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ABSTRACT

We consider the valuation and optimal exercise policy of a δ-penalty minimum guaranteed payment option in the case where the value of the underlying dividend-paying asset follows a linear diffusion. We characterize both the value and optimal exercise policy of the considered game option explicitly and demonstrate that increased volatility increases the value of the option and postpones exercise by expanding the continuation region where exercising is suboptimal. An interesting and natural implication of this finding is that the value of the embedded cancellation rights of the issuer increase as volatility increases.

JEL Classification: G12, C73, C61

Keywords: minimum guaranteed payment, δ-penalty options, Dynkin games, linear diffusions

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1 Introduction

Many pension insurance contracts as well as life annuity contracts contain minimum guarantees either in the form of a minimum guaranteed rate of return, minimum guaranteed monetary payments, or both. It is well-known from the financial literature on derivative instruments that the value of such guarantees may be relatively high (in fair value terms) and that the value of these instruments increases as further optionalities, like exercise timing flexibility, are introduced. Moreover, given the considerable long maturity of the above mentioned class of insurance contracts, the fair values of these claims are typically very sensitive with respect to unexpected parametric changes in the stochastic dynamics characterizing the inter-temporal evolution of the underlying asset value. Thus, it is naturally of importance to study how embedded potentially costly rights of the issuer to terminate the contract prior expiry (i.e. cancelation rights) affect the value and exercise strategy of contingent contracts containing minimum guarantees. Especially, delineating those circumstances under which the issuer finds optimal to exercise a costly cancelation option is important since it provides valuable information on the maximal amount the issuer is prepared to pay from the right to terminate a contract prior expiry.

In light of our previous arguments our objective in this study is to consider by following the pioneering study by Kifer (2000) and the subsequent analysis by Kyprianou (2004) the valuation of a perpetual δ-penalty minimum guaranteed payment option in the case where the value of the underlying dividend paying asset follows a linear and time homogenous diffusion. The considered game option constitutes a δ-penalized version of the minimum guaranteed payment option originally analyzed in Guo and Shepp (2001). This option guarantees to the holder that whenever the option is exercised, the holder receives the maximum of the current asset value and a pre-determined guarantee. The δ-penalized version of this contingent contract has the extra feature that it offers to the issuer an embedded costly cancelation option which permits the issuer to exercise the option as well but only at a predetermined cost which has to be added into the exercise payoff of the holder. Consequently, even though the value of this contingent contract naturally dominates the exercise payoff of the perpetual minimum guaranteed payment option its value is always majorized by the sum of this payoff and the predetermined penalty. Moreover, since the issuer has the option to terminate the contract early as well, the valuation of this contract can be interpreted as the valuation of the saddle point strategy and value of an associated Dynkin game (for mathematical references, see Friedman (1973a,b), Bensoussan and Friedman (1974, 1977), Karatzas and Wang (2001), Fukushima and Taksar (2002), Touzi and Vieille (2002), Boetius (2005), Ekström (2006), Ekström and Villeneuve (2006), and Alvarez (2006)). Instead of relying on variational inequalities, we characterize the value as well as the optimal exercise policy explicitly by focusing on saddle point strategies which can be characterized as first exit times from open intervals belonging into the state space of the underlying diffusion. Having derived this representation, we investigate how the value can be found by choosing the boundaries so that the resulting value is extremal. In this way, the resulting pair of boundaries can be derived from a pair of ordinary first order conditions. We state a set of typically satisfied conditions under which a unique pair exists and characterize the value in terms of these boundaries. As intuitively is clear, two optimal regimes arise depending on the precise magnitude of the penalty (and, therefore, the cost of protection). If the penalty exceeds a volatility dependent critical size, then it is always suboptimal to the issuer to exercise the opportunity to terminate the contract early and, consequently, in that case both the value as well as the optimal exercise strategy coincide with their corresponding counterparts in the non-strategic setting. However, if the penalty is below the above mentioned critical penalty, then it becomes optimal to the issuer to exercise the cancelation right as soon as the underlying asset value coincides with the minimum guarantee. Given that the critical penalty is a monotonically increasing function of volatility, an interesting implication of our findings is that an unexpected increase in volatility may result into a switch from the regime where the value coincides with the nonstrategic one to the corner solution case where also the issuer finds optimal to exercise early.

The contents of this study are as follows. In section two we consider the considered class of financial derivative instruments and the underlying value of the dividend paying asset. In section three the value and optimal exercise strategy is characterized in the typical case where the underlying dynamics are characterized by a geometric Brownian motion. In section four we then extend our analysis to the general setting and state a set of general conditions under which the conclusions on the sensitivity of the optimal policy with respect to volatility changes obtained in the geometric Brownian motion case are qualitatively robust and illustrate our general results explicitly in the mean reverting case. Finally, section five concludes our study.

2 The δ-penalty Minimum Guaranteed Payment Option

The main objective of this study is to characterize the value and equilibrium exercise strategy of a class of derivative instruments containing strategic elements. In order to accomplish this task we first have to characterize the state variable modeling the underlying asset value. As usually, we assume that it constitutes a linear, time homogeneous, and regular diffusion process defined on the complete filtered probability space $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ and that it evolves on $\mathbb{R}_+$ according to the dynamics described by the Itô-stochastic differential equation

$$dX_t = \mu(X_t)dt + \theta\sigma(X_t)dW_t, \quad X_0 = x, \tag{1}$$

where W_t denotes standard Brownian motion, $\theta \in \mathbb{R}_+$ is an exogenously given constant multiplier (introduced in order to consider the impact of increased volatility on the optimal policy and its value), and both the drift coefficient $\mu : \mathbb{R}_+ \mapsto \mathbb{R}$ and the diffusion coefficient $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ are assumed to be sufficiently smooth for guaranteeing the existence and uniqueness of a (weak) solution for the stochastic differential equation (1) (at least continuous, cf. Borodin and Salminen (1996), pp. 46–47). In order to avoid interior singularities, we also assume that the diffusion coefficient $\sigma(x) > 0$ for all $x \in (0, \infty)$. As usually,

$$\mathcal{A}_\theta = \frac{1}{2}\theta^2\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$$

denotes the differential operator associated to the underlying diffusion X_t. It is well-known that given the assumptions of our study, there are two linearly independent fundamental solutions $\psi_\theta(x)$ and $\varphi_\theta(x)$ (constituting the minimal r-harmonic mappings for the diffusion X) satisfying a set of appropriate boundary conditions based on the boundary behavior of the process X and spanning the set of solutions of the ordinary differential equation $(\mathcal{A}_\theta u)(x) = ru(x)$ (cf. Borodin and Salminen 2002, pp. 18 - 19). Moreover, $\psi_\theta'(x)\varphi_\theta(x) - \varphi_\theta'(x)\psi_\theta(x) = B_\theta S_\theta'(x)$, where $B_\theta > 0$ denotes the constant Wronskian of the fundamental solutions $\psi_\theta(x)$ and $\varphi_\theta(x)$ and

$$S_\theta'(x) = \exp\left(-\int \frac{2\mu(x)dx}{\theta^2\sigma^2(x)}\right)$$

denotes the density of the scale function of X.

Our purpose in this paper is to analyze the properties of the value and optimal exercise policy of the δ-penalty minimum guaranteed payment option (the δ-penalty MGP-option). The value of this option can be interpreted as the value of a infinite horizon Dynkin game characterized by the function

$$\Pi_x(\tau,\gamma) = \mathbf{E}_x\left[e^{-r(\tau\wedge\gamma)}\left(\max(X_\tau,p)1_{\{\tau\leq\gamma\}} + (\max(X_\tau,p)+\delta)1_{\{\tau>\gamma\}}\right)\right], \tag{2}$$

where $p>0$ denotes the minimum guaranteed payment and $\delta>0$ denotes the penalty that the issuer has to pay to the holder in case the issuer exercises first (i.e. in case the issuer chooses to exercise the costly cancelation right before the holder exercises the option). As usually, the associated lower and upper values are defined as $\underline{V}_\theta(x) = \sup_\tau \inf_\gamma \Pi_x(\tau,\gamma)$ and $\overline{V}_\theta(x) = \inf_\gamma \sup_\tau \Pi_x(\tau,\gamma)$, respectively. It is clear that

$$\max(x,p) \leq \underline{V}_\theta(x) \leq \overline{V}_\theta(x) \leq \max(x,p)+\delta.$$

Hence, if we also have $\underline{V}_\theta(x) \geq \overline{V}_\theta(x)$, then the considered stochastic optimal stopping game has a value and this value is denoted as $V_\theta(x) = \underline{V}_\theta(x) = \overline{V}_\theta(x)$.

Finally, a pair of stopping times (τ',γ') is said to constitute a saddle point of the considered Dynkin game whenever the condition $\Pi_x(\tau,\gamma') \leq \Pi_x(\tau',\gamma') \leq \Pi_x(\tau',\gamma)$ is satisfied for all stopping times τ,γ. In light of this inequality, it is clear that the existence of a saddle point guarantees the existence of the value for the considered game. Moreover, if the considered Dynkin game has the value $V_\theta(x)$, then the pair of stopping times

$$\tau^* = \inf\{t\geq 0 : V_\theta(X_t) \leq \max(x,p)\} \tag{3}$$

and

$$\gamma^* = \inf\{t\geq 0 : V_\theta(X_t) \geq \max(x,p)+\delta\} \tag{4}$$

constitute a saddle point for the game.

As we will later establish, the value of the δ-penalty MGP-option is closely related to the value $J_\theta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ of the standard MGP-option characterized by the ordinary optimal stopping problem

$$J_\theta(x) = \sup_\tau \mathbf{E}_x\left[e^{-r\tau}\max(X_\tau,p)\right] \tag{5}$$

which was originally considered in Guo and Shepp (2001) in the case the underlying dynamics are characterized by geometric Brownian motion.

3 The Geometric Brownian Motion Case

We begin the analysis of our study by considering the δ-penalty MGP-option in the case where the underlying diffusion evolves according to a standard geometric Brownian motion characterized by the infinitesimal coefficients $\mu(x) = \mu x$ and $\sigma(x) = \theta x$, where $\mu, \theta \in \mathbb{R}_+$ are known constants. It is well-known that in this case the fundamental solutions of the ordinary second order differential equation $(\mathcal{A}_\theta u)(x) = ru(x)$ read as $\psi_\theta(x) = x^{\eta_\theta}$ and $\varphi_\theta(x) = x^{\nu_\theta}$, where

$$\eta_\theta = \frac{1}{2} - \frac{\mu}{\theta^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\theta^2}\right)^2 + \frac{2r}{\theta^2}} > 0$$

and

$$\nu_\theta = \frac{1}{2} - \frac{\mu}{\theta^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\theta^2}\right)^2 + \frac{2r}{\theta^2}} < 0$$

denote the the roots of the characteristic equation $\theta^2 a(a-1) + 2\mu = 2r$.

Define now the functional $F_{a,b} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ as the expected present value

$$F_{a,b}(x) = \mathbf{E}_x\left[e^{-r\tau_{(a,b)}} \max(X_{\tau_{(a,b)}}, p)\right], \tag{6}$$

where $\tau_{(a,b)} = \inf\{t \geq 0 : X_t \notin (a,b)\}$ denotes the first exit time from the open interval (a,b). It is well-known from the literature on linear diffusions that this value satisfies for all $x \in (a,b)$ the ordinary differential equation $(\mathcal{A}_\theta F_{a,b})(x) = rF_{a,b}(x)$ subject to the boundary conditions $F_{a,b}(a) = \max(a,p)$ and $F_{a,b}(b) = \max(b,p)$. Hence, we observe that the expected present value $F_{a,b}(x)$ can be re-expressed explicitly as

$$F_{a,b}(x) = \begin{cases} \max(x,p) & x \geq b \\ \max(a,p)\frac{\hat{\varphi}_b(x)}{\hat{\varphi}_b(a)} + \max(b,p)\frac{\hat{\psi}_a(x)}{\hat{\psi}_a(b)} & x \in (a,b) \\ \max(x,p) & x \leq a, \end{cases} \tag{7}$$

where in the present case the functions $\hat{\psi}_a(x)$ and $\hat{\varphi}_b(x)$ are defined as

$$\hat{\psi}_a(x) = x^{\eta_\theta} - a^{\eta_\theta - \nu_\theta} x^{\nu_\theta} \tag{8}$$

and

$$\hat{\varphi}_b(x) = x^{\nu_\theta} - b^{\nu_\theta - \eta_\theta} x^{\eta_\theta}. \tag{9}$$

Given the representations (6) and (7) it is clear that for all $x \in \mathbb{R}_+$ we have $J_\theta(x) \geq F_{a,b}(x)$, where

$$J_\theta(x) = \sup_\tau \mathbf{E}_x \left[e^{-r\tau} \max(X_\tau, p)\right]. \tag{10}$$

In light of this inequality, we now investigate under which conditions the boundaries a and b can be chosen so that the value $F_{a,b}(x)$ coincides with the value of the optimal stopping problem (10). These conditions are now summarized in the following.

Lemma 3.1. *(Guo and Shepp (2001)) Assume that $r > \mu$. Then $J_\theta(x) = F_{x^*_\theta, y^*_\theta}(x)$, where*

$$x^*_\theta = \frac{\eta_\theta}{\eta_\theta - 1}\left(\frac{\nu_\theta(\eta_\theta - 1)}{\eta_\theta(\nu_\theta - 1)}\right)^{\frac{1-\nu_\theta}{\eta_\theta - \nu_\theta}} p < p \tag{11}$$

$$y^*_\theta = \frac{\eta_\theta}{\eta_\theta - 1}\left(\frac{\nu_\theta(\eta_\theta - 1)}{\eta_\theta(\nu_\theta - 1)}\right)^{\frac{-\nu_\theta}{\eta_\theta - \nu_\theta}} p > p \tag{12}$$

Proof. Since $J_\theta(x) \geq F_{x^*_\theta, y^*_\theta}(x)$ it is sufficient to demonstrate that the reverse inequality holds. To this end, we first notice that the proposed value $F_{x^*_\theta, y^*_\theta}(x)$ is continuously differentiable on $\mathbb{R}_+$, twice continuously differentiable on $\mathbb{R}_+ \backslash \{x^*_\theta, y^*_\theta\}$, and satisfies the inequalities $|F''_{x^*_\theta, y^*_\theta}(x^*_\theta\pm)| < \infty$ and $|F''_{x^*_\theta, y^*_\theta}(y^*_\theta\pm)| < \infty$. Moreover, since the value $F_{x^*_\theta, y^*_\theta}(x)$ can be re-expressed on (x^*_θ, y^*_θ) as

$$F_{x^*_\theta, y^*_\theta}(x) = \frac{p}{\eta_\theta - \nu_\theta}\left(\eta_\theta\left(\frac{x}{x^*_\theta}\right)^{\nu_\theta} - \nu_\theta\left(\frac{x}{x^*_\theta}\right)^{\eta_\theta}\right) = \frac{y^*_\theta}{\eta_\theta - \nu_\theta}\left((\eta_\theta - 1)\left(\frac{x}{y^*_\theta}\right)^{\nu_\theta} + (1 - \nu_\theta)\left(\frac{x}{y^*_\theta}\right)^{\eta_\theta}\right)$$

we find that $F_{x^*_\theta, y^*_\theta}(x)$ is strictly convex on (x^*_θ, y^*_θ) (since $\eta_\theta > 1$ when $r > \mu$). Thus, $F'_{x^*_\theta, y^*_\theta}(x^*_\theta) = 0 < F'_{x^*_\theta, y^*_\theta}(x) < 1 = F'_{x^*_\theta, y^*_\theta}(y^*_\theta)$ for all $x \in (x^*_\theta, y^*_\theta)$. In light of these observations we find that the mapping $\Delta(x) = F_{x^*_\theta, y^*_\theta}(x) - \max(x, p)$ satisfies the conditions $\Delta(x^*_\theta) = \Delta(y^*_\theta) = 0$, and $\Delta'(x^*_\theta) = \Delta'(y^*_\theta) = 0$. Combining these observations with the strict convexity of $F_{x^*_\theta, y^*_\theta}(x)$ on (x^*_θ, y^*_θ) then shows that $\Delta(x) > 0$ for all $x \in (x^*_\theta, y^*_\theta)$. Finally, since

$$(\mathcal{A}_\theta F_{x^*_\theta, y^*_\theta})(x) - rF_{x^*_\theta, y^*_\theta}(x) = \begin{cases} -(r-\mu)x & x > y^*_\theta \\ 0 & x \in (x^*_\theta, y^*_\theta) \\ -rp & x < x^*_\theta \end{cases}$$

we observe that $F_{x^*_\theta, y^*_\theta}(x)$ satisfies the sufficient variational inequalities guaranteeing that it constitutes a majorant of the value $J_\theta(x)$ and, therefore, that $F_{x^*_\theta, y^*_\theta}(x) \geq J_\theta(x)$. This completes the proof of our lemma. □

Lemma 3.1 characterize explicitly the value and exercise boundaries of the MGP-option in the present case. As intuitively is clear the optimal exercise policy is such that the holder takes the minimum guarantee at the lower boundary and the underlying stock at the upper boundary. Between these two boundaries the value of the option dominates the exercise payoff and waiting is optimal. The comparative static properties of the value and optimal policy are now summarized in our next lemma.

Lemma 3.2. *Assume that $r > \mu$. Then increased volatility increases the value of the optimal timing policy and expands the continuation region by increasing y^*_θ and decreasing x^*_θ. That is, $\partial J_\theta(x)/\partial\theta > 0$, $\partial y^*_\theta/\partial\theta > 0$, and $\partial x^*_\theta/\partial\theta < 0$.*

Proof. Denote the value of the MGP-option defined with respect to the less volatile dynamics characterized by the coefficient $\tilde{\theta} < \theta$ as $J_{\tilde{\theta}}(x)$ and let

$$\mathcal{A}_{\tilde{\theta}} = \frac{1}{2}\tilde{\theta}^2 x^2 \frac{d^2}{dx^2} + \mu x \frac{d}{dx}.$$

As was established in the proof of Lemma 3.1 the value $J_\theta(x)$ of the MGP-option is continuously differentiable on $\mathbb{R}_+$, twice continuously differentiable on $\mathbb{R}_+\setminus\{x^*_\theta, y^*_\theta\}$, convex on $\mathbb{R}_+$, and satisfies the inequality $J_\theta(x) \geq \max(x,p)$ for all $x \in \mathbb{R}_+$. Moreover, since

$$(\mathcal{A}_{\tilde{\theta}} J_\theta)(x) - rJ_\theta(x) \leq \frac{1}{2}(\tilde{\theta}^2 - \theta^2)x^2 J''_\theta(x) \leq 0 \quad \text{for all } x \in \mathbb{R}_+\setminus\{x^*_\theta, y^*_\theta\}$$

by the convexity of the value $J_\theta(x)$ we notice that $J_\theta(x)$ constitutes a r-excessive majorant of the payoff $\max(x,p)$ for the less volatile diffusion as well. Since $J_{\tilde{\theta}}(x)$ constitutes the least of these majorants, we find that $J_\theta(x) \geq J_{\tilde{\theta}}(x)$.

Denote the continuation regions associated to the stopping problems as $C_\theta = \{x \in \mathbb{R}_+ : J_\theta(x) > \max(x,p)\}$ and $C_{\tilde{\theta}} = \{x \in \mathbb{R}_+ : J_{\tilde{\theta}}(x) > \max(x,p)\}$. If $x \in C_{\tilde{\theta}}$ then $J_\theta(x) \geq J_{\tilde{\theta}}(x) > \max(x,p)$ implying that $x \in C_\theta$ as well. Hence, $C_{\tilde{\theta}} \subseteq C_\theta$ and the alleged result follows. □

Lemma 3.2 characterizes the sensitivity of the value and the optimal boundaries with respect to changes in the volatility of the underlying dividend paying stock. As usually, our results indicate that higher volatility postpones rational exercise by expanding the continuation region. Essentially, the reason for this observation is that increased volatility raises the value of waiting while leaving the exercise payoff unchanged. The optimal boundaries are explicitly illustrated as functions of the volatility coefficient θ in Figure 1 under the numerical assumptions that $r = 0.035, \mu = 0.02$, and $p = 1$.

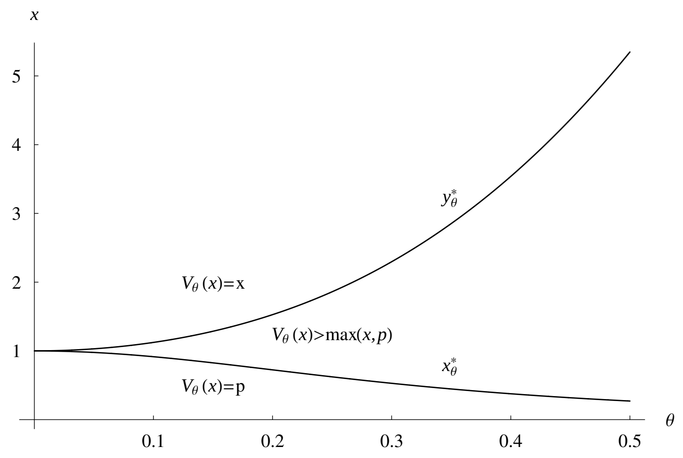

Figure 1: The Exercise Boundaries as Functions of Volatility

Our main conclusion on the value of the δ-penalty MGP-option is now summarized in the following.

Theorem 3.3. *Assume that $r > \mu$ and define the mapping $\Delta_\theta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ as $\Delta_\theta = J_\theta(p) - p$.*

(A) If $\delta \geq \Delta_\theta$ then $V_\theta(x) = J_\theta(x) = F_{x^*_\theta, y^*_\theta}(x)$.

(B) If $\delta < \Delta_\theta$ then

$$V_\theta(x) = \begin{cases} x & x \in [\bar{y}_\theta, \infty) \\ (p+\delta)\frac{\hat{\varphi}_{\bar{y}_\theta}(x)}{\hat{\varphi}_{\bar{y}_\theta}(p)} + \bar{y}_\theta \frac{\hat{\psi}_p(x)}{\hat{\psi}_p(\bar{y}_\theta)} & x \in (p, \bar{y}_\theta) \\ p+\delta & x = p \\ p\frac{\hat{\varphi}_p(x)}{\hat{\varphi}_p(\bar{x}_\theta)} + (p+\delta)\frac{\hat{\psi}_{\bar{x}_\theta}(x)}{\hat{\psi}_{\bar{x}_\theta}(p)} & x \in (\bar{x}_\theta, p) \\ p & x \in (0, \bar{x}_\theta], \end{cases} \tag{13}$$

where the functions $\hat{\varphi}_b(x)$ and $\hat{\psi}_a(x)$ are defined as in (9) *and* (8)*, respectively, and the optimal* *exercise thresholds $\bar{x}_\theta$ and $\bar{y}_\theta$ constitute the unique roots of the optimality conditions*

$$(1-\nu_\theta)\left(\frac{p}{\bar{y}_\theta}\right)^{\eta_\theta} + (\eta_\theta - 1)\left(\frac{p}{\bar{y}_\theta}\right)^{\nu_\theta} = (\eta_\theta - \nu_\theta)\left(\frac{p+\delta}{\bar{y}_\theta}\right) \tag{14}$$

$$\eta_\theta\left(\frac{p}{\bar{x}_\theta}\right)^{\nu_\theta} - \nu_\theta\left(\frac{p}{\bar{x}_\theta}\right)^{\eta_\theta} = (\eta_\theta - \nu_\theta)\left(1+\frac{\delta}{p}\right) \tag{15}$$

Proof. (A) Assume that $\delta \geq \Delta_\theta$ and consider the difference $D_\theta(x) = \max(x,p) + \delta - J_\theta(x)$. It is clear that

$$D_\theta(x) = \begin{cases} -J_\theta'(x) & x < p \\ 1 - J_\theta'(x) & x > p \end{cases}$$

and, therefore, that $p = \operatorname{argmin}\{D_\theta(x)\}$. Hence, our assumption imply that $D_\theta(x) > D_\theta(p) = p + \delta - J_\theta(p) \geq 0$ for all $x \in \mathbb{R}_+$. Consequently, $\max(x,p) \leq J_\theta(x) \leq \max(x,p) + \delta$ for all $x \in \mathbb{R}_+$. Moreover, since $J_\theta(x)$ is r-excessive for the underlying diffusion X_t, we find that the proposed value function satisfies the sufficient conditions $(\mathcal{A}_\theta V_\theta)(x) \leq rV_\theta(x)$ on $\mathbb{R}_+$ where the proposed value is smaller than $\max(x,p)+\delta$ and $(\mathcal{A}_\theta V_\theta)(x) = rV_\theta(x)$ on the set (x_θ^*, y_θ^*) where the proposed value is greater than $\max(x,p)$. Since this value is attainable by the admissible stopping policy $\tau_{(x_\theta^*, y_\theta^*)} = \inf\{t \geq 0 : X_t \notin (x_\theta^*, y_\theta^*)\}$, we find that $V_\theta(x) = J_\theta(x) = F_{x_\theta^*, y_\theta^*}(x)$.

(B) Assume that $\delta < \Delta_\theta$. It is clear from part (A) that in that case there is a nonempty open interval where $J_\theta(x) > \max(x,p) + \delta$ and, therefore, that $J_\theta(x)$ does not constitute the value of the saddle point strategy. Given this observation, we now propose that the value function is r-harmonic on a set $(\bar{x}_\theta, p) \cup (p, \bar{y}_\theta)$, satisfies the smooth fit conditions at the thresholds $\bar{x}_\theta, \bar{y}_\theta$, and coincides with $p + \delta$ at p. More precisely, we propose that

$$(\mathcal{A}_\theta V_\theta)(x) = rV_\theta(x), V_\theta(\bar{x}_\theta) = p, V_\theta'(\bar{x}_\theta) = 0$$

for all $x \in (\bar{x}_\theta, p)$ and that

$$(\mathcal{A}_\theta V_\theta)(x) = rV_\theta(x), V_\theta(\bar{y}_\theta) = \bar{y}_\theta, V_\theta'(\bar{y}_\theta) = 1$$

for all $x \in (p, \bar{y}_\theta)$. Solving these boundary value problems yield

$$V_\theta(x) = \frac{p}{(\eta_\theta - \nu_\theta)}\left[\eta_\theta\left(\frac{x}{\bar{x}_\theta}\right)^{\nu_\theta} - \nu_\theta\left(\frac{x}{\bar{x}_\theta}\right)^{\eta_\theta}\right], \tag{16}$$

for all $x \in (\bar{x}_\theta, p)$ and

$$V_\theta(x) = \frac{\bar{y}_\theta}{(\eta_\theta - \nu_\theta)} \left[(1-\nu_\theta) \left(\frac{x}{\bar{y}_\theta}\right)^{\eta_\theta} + (\eta_\theta - 1) \left(\frac{x}{\bar{y}_\theta}\right)^{\nu_\theta} \right], \tag{17}$$

for all $x \in (p, \bar{y}_\theta)$. Invoking continuity at the boundary p then yields the conditions (14) and (15). We now demonstrate that the conditions (14) and (15) have unique roots. Consider first the function

$$K_1(x) = \eta_\theta \left(\frac{p}{x}\right)^{\nu_\theta} - \nu_\theta \left(\frac{p}{x}\right)^{\eta_\theta} - (\eta_\theta - \nu_\theta) \left(1 + \frac{\delta}{p}\right).$$

It is clear that $K_1(p) = -(\eta_\theta - \nu_\theta)\delta/p < 0$, $\lim_{x \downarrow 0} K_1(x) = +\infty$, and

$$K_1'(x) = \frac{\nu_\theta \eta_\theta}{x} \left(\left(\frac{p}{x}\right)^{\eta_\theta} - \left(\frac{p}{x}\right)^{\nu_\theta} \right) < 0.$$

Thus, equation $K_1(x) = 0$ has a unique root $\bar{x}_\theta \in (0, p)$. Establishing that the function

$$K_2(x) = (1-\nu_\theta) \left(\frac{p}{x}\right)^{\eta_\theta} + (\eta_\theta - 1) \left(\frac{p}{x}\right)^{\nu_\theta} - (\eta_\theta - \nu_\theta) \left(\frac{p+\delta}{x}\right)$$

has a unique root $\bar{y}_\theta \in (p, \infty)$ is completely analogous.

In light of these observations, it is now clear that the proposed value function is non-decreasing and continuous on $\mathbb{R}_+$, continuously differentiable on $\mathbb{R}_+ \backslash \{p\}$, twice continuously differentiable on $\mathbb{R}_+ \backslash \{\bar{x}_\theta, p, \bar{y}_\theta\}$, r-harmonic on $(\bar{x}_\theta, p) \cup (p, \bar{y}_\theta)$, and r-superharmonic on $(0, \bar{x}_\theta) \cup (\bar{y}_\theta, \infty)$. Moreover, since $V_\theta(x) = p$ on $(0, \bar{x}_\theta)$ and $V_\theta(x) = x$ on $(\bar{y}_\theta, \infty)$, we find that the proposed value function is convex on $\mathbb{R}_+$. Applying now the proof of part (ii) of Theorem 2 in Kyprianou (2004) and noticing that the proposed value is attained by the admissible (Markov time) stopping policy $\tau^* = \inf\{t \geq 0 : X_t \notin (\bar{x}_\theta, p) \cup (p, \bar{y}_\theta)\}$ then completes the proof of the alleged result. □

Theorem 3.3 characterizes the value and optimal exercise policy of the δ-penalty MGP-option. According to Theorem 3.3 there is a critical penalty above which the issuer is no longer prepared to exercise the embedded costly cancelation option and the value coincides with the value of the MGP-option in the nonstrategic setting. However, below the critical penalty the issuer uses the cancelation option and terminates the contract as soon as the underlying value coincides with the minimum guarantee. As in the non-strategic setting, the exercise policy of the holder is characterized by two boundaries. The value functions in the two cases arising in Theorem 3.3 are explicitly illustrated in Figure 2 under the assumptions that $r = 0.035, \mu = 0.02, p = 1$, and $\sigma = 0.2$ (implying that $\Delta_{0.2} = 0.092$).

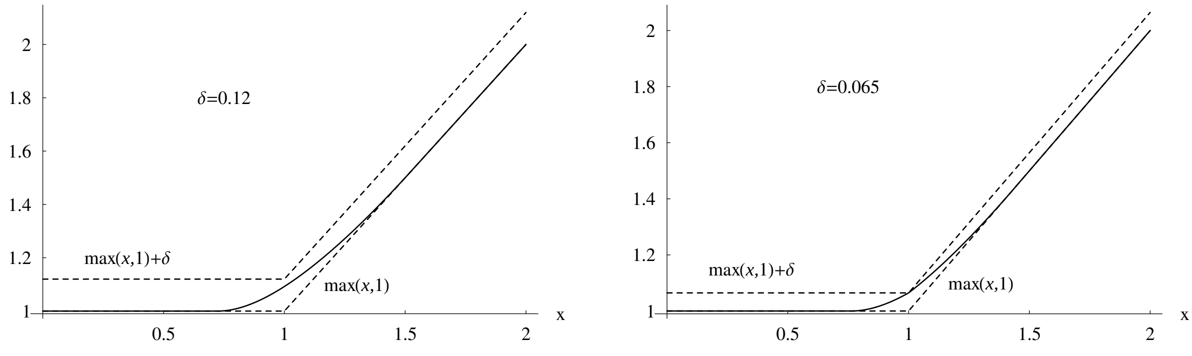

Figure 2: The Value Functions (uniform curves) and Exercise Payoffs (dashed curves)

An interesting implication of our observations characterizing the one-to-one nature of the critical penalty Δ_θ as a function of volatility is now summarized in the following.

Corollary 3.4. *Assume that* $r > \mu$. *Then,* Δ_θ *is a monotonically increasing function of volatility,* $\lim_{\theta\downarrow 0}\Delta_\theta = 0$, *and* $\lim_{\theta\to\infty}\Delta_\theta = \infty$. *Hence, for any given fixed penalty* $\delta \in \mathbb{R}_+$ *there is a unique volatility coefficient* $\theta = \Delta_\delta^{-1}$ *for which the optimal equilibrium strategy and its value can then be described as in part (A) of Theorem 3.3 as long as* $\theta \leq \Delta_\delta^{-1}$ *and as in part (B) of Theorem 3.3 whenever* $\theta > \Delta_\delta^{-1}$.

Proof. It is clear that under our assumptions we have $\eta_\theta > 1$, $\nu_\theta < 0$, $\lim_{\theta\to\infty}\nu_\theta = 0$, and $\lim_{\theta\to\infty}\eta_\theta = 1$. These observations imply that $x_\theta^* \downarrow 0$ and $y_\theta^* \uparrow \infty$ as $\theta \to \infty$. Hence, we also observe that $F_{x_\theta^*,y_\theta^*}(x) \uparrow \infty$ as $\theta \to \infty$. The alleged result is now a direct implication of Theorem 3.3. □

Corollary 3.4 demonstrates that in the present case the the critical penalty Δ_θ constitutes a bijection as a function of volatility. Thus, for any predetermined penalty δ there is a unique volatility coefficient $\theta = \Delta_\delta^{-1}$ for which the embedded cancelation option becomes valuable for the issuer as soon as volatility exceeds this value. This finding is interesting since it proves that increased volatility does not only increase the value of the optimal exercise policy for the holder, it increases the value of the embedded cancelation option as well. The critical penalty is illustrated explicitly in Figure 3 under the assumptions that $r = 0.035, \mu = 0.03$, and $p = 1$.

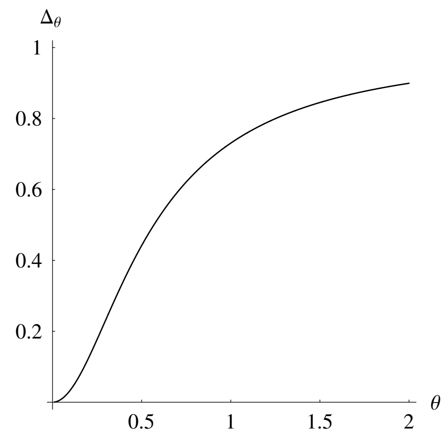

Figure 3: The Critical Penalty

4 The General Case

Having considered the valuation of the δ-penalty MGP-option in the standard case where the underlying value dynamics is characterized as a standard geometric Brownian motion, we now proceed in our analysis and consider the value of this contract in a more general setting as well. Along the lines of our previous observations, we first consider the underlying nonstrategic MGP-option and its value, and then present our main conclusions on the δ-penalty MGP-option in terms of this contingent contract. In accordance with our previous analysis, the chosen approach is based on the determination of the functional (6). In the present case, it can be expressed explicitly as in (7) subject to the obvious modification of the functionals $\hat{\psi}_a(x)$ and $\hat{\varphi}_b(x)$ which in the present case read as

$$\hat{\psi}_a(x) = \psi_\theta(x) - \frac{\psi_\theta(a)}{\varphi_\theta(a)}\varphi_\theta(x) \tag{18}$$

$$\hat{\varphi}_b(x) = \varphi_\theta(x) - \frac{\varphi_\theta(b)}{\psi_\theta(b)}\psi_\theta(x). \tag{19}$$

Before analyzing the considered δ-penalty MGP-option, we first present two auxiliary results extending previous findings to the present case. We first establish that the monotonicity of the appreciation rate and the boundary behavior of the underlying diffusion are sufficient conditions for the convexity of the minimal r-excessive mappings $\psi_\theta(x)$ and $\varphi_\theta(x)$. This task is accomplished in the following.

Lemma 4.1. *Assume that the net appreciation rate $\alpha(x) = \mu(x) - rx$ is non-increasing and that the boundaries* 0 *and* ∞ *are natural for X_t. Then, the minimal r-harmonic mappings $\psi_\theta(x)$ and $\varphi_\theta(x)$ are strictly convex on $\mathbb{R}_+$ and increased volatility raises or leaves unchanged the value of the functional*

$$\mathbf{E}_x\left[e^{-r\tau_a}\right] = \begin{cases} \frac{\psi_\theta(x)}{\psi_\theta(a)} & x \leq a \\ \frac{\varphi_\theta(x)}{\varphi_\theta(a)} & x \geq a \end{cases} \tag{20}$$

for any $0 < a < \infty$ and $x \in \mathbb{R}_+$.

Proof. Consider first the increasing function $\psi_\theta(x)$. It is clear that since $\psi_\theta(x)$ satisfies the ordinary differential equation $(\mathcal{A}_\theta\psi_\theta)(x) = r\psi_\theta(x)$ we have

$$\frac{1}{2}\theta^2\sigma^2(x)\frac{\psi_\theta''(x)}{S_\theta'(x)} = r\left(\frac{\psi_\theta(x)}{S_\theta'(x)} - x\frac{\psi_\theta'(x)}{S_\theta'(x)}\right) - \alpha(x)\frac{\psi_\theta'(x)}{S_\theta'(x)}. \tag{21}$$

Standard differentiation yields

$$\frac{d}{dx}\left(\frac{\psi_\theta(x)}{S_\theta'(x)} - x\frac{\psi_\theta'(x)}{S_\theta'(x)}\right) = \alpha(x)\psi_\theta(x)m_\theta'(x) \tag{22}$$

which, in turn, implies that

$$\left(\frac{\psi_\theta(x)}{S_\theta'(x)} - x\frac{\psi_\theta'(x)}{S_\theta'(x)}\right) = \left(\frac{\psi_\theta(a)}{S_\theta'(a)} - a\frac{\psi_\theta'(a)}{S_\theta'(a)}\right) + \int_a^x \alpha(t)\psi_\theta(t)m_\theta'(t)dt \tag{23}$$

where $a \in (0, x)$ is arbitrary. Plugging this finding in (21) and invoking the canonical form

$$\frac{\psi_\theta(x)}{S_\theta'(x)} - \frac{\psi_\theta(a)}{S_\theta'(a)} = r\int_a^x \psi_\theta(t)m_\theta'(t)dt \tag{24}$$

then shows that (21) can be re-expressed as

$$\frac{1}{2}\theta^2\sigma^2(x)\frac{\psi_\theta''(x)}{S_\theta'(x)} = r\int_a^x (\alpha(t) - \alpha(x))\psi_\theta(t)m_\theta'(t)dt + r\frac{\psi_\theta(a)}{S_\theta'(a)} - (\mu(x) - r(x-a))\frac{\psi_\theta'(a)}{S_\theta'(a)}.$$

Invoking the monotonicity of $\psi_\theta(x)$ and $\alpha(x)$ then yields the inequality

$$\frac{1}{2}\theta^2\sigma^2(x)\frac{\psi_\theta''(x)}{S_\theta'(x)} > -\mu(x)\frac{\psi_\theta'(a)}{S_\theta'(a)}.$$

Letting $a \downarrow 0$ and invoking the boundary condition $\psi_\theta'(a)/S_\theta'(a) \downarrow 0$ as $a \downarrow 0$ then implies that $\psi_\theta''(x) > 0$ for all $x \in \mathbb{R}_+$. Hence, $\psi_\theta(x)$ is strictly convex on $\mathbb{R}_+$. Establishing the strict convexity of $\varphi_\theta(x)$ is entirely analogous. The positivity of the sign of the relationship between increased volatility and the value of the functional (20) follows from Corollary 3 in Alvarez (2003). □

Lemma 4.1 states a set of conditions under which the minimal r-harmonic mappings for the underlying diffusion are strictly convex and, consequently, under which increased volatility unambiguously increases the values of the contingent contracts guaranteeing to the holder one dollar at the first time the underlying hits a predetermined boundary y. It is worth noticing that Lemma 4.1 extends part of the results of Alvarez (2003) since no integrability conditions are needed for the verification of convexity; only the local behavior of the infinitesimal characteristics count when the boundaries of the state space are natural for the underlying dynamics characterizing the value of the dividend paying stock. Our main finding on the MGP-option in the absence of strategic interaction is now summarized in the following.

Theorem 4.2. *Assume that the net appreciation rate $\alpha(x) = \mu(x) - rx$ is non-increasing and non-positive and that 0 and ∞ are natural boundaries for the underlying diffusion X_t. Then, the value of the generalized MGP-option reads as*

$$J_\theta(x) = F_{x_\theta^*,y_\theta^*}(x) = \begin{cases} x & x \in [y_\theta^*, \infty) \\ p\frac{\hat{\varphi}_{y_\theta^*}(x)}{\hat{\varphi}_{y_\theta^*}(x_\theta^*)} + y_\theta^* \frac{\hat{\psi}_{x_\theta^*}(x)}{\hat{\psi}_{x_\theta^*}(y_\theta^*)} & x \in (x_\theta^*, y_\theta^*) \\ p, & x \in (0, x_\theta^*), \end{cases} \tag{25}$$

where the optimal exercise boundaries x^*, y^* constitute the unique roots of equations

$$\frac{\varphi_\theta'(y_\theta^*)}{S_\theta'(y_\theta^*)} y_\theta^* - \frac{\varphi_\theta(y_\theta^*)}{S_\theta'(y_\theta^*)} = p\frac{\varphi_\theta'(x_\theta^*)}{S_\theta'(x_\theta^*)} \tag{26}$$

$$\frac{\psi_\theta'(y_\theta^*)}{S_\theta'(y_\theta^*)} y_\theta^* - \frac{\psi_\theta(y_\theta^*)}{S_\theta'(y_\theta^*)} = p\frac{\psi_\theta'(x_\theta^*)}{S_\theta'(x_\theta^*)}. \tag{27}$$

Proof. We first establish that if a pair satisfying the optimality conditions exist, then the resulting candidate value constitutes a r-excessive majorant of the underlying exercise payoff. To this end, we first notice that the proposed value function is continuously differentiable on $\mathbb{R}_+$, twice continuously differentiable on $\mathbb{R}_+ \setminus \{x_\theta^*, y_\theta^*\}$, r-harmonic on (x_θ^*, y_θ^*), r-superharmonic on $(0, x_\theta^*) \cup (y_\theta^*, \infty)$, and satisfies the conditions $|F''_{x_\theta^*,y_\theta^*}(x_\theta^*\pm)| < \infty$ and $|F''_{x_\theta^*,y_\theta^*}(y_\theta^*\pm)| < \infty$. Hence, $F_{x_\theta^*,y_\theta^*}(x)$ is r-excessive for the underlying diffusion X_t. We now prove that it dominates the underlying exercise payoff as well. Since $F_{x_\theta^*,y_\theta^*}(x) = \max(x, p)$ on $(0, x_\theta^*] \cup [y_\theta^*, \infty)$ we now analyze the behavior of the mapping

$$H_1(x) = \frac{F_{x_\theta^*,y_\theta^*}(x) - \max(x, p)}{\psi_\theta(x)}$$

on the continuation set (x_θ^*, y_θ^*). It is clear that we have $H_1(x_\theta^*) = H_1(y_\theta^*) = 0$ and $H_1'(x_\theta^*) = H_1'(y_\theta^*) = 0$. Moreover, since the proposed value can be expressed on (x_θ^*, y_θ^*) as $F_{x_\theta^*,y_\theta^*}(x) = c_1\psi_\theta(x) + c_2\varphi_\theta(x)$, where

$$c_1 = -\frac{p\varphi_\theta'(x_\theta^*)}{BS_\theta'(x_\theta^*)} = \frac{\varphi_\theta(y_\theta^*) - \varphi_\theta'(y_\theta^*)y_\theta^*}{BS_\theta'(y_\theta^*)}$$

and

$$c_2 = \frac{p\psi_\theta'(x_\theta^*)}{BS_\theta'(x_\theta^*)} = \frac{\psi_\theta'(y_\theta^*)y_\theta^* - \psi_\theta(y_\theta^*)}{BS_\theta'(y_\theta^*)},$$

we observe that

$$H_1'(x) = \begin{cases} \frac{S_\theta'(x)}{\psi_\theta^2(x)}\left(\frac{\psi_\theta'(x)x - \psi_\theta(x)}{S_\theta'(x)} - \frac{\psi_\theta'(y_\theta^*)y_\theta^* - \psi_\theta(y_\theta^*)}{S_\theta'(y_\theta^*)}\right) & x \in (p, y_\theta^*) \\ \frac{pS_\theta'(x)}{\psi_\theta^2(x)}\left(\frac{\psi_\theta'(x)}{S_\theta'(x)} - \frac{\psi_\theta'(x_\theta^*)}{S_\theta'(x_\theta^*)}\right) & x \in (x_\theta^*, p). \end{cases}$$

Applying (22) and (24) then show that $H_1(x)$ is increasing on (x_θ^*, p) and decreasing on (p, y_θ^*). Consequently, we find that $F_{x_\theta^*,y_\theta^*}(x) > \max(x, p)$ for all (x_θ^*, y_θ^*). Hence, the proposed value constitutes a r-excessive majorant of the payoff $\max(x, p)$ for the underlying diffusion X_t. Since the value $J_\theta(x)$ is the least of these majorants, we observe that $F_{x_\theta^*,y_\theta^*}(x) \geq J_\theta(x)$. However, since the proposed value is attained by applying the admissible stopping strategy $\tau^* = \inf\{t \geq 0 : X_t \notin (x_\theta^*, y_\theta^*)\}$ we find that $F_{x_\theta^*,y_\theta^*}(x) \leq J_\theta(x)$ as well.

It remains to establish that the ordinary first order conditions (26) and (27) have a unique root. To this end, consider the mappings

$$l_1(x, y) = \frac{\varphi_\theta'(y)}{S_\theta'(y)}y - \frac{\varphi_\theta(y)}{S_\theta'(y)} - p\frac{\varphi_\theta'(x)}{S_\theta'(x)} \tag{28}$$

$$l_2(x, y) = \frac{\psi_\theta'(y)}{S_\theta'(y)}y - \frac{\psi_\theta(y)}{S_\theta'(y)} - p\frac{\psi_\theta'(x)}{S_\theta'(x)}. \tag{29}$$

It is clear that $l_1(y, y) < 0$ and $\lim_{x\downarrow 0} l_1(x, y) = +\infty$ (since 0 is natural for X_t) for any $y \in (p, \infty)$. Since $(\partial l_1/\partial x)(x, y) = -rp\varphi_\theta(x)m_\theta'(x) < 0$ we find that equation $l_1(x, y) = 0$ has a unique root $\tilde{x}_y$ satisfying $l_1(\tilde{x}_y, y) = 0$ for any $y \in (p, \infty)$. Moreover, implicit differentiation yields

$$\frac{dx}{dy}\Big|_{l_1(x,y)=0} = -\frac{\alpha(y)\varphi_\theta(y)m_\theta'(y)}{rp\varphi_\theta(x)m_\theta'(x)} > 0.$$

Analogously, we find that $l_2(x, x) < 0$ for any $x \in (0, p)$. Applying now equation (23) and invoking the mean value theorem of integral calculus yields

$$\begin{aligned}\frac{\psi_\theta'(y)}{S_\theta'(y)}y - \frac{\psi_\theta(y)}{S_\theta'(y)} &= \frac{\psi_\theta'(x)}{S_\theta'(x)}x - \frac{\psi_\theta(x)}{S_\theta'(x)} - \int_x^y \alpha(t)\psi_\theta(t)m_\theta'(t)dt \\ &= \frac{\psi_\theta'(x)}{S_\theta'(x)}x - \frac{\psi_\theta(x)}{S_\theta'(x)} - \frac{\alpha(\xi)}{r}\left[\frac{\psi_\theta'(y)}{S_\theta'(y)} - \frac{\psi_\theta'(x)}{S_\theta'(x)}\right]\end{aligned}$$

where $\xi \in (x, y)$. Letting $y \to \infty$ then shows that $\lim_{y\to\infty} l_2(x, y) = +\infty$ (since ∞ is natural for X_t). Since $(\partial l_2/\partial y)(x, y) = -\alpha(y)\psi_\theta(y)m_\theta'(y) > 0$ we find that equation $l_2(x, y) = 0$ has a unique root $\tilde{y}_x$ satisfying $l_2(x, \tilde{y}_x) = 0$ for any $x \in (0, p)$. Moreover, implicit differentiation yields

$$\frac{dy}{dx}\Big|_{l_2(x,y)=0} = -\frac{rp\psi_\theta(x)m_\theta'(x)}{\alpha(y)\psi_\theta(y)m_\theta'(y)} > 0.$$

Combining these observations show that

$$\frac{dy}{dx}\Big|_{l_2(x,y)=0} = \left[\frac{\psi_\theta(x)\varphi_\theta(y)}{\psi_\theta(y)\varphi_\theta(x)}\right]\frac{dy}{dx}\Big|_{l_1(x,y)=0} < \frac{dy}{dx}\Big|_{l_1(x,y)=0}$$

demonstrating that if a pair $x_\theta^* < y_\theta^*$ satisfying the first order conditions (26) and (27) exists, it is unique. In order to demonstrate that such a pair indeed exists, we now show that $\tilde{y}_x$ and $\tilde{x}_y$ have an interception point $(x_\theta^*, y_\theta^*) \in (0, p) \times (p, \infty)$. Consider first the roots of the equations $l_1(p, y) = 0$ and $l_2(p, y) = 0$. Applying (22) yields

$$\begin{aligned}l_1(p, y) &= \int_p^y (rt - \mu(t))\varphi_\theta(t)m_\theta'(t)dt - \frac{\varphi_\theta(p)}{S_\theta'(p)} \\ l_2(p, y) &= \int_p^y (rt - \mu(t))\psi_\theta(t)m_\theta'(t)dt - \frac{\psi_\theta(p)}{S_\theta'(p)}.\end{aligned}$$

Denote as y_2^* the root of $l_2(p, y) = 0$. Then the monotonicity of the mapping $\psi_\theta(t)/\varphi_\theta(t)$ yields

$$\frac{\psi_\theta(p)}{S_\theta'(p)} = \int_p^{y_2^*} (rt - \mu(t))\psi_\theta(t)m_\theta'(t)dt \geq \frac{\psi_\theta(p)}{\varphi_\theta(p)}\int_p^{y_2^*} (rt - \mu(t))\varphi_\theta(t)m_\theta'(t)dt$$

proving that $l_1(p, y_2^*) \leq 0$ and, therefore, that $y_1^* \geq y_2^*$, where y_1^* denotes the root of $l_1(p, y) = 0$. Analogously, applying the canonical identity (24) shows that

$$\begin{aligned}l_1(x, p) &= rp\int_x^p \varphi_\theta(t)m_\theta'(t)dt - \frac{\varphi_\theta(p)}{S_\theta'(p)} \\ l_2(x, p) &= rp\int_x^p \psi_\theta(t)m_\theta'(t)dt - \frac{\psi_\theta(p)}{S_\theta'(p)}.\end{aligned}$$

Denote as x_2^* the root of $l_2(x_2^*, p) = 0$. Then the monotonicity of the mapping $\psi_\theta(t)/\varphi_\theta(t)$ yields

$$\frac{\psi_\theta(p)}{S_\theta'(p)} = rp\int_{x_2^*}^p \psi_\theta(t)m_\theta'(t)dt \leq \frac{\psi_\theta(p)}{\varphi_\theta(p)}rp\int_{x_2^*}^p \varphi_\theta(t)m_\theta'(t)dt$$

proving that $l_1(x_2^*, p) \geq 0$ and, therefore, that $x_1^* \geq x_2^*$, where x_1^* denotes the root of $l_1(x, p) = 0$. Combining these inequalities then demonstrate that $\tilde{x}_y$ and $\tilde{y}_x$ have a unique interception on the set $\in (0, p) \times (p, \infty)$ which completes the proof of our theorem. □

Theorem 4.2 extends the results of Lemma 3.1 to a general diffusion setting. Along the lines of the Lemma 3.1 we again observe that the optimal policy is characterized by two boundaries at which the value of an exercise policy which is characterized as a first exit time from an open interval is maximized. The comparative static properties of the value and the optimal exercise strategy are now summarized in the following.

Theorem 4.3. *Assume that the net appreciation rate $\alpha(x) = \mu(x) - rx$ is non-increasing and non-positive and that 0 and ∞ are natural boundaries for the underlying diffusion X_t. Then, the value function $J_\theta(x)$ is convex on $\mathbb{R}_+$ and strictly convex on (x_θ^*, y_θ^*). Moreover, higher volatility increases the value and expands the continuation region where exercising the option is suboptimal. That is, $\partial J_\theta(x)/\partial\theta > 0$, $\partial x_\theta^*/\partial\theta < 0$, and $\partial y_\theta^*/\partial\theta > 0$.*

Proof. As was established in the proof of Theorem 4.2, the value of the MGP-option reads on (x_θ^*, y_θ^*) as

$$J_\theta(x) = \frac{p}{BS_\theta'(x_\theta^*)} \left[\psi_\theta'(x_\theta^*)\varphi_\theta(x) - \varphi_\theta'(x_\theta^*)\psi_\theta(x)\right].$$

Since a positive affine combination of two strictly convex functions is strictly convex, we find that the alleged convexity of $J_\theta(x)$ follows from Lemma 4.1. Establishing now that increased volatility increases the value and expands the continuation region is analogous with the proof of Lemma 3.2. □

Theorem 4.3 extends the findings of Lemma 3.2 to the general setting. Interestingly, we observe that the sign of the relationship between higher volatility and the optimal policy is a process specific property which is mainly based on the behavior of the net appreciation rate of the underlying dynamics.

Having considered the MGP-option in the general setting we are now in position to proceed into the analysis of the δ-penalty MGP-option. Our main conclusion on the value of this game option is now summarized in the following.

Theorem 4.4. *Assume that the net appreciation rate $\alpha(x) = \mu(x) - rx$ is non-increasing and non-positive, that 0 and ∞ are natural boundaries for the underlying diffusion X_t, and define the mapping $\Delta_\theta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ as $\Delta_\theta = J_\theta(p) - p$.*

(A) If $\delta \geq \Delta_\theta$ then $V_\theta(x) = J_\theta(x) = F_{x^*_\theta, y^*_\theta}(x)$.

(B) If $\delta < \Delta_\theta$ then

$$V_\theta(x) = \begin{cases} x & x \in [\bar{y}_\theta, \infty) \\ (p+\delta)\frac{\hat{\varphi}_{\bar{y}_\theta}(x)}{\hat{\varphi}_{\bar{y}_\theta}(p)} + \bar{y}_\theta \frac{\hat{\psi}_p(x)}{\hat{\psi}_p(\bar{y}_\theta)} & x \in (p, \bar{y}_\theta) \\ p+\delta & x = p \\ p\frac{\hat{\varphi}_p(x)}{\hat{\varphi}_p(\bar{x}_\theta)} + (p+\delta)\frac{\hat{\psi}_{\bar{x}_\theta}(x)}{\hat{\psi}_{\bar{x}_\theta}(p)} & x \in (\bar{x}_\theta, p) \\ p & x \in (0, \bar{x}_\theta], \end{cases} \tag{30}$$

where the functions $\hat{\psi}_a(x)$ and $\hat{\varphi}_b(x)$ are defined as in (18) *and* (19)*, respectively, and the optimal exercise thresholds $\bar{x}_\theta$ and $\bar{y}_\theta$ constitute the unique roots of the optimality conditions*

$$p\frac{\psi'_\theta(\bar{x}_\theta)}{S'_\theta(\bar{x}_\theta)}\varphi_\theta(p) - p\frac{\varphi'_\theta(\bar{x}_\theta)}{S'_\theta(\bar{x}_\theta)}\psi_\theta(p) = (p+\delta)\,B \tag{31}$$

$$\frac{\varphi_\theta(\bar{y}_\theta) - \varphi'_\theta(\bar{y}_\theta)\bar{y}_\theta}{S'_\theta(\bar{y}_\theta)}\psi_\theta(p) + \frac{\psi'_\theta(\bar{y}_\theta)\bar{y}_\theta - \psi_\theta(\bar{y}_\theta)}{S'_\theta(\bar{y}_\theta)}\varphi_\theta(p) = (p+\delta)\,B \tag{32}$$

Proof. (A) It is clear from Theorem 4.2 that the proposed value constitutes the minimal r-excessive majorant of the payoff $\max(x,p)$ for the diffusion X_t. Moreover, as was observed in the proof of Theorem 3.3 $J_\theta(x) \leq p + \delta$ for all $x \in \mathbb{R}_+$ as long as $\delta \geq J_\theta(p) - p$ in this case as well.

(B) We first establish that the first order conditions (31) and (32) have unique roots. To this end, define the mappings $K_1 : (0,p) \mapsto \mathbb{R}$ and $K_2 : (p, \infty) \mapsto \mathbb{R}$ as

$$\begin{aligned} K_1(x) &= p\frac{\psi'_\theta(x)}{S'_\theta(x)}\varphi_\theta(p) - p\frac{\varphi'_\theta(x)}{S'_\theta(x)}\psi_\theta(p) - (p+\delta)\,B \\ K_2(x) &= \frac{\varphi_\theta(x) - \varphi'_\theta(x)x}{S'_\theta(x)}\psi_\theta(p) + \frac{\psi'_\theta(x)x - \psi_\theta(x)}{S'_\theta(x)}\varphi_\theta(p) - (p+\delta)\,B. \end{aligned}$$

It is clear that $K_1(p) = -\delta B < 0$, $K_1(x) \uparrow +\infty$ as $x \downarrow 0$ since $\varphi'_\theta(x)/S'_\theta(x) \downarrow -\infty$ when 0 is a natural boundary, and $K'_1(x) = -rp\psi_\theta(p)\hat{\varphi}_p(x)m'_\theta(x) < 0$ for all $x \in (0,p)$. Hence, equation $K_1(x) = 0$ has a unique root $\bar{x}_\theta \in (0,p)$. Analogously, we observe that $K_2(p) = -B\delta < 0$ and

$$K'_2(x) = -(\mu(x) - rx)\varphi_\theta(p)\hat{\psi}_p(x)m'_\theta(x) > 0$$

for all $x \in (p, \infty)$. Hence, the mean value theorem implies that

$$K_2(x) = \int_p^x (rt - \mu(t))\varphi_\theta(p)\hat{\psi}_p(t)m'_\theta(t)dt - B\delta = \varphi_\theta(p)\frac{(r\xi - \mu(\xi))}{r}\left(\frac{\hat{\psi}'_p(x)}{S'_\theta(x)} - \frac{\hat{\psi}'_p(p)}{S'_\theta(p)}\right) - B\delta$$

where $\xi \in (p, x)$. Letting $x \uparrow \infty$ and applying the result that $\psi'_\theta(x)/S'_\theta(x) \uparrow \infty$ and $\varphi'_\theta(x)/S'_\theta(x) \uparrow 0$ as $x \uparrow \infty$ demonstrates that $K_2(x) \uparrow \infty$ as $x \uparrow \infty$. Thus, equation $K_2(x) = 0$ has a unique root on (p, ∞). In light of these findings, we observe as in Theorem 3.3 that the proposed value function is continuous on $\mathbb{R}_+$, continuously differentiable on $\mathbb{R}_+ \backslash \{p\}$, twice continuously differentiable on $\mathbb{R}_+ \backslash \{\bar{x}_\theta, p, \bar{y}_\theta\}$, r-harmonic on $(\bar{x}_\theta, p) \cup (p, \bar{y}_\theta)$, and r-superharmonic on $(0, \bar{x}_\theta) \cup (\bar{y}_\theta, \infty)$. Moreover, since $V_\theta(x) = p$ on $(0, \bar{x}_\theta)$ and $V_\theta(x) = x$ on $(\bar{y}_\theta, \infty)$, we again observe that the proposed value function is convex on $\mathbb{R}_+$. Applying now the proof of part (ii) of Theorem 2 in Kyprianou (2004) and noticing that the proposed value is attained by the admissible stopping policy $\tau^* = \inf\{t \geq 0 : X_t \notin (\bar{x}_\theta, p) \cup (p, \bar{y}_\theta)\}$ then completes the proof of our theorem. □

In light of the findings of our Theorem 4.3 it is clear that if the conditions of Theorem 4.4 are satisfied then the critical penalty Δ_θ is monotonically increasing as a function of volatility. Unfortunately, it is difficult to characterize the limits $\lim_{\theta \downarrow 0} \Delta_\theta$ and $\lim_{\theta \to \infty} \Delta_\theta$ in the general setting. Numerical computations in explicitly parameterized models seem to indicate that the conclusions of Corollary 3.4 are typically satisfied in a general setting as well. Therefore, we conjecture that for any given fixed penalty $\delta \in \mathbb{R}_+$ there is a unique volatility coefficient $\theta = \Delta_\delta^{-1}$ in a general setting as well.

4.1 Explicit Illustration

In order to illustrate our general findings explicitly, assume now that the underlying dynamics are characterized by the logistic stochastic differential equation

$$dX_t = \mu X_t(1 - \gamma X_t)dt + \theta X_t dW_t, \quad X_0 = x.$$

In this case the minimal r-harmonic mappings (i.e. the fundamental solutions) read as

$$\begin{aligned} \psi_\theta(x) &= x^{\eta_\theta} M(\eta_\theta, 1 + \eta_\theta - \nu_\theta, 2\mu\gamma x/\theta^2) \\ \varphi_\theta(x) &= x^{\nu_\theta} M(\nu_\theta, 1 - \eta_\theta + \nu_\theta, 2\mu\gamma x/\theta^2), \end{aligned}$$

θ	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4
x^*_θ	1.9602	1.8488	1.6865	1.4984	1.3064	1.1254	0.9636	0.8236
y^*_θ	2.055	2.2216	2.5028	2.9019	3.419	4.0506	4.7891	5.6238
Δ_θ	0.0115	0.045	0.0976	0.1652	0.2435	0.328	0.4151	0.5021
Δ_θ/p	0.0058	0.0225	0.0488	0.0826	0.1217	0.164	0.2076	0.2511

Table 1: The Optimal Boundaries and the Critical Penalty

where $\eta_\theta > 0$ and $\nu_\theta < 0$ are defined as in Section 3 and M denotes the *Kummer confluent hypergeometric function*. If $r > \mu$, then these solutions are strictly convex and the conditions of our Theorem 4.4 are satisfied and, therefore, the MGP-option has a well-defined value $J_\theta(x)$ as well as an exercise strategy which is characterized by the stopping boundaries x^*_θ and y^*_θ. The optimal exercise boundaries as well as the critical penalty are numerically illustrated for various volatilities in Table 1 under the numeric assumptions that $r = 0.04, \mu = 0.03, \gamma = 0.1$, and $p = 2$.

It is worth noticing that in the present example the increasing fundamental solution $\psi_\theta(x)$ is locally concave on a neighborhood of the origin as long as $r \leq \mu$ and that there is a unique threshold such that $\psi_\theta(x)$ is strictly convex for all states above that threshold. Hence, an optimal policy may exist even when $r \leq \mu$. The value of the optimal exercise policy is illustrated in that case in Figure 4 under the numeric assumptions that $r = 0.03, \mu = 0.04, \gamma = 0.1, \sigma = 0.1$, and $p = 2$. The optimal boundaries and the critical penalty are, in turn, numerically illustrated for various volatilities in Table 2 under the numeric assumptions that $r = 0.03, \mu = 0.04, \gamma = 0.1$, and $p = 2$. Given the local positivity of the net appreciation rate on the set $(0, (\mu - r)/(\mu\gamma))$ it is clear that now the optimal policy is more sensitive with respect to changes in the volatility coefficient θ. Interestingly, our results indicate that the ratio between the critical penalty and the minimum guaranteed sum at low volatilities is low (1.97%). However, as the volatility coefficient becomes higher the embedded cancelation option of the issuer becomes more valuable and the ratio between the critical penalty and the minimum guaranteed sum becomes more significant (37.52%).

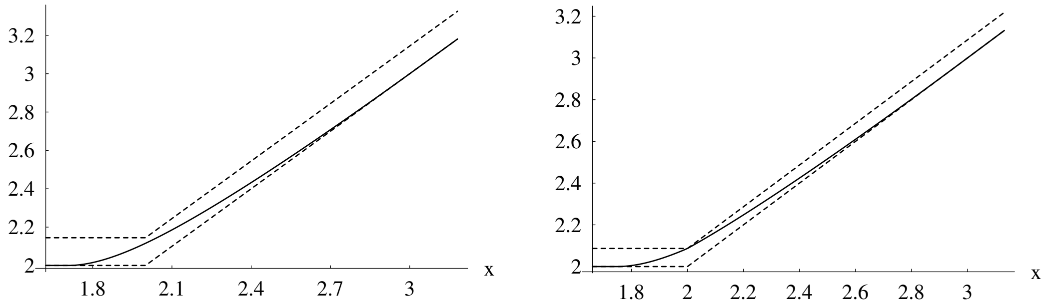

Figure 4: The Values (uniform curves) and Exercise Payoffs (dashed curves)

θ	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4
x^*_θ	1.9054	1.7077	1.4662	1.2294	1.0203	0.8452	0.7027	0.5879
y^*_θ	2.6015	3.0293	3.7028	4.5103	5.4151	6.3932	7.4264	8.501
Δ_θ	0.0394	0.1157	0.2167	0.3282	0.4414	0.5512	0.6546	0.7505
Δ_θ/p	0.0197	0.0578	0.1083	0.1641	0.2207	0.2756	0.3273	0.3752

Table 2: The Optimal Boundaries and the Critical Penalty

5 Conclusions

We considered the valuation and optimal exercise policy of a contingent contract guaranteeing the holder a minimum monetary payment at exercise and a costly right to terminate the contract before expiry to the issuer in the case where the value of the underlying dividend-paying asset follows a one dimensional but otherwise general time homogenous diffusion. Along the lines of the pioneering work by Kifer (2000) and the subsequent study by Kyprianou (2004), the considered contingent contract was modeled as a δ-penalized version of the minimum guaranteed payment option which was originally analyzed in Guo and Shepp (2001) in a non-strategic setting based on geometric Brownian motion. We presented a set of ordinary first order conditions characterizing the optimal boundaries and stated a set of typically satisfied conditions under which the pair of optimality conditions admit a unique root. The resulting saddle point strategy and value of the game option was then explicitly expressed in terms of the exercise boundaries. Interestingly, our results indicate that the sign of the relationship between increased volatility and the value of the option is positive in this case as well and that higher volatility decelerates rational exercise by expanding the continuation region where waiting is optimal.

There are several directions towards which our analysis could be extended. First, even though the assumed perpetuity of the contract is acceptable in the cases where the maturities are relatively long (say 30-40 years) it is not clear whether our findings and main conclusions would hold in a finite horizon setting. Previous studies of game options and Dynkin games indicate that the optimal policies are very sensitive with respect to the length of the time horizon and, thus, it may very well be the case that at least part of our conclusions would no longer hold in the finite horizon case. A second interesting extension (from the point of view of risk management) of our analysis would be to add spectrally negative jumps into the underlying dynamics. Such an extension would provide valuable information on the impact of downside risk on the optimal exercise policies of both the issuer and the holder. Both of these extensions are outside the scope of the present formulation and left for future research.

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