65 | 0.69 | 0.65 | **0.86** | 0.52 | 0.68 |
| t Errors | (1.00) | (0.99) | (1.00) | (0.83) | (1.00) | (0.99) |
| $\alpha$-Stable | **0.79** | **0.83** | 0.81 | **0.82** | 0.62 | 0.35 |
| | (0.92) | (0.87) | (0.98) | (0.90) | (1.00) | (1.00) |
| $\alpha$-Stable | **0.97** | **1.11** | **1.13** | **1.09** | **1.09** | **1.12** |
| GARCH(1,1) | (0.54) | (0.20) | (0.07) | (0.22) | (0.24) | (0.19) |

Figures in brackets are the estimated probability of a greater % than found based on a Poisson distribution for the number of exceedances.

Table 9: % Exceedances for 0.5% VaR given various distributional assumptions

| | Total Returns Index | | | | | |
|---|---|---|---|---|---|---|
| | ISEQ | CAC40 | DAX30 | FTSE100 | DJC | S&P500 |
| Observations | 5037 | 5056 | 12098 | 5578 | 5158 | 4559 |
| Normal | 1.55 | 1.31 | 1.17 | 1.34 | 1.00 | 1.43 |
| | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) |
| Garch(1,1) with | 0.81 | 0.91 | 0.82 | 0.95 | 1.16 | 1.16 |
| Normal Errors | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) |
| t | **0.50** | **0.53** | **0.57** | **0.52** | **0.43** | 0.20 |
| | (0.46) | (0.32) | (0.12) | (0.37) | (0.74) | (1.00) |
| d.f. | 3.4 | 4.5 | 3.9 | 4.4 | 3.8 | 3.7 |
| Garch(1,1) with | 0.33 | **0.32** | 0.34 | **0.52** | 0.29 | **0.35** |
| t Errors | (0.99) | (0.97) | (0.99) | (0.50) | 0.98 | (0.91) |
| $\alpha$-Stable | 0.18 | 0.18 | 0.31 | 0.25 | 0.21 | 0.09 |
| | (1.00) | (1.00) | (1.00) | (1.00) | 1.00 | (1.00) |
| $\alpha$-Stable | **0.32** | **0.47** | **0.51** | **0.56** | **0.47** | **0.81** |
| GARCH(1,1) | (0.96) | (0.55) | (0.39 | (0.24) | (0.59) | (0.81) |
| Figures in brackets are the estimated probability of a greater % than found based on a Poisson distribution for the number of exceedances. | | | | | | |

Table 10: % Exceedances for 0.1% VaR given various distributional assumptions

| | Total Returns Index | | | | | |
|---|---|---|---|---|---|---|
| | ISEQ | CAC40 | DAX30 | FTSE100 | DJC | S&P500 |
| Observations | 5037 | 5056 | 12098 | 5578 | 5158 | 4559 |
| Normal | 0.95 | 0.83 | 0.69 | 0.66 | 0.58 | 0.61 |
| | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) |
| Garch(1,1) with | 0.34 | 0.33 | 0.34 | 0.43 | 0.54 | 0.55 |
| Normal Errors | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) |
| t | **0.11** | **0.08** | **0.12** | **0.13** | **0.14** | **0.09** |
| | (0.32) | (0.57) | (0.24) | (0.20) | (0.15 ) | (0.48) |
| d.f. | 3.4 | 4.5 | 3.9 | 4.4 | 3.8 | 3.7 |
| Garch(1,1) with | 0.06 | **0.12** | **0.11** | **0.11** | **0.11** | **0.11** |
| t Errors | (1.00) | (0.25) | (0.32) | (0.48) | (0.26) | (0.31) |
| $\alpha$-Stable | 0.00 | 0.00 | 0.01 | **0.04** | **0.02** | 0.00 |
| | (0.99) | (0.99) | (1.00) | (0.92) | (0.96) | (0.99) |
| $\alpha$-Stable | **0.06** | **0.06** | **0.05** | **0.04** | **0.06** | **0.04** |
| GARCH(1,1) | (0.74) | (0.74) | (0.96) | (0.92) | (0.76) | (0.083) |

Figures in brackets are the estimated probability of a greater % than found based on a Poisson distribution for the number of exceedances.

The measures of VaR in the these tables include

- Static normal distribution with parameters estimated by maximum likelihood.
- Garch(1,1) with normal innovations estimated by maximum likelihood. This gives rise to a dynamic VaR estimate which may be seen as a generalization of the traditional Riskmetrics Group (1999) methodology. See appendix A.2 for details of estimates and specification tests of the GARCH(1,1) models.
- t-distribution with mean, scale and degrees of freedom estimated by maximum likelihood (see footnote (7) on page 16)
- Garch(1,1) with t-errors estimated by maximum likelihood. The resulting VaR may be compared to the Riskmetrics 2006 methodology (Zumbach (2006). See appendix A.2 for details of estimates and specification tests
- $\alpha$-stable distribution - parameters estimated by maximum liklihood. See appendix A.1 for details of estimates and specification tests
- $\alpha$-stable Garch(1,1) - This is a variation of a TS-Garch(1,1) with $\alpha$-stable innovations. See Appendix A.3 for details.

Table 11 provides a summary of Tables 6 to 10 For each VaR level and for each index it give details of

- the number of times the proportion of exceedances was significantly less than the VaR level. In these cases the estimate of the risk is too high
- the number of times that exceedances were not significantly different to the VaR level. In these cases the measure of risk can not be rejected
- the number of times the proportion of exceedances was significantly more than the VaR level. In these cases risk has been under estimated.

On the basis of these results the $\alpha$-stable GARCH(1,1) is better than all the others. The observed exceedances are not statistically different from the expected for any of the equity indices at any of the five levels considered. Figure 2 shows a typical example of VaR estimated in this way and the corresponding losses

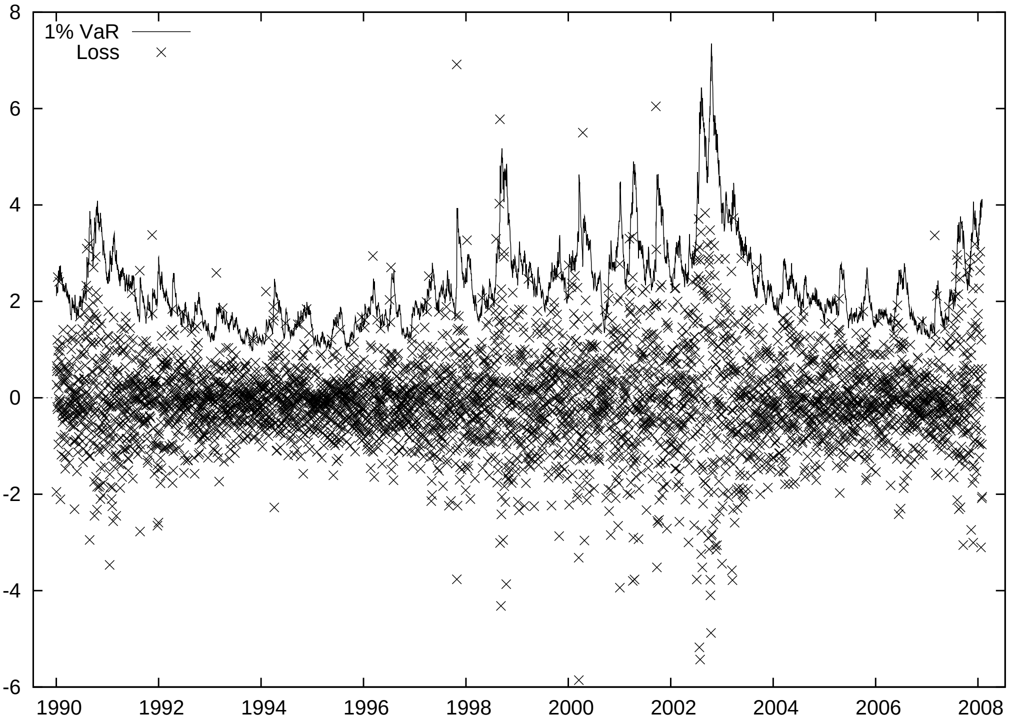


Figure 2: Losses on S&P 500 and 1% VaR based on $\alpha$-stable distributions

The static $\alpha$-stable and the t-distribution are next in order of merit. The $\alpha$-stable distribution performs best at the 10% and 5% levels and is somewhat conservative at the 1% and 0.1% level and very conservative at the 0.5% level. The t-distribution performs extremely well in the extreme tails of the distribution.

The ease of implementation of a VaR system based on a t-distribution, compared to one based on an $\alpha$-stable, combined with these results would incline many people to favour the t-distribution. If a t-distribution with about 4 degrees of freedom is appropriate for daily returns what distribution is appropriate for say hourly returns? Assuming that there are no problems with the distribution of news throughout the day then hourly returns will have a t-distribution with less than 1 degree of freedom. This does not make sense as the mean of such a distribution do not exist. It would be very difficult to make sense of any kind of theory of finance if this were the case. Aggregating a t-distribution over time would imply that returns follow a t-distribution with about 80 degrees of freedom. I have seen no evidence of this close an approximation to normality in asset returns. I also do not know of any theory in economics or finance that would lead to a t-distribution for returns. The idea that a t-distribution for asset returns results from a mixture of normal random variables with variance following an inverse gamma distribution has been argued in Weitzman (2007) is mathematically correct and as he admits has been well known to Bayesian statisticians but had no sound basis in economic theory. Many econometric models that fall down fail, not because there are problems with their econometrics, but because the economics behind the model is faulty or non-existent. The t-distribution may provide a good measure of what has been going on in the tails of the distribution but the results may be very sensitive to policy actions.

The normal distribution is conservative at the 10% level and greatly underestimates risk at at the 1% and lower levels. These quantile estimates based on the normal distribution are further evidence of the poor fit of the normal distribution to the data.

Exceedances for the two GARCH models are not good with approximately three quarters of the measures exceedances being significantly different from their expected values.

Table 11: Summary Exceedances

| VaR Level | Result | Distribution | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | Normal | Garch (Normal) | t-distr. | Garch (t-distr) | $\alpha$-Stable | $\alpha$-Stable Garch | All |
| 10% | low | 6 | 2 | 6 | 5 | 0 | 0 | 19 |
| | equal | 0 | 3 | 0 | 1 | 6 | 6 | 16 |
| | high | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 5% | low | 4 | 4 | 3 | 6 | 0 | 0 | 17 |
| | equal | 2 | 2 | 3 | 0 | 6 | 6 | 19 |
| | high | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1% | low | 0 | 0 | 0 | 5 | 3 | 0 | 8 |
| | equal | 0 | 0 | 6 | 1 | 3 | 6 | 16 |
| | high | 6 | 6 | 0 | 1 | 0 | 0 | 12 |
| 0.5% | low | 0 | 0 | 1 | 3 | 6 | 0 | 10 |
| | equal | 0 | 1 | 5 | 3 | 0 | 6 | 15 |
| | high | 6 | 5 | 0 | 1 | 0 | 0 | 11 |
| 0.1% | low | 0 | 0 | 0 | 1 | 4 | 0 | 5 |
| | equal | 0 | 0 | 6 | 5 | 2 | 6 | 19 |
| | high | 6 | 6 | 0 | 0 | 0 | 0 | 12 |
| All | low | 10 | 6 | 10 | 20 | 13 | 0 | 59 |
| | equal | 2 | 6 | 20 | 10 | 17 | 30 | 85 |
| | high | 18 | 18 | 0 | 0 | 0 | 0 | 36 |

Result -
low : % exceedances < VaR level - conservative view
equal : % exceedances not significantly different from VaR level
high : % exceedances > VaR level - liberal view

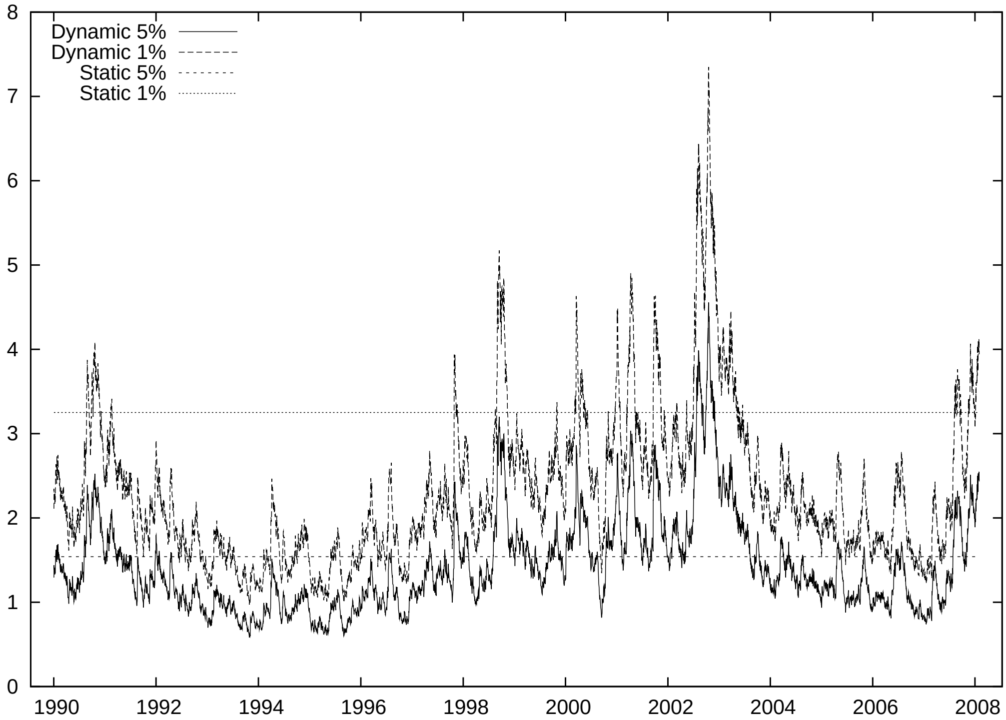


Figure 3: 5% and 1% static and Dynamic VaR of Losses on S&P 500

# 5 Conclusions

The relative performance of the various measures of VaR considered may be summarised as follows

The $\alpha$-stable GARCH(1,1) model for returns provides the best measure of VaR. It gives good estimates at all VaR levels for all the indices considered. The null hypothesis of a different rate of exceedances can not be rejected in a single case.

For the Risk manager or the supervisor I have shown that accurate measures of VaR can be obtained using an $\alpha$-stable distribution. Theoretical justification can be provided by the generalised central limit theorem and the time aggregation and domain of attraction properties which define, and are unique to, this distribution. Figure 4 compares the static and dynamic (GARCH) $\alpha$-stable 1% and 5% VaR.

The static normal distribution performs very badly even at the conventional 5% and 1% levels. It tends to over estimate VaR at the higher probability levels and under estimate at the lower. This is what one would expect given the exponential decay in the tails of the normal distribution. A normal VaR at 1% may be extremely misleading if given to management.

The static t-distribution performs very well, particularly in the tails of the distribution. In contrast to the normal and $\alpha$-stable distributions the t-distribution lacks the stability property and does not possess a domain of attraction. Aggregated t-distributions tend rapidly to a normal distribution. Disaggregation of a t-distribution of daily returns would imply that the distribution of high frequency returns would have very undesirable properties. The sometimes quoted justification for a t-distribution as a normal mixture with variances following an inverse gamma distribution is not very convincing.

The GARCH distributions with normal innovations performs somewhat better than the static normal distribution. Curiously the GARCH distribution with t-innovations does not performs worse than the static t-distribution but better than the GARCH with normal innovations.

Then $\alpha$-stable distribution performance is about equivalent to the t-distribution but is good at conventional VaR levels. Extreme VaR at levels less than 1% tends to be somewhat conservative but not always significantly so. While it is likely that the $\alpha$-stable distribution can be applied to all risk assessments it is an important measure that provides a good measure of VaR at conventional levels and perhaps conservative estimates at extreme levels. Given the likely effects of losses at these extreme levels this is probably not a bad idea.

For the Risk manager or the supervisor I have shown that accurate measures of VaR can be obtained using an $\alpha$-stable distribution. Theoretical justification can be provided by the generalised central limit theorem and the time aggregation and domain of attraction properties which define, and are unique to, this distribution. Figure 4 compares the static and dynamic (GARCH) $\alpha$-stable 1% and 5% VaR.

The volatility of the dynamic VaR may give rise to problems. Daníelsson et al. (2001) have asked if the adoption of dynamic VaR systems of risk management lead to constrains on the financial system during times of liquidity shortage. Masschelein (2007) has argued that, up to recent times, regulatory VaR requirements have not been binding. It can be argued that if regulatory requirements had been more severe

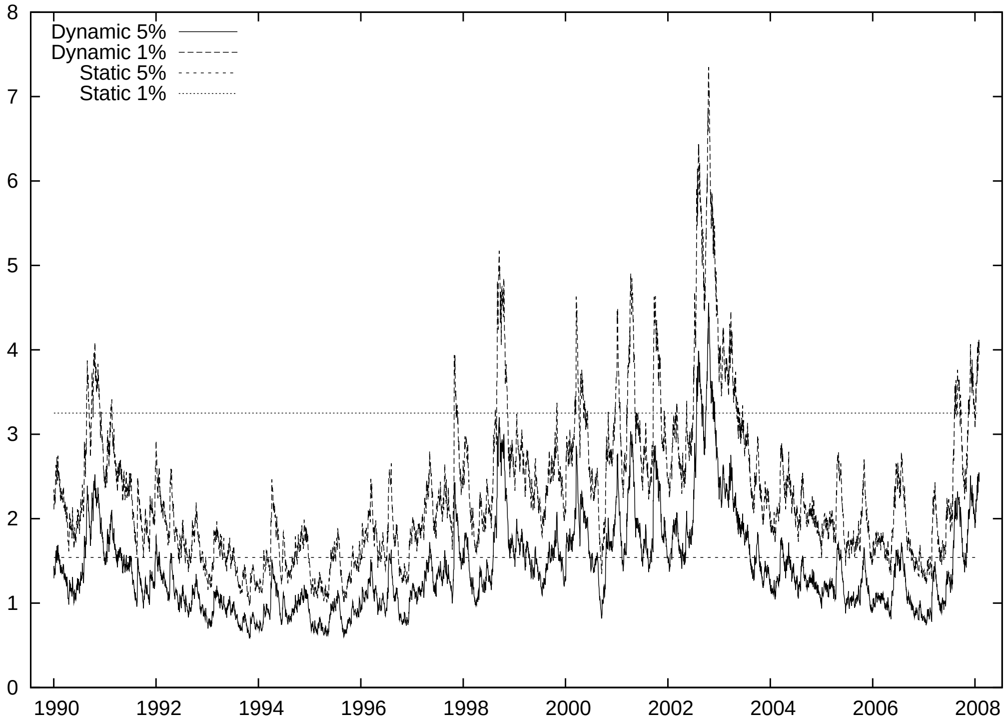


Figure 4: 5% and 1% static and Dynamic VaR of Losses on S&P 500

Table 12: Estimates of Stable parameters ISEQ Returns

| Coefficient | Estimate | $\frac{1}{2}$–confidence band |
|---|---|---|
| $\alpha$ | 1.6503 | (0.0427) |
| $\beta$ | 0.1026 | (0.1069) |
| $\gamma$ | 0.5395 | (0.0147) |
| $\delta$ | -0.0476 | (0.0263) |
| Parametrization | 1 | |
| Goodness of fit | 26.517 | $\chi^2(10)$ |
| Kolmogogorov-Smirnov | 0.0134 | 0.3259 |
| Stable Loglikelihood | -7149 | |
| Normal Loglikelihood | -6733 | |
| Likelihood Ratio Test | 832 | |
| Basic Statistics | | |
| mean | -0.0483 | |
| Standard Deviation | 1.0004 | |
| Skewness | 0.3883 | |
| Excess Kurtosis | 5.0771 | |

in less volatile times we may not have encountered the severe liquidity crisis that exists today. The use of the kind of static $\alpha$-stable VaR estimates provided here might form a useful basis for deciding appropriate levels for such an arrangement.

I have also shown that $\alpha$-stable estimates of VaR are feasible. They are a valuable and more accurate measure of VaR and would provide additional information to a risk manager. They are, of course only one aspect of risk management.

# A Appendix

## A.1 Maximum Liklihood estimates of $\alpha$-stable parameters

Tables 12 to 17 give results of maximum likelihood estimates of the $\alpha$-stable parameters of the distribution of losses on total returns indices for the ISEQ, CAC40, DAX30, FTSE100, DJAC and S&P500. Estimation is by maximum likelihood computed in C++ using the stable library functions of Nolan (2005).

Table 13: Estimates of Stable parameters CAC40 Returns

| Coefficient | Estimate | $\frac{1}{2}$–confidence band |
|---|---|---|
| $\alpha$ | 1.644 | (0.040) |
| $\beta$ | 0.170 | (0.132) |
| $\gamma$ | 0.761 | (0.020) |
| $\delta$ | -0.033 | (0.037) |
| Parametrization | 1 | |
| Goodness of fit | 26.34 | $\chi^2(10)$ |
| Kolmogogorov-Smirnov | 0.012 | 0.490 |
| Stable Loglikelihood | -8348.74 | |
| Normal Loglikelihood | -8131.47 | |
| Likelihood Ratio Test | 434.54 | |
| Basic Statistics | | |
| mean | -0.045 | |
| Standard Deviation | 1.281 | |
| Skewness | 0.150 | |
| Excess Kurtosis | 2.965 | |

Table 14: Estimates of Stable parameters DAX30 Returns

| Coefficient | Estimate | $\frac{1}{2}$–confidence band |
|---|---|---|
| $\alpha$ | 1.7100 | (0.0269) |
| $\beta$ | 0.0916 | (0.0785) |
| $\gamma$ | 0.6568 | (0.0112) |
| $\delta$ | -0.0215 | (0.0207) |
| Parametrization | 1 | |
| Goodness of fit | 19.8407 | $\chi^2(10)$ |
| Kolmogogorov-Smirnov | 0.0077 | 0.4803 |
| Stable Loglikelihood | -18031 | |
| Normal Loglikelihood | -19051 | |
| Likelihood Ratio Test | 2039 | |
| Basic Statistics | | |
| mean | -0.0254 | |
| Standard Deviation | 1.1791 | |
| Skewness | 0.2364 | |
| Excess Kurtosis | 7.8457 | |

Table 15: Estimates of Stable parameters FTSE100 Returns

| Coefficient | Estimate | $\frac{1}{2}$–confidence band |
|---|---|---|
| $\alpha$ | 1.743 | (0.038) |
| $\beta$ | 0.017 | (0.126) |
| $\gamma$ | 0.600 | (0.015) |
| $\delta$ | -0.037 | (0.027) |
| Parametrization | 1 | |
| Goodness of fit | 7.30765 | $\chi^2(10)$ |
| Kolmogogorov-Smirnov | 0.007 | 0.937 |
| Stable Loglikelihood | -8061.9 | |
| Normal Loglikelihood | -7676.9 | |
| Likelihood Ratio Test | 768.0 | |
| Basic Statistics | | |
| mean | -0.041 | |
| Standard Deviation | 1.040 | |
| Skewness | 0.700 | |
| Excess Kurtosis | 8.794 | |

Table 16: Estimates of Stable parameters DJAC Returns

| Coefficient | Estimate | $\frac{1}{2}$–confidence band |
|---|---|---|
| $\alpha$ | 1.697 | (0.042) |
| $\beta$ | 0.122 | (0.116) |
| $\gamma$ | 0.541 | (0.014) |
| $\delta$ | -0.039 | (0.026) |
| Parametrization | 1 | |
| Goodness of fit | 37.06 | $\chi^2(10)$ |
| Kolmogogorov-Smirnov | 0.017 | 0.112 |
| Stable Loglikelihood | -7306.64 | |
| Normal Loglikelihood | -6692.84 | |
| Likelihood Ratio Test | 1227.6 | |
| Basic Statistics | | |
| mean | -0.040 | |
| Standard Deviation | 1.011 | |
| Skewness | 2.449 | |
| Excess Kurtosis | 52.6741 | |

Table 17: Estimates of Stable parameters S&P500 Returns

| Coefficient | Estimate | $\frac{1}{2}$–confidence band |
|---|---|---|
| $\alpha$ | 1.676 | (0.045) |
| $\beta$ | 0.167 | (0.117) |
| $\gamma$ | 0.547 | (0.016) |
| $\delta$ | -0.030 | (0.028) |
| Parametrization | 1 | |
| Goodness of fit | 62.46 | $\chi^2(10)$ |
| Kolmogogorov-Smirnov | 0.021 | 0.021 |
| Stable Loglikelihood | -6227.4 | |
| Normal Loglikelihood | -5991.13 | |
| Likelihood Ratio Test | 472.55 | |
| Basic Statistics | | |
| mean | -0.045 | |
| Standard Deviation | 0.962 | |
| Skewness | 0.160 | |
| Excess Kurtosis | 3.637 | |

## A.2 GARCH estimates

Tables 18 to 29 give results of maximum likelihood estimates of various GARCH models of the distribution of losses on total returns indices for the ISEQ, CAC40, DAX30, FTSE100, DJAC and S&P500. I estimate ARMA(p,q)-GARCH(1,1) models for $(p, q) \in (0, 0), (1, 0), (2, 0), (1, 1)$. Although there are some problems of autocorrelation in the more parsimonious models, the number of exceedances appears to be robust with respect to the choice of ARMA components and the analysis os based on a constant mean. Specification tests in bold case are not statistically significant. Estimation testing etc. was completed using R (R Development Core Team (2007)) and the Rmetrics library (Wuertz et al. (2007)).

Table 18: Estimated ARMA(p,q) GARCH(1,1) Models with Normal innovations (CAC40)

| | ARMA model | | | |
|---|---|---|---|---|
| p | 0 | 1 | 2 | 1 |
| q | 0 | 0 | 0 | 1 |
| $\mu$ | -0.066 | -0.066 | -0.066 | -0.091 |
| | (0.015) | (0.015) | (0.015) | (0.025) |
| $\phi_1$ | | 0.179 | 0.019 | 0.363 |
| | | (0.015) | (0.015) | (0.239) |
| $\phi_2$ | | | 0.019 | |
| | | | (0.015) | |
| $\theta_1$ | | | | 0.380 |
| | | | | (0.232) |
| $\omega$ | 0.032 | 0.032 | 0.032 | 0.032 |
| | (0.006) | (0.006) | (0.006) | (0.006) |
| $\alpha_1$ | 0.086 | 0.086 | 0.086 | 0.086 |
| | (0.009) | (0.009) | (0.009) | (0.009) |
| $\beta_1$ | 0.895 | 0.895 | 0.895 | 0.895 |
| | (0.011) | (0.011) | 0.011 | (0.011) |
| Standardised Residual tests | | | | |
| J-B test | 1090.50 | 1098.59 | 1104.99 | 1093.58 |
| Residual Q10 | **17.66** | **15.70** | **15.46** | **16.16** |
| Residual Q15 | **21.08** | **19.01** | **18.84** | **19.47** |
| Residual Q20 | **24.33** | **22.52** | **22.37** | **23.04** |
| Residual ARCH tests | | | | |
| ARCH Q10 | **13.94** | **13.92** | **13.77** | **13.92** |
| ARCH Q15 | **17.02** | **17.01** | **16.85** | **17.01** |
| ARCH Q20 | **18.54** | **18.50** | **18.37** | **18.53** |
| Information Criterion Tests | | | | |
| AIC | -3.115 | -3.114 | -3.112 | -3.113 |
| BIC | -3.110 | -3.108 | -3.105 | -3.106 |
| 5% critical points for $\chi^2$ distribution with 2 ,10, and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively | | | | |

Table 19: Estimated ARMA(p,q) GARCH(1,1) Models with t innovations (CAC40)

| | ARMA model | | | |
|---|---|---|---|---|
| p | 0 | 1 | 2 | 1 |
| q | 0 | 0 | 0 | 1 |
| $\mu$ | -0.074 | -0.063 | -0.061 | -0.051 |
| | (0.011) | (0.010) | (0.010) | (0.011) |
| $\phi_1$ | | 0.144 | 0.140 | 0.292 |
| | | (0.014) | (0.015) | (0.091) |
| $\phi_2$ | | | 0.015 | |
| | | | (0.014) | |
| $\theta_1$ | | | | -0.153 |
| | | | | (0.094) |
| $\omega$ | 0.022 | 0.021 | 0.020 | 0.021 |
| | (0.006) | (0.006) | (0.006) | (0.006) |
| $\alpha_1$ | 0.095 | 0.097 | 0.096 | 0.097 |
| | (0.016) | (0.016) | (0.016) | (0.016) |
| $\beta_1$ | 0.886 | 0.885 | 0.886 | 0.885 |
| | (0.019) | (0.019) | (0.019) | (0.019) |
| $\nu$ | 5.236 | 5.258 | 5.245 | 5.262 |
| | (0.373) | (0.372) | (0.372) | (0.374) |
| Standardised Residual tests | | | | |
| Residual Q10 | 120.66 | 22.51 | 19.15 | **17.57** |
| Residual Q15 | 128.44 | 28.66 | 25.15 | **23.44** |
| Residual Q20 | 134.82 | 33.24 | **29.97** | **28.16** |
| Residual ARCH tests | | | | |
| ARCH Q10 | **3.08** | **3.74** | **3.76** | **3.74** |
| ARCH Q15 | **5.82** | **6.53** | **6.55** | **6.53** |
| ARCH Q20 | **7.34** | **7.93** | **7.93** | **7.92** |
| Information Criterion Tests | | | | |
| AIC | -2.521 | -2.499 | -2.498 | -2.499 |
| BIC | -2.514 | -2.492 | -2.489 | =2.489 |
| 5% critical points for $\chi^2$ distribution with 2 ,10, and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively | | | | |

Table 20: Estimated ARMA(p,q) GARCH(1,1) Models with Normal innovations (DAX 30)

| | ARMA model | | | |
|---|---|---|---|---|
| p | 0 | 1 | 2 | 1 |
| q | 0 | 0 | 0 | 1 |
| $\mu$ | -0.036 | -0.032 | -0.034 | -0.044 |
| | (0.008) | (0.008) | (0.008) | (0.011) |
| $\phi_1$ | | 0.099 | 0.104 | 0.241 |
| | | (0.010) | (0.010) | (0.072) |
| $\phi_2$ | | | -0.051 | |
| | | | (0.010) | |
| $\theta_1$ | | | | 0.347 |
| | | | | (0.069) |
| $\omega$ | 0.031 | 0.030 | 0.030 | 0.030 |
| | (0.003) | (0.003) | (0.003) | (0.003) |
| $\alpha_1$ | 0.130 | 0.131 | 0.132 | 0.132 |
| | (0.008) | (0.008) | (0.008) | (0.007) |
| $\beta_1$ | 0.851 | 0.850 | 0.850 | 0.850 |
| | (0.009) | (0.008) | 0.008 | (0.008) |
| Standardised Residual tests | | | | |
| J-B test | 24402 | 22917 | 21696 | 17038 |
| Residual Q10 | 126.84 | 27.28 | 24.36 | 20.53 |
| Residual Q15 | 133.29 | 32.64 | 30.09 | 25.95 |
| Residual Q20 | 43.87 | 41.36 | 39.61 | 34.95 |
| Residual ARCH tests | | | | |
| ARCH Q10 | **4.65** | **4.38** | **4.44** | **4.50** |
| ARCH Q15 | **6.37** | **6.47** | **6.58** | **6.66** |
| ARCH Q20 | **7.67** | **7.97** | **8.15** | **8.24** |
| Information Criterion Tests | | | | |
| AIC | -2.842 | -2.834 | -2.814 | -2.832 |
| BIC | -2.840 | -2.831 | -2.827 | -2.828 |
| 5% critical points for $\chi^2$ distribution with 2 ,10, and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively | | | | |

Table 21: Estimated ARMA(p,q) GARCH(1,1) Models with t innovations (DAX 30)

| | ARMA model | | | |
|---|---|---|---|---|
| p | 0 | 1 | 2 | 1 |
| q | 0 | 0 | 0 | 1 |
| $\mu$ | -0.041 | -0.037 | -0.039 | -0.050 |
| | (0.008) | (0.008) | (0.008) | (0.011) |
| $\phi_1$ | | 0.093 | 0.098 | -0.220 |
| | | (0.009) | (0.010) | (0.070) |
| $\phi_2$ | | | -0.048 | |
| | | | (0.009) | |
| $\theta_1$ | | | | 0.320 |
| | | | | (0.068) |
| $\omega$ | 0.022 | 0.022 | 0.021 | 0.022 |
| | (0.003) | (0.003) | (0.003) | (0.003) |
| $\alpha_1$ | 0.109 | 0.112 | 0.111 | 0.111 |
| | (0.008) | (0.008) | (0.008) | (0.008) |
| $\beta_1$ | 0.876 | 0.873 | 0.874 | 0.873 |
| | (0.008) | (0.008) | (0.008) | (0.008) |
| $\nu$ | 10.814 | 10.875 | 10.773 | 10.804 |
| | (0.826) | (0.831) | (0.821) | (0.823) |
| Standardised Residual tests | | | | |
| Residual Q10 | 127.78 | 28.74 | 24.24 | 20.48 |
| Residual Q15 | 134.36 | 34.26 | 30.15 | 26.08 |
| Residual Q20 | 144.88 | 43.00 | 39.60 | 35.03 |
| Residual ARCH tests | | | | |
| ARCH Q10 | **4.89** | **4.90** | **5.05** | **5.08** |
| ARCH Q15 | **7.04** | **7.41** | **7.61** | **7.64** |
| ARCH Q20 | **9.13** | **9.67** | **9.98** | **9.99** |
| Information Criterion Tests | | | | |
| AIC | -2.800 | -2.792 | -2.790 | -2.790 |
| BIC | -2.797 | -2.788 | -2.785 | =2.786 |
| 5% critical points for $\chi^2$ distribution with 2 ,10, and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively | | | | |

Table 22: Estimated ARMA(p,q) GARCH(1,1) Models with Normal innovations (FTSE100)

| | ARMA model | | | |
|---|---|---|---|---|
| p | 0 | 1 | 2 | 1 |
| q | 0 | 0 | 0 | 1 |
| $\mu$ | -0.065 | -0.064 | -0.064 | -0.064 |
| | (0.011) | (0.011) | (0.011) | (0.022) |
| $\phi_1$ | | 0.024 | 0.024 | 0.019 |
| | | (0.014) | (0.014) | (0.302) |
| $\phi_2$ | | | -.002 | |
| | | | (0.014) | |
| $\theta_1$ | | | | -0.004 |
| | | | | (0.303) |
| $\omega$ | 0.018 | 0.018 | 0.018 | 0.018 |
| | (0.004) | (0.004) | (0.004) | (0.004) |
| $\alpha_1$ | 0.091 | 0.091 | 0.091 | 0.091 |
| | (0.009) | (0.009) | (0.009) | (0.009) |
| $\beta_1$ | 0.893 | 0.893 | 0.893 | 0.893 |
| | (0.011) | (0.011) | 0.011 | (0.011) |
| Standardised Residual tests | | | | |
| J-B test | 10791 | 10856 | 10885 | 10857 |
| Residual Q10 | **17.05** | **10.09** | **10.01** | **10.08** |
| Residual Q15 | **23.38** | **16.10** | **16.04** | **16.09** |
| Residual Q20 | **28.31** | **20.95** | **20.96** | **20.95** |
| Residual ARCH tests | | | | |
| ARCH Q10 | **8.79** | **8.10** | **8.09** | **8.10** |
| ARCH Q15 | **11.49** | **10.80** | **10.80** | **10.80** |
| ARCH Q20 | **15.83** | **15.21** | **15.20** | **15.21** |
| Information Criterion Tests | | | | |
| AIC | -2.644 | -2.643 | -2.642 | -2.642 |
| BIC | -2.639 | -2.637 | -2.635 | -2.635 |
| 5% critical points for $\chi^2$ distribution with 2 ,10, and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively | | | | |

Table 23: Estimated ARMA(p,q) GARCH(1,1) Models with t innovations (FTSE100)

| | ARMA model | | | |
|---|---|---|---|---|
| p | 0 | 1 | 2 | 1 |
| q | 0 | 0 | 0 | 1 |
| $\mu$ | -0.068 | -0.067 | -0.068 | -0.079 |
| | (0.014) | (0.011) | (0.011) | (0.025) |
| $\phi_1$ | | 0.021 | 0.022 | -0.155 |
| | | (0.014) | (0.014) | (0.313) |
| $\phi_2$ | | | -0.017 | |
| | | | (0.014) | |
| $\theta_1$ | | | | 0.178 |
| | | | | (0.312) |
| $\omega$ | 0.014 | 0.014 | 0.014 | 0.014 |
| | (0.003) | (0.003) | (0.003) | (0.003) |
| $\alpha_1$ | 0.080 | 0.078 | 0.079 | 0.080 |
| | (0.009) | (0.008) | (0.008) | (0.008) |
| $\beta_1$ | 0.906 | 0.906 | 0.907 | 0.906 |
| | (0.009) | (0.010) | (0.010) | (0.009) |
| $\nu$ | 12.397 | 12.432 | 12.272 | 12.404 |
| | (1.549) | (1.558) | (1.532) | (1.553) |
| Standardised Residual tests | | | | |
| Residual Q10 | **17.04** | **10.53** | **11.277** | **10.26** |
| Residual Q15 | **23.36** | **16.57** | **17.39** | **16.28** |
| Residual Q20 | **28.36** | **21.49** | **22.51** | **21.24** |
| Residual ARCH tests | | | | |
| ARCH Q10 | **12.30** | **11.37** | **11.33** | **11.29** |
| ARCH Q15 | **14.99** | **14.06** | **14.02** | **13.98** |
| ARCH Q20 | **19.14** | **18.28** | **18.16** | **18.18** |
| Information Criterion Tests | | | | |
| AIC | -2.608 | -2.607 | -2.606 | -2.606 |
| BIC | -2.602 | -2.600 | -2.598 | =2.2.598 |
| 5% critical points for $\chi^2$ distribution with 2 ,10, and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively | | | | |

Table 24: Estimated ARMA(p,q) GARCH(1,1) Models with Normal innovations (ISEQ)

| | ARMA model | | | |
|---|---|---|---|---|
| p | 0 | 1 | 2 | 1 |
| q | 0 | 0 | 0 | 1 |
| $\mu$ | -0.075 | -0.062 | -0.061 | -0.048 |
| | (0.012) | (0.012) | (0.012) | (0.012) |
| $\phi_1$ | | 0.150 | 0.146 | 0.341 |
| | | (0.016) | (0.016) | (0.099) |
| $\phi_2$ | | | 0.018 | |
| | | | (0.016) | |
| $\theta_1$ | | | | -0.197 |
| | | | | (0.104) |
| $\omega$ | 0.034 | 0.033 | 0.033 | 0.033 |
| | (0.012) | (0.006) | (0.006) | (0.006) |
| $\alpha_1$ | 0.090 | 0.089 | 0.089 | 0.089 |
| | (0.011) | (0.011) | (0.011) | (0.011) |
| $\beta_1$ | 0.877 | 0.877 | 0.876 | 0.877 |
| | (0.016) | (0.015) | 0.015 | (0.015) |
| Standardised Residual tests | | | | |
| J-B test | 14401.46 | 15342.95 | 15272.99 | 15182.22 |
| Residual Q10 | 119.87 | 21.10 | **17.56** | **15.47** |
| Residual Q15 | 127.60 | 27.28 | **23.58** | **21.27** |
| Residual Q20 | 134.68 | 32.47 | **29.06** | **27.72** |
| Residual ARCH tests | | | | |
| ARCH Q10 | **2.55** | **3.69** | **3.80** | **3.63** |
| ARCH Q15 | **4.58** | **5.77** | **5.93** | **5.73** |
| ARCH Q20 | **5.57** | **6.63** | **6.70** | **6.59** |
| Information Criterion Tests | | | | |
| AIC | -2.633 | -2.613 | -2.612 | -2.612 |
| BIC | -2.628 | -2.606 | -2.604 | -2.604 |
| 5% critical points for $\chi^2$ distribution with 2 ,10, and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively | | | | |

Table 25: Estimated ARMA(p,q) GARCH(1,1) Models with t innovations (ISEQ)

| | ARMA model | | | |
|---|---|---|---|---|
| p | 0 | 1 | 2 | 1 |
| q | 0 | 0 | 0 | 1 |
| $\mu$ | -0.074 | -0.063 | -0.061 | 0.051 |
| | (0.011) | (0.010) | (0.010) | (0.011) |
| $\phi_1$ | | 0.144 | 0.140 | 0.292 |
| | | (0.014) | (0.015) | (0.091) |
| $\phi_2$ | | | 0.015 | |
| | | | (0.014) | |
| $\theta_1$ | | | | -0.153 |
| | | | | (0.094) |
| $\omega$ | 0.022 | 0.021 | 0.020 | 0.021 |
| | (0.006) | (0.006) | (0.006) | (0.006) |
| $\alpha_1$ | 0.095 | 0.097 | 0.096 | 0.097 |
| | (0.016) | (0.016) | (0.016) | (0.016) |
| $\beta_1$ | 0.886 | 0.885 | 0.886 | 0.885 |
| | (0.019) | (0.019) | (0.019) | (0.019) |
| $\nu$ | 5.236 | 5.258 | 5.245 | 5.262 |
| | (0.373) | (0.372) | (0.372) | (0.374) |
| Standardised Residual tests | | | | |
| Residual Q10 | 120.66 | 22.51 | 19.15 | **17.57** |
| Residual Q15 | 128.44 | 28.66 | 25.15 | **23.44** |
| Residual Q20 | 134.82 | 33.24 | **29.97** | **28.16** |
| Residual ARCH tests | | | | |
| ARCH Q10 | **3.08** | **3.74** | **3.76** | **3.74** |
| ARCH Q15 | **5.82** | **6.53** | **6.55** | **6.53** |
| ARCH Q20 | **7.34** | **7.93** | **7.93** | **7.92** |
| Information Criterion Tests | | | | |
| AIC | -2.521 | -2.499 | -2.498 | -2.499 |
| BIC | -2.514 | -2.492 | -2.489 | =2.489 |
| 5% critical points for $\chi^2$ distribution with 2 ,10, and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively | | | | |

Table 26: Estimated ARMA(p,q) GARCH(1,1) Models with Normal innovations (S&P500)

| | ARMA model | | | |
|---|---|---|---|---|
| p | 0 | 1 | 2 | 1 |
| q | 0 | 0 | 0 | 1 |
| $\mu$ | -0.063 | -0.059 | -0.059 | -0.062 |
| | (0.011) | (0.011) | (0.011) | (0.018) |
| $\phi_1$ | | 0.059 | 0.059 | 0.009 |
| | | (0.016) | (0.016) | (0.227) |
| $\phi_2$ | | | -0.001 | |
| | | | (0.016) | |
| $\theta_1$ | | | | -0.049 |
| | | | | (0.227) |
| $\omega$ | 0.009 | 0.009 | 0.009 | 0.009 |
| | (0.002) | (0.002) | (0.002) | (0.002) |
| $\alpha_1$ | 0.066 | 0.066 | 0.066 | 0.066 |
| | (0.008) | (0.008) | (0.008) | (0.008) |
| $\beta_1$ | 0.925 | 0.926 | 0.926 | 0.926 |
| | (0.009) | (0.008) | (0.009) | (0.009) |
| Standardised Residual tests | | | | |
| J-B test | 1129 | 1139 | 1144 | 1 1148 |
| Residual Q10 | 31.02 | **10.24** | **10.25** | **10.19** |
| Residual Q15 | 43.71 | **21.86** | **21.79** | **21.79** |
| Residual Q20 | 44.43 | **22.61** | **22.54** | **22.54** |
| Residual ARCH tests | | | | |
| ARCH Q10 | **5.01** | **4.85** | **4.85** | **4.85** |
| ARCH Q15 | **7.28** | **7.12** | **7.12** | **7.11** |
| ARCH Q20 | **8.91** | **8.49** | **8.49** | **8.48** |
| Information Criterion Tests | | | | |
| AIC | -2.525 | -2.521 | -2.520 | -2.520 |
| BIC | -2.519 | -2.514 | -2.512 | -2.512 |
| 5% critical points for $\chi^2$ distribution with 2 ,10, and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively | | | | |

Table 27: Estimated ARMA(p,q) GARCH(1,1) Models with t innovations (S&P500)

| | ARMA model | | | |
|---|---|---|---|---|
| p | 0 | 1 | 2 | 1 |
| q | 0 | 0 | 0 | 1 |
| $\mu$ | -0.075 | -0.071 | -0.073 | -0.090 |
| | (0.011) | (0.011) | (0.011) | (0.020) |
| $\phi_1$ | | 0.046 | 0.047 | -0.201 |
| | | (0.015) | (0.015) | (0.188) |
| $\phi_2$ | | | -0.025 | |
| | | | (0.015) | |
| $\theta_1$ | | | | 0.250 |
| | | | | (0.187) |
| $\omega$ | 0.005 | 0.005 | 0.005 | 0.005 |
| | (0.002) | (0.002) | (0.002) | (0.002) |
| $\alpha_1$ | 0.059 | 0.060 | 0.060 | 0.060 |
| | (0.008) | (0.008) | (0.008) | (0.008) |
| $\beta_1$ | 0.937 | 0.936 | 0.936 | 0.936 |
| | (0.009) | (0.009) | (0.009) | (0.009) |
| $\nu$ | 7.39 | 7.537 | 7.438 | 7.507 |
| | (0.769) | (0.795) | (0.779) | (0.789) |
| Standardised Residual tests | | | | |
| Residual Q10 | 31.85 | **13.03** | **14.42** | **12.79** |
| Residual Q15 | 44.57 | **24.88** | 26.14 | **24.52** |
| Residual Q20 | 45.31 | **25.62** | **26.90** | **25.25** |
| Residual ARCH tests | | | | |
| ARCH Q10 | **5.93** | **5.59** | **5.58** | **5.60** |
| ARCH Q15 | **8.26** | **8.01** | **7.90** | **7.96** |
| ARCH Q20 | **10.32** | **9.86** | **9.71** | **9.79** |
| Information Criterion Tests | | | | |
| AIC | -2.484 | -2.480 | -2.479 | -2.479 |
| BIC | -2.476 | -2.472 | -2.469 | =2.469 |
| 5% critical points for $\chi^2$ distribution with 2 ,10, and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively | | | | |

Table 28: Estimated ARMA(p,q) GARCH(1,1) Models with Normal innovations (Dow Jones Composite)

| | ARMA model | | | |
|---|---|---|---|---|
| p | 0 | 1 | 2 | 1 |
| q | 0 | 0 | 0 | 1 |
| $\mu$ | -0.068 | -0.066 | -0.068 | -0.107 |
| | (0.011) | (0.011) | (0.011) | (0.021) |
| $\phi_1$ | | 0.026 | 0.026 | -0.571 |
| | | (0.015) | (0.015) | (0.144) |
| $\phi_2$ | | | -0.020 | |
| | | | (0.015) | |
| $\theta_1$ | | | | 0.603 |
| | | | | (0.141) |
| $\omega$ | 0.022 | 0.023 | 0.023 | 0.022 |
| | (0.003) | (0.004) | (0.004) | (0.004) |
| $\alpha_1$ | 0.091 | 0.092 | 0.091 | 0.092 |
| | (0.008) | (0.008) | (0.007) | (0.008) |
| $\beta_1$ | 0.889 | 0.889 | 0.889 | 0.889 |
| | (0.010) | (0.010) | (0.010) | (0.010) |
| Standardised Residual tests | | | | |
| J-B test | 11353 | 11939 | 11887 | 11831 |
| Residual Q10 | 24.41 | **14.86** | **15.19** | **13.35** |
| Residual Q15 | 32.11 | **22.91** | **23.21** | **21.32** |
| Residual Q20 | 37.65 | **28.47** | **28.78** | **26.82** |
| Residual ARCH tests | | | | |
| ARCH Q10 | **2.08** | **2.70** | **2.74** | **1.97** |
| ARCH Q15 | **4.08** | **5.29** | **5.35** | **4.31** |
| ARCH Q20 | **5.76** | **6.84** | **6.90** | **5.87** |
| Information Criterion Tests | | | | |
| AIC | -2.584 | -2.584 | -2.583 | -2.582 |
| BIC | -2.579 | -2.577 | -2.575 | -2.575 |
| 5% critical points for $\chi^2$ distribution with 2 ,10, and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively | | | | |

Table 29: Estimated ARMA(p,q) GARCH(1,1) Models with t innovations (Dow Jones Composite)

| | ARMA model | | | |
|---|---|---|---|---|
| p | 0 | 1 | 2 | 1 |
| q | 0 | 0 | 0 | 1 |
| $\mu$ | -0.071 | -0.069 | -0.072 | -0.107 |
| | (0.010) | (0.010) | (0.010) | (0.024) |
| $\phi_1$ | | 0.019 | 0.020 | -0.513 |
| | | (0.014) | (0.014) | (0.250) |
| $\phi_2$ | | | -0.035 | |
| | | | (0.014) | |
| $\theta_1$ | | | | 0.541 |
| | | | | (0.094) |
| $\omega$ | 0.014 | 0.014 | 0.014 | 0.014 |
| | (0.003) | (0.003) | (0.003) | (0.003) |
| $\alpha_1$ | 0.059 | 0.059 | 0.059 | 0.059 |
| | (0.007) | (0.008) | (0.008) | (0.008) |
| $\beta_1$ | 0.925 | 0.925 | 0.926 | 0.925 |
| | (0.009) | (0.009) | (0.009) | (0.009) |
| $\nu$ | 6.172 | 6.191 | 6.112 | 6.211 |
| | (0.500) | (0.503) | (0.494) | (0.504) |
| Standardised Residual tests | | | | |
| Residual Q10 | 24.22 | **16.42** | 19.03 | **13.47** |
| Residual Q15 | 32.15 | **24.89** | 27.45 | **21.82** |
| Residual Q20 | 37.56 | **30.28** | 32.88 | **27.19** |
| Residual ARCH tests | | | | |
| ARCH Q10 | **5.77** | **6.45** | **6.77** | **5.99** |
| ARCH Q15 | **7.12** | **8.25** | **8.58** | **7.61** |
| ARCH Q20 | **8.46** | **9.51** | **9.83** | **8.85** |
| Information Criterion Tests | | | | |
| AIC | -2.496 | -2.499 | -2.493 | -2.494 |
| BIC | -2.489 | -2.492 | -2.484 | =2.485 |
| 5% critical points for $\chi^2$ distribution with 2 ,10, and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively | | | | |

## A.3 $\alpha$-stable GARCH Estimates and VaR

The usual GARCH(p,q) model takes the form

$$\varepsilon_t = z_t \sigma_t$$

where $z_t$ is an *iid* process with zero mean and unit variance. The conditional variance of this process is $\sigma_t^2$. $\sigma_t^2$ is taken to follow various stochastic processes. The GARCH process is defined as the following process.

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2$$

In the GARCH estimates above $z_t$ was taken to follow either a normal or a t-distribution. The residuals in both the normal and t-distributions for $z_t$ showed considerable excess kurtosis.

It would be attractive to model the $z_t$ with an $\alpha$-stable distribution. The exact formulation can not be followed in the general case when $a < 2$ as the second moment of the distribution of $z_t$ does not exist. Following Panorska et al. (1995) or Rachev and Mittnik (2000) we say that x follows a stable GARCH($\alpha$,p,q) if $X_t$ is $\alpha$-stable with parameters $\alpha$, $\beta$, $\gamma = \gamma_t$ and $\delta$ where

$$\gamma_t = \omega + \sum_{i=1}^{q} \alpha_i |x_{t-i} - \delta| + \sum_{i=1}^{p} \beta_i \gamma_{t-i} \tag{3}$$

and $a_i,\ i = 1, \ldots, q$ and $b_j,\ i = 1, \ldots, p$ and $\omega > 0$. Panorska et al. (1995) establishes stationarity conditions for the process in equation (3). For the stable GARCH(1,1) process, estimated here, we require that $\beta_1 + \lambda\alpha_1 < 1$ where $\lambda$ is a function of $\alpha$ and, for example, $\lambda = 1.5091,\ 1.3709,$ and $1.2687$ for $\alpha = 1.6,\ 1.7,$ and $1.8$ respectively. All $\alpha$-stable processes estimates here satisfy these restrictions and may be taken to be stationary.

Parameters were estimated by maximum likelihood using C++ and the STABLE

Table 30: Estmated Parameters of $\alpha$-stable GARCH loss distributions

| | ISEQ | CAC40 | DAX30 | FTSE100 | DJC | S&P500 |
|---|---|---|---|---|---|---|
| $\alpha$ | 1.80 (0.020) | 1.95 (0.0084) | 1.94 (0.0085) | 1.95 (0.016) | 1.88 (.023) | 1.91 (0.0024) |
| $\beta$ | 0.175 (0.035) | 0.727 (0.0041) | 0.362 (0.0078) | 0.851 (0.165) | 0.438 (0.066) | 0.703 (0.0014) |
| $\delta$ | -0.0581 (0.047) | -0.0657 (0.00026) | -0.0315 (0.00070) | -0.0522 (0.0098) | -0.513 (0.022) | -0.550 (2.7e-5) |
| $\omega$ | 0.00984 (.00028) | 0.0104 (2.4e-05) | 0.0128 (0.00018) | .00862 (0.0054) | 0.00761 (0.00088) | 0.00463 (5.8e-6) |
| $\alpha_1$ | 0.0599 (0.0024) | 0.0570 (9.7e-05) | 0.0738 (0.00040) | 0.0584 (0.0085) | .0426 (0.00073) | 0.0471 (5.0e-5) |
| $\beta_1$ | 0.911 (0.0033) | 0.922 (0.0013) | 0.897 (0.0012) | 0.919 (0.173) | 0.937 (.0016) | 0.939 (0.00053) |

library functions of Nolan (2005). The optimisation [9] process was started using the Nelder-Mead minimisation algorithm and continued to completion using the BFGS algorithm. Standard errors of the estimates were derived from the inverse Hessian matrix calculated during the minimisation process.

9. maximisation was completed by minimizing the negative of the log likelihood

Table 31: Exceedances and percentage exceedances for $\alpha$-stable GARCH VaR estimates

| Index | | | VaR Level | | | | |
|---|---|---|---|---|---|---|---|
| | | Observations | 10.00% | 5.00% | 1.00% | 0.50% | 0.10% |
| ISEQ | Count | 5037 | 513 | 267 | 49 | 16 | 3 |
| | % | | 10.18 | 5.30 | 0.97 | 0.32 | 0.06 |
| CAC40 | Count | 5056 | 530 | 282 | 56 | 24 | 3 |
| | % | | 10.48 | 5.58 | 1.11 | 0.47 | 0.06 |
| DAX30 | Count | 12098 | 1232 | 634 | 137 | 62 | 6 |
| | % | | 10.18 | 5.24 | 1.13 | 0.51 | 0.05 |
| DJC | Count | 5156 | 547 | 268 | 56 | 24 | 3 |
| | % | | 10.61 | 5.20 | 1.09 | 0.47 | 0.06 |
| FTSE100 | Count | 5575 | 582 | 304 | 61 | 31 | 2 |
| | % | | 10.44 | 5.45 | 1.09 | 0.56 | 0.04 |
| S&P500 | Count | 4557 | 497 | 251 | 51 | 18 | 2 |
| | % | | 10.91 | 5.51 | 1.12 | 0.39 | 0.04 |
| All | Count | 37479 | 3901 | 2006 | 410 | 175 | 19 |
| | % | | 10.41 | 5.35 | 1.09 | 0.47 | 0.05 |

# B Data and Software

## B.1 Data

The total returned indices used in this analysis were downloaded from the Reuters EcoWin database. The series used were

- France, Paris SE, CAC 40 Index, Total Return, Close, EUR, (ew:fra15660).
- Germany, Deutsche Boerse, DAX 30, Index, Total Return, Close, EUR, (ew:deu15500).
- United States, Dow Jones, Averages, Composite Index, Total Return, Close, USD, (ew:usa15575200).
- United Kingdom, FTSE, 100, Index, Total Return, Close, GBP, (ew:gbr15500200).
- United States, Standard & Poors, 500 Composite, Equal Weighted Index, Total Return, Close, USD, (ew:usa15508200).
- Ireland, Irish SE, ISEQ Index, Total Return, Close, EUR, ew:irl15550.

## B.2 Software

The parameters of $\alpha$-stable distributions were estimated by Maximum Likelihood using C++ and the Dynamic link Libraries of Nolan (2005). Other statistical analysis was completed in R (R Development Core Team (2007)) (using the Rmetrics (Wuertz et al. (2007)), QRMlib (McNeil and Ulman (2007)) and related R packages), Gretl (Cottrell and Lucchetti (2007)) and Mathematica (Wolfram (2003)).

# C Bibliography

## References

Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath (1999). Coherent measures of risk. *Mathematical Finance 9*(3), 203–228.

Bachelier, L. (1900). *Théorie de la Spéculation, théorie mathematique du jeu*. Ph. D. thesis, Ecole Normale Sup.

Cootner, P. H. (Ed.) (1964). *The Random Character of Stock Market Prices*. M.I.T. Press. Reprinted 2000 by Risk Publications.

Cottrell, A. and R. J. Lucchetti (2007). *Gnu Regression, Econometrics and Time Series Library*.

Crouhy, M., D. Galai, and R. Mark (2006). *The essentials of risk management*. McGraw-Hill.

Daníelsson, J., P. Embrechts, C. Goodhart, C. Keating, F. Muennich, O. Muennich, O. Renault, and H. S. Shin (2001). An academic response to basel II. LSE Financial Markets Group, An ESRC Research Centre, special Paper Series, May2001.

Daníelsson, J., B. N. Jorgrnsen, G. Samorodinitsky, M. sarma, and G. G. de Vries (2005). Subadditivity re-examined: The case for value at risk.

Dowd, K. (1998). *Beyond Value at Risk - The new Science of Risk Management*. Wiley.

Dowd, K. (2002). *Measuring Market Risk*. Wiley.

Fama, E. F. (1964). *Mandelbrot and the Stable paretian Hypothesis*, Chapter 14, Cootner (1964), pp. 297–306. M.I.T. Press.

Fama, E. F. (1965, Jan.). The bahavior of stock-market prices. *Journal of Business 38*(1), 34–105.

Fama, E. F. (1976). *Foundations of Finance*. Basic Books.

Feller, W. (1971). *An Introduction to Probability Theory and its Applications, Volume II* (Second Edition ed.). Wiley.

Frain, J. C. (2006). Total returns on equity indices, fat tails and the $\alpha$-stable distribution. http://www.tcd.ie/Economics/staff/frainj/main/Stable_distribution/index.htm.

Frain, J. C. and C. Meegan (1996). Market risk: An introduction to the concept and analytics of value-at-risk. Technical Report 7/RT/96, Economic Analysis, Research and publications Department, Central bank of ireland.

Harrell, F. and C. E. Davis (1982). A new distribution free quantile estimator. *Biometrica 69*, 635–640.

Janicki, A. and A. Weron (1994). *Simulation and ChaoticBehavior of $\alpha$-Stable Stochastic Processes*. Dekker.

Jorian, P. (2007). *Value at Risk* (Third ed.). McGraw-Hill.

Mandelbrot, B. B. (1962, March). The variation of certain speculative prices. Technical Report NC-87, IBM Research Report.

Mandelbrot, B. B. (1964). *The Variation of Certain Speculative Prices*, Chapter 15, Cootner (1964), pp. 307–332. M.I.T. Press.

Mandelbrot, B. B. (1967). The variation of the prices of cotton, wheat, and railroad stocks, and of some financial rates. *The Journal of Business 40*, 393–413.

Mandelbrot, B. B. (1997). *Fractals and Scaling in Finance*. Springer.

Mandelbrot, B. B. and R. L. Hudson (2004). *The (mis)Behaviour of Markets*. Profile Books.

Masschelein, N. (2007). Monitoring pro-cyclicality under the capital requirements directory: preliminary concepts for developing a framework. Working Paper Document no 120, National Bank of Belgium.

McNeil, A. and S. Ulman (2007). *QRMlib: Provides R-language code to examine Quantitative Risk Management concepts*. R package version 1.4.2, Alexander McNeil for S-Plus original; R port by Scott Ulman.

Nolan, J. P. (2005). User manual for the stable 3.14 library, library version. Robust Analysis Inc.

Nolan, J. P. (2007). *Stable Distributions - Models for Heavy Tailed Data*. Boston: Birkhäuser. In progress, Chapter 1 online at academic2.american.edu/∼jpnolan.

Panorska, A. K., S. Mittnik, and S. Rachev (1995). Stable garch models for financial time series. *Applied Mathematics Letters 8*(5), 33–37.

R Development Core Team (2007). *R: A Language and Environment for Statistical Computing*. Vienna, Austria: R Foundation for Statistical Computing. ISBN 3-900051-07-0.

Rachev, S. and S. Mittnik (2000). *Stable Paretian Models in Finance*. Wiley.

Riskmetrics Group (1999). Risk management: A practical guide.

Samorodnitsky, G. and M. S. Taqqu (1994). *Stable Non-Gaussian Random Processes Stochastic Models with Infinite Variance*. Chapman and Hall/CRC.

Taylor, S. J. (2005). *Asset Price Dynamics, Volatility and Prediction.* Princeton University Press.

Uchaikin, V. V. and V. M. Zolotarev (1999). *Chance and Stability*. VSP, Uthrecht.

Weitzman, M. L. (2007). Subjective expectations and asset-return puzzles. *American Economic Review 97*(4), 1102–1130.

Wolfram, S. (2003). *The Mathematica Book* (5th ed.). Wolfram Media.

Wuertz, D., many others, and see the SOURCE file (2007). *fSeries: Rmetrics - Financial Time Series Objects*. R package version 260.72.

Zolotarev, V. M. (1986). *One-dimensional Stable Distributions*. Translations of Mathematical Monographs, Volume 65, American Mathematical Society.

Zumbach, G. (2006). A gentle introduction to the rm 2006 methodology. Riskmetrics Group.