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ABSTRACT

We study infinitely repeated Prisoners' Dilemma, where one of the players may be demented. If a player gets demented in period t after his choice of action, he is stuck to this choice for the rest of the game. So if his last choice was "cooperate" just before dementia struck him, then he's bound to cooperate always in the future. Even though a demented player cannot make choices any more he enjoys the same payoffs from strategy profiles as he did when healthy. A player may prove he is still healthy by showing a (costly) health certificate. This is possible only as long as the player really is healthy: a demented player cannot get a clean bill of health. We study an asymmetric information game where it is known that player 1 cannot get demented but player 2 may be either a "healthy" type who will never be demented or a "dementible" type who eventually will get demented. We study when cooperation can be maintained in a perfect Bayesian equilibrium with at most health check.

JEL Classification: C72

Keywords: prisoners' dilemma, dementia, co-operation

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1 Introduction

Asymmetric information easily leads to inefficienct outcomes; either to signalling or selection. We consider an infinitely repeated prisoners' dilemma where one of the players may be of a dementible type. This means that whenever he gets demented he keeps on making the same choice forever. It is clear that this reduces the possibility of co-operation in two ways. First, the dementible type is afraid that he gets demented and will be exploited by his opponent. This fear is warranted as the other player, the non-dementible type, has an incentive to find out whether his opponent has demented.

In this setting co-operation can be supported by a strategy where the dementible type signals that he has not demented. This happens in designated periods when both players play Defect for one period and then return to the co-operative choices. If the demented player fails to play Defect in a designated period the other player finds out that his opponent has demented. Signalling is costly and the players would like to avoid it as far as possible.

We determine conditions for an equilibrium where this kind of signalling takes place only once as well as for an equilibrium where health is not checked at all. To further simplify we assume that the dementible player can present a costly health certificate. This has exactly the same effect as both players choosing Defect but leads to easier analysis.

As one expects an equilibrium where health is not checked at all requires that the probability of being of a dementible type is low enough. In an equilibrium where health is checked exactly once the player who is healthy for sure cares only about the probability of his opponent being healthy. As being dementible is private information it is the probability of getting demented (conditional on being of a dementible type) that determines the behavior of the player whose health is uncertain. It turns out that the further in the future the time for health check the smaller the conditional probability of getting demented has to be.

There is a large literature on repeated games but this kind of applications we are not aware of. Superficially somewhat related work can be found in repeated games with forgetful players. An example with a good list of references is Özyurt (2008); this literature, however, focuses on the consequences of imperfect monitoring, and there are no methodological similarities with the present work.

The seminal paper dealing with asymmetric information in Prisoners' Dilemma is Kreps *et.al.* (1982). They study when cooperation can be maintained in a finitely repeated game when rationality is not common knowledge. In their model one of the players may be irrational who plays *tit for tat*. The rational player in this model faces the same problem as the healthy type in our model: to continue cooperation or to find out whether the opponent is irrational. However the analyses in these papers are totally different, and we cannot utilize their methods or results in our work.

2 The Model

Players 1 and 2 play infinitely repeated Prisoners' Dilemma with the following stage game:

	c	d
c	$2, 2$	$0, 3$
d	$3, 0$	$1, 1$

Figure 1: Stage game payoff matrix.

Players have the same discount factor δ, $0 < \delta < 1$, and they maximize expected discounted sum of per period payoffs. It is common knowledge that player 1 is healthy, and that player 2 is healthy with prior probability $1 - q$ and dementible with prior probability q. A dementible player gets demented with probability p, $0 < p < 1$, identically and independently each period as long as he is healthy. Once struck by a dementia, he will stay demented for the rest of the game. We assume that player 2 may get demented just before period 0 when the game starts. If player 2 is still healthy in the beginning of period $t = 0, 1, \ldots$, he may get demented just after the stage game actions in period t have been chosen. A healthy person cannot get the dementia.

Player's type (healthy or dementible) is his private information. Denoting by T_i

the type set of player i, we have that $T_1 = \{1\}$ and $T_2 = \{H, D\}$, where H is the type of player 2 who cannot get dementia. Each period player 2 may prove he is not yet demented by showing a health certificate that costs C units of utility, $0 < C \leq 1$. This action is not available to a player who already is demented, and as player 1 is known to healthy, he doesn't have to prove it any more.

We assume that health checks are performed at the same time when the stage game is played, so the news that player is demented or healthy today cannot yet affect the choices made in the stage game. A clean bill of health makes it possible to update beliefs for the next period's game. But it is possible that player gets demented before the next period, although he has been verified to be non-demented today.

The problem of a demented player is that he cannot make choices any more although he gets the same payoffs as he did when he was healthy. We model this by assuming that if player chose action a_t in the stage game of period t and he gets demented immediately after that, then his choice will be a_t in all subsequent periods.

The prior probability that player 2 gets demented already before the stage game in period $t = 0$ is played is qp. The prior probability that he gets a dementia before stage game in period t is played is

$$qP_{t+1} = q[1 - (1 - p)^{t+1}]. \tag{1}$$

These probabilities are the beliefs of player 1. Player 1 knows in the beginning of each period that he is healthy if he is healthy. So for example in the beginning of period 0 player 2 believes that he will get demented before the game in period t is played is P_t, while it is given by equation (1) for player 1.

If player 2 is able to show in period $t = 0, 1, \ldots$ that he is still healthy, then the updated belief that he is dementible is

$$q_t = \frac{q[1 - P_{t+1}]}{q[1 - P_{t+1}] + 1 - q} = \frac{q(1 - p)^{t+1}}{q(1 - p)^{t+1} + 1 - q}. \tag{2}$$

which goes to zero as t tends to infinity, for any prior $q \in (0, 1)$. Note that this is the updated belief both in the case that the health is checked the first time in period

t and in the case when health has already been checked in periods $k = 0, \ldots, t-1$ and again in period t.

3 Results

We will study the existence of perfect Bayesian equilibria such that health is checked at most once and there is cooperation until somebody has deviated or it turns out that player 2 has got the dementia. Deviations (choosing d when c should have been chosen, or not taking the health examination in the agreed upon date) are punished by playing d ever after. Let us study what kind of restrictions the parameters q, p, C and δ must satisfy in order to have this kind of cooperative equilibrium. We proceed by analysing separately each type's incentives to cooperate.

3.1 Incentives of player 1

Recall that player 1 is known to be healthy, he never gets the dementia and he doesn't have to take the health exam. The payoff from (c, c) every period is 2 (we normalize each periods payoffs by multiplying by $1-\delta$ as usual). If there are no health checks player 1 prefers cooperation to deviation in period t if

$$\begin{aligned} 2 \geq 3(1-\delta) + \delta\{&[1-q+q(1-P_{t+1})(1-p)] + \\ &+ [q(1-P_{t+1})p + qP_{t+1}]3\}. \end{aligned} \tag{3}$$

On the right hand side of the inequality, the term $[1-q+q(1-P_{t+1}(1-p)]$ is the probability that player 2 is not demented when the period t game is played and doesn't get the dementia during the next night. In these cases the deviation of player 1 is punished by 2 by reverting to d forever. The other term is the complement of this probability times the utility 3 because in this case player 2 may exploit his demented partner. Equation 3 is equivalent to

$$\delta[2 - 2q[(1-P_{t+1})p + P_{t+1}]] \geq 1 \tag{4}$$

From this we can deduce the critical value for δ:

$$\delta \geq \frac{1}{2 - 2q[(1-P_{t+1})p + P_{t+1}]} \tag{5}$$

Since P_{t+1} goes to 1 as t goes to infinity, the denominator of the right hand side of this inequality is decreasing in t. Hence this inequality must hold in the limit as t goes to infinity and we get

$$\delta \geq \frac{1}{2(1-q)} \quad \text{and} \quad q \leq \frac{2\delta - 1}{2\delta} \tag{6}$$

Note that (6) gives gives also the upper bound for an *updated* prior (replace q by q_1) so that exactly one health check is enough at least as far as player 1 is concerned. If the prior already satisfies this inequality (equivalently, the discount factor satisfies (6)) then player 1 would be happy to cooperate without any health checks.

For expressions (3) and (6) one should also check that deviation is not profitable if health has been checked once in period $k < t$. In this case in expression (3) q is replaced by q_k and P_{t+1} by P_{t-k}. One then gets condition

$$\delta \geq \frac{q(1-p)^{k+1} + 1 - q}{2\left[q(1-p)^{t+2} + 1 - q\right]}$$

This is hardest to satisfy when t grows indefinately, and in the limit

$$\delta \geq \frac{q(1-p)^{k+1} + 1 - q}{2(1-q)}$$

This is a less stringent condition than (6).

Suppose now that there is one health check and this takes place in period $t = 0, 1, \ldots$. We will now compute the critical values of discount factors and probabilities such that player 1 has proper incentives to cooperate. First, let us check when player 1 wants to cooperate in the period t when health is checked:

$$\begin{aligned} &2(1-\delta) + \delta\{[1 - q + q(1 - P_{t+1})]2 + 3qP_{t+1}\} \geq \\ &\quad 3(1-\delta) + \delta\{[1 - q + q(1 - P_{t+1})(1-p)] + [q(1 - P_{t+1})p + qP_{t+1}]3\}. \end{aligned} \tag{7}$$

Note that continuation values in the left and right hand sides are different. This is because if player 1 cooperates and learns that player 2 is still healthy, he doesn't learn whether or not player 2 gets demented the following night. But if player 1 cheats, he learns also if player 2 gets demented the following night, because in this case player 2 is unable to stop cooperation but chooses c always.

Simplifying expression (7) we get the following

$$\begin{aligned}
& \delta \{2(1-qP_{t+1})+3qP_{t+1}\} \\
\geq \; & 1-\delta+\delta\{1-qP_{t+1}-pq(1-P_{t+1})+3pq(1-P_{t+1})+3qP_{t+1}\}
\end{aligned}$$

which simplifies to

$$\delta\{2+qP_{t+1}\} \geq 1+\delta 2qP_{t+1}+2pq(1-P_{t+1})$$

which is equivalent to

$$\delta\{2+qP_{t+1}-2qP_{t+1}-2pq(1-P_{t+1})\} \geq 1$$

which is equivalent to

$$\delta\{2-q[2p(1-P_{t+1})+P_{t+1}]\} \geq 1 \tag{a1}$$

On the left hand side of (a1) the coefficient of P_{t+1}, namely $-q(-2p+1)$, is positive if $p > \frac{1}{2}$. As P_{t+1} is increasing in t, the condition is most difficult to satisfy when $t = 0$. Then we get

$$\delta \geq \frac{1}{2-qp(3-2p)} \tag{a2}$$

If $p < \frac{1}{2}$ the coefficient of P_{t+1} is negative, and (a1) is most difficult to satisfy when t grows indefinitely. In the limit we get

$$\delta \geq \frac{1}{2-q}$$

One gets this condition also when $p = \frac{1}{2}$.

It is easy to show that the denominator in (a2) is greater than the denominator in (6), namely $2(1-q)$,and consequently (6) is the most stringent of conditions (6), (a1) and (a2).

We must still check that if the only health check is in period t, then player 1 doesn't want to deviate at any period $k < t$. The following inequality must hold.

$$\begin{aligned}
& 2(1-\delta^{t-k})+\delta^{t-k}\{[1-q+q(1-P_{t+1})]2+3qP_{t+1}\} \geq \\
& \quad 3(1-\delta)+\delta\{[1-q+q(1-P_{k+1})(1-p)]+[q(1-P_{k+1})p+qP_{k+1}]3\}.
\end{aligned} \tag{8}$$

This inequality describes the utilities from cooperation and defection just before the period k stage game is played. There has been no prior health checks so player 1 hasn't been able to update his prior q. However, he knows that the dementible type of player 2 will get the dementia sooner or later, and so the greater is k the greater is the probability that 2 is already demented. If player 1 cooperates, he learns after health check in period t whether 2 has already got the dementia. But if he defects already in period k, he will learn after the stage game of period $k+1$ if player 2 has demented some time before $k+1$. Inequality (8) simplifies to

$$2(1-\delta^{t-k}) + \delta^{t-k}[2 + qP_{t+1}] \geq 3(1-\delta) + \delta\{1 + 2q[P_{k+1} + p(1-P_{k+1})]\}. \tag{9}$$

The following is an equivalent formulation

$$\delta^{t-k} q P_{t+1} \geq 1 + 2\delta \left\{-1 + pq + (1-p)qP_{k+1}\right\}$$

Now LHS is increasing in k and attains the smallest value when $k = 0$. The right hand side is increasing in k and attains the highest value when $k = t-1$ (this is the case where $k < t$). With these values we get

$$\delta^t q P_{t+1} \geq 1 + 2\delta \left\{-1 + pq + (1-p)qP_t\right\}$$

which is equivalent to

$$q\left\{\delta^t\left[1-(1-p)^{t+1}\right] - 2\delta(1-p) + 2\delta(1-p)^{t+1}\right\} \geq 1 + 2\delta\left\{-1+pq\right\} \tag{a3}$$

Evaluating the LHS of the above expression at $t+1$ one gets

$$q\left\{\delta\delta^t\left[1-(1-p)^{t+1} + p(1-p)^{t+1}\right] - 2\delta(1-p) + 2\delta(1-p)^{t+1} - 2\delta p(1-p)^{t+1}\right\}$$

which is equivalent to

$$q\left\{\delta\delta^t\left[1-(1-p)^{t+1}\right] - 2\delta(1-p) + 2\delta(1-p)^{t+1} + \delta p(1-p)^{t+1}(\delta^t - 2)\right\}$$

Comparing this to the LHS of (a3) evaluated at t one sees that in the curly brackets the first three terms are less than the LHS of (a3) and the last term is negative. Thus, increasing t decreases the LHS of (a3), and the most stringent conditions for δ is got when t increases indefinately. In the limit

$$\delta \geq \frac{1}{2(1-q)}$$

which is the same as expression (6).

Lemma 1 *Player 1 is willing to cooperate for all $C \leq 1$ and $p < 1$ if condition (6) holds for δ and q. In all cases when one health check is required, the updated prior must satisfy* (6) *if player 2 has tested healthy.*

3.2 Incentives for the healthy type H of player 2

If there are no health checks or the only health check has already been performed, and 2 has been deemed non-demented at that time, then the constraint for the healthy type H is the following.

$$2 \geq 3(1-\delta) + \delta \tag{10}$$

which doesn't set any new constraints for the existence of equilibrium, since (15) demands that $\delta \geq 1/2$ which is less than what (6) demands.

If the only health check is in period t, then the following inequality must hold for H so that he doesn't cheat in that period.

$$(2-C)(1-\delta) + \delta 2 \geq 3(1-\delta) + \delta. \tag{11}$$

This is equivalent to

$$\delta \geq \frac{1+C}{2+C}. \tag{12}$$

Let's no set a condition such that H doesn't want deviate in any earlier period $k < t$ either.

$$2(1-\delta^{t-k}) + \delta^{t-k}[2-(1-\delta)C] \geq 3(1-\delta) + \delta. \tag{13}$$

This is equivalent to

$$2 - \delta^{t-k}(1-\delta)C \geq 3 - 2\delta,$$

and we see that the left hand side is decreasing in k. Hence by setting $k = t-1$ gives the tightest constraint. Doing this substitution gives us after some simplifying steps

$$C\delta^2 + (2-C)\delta - 1 \geq 1. \tag{14}$$

Solving for feasible δ's gives us

$$\delta \geq \frac{\sqrt{4+C^2} - (2-C)}{2C}. \tag{15}$$

Since the left hand side of (19) is decreasing in C, inserting $C = 1$ in (20) gives the condition that H doesn't deviate in any period $k < t$ no matter what is C.

$$\delta \geq \frac{\sqrt{5}-1}{2}. \tag{16}$$

Let's summarize our findings concerning the healthy type H of player 2.

Lemma 2 *The healthy type H is willing to cooperate for all values of $C \leq 1$, $p, q \leq 1$ and no matter when the health check is performed, if* (12) *and* (16) *hold:*

$$\delta \geq max\left\{\frac{1+1}{2+1}, \frac{\sqrt{5}-1}{2}\right\} = \frac{2}{3}.$$

3.3 Incentives for the dementible type D of player 2

If there are no health checks or the only check has already been done, then the type D has the same incentive constraint (10) as type H.

If the only health check is planned for day t, then D has the same incentive constraint (12) as type H for not cheating at t, since D cannot get demented in the beginning of period t before his health is checked.

If health is checked in period t then type D doesn't want to deviate in any period $k < t$ if the following holds.

$$2(1-\delta^{t-k}) + \delta^{t-k}(1-P_{t-k})(2-(1-\delta)C) \geq 3(1-\delta) + \delta.$$

Recall that $P_{t-k} = 1-(1-p)^{t-k}$. Making this substitution gives us

$$2(1-\delta^{t-k}) + \delta^{t-k}(1-p)^{t-k}(2-(1-\delta)C) \geq 3(1-\delta) + \delta. \tag{17}$$

Note that when $t-k$ increases it becomes more difficult to satisfy inequality (17). Since (17) must hold for all $k < t$, this inequality gives an upper bound for t, the date at which the only health check is performed, for fixed parameter values of δ, p and C. Let's study more carefully the case when the discount factor goes to 1.

Lemma 3 *Given $t > 0$, there exists $\underline{\delta} < 1$ such that inequality (17) holds for all $k < t$ and all $\delta > \underline{\delta}$, if $p < 1 - 2^{-1/t}$.*

Proof. Let δ go to 1 in (17) and require that the resulting inequality holds strictly:

$$(1-p)^{t-k} > \frac{1}{2}.$$

This inequality is most difficult to satisfy when $k = 0$. Making this substitution gives us after some simplifications

$$p < 1 - 2^{-1/t}.$$

When p satisfies this inequality, there exists a least $\underline{\delta}$ such that (17) holds for all $k < t$ and all $\delta > \underline{\delta}$. ■

3.4 Existence of equilibrium

A perfect Bayesian equilibrium specifies beliefs and actions of each player after all histories. We haven't yet discussed what happens if player 2 presents a health certificate already *before* the planned testing day. One may wonder if it is not possible that the healthy type H of player 1 could try to signal in this way that he is not dementible.

But type H has no such incentives in the equilibria satisfying the conditions of Proposition 1. The best thing that could happen to H after such a deviation is that player 1 believes that player 2 is not dementible and then they cooperate forever. But H gets the same outcome for certainty by waiting to period t and then presenting his health certificate. Since testing is costly, type H likes to postpone it.

However, if such a deviation is rewarded by the belief that player 2 cannot be dementible, the the dementible type D of player 2 may have an incentive to present a health certificate already before period t.

This gives the clue how to define beliefs if an unexpected health certificate is presented: player 1 believes that player 2 is dementible with probability 1. These beliefs make sense since the dementible type only could possibly benefit from such a deviation.

If player 2 is believed with probability 1 to be dementible, then the belief of player 1 is $P_{t+1} = 1 - (1-p)^{t+1}$ that 2 gets dementia before period $t = 0, 1, \ldots$. If player 1 had never any incentives to check if player 2 is already demented, then cooperation could be maintained. Player 1 could check whether or not player 2 is already demented by defecting and seeing how 2 reacts.

But we see from equations (5) and (6) immediately that player 1 will do such a check when t is sufficiently high. But then cooperation ends at that period whether or not player 2 has already get the dementia. Hence the off the equilibrium path beliefs as specified above really prevent player 2 from showing a health certificate too early. We have the following result.

Proposition 1 *If the prior probability q satisfies (6), i.e., $\delta \geq 1/(2(1-q))$, then no health checks are need in a perfect Bayesian equilibrium in which cooperation takes place every period. If (6) does not hold for the prior q, then let $t > 0$ any period such that the updated prior q_1 satisfies* (6)*: $\delta \geq 1/(2(1-q_1))$. There exists a $\underline{\delta} < 1$ such that for all cost levels $C \leq 1$, for all $\delta > \underline{\delta}$, and for all probabilities p satisfying $p < 1-2^{-1/t}$ there exists a perfect Bayesian equilibrium with cooperation every period and exactly one health check that takes place in period t.*

Proof. The result follows immediately from Lemmata 1 - 3. ■

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Aboa Centre for Economics (ACE) was founded in 1998 by the departments of economics at the Turku School of Economics, Åbo Akademi University and University of Turku. The aim of the Centre is to coordinate research and education related to economics in the three universities.

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Aboa Centre for Economics (ACE) on Turun kolmen yliopiston vuonna 1998 perustama yhteistyöelin. Sen osapuolet ovat Turun kauppakorkeakoulun kansantaloustieteen oppiaine, Åbo Akademin nationalekonomi-oppiaine ja Turun yliopiston taloustieteen laitos. ACEn toiminta-ajatuksena on koordinoida kansantaloustieteen tutkimusta ja opetusta Turun kolmessa yliopistossa.

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