Hannu Vartiainen

A conflict-free arbitration scheme in a large population

Aboa Centre for Economics

Discussion Paper No. 34
Turku 2008

ISSN 1796-3133

Turun kauppakorkeakoulun monistamo
Turku 2008

Hannu Vartiainen

A conflict-free arbitration scheme in a large population

Aboa Centre for Economics
Discussion Paper No. 34
June 2008

ABSTRACT

This paper studies allocations that can be implemented by an arbitrator subject to the constraint that the agents' outside option is to start bargaining by themselves. As the population becomes large, the set of implementable allocations shrinks to a singleton point - the conflict-free allocation. Finally, the conflict-free allocation can be implemented via a simple "lobbying" game where parties composed of agents with similar preferences bid for the right to be the first proposer in a bargaining game among the parties, i.e. in the "political game".

JEL Classification: C72, C78

Keywords: non-cooperative bargaining, arbitration, implementation

Contact information

Turku School of Economics, Department of Economics,
Rehtorinpellonkatu 3, FIN-20500 Turku.
E-mail: hannu.vartiainen(a)tse.fi.

Acknowledgements

I thank Klaus Kultti and Hannu Salonen for comments and discussions.

1 Introduction

An arbitrator's problem is how to divide a pie fairly among a society of agents. In the standard bargaining scenario, which is often motivated via the arbitration metaphor, fairness of the outcome is defined with respect to a prespecified disagreement outcome.[1] But it is not clear why the agents should implement the disagreement point in the case they cannot agree on the division. Namely the agents can always - since they *form* the society - take the decision into their own hands. Disagreement should therefore reflect the potentially arduous process of bargaining among the individuals.

This paper studies allocations that can be implemented by the arbitrator subject to the constraint that the agents' outside option is to start bargaining by themselves. More precisely, the arbitrator serves as the first proposer in the game where the agent who rejects the arbitrator's proposal becomes the first proposer in the unanimity bargaining game (a multiplayer extension of Rubinstein, 1982). The arbitrator's objective is to implement a Pareto optimal outcome.

Our main focus is in large societies. Assuming stationary equilibrium in the bargaining game, we show that the set of feasible allocations shrinks to a singleton set as the number of agents becomes large. This is due to the fact that as the number of agents' grows, the bargaining power of a single agent (measured in as the first mover advantage) becomes small. The limit allocation can be interpreted as the *conflict-free* outcome: no individual wants to challenge it by starting to bargain. We give a simple finite characterization of the conflict-free allocation (under the hypothesis that the agents' time-preferences are drawn from a finite set of preferences).

Admittedly, implementing a rule in a large society that is contingent on the individuals' time-preferences is too unrealistic. The second task of the paper is to offer a practical solution the implementation problem. We identify a simple mechanism that induces the conflict-free solution in a unique equilibrium. The content of the mechanism is the following. First all agents with similar preferences are grouped together to form a "party". The party, or its representative, acts on behalf of all its members - gains and losses of the party are divided evenly among the members of the group. The task of parties is to engage into bargaining over common resources - they do "politics". However, before doing that they compete over the right to make the first proposal in the bargaining game - they give promises of how much of their own good they will give up for the common good. The winning bid is added to the pool of common resources that is later shared via bargaining.

It is shown that the unique equilibrium outcome (under the stationarity as-

[1] For the arbitration interpretation of bargaining solutions, see e.g. Luce and Raiffa (1957).

sumption) coincides with the conflict-free allocation. Hence we obtain a version of the Core convergence result: the outcome that is obtained in large bargaining markets with negligent bargaining power can be simulated in a small market with a "Walrasian" auctioneer.

This paper is related to Kultti and Vartiainen (2007a) who study convergence of bargaining outcomes in a related model of large population. The driving force behind the convergence in there as well as in here is that as the number of players increases the bargaining power of individual players vanishes. An important observation is that convergence has different characteristic than when the bargaining power vanishes due to speeding up the bargaining process (Binmore et al., 1986). In particular, the convergence point under large population is not related to the Nash bargaining solution.[2]

First we define the set up and specify the bargaining game. Then we establish the feasible arbitrations schemes. Finally, the implementation result is proven. All omitted proofs are in the appendix.

2 The set up

There is a society of agents, distributing common resources. The primitive of the model is the set of agents, their time-preferences and resources. There are $1, ..., n$ agents, each of them endowed with one unit of resources. As an agent enters the society, his resources become part of the common pool of resources.

Time preferences of the agent i has the representation $u_i(x_i)\delta^t$, where $x_i \in \mathbb{R}_+$ is the agent's consumption. We assume that the publicly observable utility functions $u_1, ..., u_n$ are drawn independently from a *finite* set U, whose cardinality is also denoted by U. The probability of $u \in U$ is λ_u, a rational number. We assume that each $u \in U$ is concave and continuously differentiable function and that $\delta \in (0, 1)$.[3]

Given u_i, define a function v_i that specifies the present consumption value of x_i in date 1 such that

$$u_i(v_i(x_i)) = u_i(x_i)\delta, \quad \text{for all } x_i \in [0, 1]. \tag{1}$$

By the concavity of u_i, $u_i'(x_i)/u_i(x_i)$ is a monotonically decreasing, strictly positive

[2]However, see Thomson and Lensberg (1989).

[3]Weaker conditions would suffice (see Fishburn and Rubinstein, 1981, or Kultti and Vartiainen 2007a,b). The current choice is for simplicity.

function under all $x_i > 0$, and hence (since $v_i(x_i) < x_i$), for all $x_i > 0$,

$$\begin{aligned} v_i'(x_i) &= \frac{u_i'(x_i)\delta}{u_i'(v_i(x_i))} \\ &= \frac{u_i'(x_i)/u_i(x_i)}{u_i'(v_i(x_i))/u_i(v_i(x_i))} \\ &\in (0,1). \end{aligned} \tag{2}$$

That is,

$$\frac{dv_i^{-1}(x_i)}{dx_i} = \frac{1}{v_i'(x_i)} > 1, \text{ for all } x_i \geq 0. \tag{3}$$

This property will be used when we prove the existence of a stationary equilibrium.

3 The unanimity bargaining game

For later purposes, we discuss of the bargaining game in a more general level than the current set up requires. Let the size of shareable resources be $X > 0$, and the group of agents finite set N (whose cardinality we also denote by N). The set of allocations is

$$S = \left\{ x \in \mathbb{R}_+^N : \textstyle\sum_{i \in N} x_i \leq X \right\}.$$

Given N and X, we define a unanimity bargaining game $\Gamma^N(X, i)$ as follows: At any stage $t = 0, 1, 2, ...,$

- Player $i(t) \in N$ makes an offer $x \in S$. Players $j \neq i(t)$ accept or reject the offer in the ascending order of their index.[4]
- If all $j \neq i(t)$ accept, then x is implemented. If j is the first who rejects, then j becomes $i(t+1)$.
- $i(0) = i$.

We focus on the stationary subgame perfect equilibria, simply equilibria or SPE in the sequel, of the game, where:

1. Each $i \in N$ makes the same proposal $x(i)$ whenever he proposes.
2. Each i's acceptance decision in period t depends only on x_i that is offered to him in that period.

[4]The order in which players response to a proposal does not affect the results.

We now characterize equilibria.[5] We first state an important intermediate result.

Lemma 1 *For any* $Y > 0$ *and* $c \in \mathbb{R}^N_{++}$*, there is a unique* $x = (x_i)_{i \in N}$ *and* $d > 0$ *such that*

$$\begin{aligned} c_i x_i &= v_i(c_i(x_i + d)), \text{ for all } i \in N, & (4) \\ \sum_{i \in N} x_i &= Y. & (5) \end{aligned}$$

Lemma 2 x^i *is a stationary equilibrium outcome of* $\Gamma^N(X, i)$ *if and only if* $x^i = (x_i + d, x_{-i})$*, for* x *and* $d > 0$ *such that*

$$\begin{aligned} x_i &= v_i(x_i + d), \text{ for all } i \in N, & (6) \\ \sum_{i \in N} x_i &= X - d. & (7) \end{aligned}$$

Choosing $Y = X - d$ and combining Lemmata 1 and 2 the following result is obtained.

Proposition 1 *A stationary equilibrium of* $\Gamma^N(X, i)$ *exists. Moreover, it is unique.*

Thus in our $n-$player set, the pool of shareable resources is n and the game is $\Gamma^{\{1,\ldots,n\}}(n)$.

Corollary 1 *A stationary equilibrium of* $\Gamma^{\{1,\ldots,n\}}(n, i)$ *exists. Moreover, it is unique.*

4 Arbitrator's problem

Given the set of agents $1, ..., n$ and the amount of shareable resources n, let there be an arbitrator who suggests an allocation to the society subject to the constraint that every agent must accept the proposal. A rejection triggers a bargaining game. That is, an agent i who rejects the offer becomes the first proposer in the unanymity game $\Gamma^{\{1,\ldots,n\}}(n, i)$, starting.with one period delay. Given that the arbitrator wants to induce a Pareto optimal outcome, the indeucable outcomes must be such that all agents accept the proposal.

Proposition 2 *Let* x *satisfy (6) and (7) for some* d*. Then allocation* y *is a feasible arbitration scheme if and only if* $\sum_i x_i \leq n$ *and* $x_i \geq y_i$*, for all* $i = 1, ..., n$*.*

By (7), the set of feasible allocations y is nonempty.

[5] Our treatment draws on Krishna and Serrano (1996).

4.1 Large Population: Conflict-free allocation

We now establish that when the number of agents increases, the set of feasible outcomes shrinks and reaches, in the limit, a singleton set. To characterize the limit, let $y^* \in \mathbb{R}^U$ and $d^* > 0$ satisfy

$$y_u^* = v\left(y_u^* + d^*\right), \text{ for all } u \in U, \tag{8}$$
$$\sum_{u \in U} \lambda_u y_u^* = 1. \tag{9}$$

See Figure 1 where $U = \{u, \hat{u}\}$ and $\lambda_u = \lambda_{\hat{u}}$. By construction, $u(y_u^*) = \delta u(y_u^* + d^*)$ and $\hat{u}(y_{\hat{u}}^*) = \delta \hat{u}(y_{\hat{u}}^* + d^*)$.

Then construct the *conflict-free* allocation $x^* = (x_1^*, x_2^*, ...) \in \mathbb{R}^\infty$ as follows:

$$x_i^* = y_u^* \text{ if } u = u_i, \text{for all } i = 1, 2, ..., \text{ for all } u \in U. \tag{10}$$

Since each λ_u is rational, it is not diffcult to see from Lemma 1 that the desired y^* and d^* do exist.

Proposition 3 *As $n \to \infty$, an allocation y is a feasible arbitration scheme if and only if it coincides with the conflict-free allocation x^*.*

4.2 Implementation procedure: Competitive lobbying

We now construct a simple mechanism that implements the conflict-free allocation x^* when n is large. Let the agents form homogenous groups - "parties" - based on their preferences. That is for each $u \in U$ all the agents of type u constitute a group. By the law of large numbers, the share of the agents in the $u-$group is λ_u of the set of all agents as n becomes large. Let each group select one agent as the representative of the group who is entitled to bargain and trade on behalf of the whole group. Gains and losses of the group are divided equally among its members.

Consider then a market where the right to be the first proposer in a bargaining game is sold after a bidding contest to one of the U groups (or their representatives). The right is sold to the group that makes the highest bid (break ties by using randomization). Once the price p is paid by the winner it is added to the pool of resources over which bargaining then takes places.

The bidding contest can be interpreted as a "lobbying" game where all the groups, "political parties", bid for the right to be the leader in the bargaining game, "political process", that follows the bidding contest. Only one group can serve as the initial proposer and hence enjoy from the bargaining power that comes with it.

More formally, since the agents' utility functions are i.i.d, λ_u is the limit share of type u agents in the population as the population becomes large. Since all gains and losses of the group are divided equally among its members, if z_u is the $u-$group's relative share of the total shareable resources, an $u-$type agent's consumption is approximated by $\lambda_u^{-1} z_u$ as n becomes large. It is convenient to describe the $u-$group's agents utilities directly in terms of z_u. The utility function $\bar{u}$ of the representative of the $u-$group with respect to z_u is:

$$\bar{u}(z_u) = u\left(\lambda_u^{-1} z_u\right), \text{ for all } z_u \in [0,1]. \tag{11}$$

Function $\bar{u}$ is convex and continuous since u is. Define the function $\bar{v}$ such that

$$\bar{u}(\bar{v}(z_u)) = \bar{u}(z_u)\delta, \text{ for all } z_u \in [0,1].$$

Note that (3) applies to $\bar{v}$ as well.

Denote the set of normalized utility functions by $\bar{U}$. The rules of the bidding mechanism Γ^* are formally as follows: Players in the set $\bar{U}$ first cast their bids. Given the normalized resources 1, if $i \in \bar{U}$ wins the bidding contest with bid p, then the bargaining game $\Gamma^{\bar{U}}(1+p:i)$, with i as the first proposer, is initiated.

Our claim is that this mechanism implements the conflict-free arbitration scheme.

First, let $z_j(X)$ be what a receiver j gets in the game $\Gamma^{\bar{U}}(X:i)$. By (6) and (7) and Proposition 1, there is $z(X) = (z_1(X), ..., z_n(X))$ that is the unique solution to

$$z_i(X) = \bar{v}_i\left(X - \textstyle\sum_{j\neq i} z_j(X)\right) \text{ for all } i. \tag{12}$$

By the Implicit Function Theorem, $z_i(\cdot)$ is continuous.

Lemma 3 *$z_i(X)$ is strictly increasing in X, for all i.*

Proof. Rewrite condition (12) as

$$v_i^{-1}(z_i(X)) - z_i(X) = X - \sum_{i\in U} z_j(X).$$

By (3), and since z_i is a continuous function, z_i is strictly increasing if $X - \sum z_j(X)$ is. Since this applies to all i, $\sum z_j(X)$ is strictly increasing if $X - \sum z_j(X)$ is. But then, since $\sum z_j(X)$ being weakly decreasing means that $X - \sum z_j(X)$ is strictly increasing, it cannot be the case that $\sum z_j(X)$ is not strictly increasing. Thus $\sum z_j(X)$ is strictly increasing and hence z_i is strictly increasing. ■

By (6) and (7) there is a unique $(z_i^*)_{i\in\bar{U}}$ and $p^* > 0$ such that

$$z_i^* = \bar{v}(z_i^* + p^*), \text{ for all } i \in \bar{U}, \tag{13}$$

$$\sum_{i\in\bar{U}} z_i^* = 1. \tag{14}$$

Lemma 4 *In equilibrium of the bidding mechanism* Γ^*, p^* *is the winning bid and* $z^*_{\bar{u}}$ *is the* $u-$*group's share of resources, for* z^* *and* p^* *as specified in (13) and (14).*

Proof. *Only if:* First we argue that there are at least two highest bids. Suppose that there is a single highest bid. Then buying the proposing right with price p must be at least profitable as the opportunity cost of lowering the bid by a small $\varepsilon > 0:$

$$\left[1 + p - \sum_{j \neq i} z_j(1+p)\right] - p \geq \left[1 + p - \varepsilon - \sum_{j \neq i} z_j(1+p-\varepsilon)\right] - (p - \varepsilon).$$

That is

$$0 \geq \sum_{j \neq i} [z_j(1+p) - z_j(1+p-\varepsilon)].$$

But by Lemma 3 this cannot hold.

Thus at least two bidders bid the winning bid p. Then buying the proposing right under p must be at least profitable as the opportunity cost of letting the other highest bidder win with price $p:$

$$\left[1 + p - \sum_{j \neq i} z_j(1+p)\right] - p \geq z_i(1+p). \tag{15}$$

Since increasing ones bid is not profitable for the losing bargainer j that bids p,

$$\left[1 + p + \varepsilon - \sum_{k \neq j} z_k(1+p+\varepsilon)\right] - (p+\varepsilon) \leq z_j(1+p), \text{ for all } \varepsilon > 0. \tag{16}$$

Since z_k is continuous and (16) holds for all $\varepsilon > 0$, it follows that

$$\left[1 + p - \sum_{k \neq j} z_k(1+p)\right] - p \leq z_k(1+p). \tag{17}$$

Combining (15) and (17) gives

$$1 = \sum_{i \in \bar{U}} z_i(1+p).$$

Thus by (12),

$$z_i(1+p) = \bar{v}_i(z_i(1+p) + p), \text{ for all } i = 1, ..., n.$$

By Lemma 5, this yields $z_i(1+p) = z_i^*$ for all i, and $p = p^*$.

If: Let all U bargainers bid $p = p^*$. By construction, $z_i(1+p^*) = z_i^*$ for all $i \in \bar{U}$. We show this does constitute an equilibrium. Since $n > 1$ and

$$1 = \sum_{i \in \bar{U}} z_i(1+p^*), \tag{18}$$

it follows that

$$\left[1 + p^* - \sum_{j \neq i} z_j(1+p^*)\right] - p^* = z_i(1+p^*).$$

Thus *decreasing* one's bid does not have payoff consequences. *Increasing* one's bid by $\varepsilon > 0$ is strictly profitable if

$$\left[1 + p^* + \varepsilon - \sum_{j \neq i} z_j(1+p^*+\varepsilon)\right] - (p^* + \varepsilon) > z_i(1+p^*).$$

That is, by (18),

$$1 - \sum_{j \neq i} z_j(1+p^*+\varepsilon) > 1 - \sum_{j \neq i} z_j(1+p^*),$$

which is in conflict with Lemma 3. Thus all players bidding p^* does constitute an equilibrium. ■

We now argue that from the viewpoint of a single agent, the outcome of the auction among the representatives is the same as the limit outcome of the arbitration process - the conflict-free allocation. Hence the auction mechanism implements the desired arbitration scheme.

Proposition 4 *The conflict-free allocation x^* is the unique equilibrium allocation of the bidding mechanism Γ^*.*

Proof. Since $\bar{u}(\bar{v}(z_u)) = \bar{u}(z_u)\delta$ and (11) imply $u(\lambda_u^{-1}\bar{v}(z_u)) = u(\lambda_u^{-1}z_u)\delta$ and the definition of v implies $u(\lambda_u^{-1}z_u)\delta = u(v(\lambda_u^{-1}z_u))$ we have $\bar{v}(z_u) = \lambda_u v(\lambda_u^{-1}z_u)$. Thus (13) and (14) can be written

$$\begin{aligned} \lambda_u^{-1}z_u^* &= v\left(\lambda_u^{-1}(z_u^* + p^*)\right), \text{ for all } u \in U, \\ \sum_{u \in U} z_u^* &= 1. \end{aligned}$$

By Lemma 4, this characterizes the equilibrium. Letting $y_u^* = \lambda_u^{-1}z_u^*$ for all u, and $d^* = \lambda^{-1}p^*$, this transforms into

$$\begin{aligned} y_u^* &= v\left(y_u^* + d^*\right), \\ \sum_{u \in U} y_u^* \lambda_u &= 1. \end{aligned}$$

Constructing x^* as in (10) now gives the result. ■

5 Concluding remarks

An arbitration scheme should be such that the players cannot do better by rejecting the scheme. We model the situation by assuming that the rejection triggers a bargaining game in which the rejecting agent is the first proposal. When the population becomes large, only one allocation scheme remains feasible. We call such scheme conflict-free. This outcome reflects fairness in a sense that it is an outcome of an imaginary bargaining game in which no player benefits unfairly from the first mover advantage.

With large population arbitration is hard as the optimal outcome is responsive to the agents' preferences. We construct a natural and simple mechanism that implements the conflict-free allocation. Such mechanism has the following interpretation: All agents with similar preferences group to form a "party". The party, or its representative, acts on behalf of its members - gains and losses of the party are divided evenly. The parties engage into bargaining over common resources. However, before doing that they compete over the right to make the first proposal in the bargaining game. The winning bid is added to the pool of common resources that is later shared via bargaining. The equilibrium outcome of this process is precisely the conflict-free allocation. Thus the simple market game (lobbying?) with small number of players can be used to induce a fair and potentially complex allocation with many agents.

A Appendix: Proofs

Proof of Lemma 1. Recall that $c_i > 0$ for all i and $Y \geq 0$. By (3), $v_i^{-1}(x_i) - x_i$ is a continuous and monotonically increasing function. Thus, the function $e_i(\cdot)$ defined by

$$e_i(x_i) := \frac{v_i^{-1}(c_i x_i)}{c_i} - x_i, \text{ for any } x_i \geq 0, \tag{19}$$

is continuous and monotonically increasing.

Define $\bar{e}_i \in (0, \infty]$ by

$$\sup_{x_i \geq 0} e_i(x_i) := \bar{e}_i.$$

Since $e_i(\cdot)$ is continuous and monotonically increasing, also its inverse

$$x_i(e) := e_i^{-1}(e), \text{ for all } e \in [0, \bar{e}_i],$$

is continuous and monotonically increasing in its domain $[0, \bar{e}_i]$. Condition (19) can now be stated in the form

$$x_i(e) = \frac{v_i(c_i(x_i(e) + e))}{c_i}, \text{ for all } e \in [0, \bar{e}_i]. \tag{20}$$

Moreover, since $0 = x_i(0)$ and $\infty = x_i(\bar{e}_i)$, there is, by the Intermediate Value Theorem, a unique $d > 0$ such that

$$\sum_{i=1}^{n} x_i(d) = Y.$$

■

Proof of Proposition 2:. Only if: In a stationary SPE the game ends in finite time. Assume that it never ends. Then each player receives zero. This means that in all subgames each player must get zero. Otherwise there would be a subgame where some offer $y = (y_1, ..., y_n)$ is accepted. Because of stationarity this offer is accepted in every subgame. In particular, player 1 can deviate in the first period and offer $y = (y_1, ..., y_n)$. This is a profitable deviation and constitutes a contradiction with the assumption that there is a stationary SPE where the game never ends.

Assume next that there is a stationary SPE where an offer $x(i)$ by some player $i \in \{1, 2, ..., n\}$, is not accepted immediately. Denote by $z(i)$ the equilibrium outcome in a subgame that starts with an offer $x(i)$ of player i. But now player i could offer $z(i)$ instead of $x(i)$; everyone else would accept the offer as in the stationary equilibrium acceptance depends only on the offer.

Thus, in any equilibrium, $i(t)$'s offer $x(i(t)) = (x_j(i(t)))_{j \in N}$ is accepted at stage $t \in \{0, 1, 2, ..\}$. In stationary equilibrium the time index t can be relaxed from $x(i(t))$. An offer x by i is accepted by all $j \neq i$ if

$$x_j(i) \geq v_j(x_j(j)), \text{ for all } j \neq i. \tag{21}$$

Player i's equilibrium offer $x(i)$ maximizes his payoff with respect to constraint (21) and the resource constraint. By A3, all constraints in (21) and the resource constraint must bind. That is,

$$x_j(i) = v_j(x_j(j)), \text{ for all } j \neq i, \tag{22}$$

and

$$\sum_{i=1}^{n} x_i(j) = X, \text{ for all } j. \tag{23}$$

Since player i's acceptance decision is not dependent on the name of the proposer, there is $x_i > 0$ such that $x_i(j) = x_i$ for all $j \neq i$. By (22), $x_j(i) < x_j(j)$ for all j. Hence there is $d > 0$ such that

$$\sum_{i=1}^{n} x_i = X - d. \tag{24}$$

By (22) and (24), x and d do meet (6) and (7). Since 1 is the first proposer, the resulting outcome is $x(1) = (x_1 + d, x_2, ..., x_n)$.

If: Let x and d meet (6) and (7). Construct the following stationary strategy: Player i always offers x_{-i} and does not accept less than x_i. Player i's offer y is accepted by all $j \neq i$ only if

$$y_j \geq v_j(X - \textstyle\sum_{k \neq j} x_k) = v_j(x_j + e_1(x_1)), \text{ for all } j \neq i. \tag{25}$$

Since v_j is increasing, and since

$$x_j = v_j(x_j + d), \text{ for all } j \neq i,$$

i's payoff maximizing offer to each j is x_j. ■

Proof of Proposition 5:

Lemma 5 *For any n, there are unique $y(n) \in \mathbb{R}^n$ and $d(n) > 0$ such that*

$$\begin{aligned} y_i(n) &= v_i(y_i(n) + d(n)), \text{ for all } i = 1, ..., n, && (26) \\ \sum_{u=1}^{n} y_i(n) &= n - d(n). && (27) \end{aligned}$$

Proof. By Lemma 1. ■

By Lemma 2, the set of allocations the planner can implement under n agents is

$$\left\{x : \frac{1}{n}\sum_{i=1}^{n} x_i \leq 1, \text{ and } x_i \geq y_i(n), \text{ for all } i = 1, ..., n\right\}.$$

Lemma 6 *Let $y(n)$ and $d(n)$ be defined as in Lemma 5. Then there is $y^* \in \mathbb{R}^U$ and $d^* > 0$ such that $y_i(n) \rightarrow_n y_u^*$, for all $u_i = u$ and $u \in U$, and $d(n) \rightarrow_n d^*$, where*

$$y_u^* = v(y_u^* + d^*), \text{ for all } u \in U, \tag{28}$$

$$\sum_{u \in U} \lambda_u y_u^* = 1. \tag{29}$$

Proof. By Lemma 5, for any $n = 1, 2, ...,$

$$y_i(n) = v_i(y_i(n) + d(n)), \text{ for all } i = 1, ..., n, \tag{30}$$

$$\sum_{i=1}^{n} y_i(n) = n - d(n). \tag{31}$$

Dividing both sides of (31) by n,

$$\frac{1}{n}\sum_{i=1}^{n} y_i(n) = 1 - \frac{d(n)}{n}. \tag{32}$$

Define a function $i : U \rightarrow \{1, 2, ...\}$ such that $u_{i(u)} = u$, for all $u \in U$. By stationarity, $y_{i(u)}(n) = y_j(n)$ if $u = u_j$. The left hand side of (32) can now be written

$$\frac{1}{n}\sum_{i=1}^{n} y_i(n) = \frac{1}{n}\sum_{u \in U} y_{i(u)}(n) \sum_{i=1}^{n} 1_{(u_i = u)}.$$

By the law of large numbers,

$$\lim_n \frac{1}{n}\sum_{i=1}^{n} 1_{(u_i = u)} = \lambda_u. \tag{33}$$

Take any subsequence $\{n'\}$ under which $\lim_{n'} y_{i(u)}(n')$ for all u and $\lim_{n'} d(n')$ exist (the limit can be either finite or infinite). Then (30) can be written

$$\lim_{n'} \sum_{u \in U} \lambda_u y_{i(u)}(n') = 1 - \lim_{n' \rightarrow \infty} \frac{d(n')}{n'} \tag{34}$$

By (30) $\lim_{n'} d(n') = \infty$ if and only if $\lim_{n'} y_{i(u)}(n') = \infty$ for all u. Thus, by (34), it must be the case that $\lim_{n'} y_{i(u)}(n') = y_u^*$ and $d(n) = d^*$, for some $(y^*, d^*) \in \mathbb{R}_{++}^{|U|} \times \mathbb{R}_{++}$. By (30), (34) becomes

$$\lim_{n'} \sum_{u \in U} \lambda_u y_{i(u)}(n') = \sum_{u \in U} \lambda_u y_u^* = 1. \tag{35}$$

By Lemma 1 and (30), $y_{i(u)}^*$ is the limit of any converging subsequence $\{y_{i(u)}(n'')\}$, and d^* is the limit of any converging subsequence $\{d(n'')\}$. Thus (y^*, d^*) is the unique limit and by (35), continuity, and (30) it meets the conditions imposed by the lemma. ■

Proposition 5 *As $n \to \infty$, allocation x is implementable by the planner if and only if $x = x^*$.*

Proof. Again, define a function $i : U \to \{1, 2, ...\}$ such that $u_{i(u)} = u$, for all $u \in U$. By stationarity, $y_{i(u)}(n) = y_j(n)$ if $u = u_j$, for all $j = 1, ..., n$. The set of implementable allocations can be written

$$\left\{ x \in \mathbb{R}_+^n : \frac{1}{n} \sum_{i=1}^n x_i \leq 1, \text{ and } x_i \geq y_i(n), \text{ for all } i = 1, ..., n \right\}$$

$$= \left\{ x \in \mathbb{R}_+^n : \frac{1}{n} \sum_{u \in U} x_{i(u)} \sum_{i=1}^n 1_{(u_i = u)} \leq 1, \text{ and } x_j = x_{i(u)} \geq y_{i(u)}(n) \text{ if } u = u_j, \text{ for all } u \in U \right\}$$

Taking the limit,

$$\lim_n \left\{ x \in \mathbb{R}_+^n : \frac{1}{n} \sum_{u \in U} x_{i(u)} \sum_{i=1}^n 1_{(u_i = u)} \leq 1, \text{ and } x_j = x_{i(u)} \geq y_{i(u)}(n) \text{ if } u = u_j, \text{ for all } u \in U \right\}$$

$$= \left\{ x \in \mathbb{R}_+^\infty : \sum_{u \in U} x_{i(u)} \lambda_u \leq 1, \text{ and } x_j = x_{i(u)} \geq y_{i(u)}^* \text{ if } u = u_j, \text{ for all } u \in U \right\}.$$

By (29), this reduces to

$$\left\{ x \in \mathbb{R}_+^\infty : x_j = y_{i(u)}^* \text{ if } u = u_j, \text{ for all } u \in U \right\},$$

which is a singleton $\{x^*\}$, as required by the proposition. ■

References

[1] Binmore, K., A. Rubinstein and A. Wolinsky 1986, The Nash bargaining solution in economic modelling, *Rand Journal of Economics 17*, 176-188

[2] Fishburn, P. and Rubinstein, A. 1982, Time preference, *International Economic Review 23*, 677-95

[3] Krishna V. and R. Serrano 1996, Multilateral bargaining, *Review of Economic Studies 63*, 61-80

[4] Kultti, K. and H. Vartiainen 2007a, Von Neumann-Morgenstern stable set bridges time-preferences to the Nash solution , *B.E. Journal of Theoretical Economics 7:1 (Contributions), Article 41*

[5] Kultti, K. and H. Vartiainen 2007b, Bargaining with many players: a limit result, manuscript

[6] Luce R and H. Raiffa 1957, *Games and decisions:. introduction and critical survey.* New York: Wiley

[7] Thomson W. and T. Lensberg 1989, *Axiomatic theory of bargaining with a variable number of agents*, Cambridge University Press, Cambridge

Aboa Centre for Economics (ACE) was founded in 1998 by the departments of economics at the Turku School of Economics, Åbo Akademi University and University of Turku. The aim of the Centre is to coordinate research and education related to economics in the three universities.

Contact information: Aboa Centre for Economics, Turku School of Economics, Rehtorinpellonkatu 3, 20500 Turku, Finland.

Aboa Centre for Economics (ACE) on Turun kolmen yliopiston vuonna 1998 perustama yhteistyöelin. Sen osapuolet ovat Turun kauppakorkeakoulun kansantaloustieteen oppiaine, Åbo Akademin nationalekonomi-oppiaine ja Turun yliopiston taloustieteen laitos. ACEn toiminta-ajatuksena on koordinoida kansantaloustieteen tutkimusta ja opetusta Turun kolmessa yliopistossa.

Yhteystiedot: Aboa Centre for Economics, Kansantaloustiede, Turun kauppakorkeakoulu, 20500 Turku.

www.ace-economics.fi

ISSN 1796-3133