Dynamically Optimal and Approximately Optimal Beef Cattle Diets Formulated by Nonlinear Programming

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Cattle purchasing, feeding, and selling decisions are described by a free-time optimal control model. The nutrient constraints of the National Research Council and a recently published dry matter intake constraint augment the model and make it nonlinear in the feed ingredients, the daily gain, and the weight of the cattle. Optimal feeding programs are calculated by nonlinear programming under two scenarios: first, when the feedlot has excess capacity and, second, when animals must be sold to make room in the feedlot before more can be purchased. An approximately optimal feeding program is calculated by nonlinear programming and is all but identical to the dynamically optimal programs.

Key words: beef cattle, diets, nonlinear programming, optimal control.

Since the mid-1970s, net returns from beef cattle feeding have been volatile. Planning of marketing strategies and feeding programs has become crucial to the economic survival of cattle feeders. Cattle feeders must determine which cattle to purchase, what diet to feed the cattle for producing daily gain, and when to sell the cattle.

Kennedy, and Meyer and Newett were the first to solve dynamic optimization models of cattle feeding and marketing decisions. Using a "hybrid" solution algorithm of dynamic programming and linear programming, they calculated optimal beef diets and rates of gain over a feeding program of fixed length. Apland developed a linear programming model to minimize the cost for beef cattle to reach a fixed selling weight. Chavas, Kliebenstein, and Crenshaw used nonlinear programming to find the diets and rates of gain for swine when neither the length of the feeding program nor the market weight were fixed. They also considered a Faustmann rule for replacing animals when space in the feeding facilities is limited and the current animals have to be sold before a new lot can be started. This study extends the methods of Chavas, Kliebenstein, and Crenshaw to beef cattle and devises a method to closely approximate dynamically optimal decisions.

Embedded within a dynamic model for cattle feeding must be a diet model for the production of daily gain from the feeds in the diet. The advent of the net energy system (Lofgren and Garrett) was the first in a series of refinements to make diet models for beef cattle nonlinear. The net energy system was successfully incorporated into linear programming by Brokken. Brokken's Model II for least-cost diets has been widely used and adapted (Ladd and Williams; Olson, Willham, and Boehlje; Rozzi et al.; Glen). His Model III for optimal-return diets (choosing the daily gain as well as the feeds) is more difficult to implement because the energy requirement is significantly nonlinear with respect to gain.

In addition to energy, the National Research Council (NRC 1984) has since changed the requirements of other major nutrients to depend nonlinearly on gain. Dry matter intake restrictions (NRC 1987, Plegge et al.) have been proposed to adjust for the energy concentra-

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tion of the diet and make the diet model significantly nonlinear with respect to the feed ingredients as well. For a dynamic decision problem, it is also important that the nutrient requirements and the dry matter restriction are nonlinear in the weight of the cattle. In short, a modern diet model is significantly nonlinear in every variable.

One purpose of this paper is to construct and solve a general model of the decision problem facing a cattle feeder and determine the dynamically optimal purchasing, feeding, daily gain, and selling decisions. Unfortunately, the solution process is quite involved, which limits its practical application by cattle feeders and nutritionists. Another purpose, then, is to develop a quick and easy approximation method.

The general model is especially difficult to solve because it is a free-time optimal control problem. The model's high degree of nonlinearity adds to the difficulties of finding a solution. Fortunately, a dynamically optimal feeding program can be approximated by a series of optimal-return diets at increasing weights of cattle. Each optimal-return diet is "static" in the sense that the effects of current feeding decisions on future decisions are ignored. However, the dynamically optimal selling weight can still be determined and the calculated daily gains and diets are close to optimal.

Optimal Feeding Program

Consider a model of the decisions made by a farmer-feeder who does not feed continuously or by a commercial feeder whose feedlot is not operating at full capacity. Modeling the optimal decisions of a commercial feeder whose feedlot has no empty pens requires a modification to include the opportunity cost of limited space. The objective is to maximize the discounted revenue from selling the animals minus the feed costs, variable costs, marketing costs, and initial expenditures for purchasing the animals. Two state variables in the model will evolve over time. The weight of the animals will increase with gain, and the number of animals will decline with death loss. Formally, the model is

$$(1a)\quad J(W_0, N_0) = \max\left(\frac{1}{1+i}\right)^T [P_T W_T - M] N_T - \sum_{t=0}^{T-1}\left(\frac{1}{1+i}\right)^t [F(W_t, G_t) + V_t] N_t - P_0 W_0 N_0;$$

subject to

$$(1b)\quad W_{t+1} - W_t = G_t \qquad t = 0, \ldots, T-1;$$

$$(1c)\quad N_{t+1} - N_t = -\delta N_t \qquad t = 0, \ldots, T-1,$$

where J is the optimal value function, W is the weight of each animal in kilograms, N is the number of animals, P is the expected price of an animal per kilogram of weight, M is a marketing charge per head, F is feed costs per day, G is gain in kilograms per day, V is variable costs per day, T is the length of the feeding program in days, i is the daily interest rate, and δ is the daily rate of death loss.

To simplify the presentation, feed costs are specified as a function of weight and gain. Conceptually, a least-cost diet has been calculated for each possible combination of weight and gain and the costs summarized as F. Once the weight and gain are determined by the dynamic model, the optimal feeds in the diet are known. For the actual application to follow, feeds and gain are determined simultaneously by including the nutrient and dry matter constraints of a beef diet model for each day of the feeding program. Other constraints for the availability of feeds, the size of the pen, the number of cattle available, and the availability of capital could be added as well.

Variable costs, V, are incurred daily and include fuel, electricity and perhaps labor costs but do not include interest expenses nor death losses. It might seem natural to include the actual interest paid as a variable cost. However, in a dynamic planning model such as this, short-term interest expenses are the opportunity costs of investing in cattle production and are incorporated automatically through discounting. This point will be discussed in more detail later. Death losses are incorporated into the model through the rate of death loss. Both the interest rate and rate of death loss could be allowed to change over time if needed. The fixed costs of depreciation, interest, property taxes, and insurance on corrals and equipment cannot be avoided no matter what cattle feeding program is chosen and are not considered in the model.

The cattle purchasing decision is an all-or-none proposition. To see this, first solve difference equation (1c) for the number of animals at time t:

$$N_t = (1 - \delta)^t N_0.$$

Then substitute the result into the objective function, equation (1a):

$$J(W_0, N_0) = \max\left[\left(\frac{1 - \delta}{1 + i}\right)^T [P_T W_T - M] - \sum_{t=0}^{T-1} \left(\frac{1 - \delta}{1 + i}\right)^t [F(W_t, G_t) + V_t] - P_0 W_0\right] N_0.$$

The final result discounts the future more heavily to account for death loss and separates the objective function into the net returns per animal over the feeding program multiplied by the initial number of animals purchased. If net returns per animal are positive, the number of animals purchased should be as large as possible. Presumably there is some upper bound corresponding to the availability of space in the feedlot, cattle, or capital. Moreover, different types of animals will each have different net returns. The type of animals with the largest net return should be purchased.

Let the interest rate adjusted for death loss be $r = (i + \delta)/(1 - \delta)$. For a given group of animals, the optimal feeding program can be determined by maximizing the net returns per animal:

$$\text{(2a)} \quad J(W_0) = \max\left(\frac{1}{1 + r}\right)^T [P_T W_T - M] - \sum_{t=0}^{T-1} \left(\frac{1}{1 + r}\right)^t [F(W_t, G_t) + V_t] - P_0 W_0;$$

subject to

$$\text{(2b)} \quad W_{t+1} - W_t = G_t; \qquad t = 0, \ldots, T - 1.$$

Although the initial purchase expenditure, $P_0 W_0$, is a constant, it affects the decisions in a commercial feedlot operating continuously at full capacity because the net returns per head determine when the current group of cattle should be sold to make room for a new group. In this situation, a sequence of future feeding programs for successive groups of cattle must be modeled. If it is assumed that there are an infinite number of feeding programs and that the feeding program for each new group of cattle in this sequence is identical to all others, a Faustmann-type problem results (Chavas, Kliebenstein, and Crenshaw). The objective function in equation (2a) would simply be multiplied by an additional discount term, $1/(1 - (1 + i)^{-T})$, where T is the length of each of the feeding programs. Each feeding program would be shorter than the length of a feeding program in a feedlot with discontinuous feeding or with excess capacity, and the optimal diets and daily gain would be indirectly affected as well.

The optimal daily gain at any time during the feeding program can be found by maximizing the current-value Hamiltonian (Kamien and Schwartz, p. 151). The current-value hamiltonian is a dynamic measure of returns above costs for a single day. It is formed by first multiplying the daily gain on the right-hand side of equation (2b) by a costate variable, say λ, interpreted as the implicit price for gain. This gives a dynamic measure of total returns. Then the total feed and total variable costs for day t are subtracted.

$$\text{(3)} \quad H_t = \lambda_{t+1} G_t - [F(W_t, G_t) + V_t]; \quad t = 0, \ldots, T - 1.$$

The three kinds of first-order conditions equate the marginal feed costs with respect to gain to the costate, describe the change over time in the costate due to discounting and due to the marginal feed costs with respect to weight, and relate the costate at the end of the feeding program with the expected selling price.

$$\text{(4a)} \quad \frac{\partial H_t}{\partial G_t} = 0 = \lambda_{t+1} - \frac{\partial F_t}{\partial G_t}; \quad t = 0, \ldots, T - 1;$$

$$\text{(4b)} \quad -\frac{\partial H_t}{\partial W_t} = \lambda_{t+1} - (1 + r)\lambda_t = \frac{\partial F_t}{\partial W_t}; \quad t = 0, \ldots, T - 1;$$

$$\text{(4c)} \quad \lambda_T = \frac{1}{1 + r} P_T.$$

The current-value costate must be known for the optimal gain in equation (4a) to be chosen. At the end of the feeding program, terminal condition (4c) equates the costate to the expected sale price discounted one day. Earlier in the feeding program, however, the expected sale price is discounted to day t, as seen in condition (4b). Further, increasing the weight of the animals will increase feed costs in the future, making gain in the present relatively less profitable by making the costate even smaller. In short, the costate is a dynamic price which accounts for the effects that current decisions will have on the future.

The Hamiltonian is also important in characterizing the optimal length of the feeding program. To see this, first notice that T could initially be fixed at a relatively small value in the maximization problem of equations (2a) and (2b). The optimal value function, J, would increase with successively longer lengths of the feeding program until the optimal length was reached and then would decrease thereafter. It follows that a procedure for solving the free-time optimal control problem would start with a short length of time and then repeatedly solve a series of fixed-time optimal control problems for increasingly longer lengths. The optimal length of the feeding program and the optimal daily gains are found when the optimal value function no longer increases.

Suppose that G^* and T^* are the optimal gains and length for a feeding program in a feedlot with excess capacity. Further, suppose that, after determining the optimum, the cattle are sold one day too early. The loss of discounted net revenues would be the difference in J, the optimal value function of equation (2a), evaluated at T^* and J evaluated at $T^* - 1$:

$$\left(\frac{1}{1+r}\right)^{T^*}[P_T W_T - M] - \left(\frac{1}{1+r}\right)^{T^*-1}[F(W_{T-1}, G^*_{T-1}) + V_{T-1}] - \left(\frac{1}{1+r}\right)^{T^*-1}[P_{T-1}W_{T-1} - M] > 0.$$

Substituting the relationship $W_{T-1} + G^*_{T-1}$ for W_T and rearranging slightly gives the transversality condition for the variable T (Kamien and Schwartz, p. 143). The dynamic profitability of feeding the animals on day $T^* - 1$ exceeds the opportunity returns from selling them.

$$(5)\quad \left(\frac{1}{1+r}\right)P_T G^*_{T-1} - [F(W_{T-1}, G^*_{T-1}) + V_{T-1}] > \left(\frac{r}{1+r}\right)[P_{T-1}W_{T-1} - M] - \left(\frac{1}{1+r}\right)[P_T - P_{T-1}]W_{T-1}.$$

With the help of equation (4c), the left-hand side of equation (5) is seen to be the Hamiltonian for day $T^* - 1$. The right-hand side has two terms. The first is the opportunity of earning interest from selling the animals at day $T^* - 1$ and investing the proceeds at the rate r. The second term on the right-hand side adjusts for any changes in the expected sale price. If a Faustmann rule were applicable, the right-hand side would also contain a third term for the opportunity return to scarce feedlot space.

Reconsider the question of whether interest expenses are properly accounted for by the opportunity returns on the market value of the animals, as in equation (5), or whether a daily interest charge should be levied instead on the initial investment to purchase the animals and on the accrued feed and variable costs. The latter method may be the more common cost-accounting approach for tracking actual expenses. However, in planning for the future, past purchase and feeding decisions already have been made and the interest on the initial investment plus accrued costs is like any other fixed cost. Instead, the optimal selling decision at day $T^* - 1$ should compare the opportunity return on the animal's market value to the dynamic profitability of continued feeding.

In summary, the Hamiltonian in equation (3) is the key to determining the optimal diets, daily gains, and the optimal selling weight during a feeding program. Approximating an optimal feeding program requires an approximation of the Hamiltonian for every day except the last. More specifically, some assumption about the costate variable must be made. Suppose the marginal feed costs with respect to weight were to be ignored in equation (4b) and the rate of interest adjusted for death loss were ignored as well. Then the costate would be assumed constant over time and, according to equation (4c), would be equal to the expected selling price discounted one day. This seemingly crude approximation will be shown to be surprisingly accurate. Notice, however, that no approximation is needed to find the optimal selling weight according to equation (5), although the length of the feeding program will, of course, depend upon previous daily gains.

Optimal-Return Diet Model

To keep the notation uncluttered, the dynamic model in equations (2a) and (2b) summarizes the diet for any day during the feeding program as the least-cost function, F. In applying the model, the feed cost function can be made explicit and the model can be augmented by a diet model for each day comprised of a set of nutrient constraints and a dry matter constraint. For presentation purposes, it is less cumbersome and just as informative to augment the Hamiltonian in equation (3) for a single day. An optimal-return diet model will be defined as an approximation to the augmented Hamiltonian with a simplifying assumption about the costate variable.

In a diet model, nutrient constraints compare the available nutrients in the diet with the nutrient requirements of an animal, and a dry matter constraint compares the weight of the feeds in the diet with a maximum dry matter intake for the animal. Net energy for maintenance (NE_m), net energy for gain (NE_g), crude protein (CP), calcium (Ca), phosphorus (Ph) and dry matter (DM) in the diet will be compared with the NRC (1984, p. 38) nutrient requirements and the dry matter intake restriction of Plegge et al. (see also NRC 1987). The nutrient requirements are all nonlinear in the weight of the animal and in daily gain. The DM intake restriction and the availability of NE_g are nonlinear with respect to the animal's weight and the feed ingredients.

The estimation of dry matter intake is, at present, an active area of research in the field of animal science (NRC 1987). The reason that the DM restriction of Plegge et al. was chosen for the diet model is that it adjusts intake both for the energy concentration of the diet and for the weight of the animal. An accurate adjustment of intake for increases in the animal's weight is crucial in a dynamic model.

The Hamiltonian is to be maximized subject to the nutrient and dry matter constraints. Formally, the maximization problem for a medium-frame steer is

(6a) Hamiltonian (\$/d) $\max \lambda_{t+1} G_t - \left[\sum_i C_i F_i + V_t\right]$,

subject to

(6b) NE_g (Mcal/d) $\sum_i NE_{gi} F_i \left[1 - .077 W^{.75} / \sum_i NE_{mi} F_i\right] \geq .0557 W^{.75} G^{1.097}$;

(6c) CP (g/d) $\sum_i CP_i F_i \geq \left[33.4 \sum_i F_i + 2.75 W^{.75} + .2 W^{.6} + 268G - 29.4 \cdot (.0557 W^{.75} G^{1.097})\right] / .594$;

(6d) Ca (g/d) $\sum_i Ca_i F_i \geq [.0154 W + .071 \cdot (268G - 29.4 \cdot (.0557 W^{.75} G^{1.097}))] / .50$;

(6e) Ph (g/d) $\sum_i Ph_i F_i \geq [.0280 W + .039 \cdot (268G - 29.4 \cdot (.0557 W^{.75} G^{1.097}))] / .85$;

(6f) DM (kg/d) $\sum_i F_i \leq -42.925 - .004(250) + .00003(250)^2 + 36.8326(W/500) - 20.8356(W/500)^2 + 24.5011 \cdot \left(\sum_i ME_i F_i / \sum_i F_i\right) - 4.4019 \cdot \left(\sum_i ME_i F_i / \sum_i F_i\right)^2$;

where λ is the costate, or implicit selling price of the steer in dollars per kilogram; G is the daily gain in kilograms per day; C_i is the price of the i^{th} feed in dollars per kilogram of DM; F_i is the quantity of the i^{th} feed in kilograms per day of DM; V is the daily variable costs excluding interest expenses and death losses in dollars per day; W is the animal's live weight in kilograms; NE_{mi}, NE_{gi}, and ME_i are the net energy for maintenance, net energy for gain, and metabolizable energy of the i^{th} feed in thousand calories per kilogram of DM; and CP_i, Ca_i, and Ph_i are the crude protein, calcium, and phosphorus of the i^{th} feed in gain per kilogram of DM.

A separate constraint for NE_m is not included because the NE_m requirement is satisfied implicitly through the NE_g constraint for nonnegative G. Nutrient availabilities are on the left-hand sides, and nutrient requirements and the DM intake restriction are on the right-hand sides of the constraints. The nutrient requirements for animals other than a medium-frame steer will have different coefficients. For example, a medium-frame heifer has an NE_g requirement of $.068 W^{.75} G^{1.119}$ (NRC 1984, p. 38). In the DM restriction, 250 kilograms is the weight at which the animals were first placed on feed (which may be less than initial weight W_0 if the animals were fed by a previous owner) and the finished weight at which 70% of similar animals will grade low-choice has been set to 500 kilograms.

The control variables are gain, G, and the feeds, F_i. Parameters to be assigned values are the costate, λ, the feed costs, C_i, daily variable costs, V, the nutrients in each feed, NE_{mi}, NE_{gi}, ME_i, CP_i, Ca_i, and Ph_i, and the current weight of the steer, W.

Suppose the assumption is made that the costate is approximately equal to the expected selling price discounted one day,

$$\lambda_{t+1} = \frac{1}{1 + r} P_T, \tag{7}$$

where P is the expected selling price in dollars per kilogram and r is the death-loss adjusted interest rate. Then a sequence of optimal-return diets can be calculated at increasing weights to approximate the dynamically optimal sequence of diets. Early in the feeding program, the assumption in equation (7) is an overestimate of the costate causing the optimal-return diets to prescribe faster than optimal daily gains. Toward the end of the feeding program, the approximation becomes better until equation (7) gives the actual value of the costate at the optimal selling weight. The optimal selling weight is determined when the objective function from the optimal-return diet in equation (6a) no longer exceeds the opportunity costs of selling on the right-hand side of equation (5).

Application

The dynamic model of optimal cattle feeding, complete with nutrient and dry matter constraints for each day, can be solved as a large-scale nonlinear programming problem by the MINOS software for mainframe computers (Murtagh and Saunders). MINOS uses sparse matrix techniques that can easily accept the large but sparse matrices of optimal control models like the cattle feeding model. Nonlinear objective functions are solved by a reduced gradient algorithm and nonlinear constraints by a projected lagrangian method. MINOS is intended for use by researchers, and no attempt has been made to make the software user-friendly.

In contrast, the optimal-return diet model is a small-scale nonlinear programming problem which can be quickly and easily solved by the GINO software for microcomputers (Liebman et al.). Nonlinear objective functions are solved by a reduced gradient algorithm and nonlinear constraints by a generalized reduced gradient algorithm. Although GINO can solve only small problems the software is intended for use by nonprofessionals and is user-friendly.

One question remaining is how often diets should be calculated. Over a 200-day feeding program, for example, up to 200 different diets could be determined. Say there are 5 possible feeds in each day's diet. Then a nonlinear program to solve for the optimal 200-day feeding program would have 1,401 nonlinear variables, 1,000 nonlinear constraints, and 201 linear constraints. Such a problem would be considered medium-sized by the authors of MINOS. It is questionable, however, whether diet models have enough precision to warrant daily diet calculations. In this application diets are calculated weekly, easing the computational burden by a factor of seven.

Nutrients analyses for alfalfa hay, corn for grain, corn silage, soybean meal, and limestone are shown in table 1 (NRC 1984, pp. 48 and 62). Hay costs 6¢ per kilogram of dry matter; corn 13¢ per kilogram; corn silage, 7¢ per kilogram; soybean meal, 16¢ per kilogram, and limestone 6¢ per kilogram. These costs are estimates of market prices rounded to the nearest penny, except for corn silage. Corn silage has a limited market and so has been priced at 55% of corn for grain. The expected selling price will be set conservatively at $1.30 per kilogram so that the model does not always push for the maximum rate of gain. The marketing charge will be set at $4.00 per head, the daily variable costs will be 4¢ per day, the annual interest rate will be 13.0%, the annual interest rate adjusted for death losses will be 14.6%, and the purchase price of a medium-frame steer at an initial weight of 350 kilograms will be $1.35 per kilogram.

Table 2 contains a comparison of feeding

Table 1. Nutrient Analyses

	Midbloom, Sun-cured Alfalfa Hay	Grade 2 Corn	Corn, Silage, Well-eared	Soybean Meal Solv. Extract	Limestone
ME_i (Mcal/kg)	2.10	3.25	2.53	3.04	
NE_{mi} (Mcal/kg)	1.24	2.24	1.63	2.06	
NE_{gi} (Mcal/kg)	0.68	1.55	1.03	1.40	
CP_i (g/kg)	170	101	81	499	
Ca_i (g/kg)	14.1	0.2	2.3	3.3	340
Ph_i (g/kg)	2.4	3.5	2.2	7.1	0.2
DM (kg/kg)	1	1	1	1	1

Source: NRC 1984, pp. 48, 62.

programs and a selection from the many diets actually calculated. Three different models were run: (*a*) a dynamic model for the decisions of a farmer-feeder or a feedlot operator with excess feedlot capacity, (*b*) a dynamically optimal Faustmann model for the decisions of a cattle feeder who feeds continuously with no excess feedlot capacity, and (*c*) an optimal-return model as an approximation to the two dynamically optimal models. The diets from the beginning and ending weeks of each feeding program are reported. All of the models were run on MINOS to make the results as comparable as possible.

In the dynamic model, the steer is fed from a weight of 350 kilograms to 502.6 kilograms during a nineteen-week period, netting a return before fixed costs of $23.40 per head. The implicit price of gain, which in this case is the costate variable, starts at 84.4¢ per kilogram and increases during the feeding program to equal the selling price discounted by one week, or $1.296 per kilogram. Daily gain, on the other hand, starts fairly high and declines, even

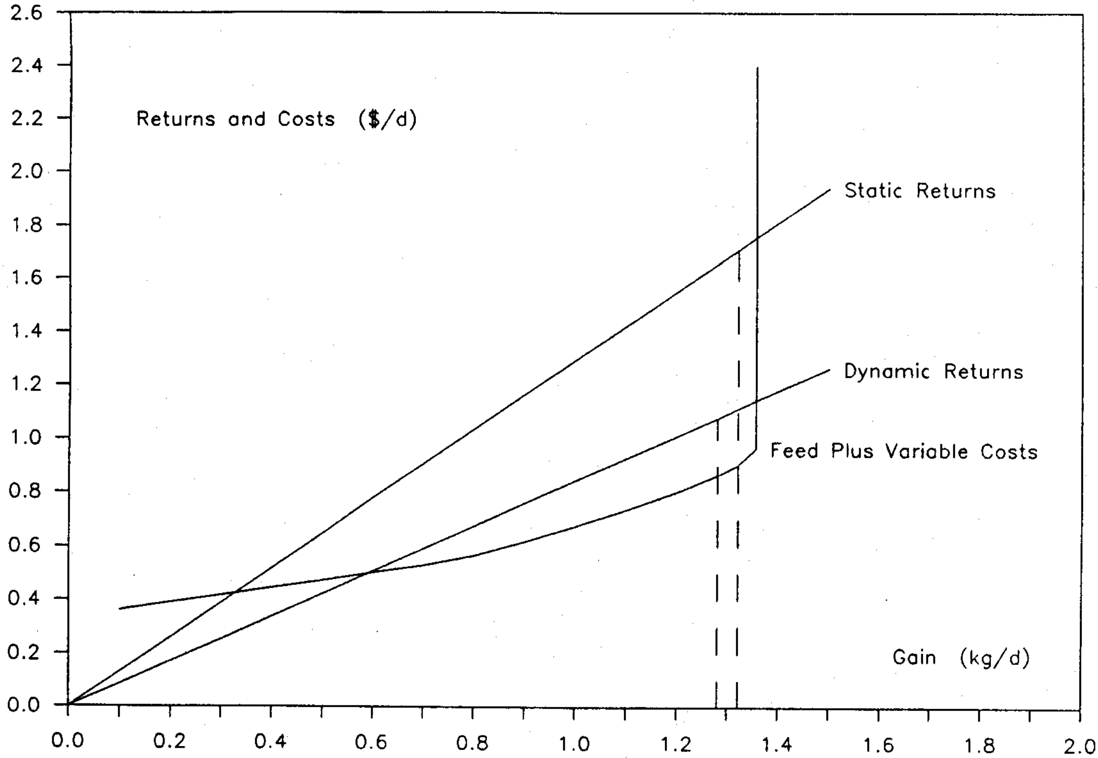

Figure 1. Returns above feed plus variable costs for a medium-frame, 350-kilogram steer

Table 2. Comparison of Feeding Programs and the Beginning and Ending Diets Fed to a Medium-Frame Steer

	Dynamic Model				Faustmann Model	
Net present value ($)	23.40/head				18.83/head	
Length of program	19 weeks				13 weeks	
	350–359.0-kg Steer		496.0–502.6-kg Steer		350–359.1-kg Steer	
Implicit price of gain ($/kg)	.844		1.296		1.001	
Gain (kg/d)	1.280		.931		1.306	
	Feeds	Reduced Cost	Feeds	Reduced Cost	Feeds	Reduced Cost
Hay ($/kg)		−.016		−.027		−.017
Corn (kg/d)	4.638		5.641		5.167	
Corn silage (kg/d)	2.480		2.105		1.846	
Soybean meal (kg/d)	.268			$−.045	.262	
Limestone (kg/d)	.080		.065		.086	
	Excess Nutrients	Shadow Price	Excess Nutrients	Shadow Price	Excess Nutrients	Shadow Price
NE_g		$−.161/Mcal		$−.193/Mcal		$−.192/Mcal
CP		$−.000/g	31.788 g			$−.000/g
Ca		$−.000/g		$−.000/g		$−.000/g
Ph	4.305 g		3.901 g		4.582 g	
DM		$.115/kg		$.163/kg		$.160/kg

as the energy concentration of the diet is increasing. Hay is never fed because soybean meal supplements the protein and because limestone supplements the calcium. For the diet fed to a 350-kilogram steer, hay has a reduced cost of 1.6¢ per kilogram, meaning that the price of hay must be reduced to 6 − 1.6 = 4.4¢ per kilogram before it will be included. In this same diet, the nutrients, NE_g, *CP*, and *Ca* are limiting and valuable. An additional thousand calories of NE_g in a feed is worth 16.1¢ per thousand calories. Dry matter intake, *DM*, is restrictive and an extra kilogram of dry matter in a feed costs 11.5¢ per kilogram. The composition of the diets changes gradually until, by the end of the feeding program, the diet fed to a 496-kilogram steer has excess *CP* and soybean meal has been excluded.

In the Faustmann model for limiting feedlot capacity, the diets and daily gains are very similar to those of the dynamic model, even though the costate variables (reported as the implicit prices of gain) are greater. However, there are substantial differences in the length of the feeding program and the selling weight. When feedlot capacity is limiting, the steer is fed for only thirteen weeks to a weight of 462.2 kilograms, compared with nineteen weeks and 502.6 kilograms when there is excess capacity. The net returns per head are reduced from $23.40 to $18.83, but the returns per week are higher, $1.45 in the Faustmann model and only $1.23 in the dynamic model. Each steer is to be sold fairly quickly and replaced by a younger, faster growing steer. However, each steer is sold so quickly that it will surely be of a lower quality grade and may be unacceptable to packer buyers. The Faustmann example does illustrate that animals should be rotated as quickly as possible out of a feedlot with limiting capacity.

The surprising result in table 2 is that the net returns, the length of the feeding program, the daily gains, and the individual diets from the optimal-return model are, for all practical purposes, the same as those from the dynamic model. If the cattle were sold at thirteen weeks, the optimal-return model would also closely

Table 2. Extended

Faustmann Model		Optimal-Return Model			
18.83/head		23.03/head			
13 weeks		19 weeks			
454.5–462.2-kg Steer		350–359.2-kg Steer		498.4–504.9-kg Steer	
	1.296		1.296		1.296
	1.109		1.321		.920
Feeds	Reduced Cost	Feeds	Reduced Cost	Feeds	Reduced Cost
	−.029		−.020		−.029
6.100		5.572		5.604	
1.812		1.337		2.126	
.004		.257			$−.047
.075		.090		.065	
Excess Nutrients	Shadow Price	Excess Nutrients	Shadow Price	Excess Nutrients	Shadow Price
	$−.206/Mcal		$−.250/Mcal		$−.202/Mcal
4.838 g			$−.000/g	32.912 g	
	$−.000/g		$−.000/g		$−.000/g
4.984 g		4.761 g		3.805 g	
	$.181/kg		$.246/kg		$.170/kg

approximate the Faustmann model. This result is surprising because a very crude approximation to the costate variable was used as the implicit price of gain in the optimal-return model. The first-order condition for choosing gain, condition (4a), compares the implicit price of gain with marginal feed costs, and, in general, substituting the wrong implicit price should do great damage to optimality.

The reason that the optimal-return model approximates the dynamic optimum so well is the unique shape of the least-cost feed function. In figure 1, a series of least-cost diets at increasing rates of gain have been calculated for a 350-kilogram steer. The variable costs per day, which are constant with respect to gain, have been added to feed costs to give the feed plus variable costs curve. These costs increase slowly and then abruptly to infinity at a high rate of gain. Beyond a maximum of 1.357 kilograms per day, the nutrient and dry matter constraints of the diet model cannot be satisfied and the feed costs must become infinite.

The dynamic returns line in figure 1 represents the costate variable times the rate of gain, making the difference between it and the feed plus variable costs curve equal to the hamiltonian of equation (3). The dynamically optimal diet is that least-cost diet producing 1.280 kilograms per day of gain. For the optimal-return model, the static returns line represents the selling price discounted one week multiplied by the rate of gain. The optimal-return diet is that least-cost diet producing 1.321 kilograms per day of gain. Because daily gain is chosen within a very narrow range, the dynamically optimal choice is insensitive to crude approximations for the implicit price and optimal-return diets are nearly optimal, even in the early weeks of a feeding program.

Conclusions

A model of dynamically optimal cattle purchasing, feeding, and selling decisions was constructed and a simple static model developed to closely approximate dynamically optimal decisions. The models were solved using the nonlinear programming software, MINOS.

Dynamically optimal purchasing decisions are an all-or-none proposition. If the net present value of the feeding program is positive, as many cattle should be purchased as possible, subject to constraints on the size of the pen, the availability of cattle, and the availability of capital. Among different types of cattle, those with the greatest net present value should be purchased. The dynamically optimal choice of feeds in the diets and the daily gains during the feeding program will be determined by the implicit price of gain, i.e., the current-value costate variable. The costate variable will be less than the expected selling price of the animals for two reasons. The first is discounting; the second is the fact that gain produced today will make the cattle heavier in the future, which will increase future feed costs. Cattle fed dynamically optimal diets will gain more slowly in the early weeks of the feeding program than cattle fed optimal-return diets calculated using the expected selling price. The optimal selling weight is chosen when the profitability of feeding one more week no longer exceeds the opportunity returns from selling the animals plus the opportunity returns, if any, to scarce feedlot space.

The dynamic model is a free-time optimal control problem which makes its solution more difficult than if the length of the feeding program were fixed. Essentially, a series of fixed-time optimal control problems must be solved for different lengths of the feeding program. The optimal solution is the one with the largest net present value. Another difficulty in solving the model is the high degree of nonlinearity of the nutrient and dry matter constraints. Dynamically optimal solutions were obtained, however, for two scenarios. The first scenario was for a farmer-feeder who does not feed continuously or for a commercial feeder who has excess capacity in his feedlot. The second scenario was for a commercial feeder whose feedlot is filled to capacity, so that an opportunity return can be earned if feedlot space is made available by selling animals. The length of the feeding program in the second scenario was almost one-third shorter than under the first scenario because the feedlot operator substitutes younger, faster growing animals to maximize the return per week rather than the return per head. However, severe marketing penalties for lighter, less finished cattle may preclude such a short feeding program.

Finally, the interesting result was obtained that the dynamically optimal feeding program can be closely approximated by a series of static optimal-return diets. The reason for the goodness of the approximation is not the same as that for other approximately optimal decisions rules which require accurate assumptions about the costate. Rather, the optimal daily gain is insensitive to assumptions about the costate and will usually lie within a narrow range close to the maximum feasible rate of gain. The implication of this finding is that approximately optimal cattle purchasing, feeding, and selling decisions can be made by cattle feeders and nutritionists with access to the new generation of user-friendly nonlinear programming software, such as GINO for microcomputers.

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