A Critique of the Constant Elasticity of Transformation (CET) Linear Supply System

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An elusive restriction maintained in earlier CET supply models with three or more products is shown to result in a potentially serious misspecification. Its impact on empirical estimates is found to be substantial, and an alternative formulation is presented which overcomes the problem while still maintaining the CET hypothesis.

In a 1968 article, Powell and Gruen (P&G) developed the constant elasticity of substitution (CES) analog on the production possibilities surface and demonstrated how it permits estimation of a linear approximation to supply response along the surface. When applied to perfectly competitive firms, considerable estimation appeal occurred because of its parsimony in number of parameters requiring estimation for an n-product system.

Their supply model was very much in the spirit of the Rotterdam demand models in which the specified demand (supply) equations are intended to map only locally back to the utility (production) functions of the space in which agents optimize utility (costs). Global mapping was not intended nor claimed. The authors further noted that the supply system was limited as an empirical device to measuring supply response in the very short run since its scope was restricted to movements along the production possibilities surface.

Because some researchers desire to impose the CET constraint in estimating locally-dual supply relations and because there is a subtle but important misspecification that may be introduced by P&G's restrictions, this critique is written to document the problem, give a possible resolution, and identify the magnitude of error caused in one set of empirical estimates. The problem occurs only in generalization of their original model for estimation of n-output (where $n \geq 3$) supply systems. After documentation, the problem may seem sufficiently transparent that it should be obvious to anyone who tried to use the P&G model for larger systems. However, the fact that it has gone unreported for more than a decade, during which time several unwary researchers in addition to P&G have used it inappropriately, suggests the importance of providing a less restrictive framework for analysis.

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This research was supported in part by a grant from the United States-Israel (Binational) Agricultural Research and Development Fund (BARD). Texas Agricultural Experiment Station Technical Article No. 18894. Appreciation is extended to John Stoll, Rulon Pope, Jean-Paul Chavas, and anonymous *Journal* reviewers for comments on earlier drafts of this paper.

Western Journal of Agricultural Economics, 9(2): 314–321

The Problem

To identify the problem, one needs to start with P&G equation (18). Assuming competitive equilibrium, constant aggregate input level, and constant technology, the authors noted that the compensating small change in one product equals the negative change in a second product multiplied by the price ratio. This equation was used to derive the partial cross-price parameter in P&G (20) for a two-commodity system with constant aggregate input level and technology. The assumption of compensating change implied that either the firm produces only two products or all other output levels are held constant. This restriction was implicitly maintained in the development of the n-commodity supply model (P&G 27) where the cross-price parameters were specified assuming that all other output prices were also held constant. This resulted in a highly restrictive n-commodity estimation system, P&G (28), derived from P&G (20) and P&G (27), because of the assumption that each cross-price parameter with only two variable outputs is the same as when all outputs are variable. No problem is created when we model firms that produce only two commodities. A misspecification is often introduced, however, in models of firms with a potentially larger line of products. The cause of this misspecification is now formally documented.

Following P&G, let x_i be output of product i and π_i be the price of i. Assume input-output separability (Hasenkamp), so that P&G's scalar index in inputs, v, is well defined. Further, assume products i, j, and k (where k is a vector which includes all products other than i and j). The factor requirements function is:

$$v = f(x_i, x_j, x_k), \tag{1}$$

in which x_k possibly is a (column) vector. On the production possibilities surface,

$$dv = f_i\, dx_i + f_j\, dx_j + f_k\, dx_k = 0, \tag{2}$$

where f_s is $\partial v/\partial x_s$, s = i, j, k, and f_k possibly is a (row) vector of such derivatives. Then

$$dx_j = -(f_i/f_j)\, dx_i - (f_k/f_j)\, dx_k. \tag{3}$$

In competitive equilibrium,

$$dx_j = -(\pi_i/\pi_j)\, dx_i - (\pi_k/\pi_j)\, dx_k. \tag{4}$$

For more than two products, (4) reduces to P&G's equation (18),

$$dx_j = -(\pi_i/\pi_j)\, dx_i, \qquad \text{(P\&G 18)}$$

only if the inner product of π_k and dx_k vanishes. Their equation (19),

$$(dx_i/x_i)(1 + \pi_i x_i/\pi_j x_j) = \tau_{ij}(d\pi_j/\pi_j) \qquad \text{(P\&G 19)}$$

(where τ_{ij} is the direct partial elasticity of transformation between i and j), is a simple algebraic manipulation of their equation (18) given the following equality (taken from their equations (16) and (17)) which characterizes the maintained hypothesis that the partial transformation elasticity τ_{ij} is constant:

$$dx_i/x_i - dx_j/x_j = \tau_{ij}(d\pi_j/\pi_j). \tag{5}$$

It is clear that P&G (19) can be derived as written only if $\pi_k dx_k = 0$, for only then does (4) reduce to P&G (18). With positive prices ($\pi_k > 0$), the only conditions under which P&G (19) can be generalized for a firm producing more than two outputs are for either (a) each output in the vector x_k to remain constant ($dx_k = 0$) or (b) the price-weighted changes to exactly offset each other ($\pi_k dx_k = 0$). Since (b) is not an implication either of the CET technology or the behavioral objective assumed by P&G, it would be satisfied only coincidentally at any particular data point. That leaves (a), i.e., constant levels of all outputs other than i and j, as the apparent maintained hypothesis in any generalization of P&G (19).[1] Otherwise, when there

[1] It is apparent from the following statement that P&G recognized that they were maintaining a serious restriction when they generalized their model to n outputs: "Equation (28) [the n-output generalization of the estimation system—see text] consequently can stand as it is, *provided* all partial transformation frontiers between any given pair of products i and j retain the same, constant, partial transformation elasticity τ_{ij} *irrespective* of shifts induced by investment, technology *or changes in the output levels of other products*" (Powell and Gruen, p. 321, italics theirs).

are more than two outputs, that equation would be written as:

$$(dx_i/x_i)(1 + \pi_i x_i/\pi_j x_j) + \pi_k dx_k/\pi_j x_j = \tau_{ij}(d\pi_j/\pi_j). \quad (6)$$

Thus, the assumption of constant levels of all outputs other than i and j could be added to the assumption of constant resources and constant technology in any generalization of P&G (20) in which $dx_i/d\pi_j$ from P&G (19) is rewritten as $\partial x_i/\partial \pi_j$:

$$\left.\frac{\partial x_i}{\partial \pi_j}\right|_{\substack{\text{constant resources} \\ \text{constant technology}}} = x_i\tau_{ij}/\pi_j\omega_{ij}, \quad \text{(P\&G 20)}$$

where $\omega_{ij} = 1 + \pi_i x_i/\pi_j x_j$.

The above restriction was implicitly maintained in subsequent equations when variables were transformed to formulate the system of estimation equations for n distinct output supplies. In particular, P&G (20) was taken as the parameter a_{ij} measured at the means of x_i, π_j, and ω_{ij} (i.e., $\bar{x}_i$, $\bar{\pi}_j$, $\bar{\omega}_{ij}$) in P&G's linear supply model,

$$x_{it} = \phi_{it} + \sum_{j \neq i} a_{ij}\pi_{jt} + a_{ii}\pi_{it}, \quad \text{(P\&G 27)}$$

where "ϕ_{it} is a shift variable incorporating the effects of investment and technological advances," (P&G p. 319) but not changes in other output levels. The parameters a_{ii} and a_{ij} in this equation are the partial derivatives of output i with respect to changes in individual prices when all other output *prices* are constant (and in this case when input level and technology are also constant).

We are consequently left with trying to derive economic intuition from the n-output estimation system, represented for each output i (i = 1, . . . , n) by

$$x_{it} = \phi_{it} + \bar{x}_i \sum_{j \neq i} (\tau_{ij}/\bar{\omega}_{ij})(\pi_{jt}/\bar{\pi}_j - \pi_{it}/\bar{\pi}_i), \quad \text{(P\&G 28)}$$

in which each partial derivative assuming constant *levels* of all other outputs (P&G 20) is identically equal to its corresponding partial derivative in P&G (27) assuming constant *prices* of all other outputs.[2] This situation is precisely analogous to the requirement that each cross-price input-demand parameter for a cost-minimizing CES production function with output and all but two inputs held constant is the same as when all inputs can adjust simultaneously to their cost-minimizing levels.

For P&G (28) to be a valid estimation system for n outputs, the inner product of π_k and dx_k in (4) must vanish. It is obvious that extremely strong restrictions must be imposed on the technology for this to occur. To document this, it is noted that neither strong homogeneous separability nor additive separability of the factor requirements function (1) is sufficient (along with P&G's other assumptions of constant resources, constant technology, CET transformation surface, perfect competition, and local correspondence) to render P&G (28).[3] It is thus apparent that the requisite restrictions are extremely serious and likely result in an important misspecificiation of the estimation system. This potential misspecification affects the empirical estimates reported in Gruen *et al.*; Powell and Gruen; Scobie and Johnson; Shumway and Chang; Shumway and Green; Whittaker; Wilson; and other studies using the CET linear supply model for more than two commodities.

Proposed Solution

Two possible solutions for this problem might be considered. Either the recently

[2] Homogeneity of degree zero in output prices is also a maintained hypothesis, derived from the behavioral objective, in each supply equation of P&G (28).

[3] The following examples may be examined to verify this statement: (a) additively-separable CET factor requirements function:

$$v = b_1x_1^2 + b_2x_2^2 + b_3x_3^2,$$

and (b) a strong homogeneously-separable CES production function (analog to the factor requirements function):

$$x = b_0 v_1^{b_1} v_2^{b_2} v_3^{b_3}.$$

TABLE 1. Original CET Linear Supply Model Estimates.[a]

Commodity	Unit (Million)	Intercept	CET Price Variables[b] Corn	Cotton	Hay	Rice	Sorghum	Wheat	Shift Variables Land Input	Lagged Output (Technology)	Weighted Diversion Payment	Weather Index
Corn	bu.	−36.98		−.739	1.404	−.118	−1.196	.584	.53	1.48	11.7	.122
		(17.46)		(.218)	(.337)	(.306)	(.561)	(.483)	(.86)	(.11)	(22.4)	(.078)
Cotton	lb.	−145.82	−.739		.118	−.034	−.467	−1.160	19.43	−.05	−6,089.4	17.575
		(322.89)	(.218)		(.153)	(.099)	(.146)	(.173)	(13.24)	(.06)	(1,072.2)	(1.582)
Hay	ton	.22	1.404	.118		−.460	−.493	−.366	−.08	.67		.026
		(.81)	(.337)	(.153)		(.196)	(.256)	(.300)	(.03)	(.08)		(.006)
Rice	cwt.	−11.65	−.118	−.034	−.460		.143	−.464	.05	.65		.172
		(4.89)	(.306)	(.099)	(.196)		(.182)	(.213)	(.14)	(.06)		(.025)
Sorghum	bu.	−20.26	−1.196	−.467	−.493	.143		.455	−7.36	.58	−117.8	2.782
		(63.13)	(.561)	(.146)	(.256)	(.182)		(.347)	(2.83)	(.06)	(74.6)	(.457)
Wheat	bu.	−94.05	.584	−1.160	−.366	−.464	.455		2.70	.15	−9.2	.976
		(14.06)	(.483)	(.173)	(.300)	(.213)	(.347)		(.68)	(.05)	(6.0)	(.066)

[a] Chang, p. 136. Estimated standard errors are in parentheses.
[b] The parameter estimates reported in the price variable columns are the uncorrected estimates of the elasticities of transformation.

TABLE 2. Mean Short Run Supply Elasticities, Original Model Estimates.

Commodity	Elasticity with Respect to the Price of					
	Corn	Cotton	Hay	Rice	Sorghum	Wheat
Corn	.57	−.66	.76	−.07	−.97	.37
Cotton	−.08	.44	.01	−.01	−.16	−.21
Hay	.65	.10	.12	−.26	−.39	−.22
Rice	−.05	−.03	−.20	.43	.11	−.25
Sorghum	−.22	−.31	−.10	.04	.47	.13
Wheat	.20	−.95	−.14	−.21	.32	.78

developed CRESH/CRETH production system (Vincent *et al.*) could be used or the following changes could be made in the CET linear supply model:

a. Choose a loglinear supply system rather than one that is linear in variables.
b. Use Allen-Uzawa partial elasticities of transformation (see Allen, p. 504, for the analogous elasticities of substitution) rather than direct partial elasticities of transformation. Since Allen-Uzawa elasticities of transformation maintain the hypothesis that other output *prices*, not *quantities*, are constant, they are conceptually compatible with an n-variable-output supply system.

These changes permit us to write the supply equation for commodity i (i = 1, . . . , n) as

$$\log x_{it} = \phi_{it} + \sum_{j=i}^{n} a_{ij} \log \pi_{jt}. \qquad (7)$$

If ϕ_{it}, as in P&G's equation (27), locates the position of the production possibilities frontier, so that the price term is to capture only movements around the frontier, then

$$a_{ij} = \left.\frac{\partial \log x_i}{\partial \log \pi_j}\right|_{\pi_m,\, m \neq j;\, \phi_i} \qquad (8)$$

is the cost-compensated cross-price supply elasticity of i with respect to the expected price of j. From demand/production theory, we know that the Allen-Uzawa partial transformation elasticities are:

$$\tau_{ij} = \left.\frac{\partial \log x_i}{\partial \log \pi_j}\right|_{\pi_m,\, m \neq j;\, \phi_i} \Big/ S_j = a_{ij}/S_j \quad (i \neq j), \qquad (9)$$

where S_j is the share of j in expected total revenue, viz.,

$$S_j = \pi_j x_j / \sum_{m=1}^{n} \pi_m x_m . \qquad (10)$$

Substituting from (9) into (7), we get:

$$\log x_{it} = \phi_{it} + \sum_{j \neq i} \tau_{ij} S_j \log \pi_{jt} + a_{ii} \log \pi_{it}. \qquad (11)$$

Maintaining homogeneity of degree zero in prices, the parameters of each supply equation are subject to the restriction

$$\sum_{j=1}^{n} a_{ij} = 0, \qquad i = 1, \ldots, n, \qquad (12)$$

so that

$$a_{ii} = -\sum_{j \neq i} \tau_{ij} S_j, \qquad i = 1, \ldots, n. \qquad (13)$$

The final loglinear supply system then is

$$\log x_{it} = \phi_{it} + \sum_{j \neq i} \tau_{ij} S_j (\log \pi_{jt} - \log \pi_{it}), \quad i = 1, \ldots, n. \qquad (14)$$

Equation (14) permits linear estimation of a system of supply equations having the same parameter parsimony as in P&G's original model. It also maintains the hypothesis of locally constant elasticities of transformation, but does not suffer from the restrictive assumption that quantities of all products other than i and j either do not change in response to changes in π_i and π_j or, if they do, they have no influence on τ_{ij}.

TABLE 3. New CET Loglinear Supply Model Estimates.[a]

Commodity	Unit (Million)	Intercept	CET Price Variables[b]						Shift Variables			
			Corn	Cotton	Hay	Rice	Sorghum	Wheat	Land Input	Lagged Output (Technology)	Weighted Diversion Payment[c]	Weather Index
Corn	bu.	−1.537		−.649	.857	−.486	−2.963	4.746	−.616	.671	−.0013	.871
		(1.515)		(.262)	(.455)	(.726)	(1.859)	(2.551)	(.551)	(.182)	(.0007)	(.206)
Cotton	lb.	3.762	−.649		.309	.077	−.593	−.114	.180	−.059	−.0032	.838
		(1.012)	(.262)		(.173)	(.211)	(.159)	(.202)	(.214)	(.075)	(.0007)	(.106)
Hay	ton	−1.316	.857	.309		.106	.305	.119	−1.015	.307		1.327
		(1.826)	(.444)	(1.69)		(.801)	(.291)	(.388)	(.510)	(.165)		(.285)
Rice	cwt.	−3.156	−.486	.077	.106		−.091	−.030	.167	.665		.799
		(2.162)	(.709)	(.206)	(.801)		(.359)	(.779)	(.354)	(.122)		(.273)
Sorghum	bu.	−2.261	−2.963	−.593	.305	−.091		−1.112	−1.327	.699	−.0018	1.662
		(1.703)	(1.859)	(.159)	(.298)	(.368)		(1.001)	(.432)	(.081)	(.0006)	(.312)
Wheat	bu.	−8.369	4.746	−.114	.119	−.030	−1.112		1.097	.383	−.0001	1.636
		(1.568)	(2.551)	(.202)	(.398)	(.798)	(1.001)		(.454)	(.091)	(.0002)	(.195)

[a] Estimated standard errors are in parentheses.
[b] Price variables are defined as in equation (14). The parameter estimates reported in these columns are the corrected Allen elasticities of transformation.
[c] Because diversion payments were not available in all years, this variable was not converted to logarithms.

TABLE 4. Mean Short Run Supply Elasticities, New Model Estimates.

Commodity	Elasticity with Respect to the Price of					
	Corn	Cotton	Hay	Rice	Sorghum	Wheat
Corn	.50	−.31	.05	−.04	−.66	.46
Cotton	−.04	.15	.02	.01	−.13	−.01
Hay	.05	.15	−.28	.01	.07	.01
Rice	−.03	.04	.01	.01	−.02	−.003
Sorghum	−.16	−.28	.02	−.01	.54	−.11
Wheat	.26	−.05	.01	−.002	−.25	.03

Magnitude of Error in Empirical Estimation

The most comprehensive use of the P&G CET linear supply model appears to be the work by Shumway and Chang. Twelve model specifications were considered in estimation of Texas field crop supply response for the period 1946–1976. Model 10 (Shumway and Chang, pp. 158–59) was most in the spirit of the original P&G supply model. It consisted of a system of six seemingly unrelated equations for corn, cotton, hay, rice, sorghum, and wheat supply estimated subject to homogeneity and symmetry restrictions by Zellner's generalized least squares. Variables that shift the production possibilities surface were specified to include total acreage, lagged output (as proxy for technology), weighted diversion payments (Houck and Ryan), and a weather index (Stallings). The empirical estimates (Chang, p. 136) are presented in Table 1. Elasticities of supply computed at the means are reported in Table 2.

Corresponding estimates and mean supply elasticities for the loglinear supply system, in which the earlier strong restriction is deleted (equation 14), are reported in Tables 3 and 4, respectively. The parameters on price variables are the estimated Allen-Uzawa elasticities of transformation and are constant.

Differences in both the elasticities of transformation and mean supply elasticities between the two sets of estimates are substantial. The range of transformation elasticities in these Allen-Uzawa estimates (−2.96 to +4.75) is considerably wider than in the earlier direct estimates (−1.20 to +1.40), but the new supply elasticities are generally smaller in absolute magnitude than the earlier estimates (28 of 36 elasticities are smaller). Some 75 percent of the new supply elasticities lie between ±0.2 as compared to 39 percent of the earlier estimates. The new supply elasticities are also much closer to the 1979 elasticities reported by Shumway using a model dual to a quadratic production function.

Conclusions

This paper has documented the strong nature of one restriction in the P&G CET linear supply model and the serious misspecification that can result when it is applied to three or more commodities. The empirical magnitude of this problem has been shown to be substantial in actual estimation. A convenient method for relaxing the restriction while still retaining the CET foundation has also been presented. This revised locally-dual model is obviously preferred to the former for estimating supply relationships along the production possibilities surface when the elasticities of transformation are expected to be approximately constant.

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