DETERMINING OPTIMAL FERTILIZATION RATES UNDER VARIABLE WEATHER CONDITIONS

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This paper presents a theoretical framework for incorporating the following sources of risk into the determination of optimal fertilization rates: (a) the influence of weather and other stochastic factors on the marginal product of fertilizer, and (b) uncertainty about the coefficients of the response function. The decision criterion considered is the maximization of profit subject to a risk constraint on the probability of not recovering the cost of the fertilizer. The theoretical framework is applied to the fertilization of dryland grain sorghum in the Texas Blacklands. Results indicate that the risk averse producer should substantially lower his fertilization rate if soil moisture at fertilization time is low.

The decision criterion commonly used in making fertilizer recommendations is expected profit maximization. However, the risk averse producer who bases his fertilization program on this criterion may experience a serious misallocation of resources if he is uncertain about the influence of weather on the marginal productivity of fertilizer and about the response function.

In a pioneering article, de Janvry presented a model that accounted for risk due to weather variability. However, he implicitly assumed that the response function was known with certainty. This article extends the de Janvry framework to include uncertainty in the response function. This extended model is applied to the fertilization of dryland grain sorghum in the Texas Blacklands. Weather risk is appraised with historical records, while the response function is appraised with experimental data on the yield response to different fertilizer rates.

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The Decision Model

In a recent article, R. H. Day, et. al., provide an excellent discussion about firm behavior under risk. In particular, they explore variations of the "safety-first principle" originally developed by Roy. One variation is the "strict safety-first principle" advocated by Shackle and applied by Telser. This criterion assumes that the decision maker will apply his resources to maximize expected profits subject to a constraint on the probability (δ) of experiencing a loss. Day, et. al., show that this criterion involves a minimum acceptable safety margin. If the safety margin is less than the decision maker's subjectively specified δ, resource use is constrained at a level just securing the acceptable safety margin. The appeal of this criterion becomes apparent by recognizing that it "...represents a compromise between expected profit maximization and safety margin maximization" [Day, et. al., p. 1296]. Robinson and Day have shown that this principle reflects a utility function with a lexicographic ordering in the expected value-risk space. Therefore, this principle can be rationalized by a set of consistent axioms of behavior.

This strict safety-first decision criterion is used to evaluate optimal fertilization rates under variable weather conditions. Formally, the objective is to maximize the profit to fertilizer for a crop producer who operates in competitive markets, subject to a risk constraint defined as the probability of not recovering the cost of fertilizer. Stated mathematically, the decision model is:

$$1)\ \ \text{MAX}\ E[P\cdot Y(N,W)] - mN$$

subject to

$$2)\ \ \Pr[P\cdot D(N,W) \geqslant mN] \geqslant \delta$$

where: E = expected value operator; P = unit price of the product; Y = yield response function; N = fertilization rate; W = weather variable; m = fertilizer price; Pr = probability; D = Y(N,W) − Y(N=O,W) = increment in yield attributed to fertilizer; and δ = subjective loss probability threshold (maximum risk).

To find the fertilization rate that satisfies this decision criterion, we must first find the probability distribution given by expression (2). The stochastic variables in the above model are weather (W) and the increment in yield attributed to fertilizer (D) which are expressed by the following conditional probability density functions:

$$3)\ \ f_1(D|N,W)$$
$$4)\ \ f_2(W|A)$$

where W is specified as the number of stress days after planting and fertilization, and A is available soil moisture at fertilization time.[1]

From the definition of conditional probability, it is known that the joint conditional probability distribution of D and W is:

$$5)\ \ f_3(D,W|N,A) = f_1(D|N,W) \cdot f_2(W|A)$$

Now note that integrating this distribution over all values of the weather variable gives:

$$6)\ \ f_4(D|N,A) = \int_0^\infty f_3(D,W|N,A)dW = \int_0^\infty f_1(D|N,W) \cdot f_2(W|A)dW$$

which is the conditional probability distribution for the increment in yield attributable to fertilizer. Since the total fertilizer cost, mN, is known with certainty, the problem reduces to finding the probability distribution of the net revenue (R), where R = P·D. For this application of the model, it is assumed that price (P) is known with certainty. This assumption is approximately valid for a farmer who has a forward market contract for the product or copes with price risks by other means. With price known, the probability distribution of R is obtained by transforming the probability distribution (6). Applying a theorem from mathematical statistics for obtaining the probability distribution of a function of a random variable [see, for example, Meyer, p. 88] gives:

$$7)\ \ f_5(R|N,A) = \frac{1}{P} \cdot f_4(D|N,A)$$

By integrating this probability distribution from an infinite loss (R = −∞) to the cost of the fertilizer (R = mN), the probability of not recovering the cost of the fertilizer as expressed in equation (2) is obtained

$$8)\ \ \Pr(R \leqslant mN|N,A) = \int_{-\infty}^{mN} f_5(R|N,A)dR = \frac{1}{P}\int_{-\infty}^{mN} f_4(D|N,A)dD$$

Referring back to (6), it can be seen that an alternative expression is:

$$9)\ \ \Pr(R \leqslant mN|N,A) = \frac{1}{P}\int_{-\infty}^{mN/P} \left[\int_0^\infty f_1(D|N,W)\, f_2(W|A)dW\right] dD.$$

As de Janvry has shown, the solution to this type of decision model is characterized

[1]The amount of water deficit experienced by the crop is described by the number of "stress days" during the growing season. Formally, the number of stress days, W, was calculated by Kissel, Ritchie, and Richardson as

$$W = \sum_{n=1}^{n^*} \left(1 - \frac{E_n}{E_o}\right),$$

where n^* is the number of days from crop emergence to harvest, E_n is daily evapotranspiration, and E_o is daily evapotranspiration, and E_o is daily potential evaporation rate above the plant canopy. For further discussion of the stress day concept, see Kissel, et. al.

by two regions. The characteristic of one region is that $Pr(mN_e | N_e, A) \geq \delta$, where N_e = the fertilization rate (N) that maximizes expected profit. In this region, the "strict safety-first" level of fertilization (N*) is N ; that is, $N^* = N_e$. The characteristic of the other region is that $Pr(mN_e | N_e, A) < \delta$. Here the minimum acceptable safety margin is not met by applying the expected profit maximizing rate, N_e. For this region, the strict safety-first level of fertilization (N*) is below N_e, and N* is found by setting expression (9) equal to δ and solving for the N that gives the highest expected profit.

An Application

This section presents the results of applying the model to evaluating fertilization rates for dryland grain sorghum in the Texas Blacklands. Both weather and the response function are assumed to be random variables. In the sections that follow, response uncertainty is considered first, then the weather uncertainty, and finally the two types of probability information are combined with the use of equation (9) for joint evaluation.

Response Uncertainty

Using experimental data presented by Kissel, Ritchie, and Richardson, the following response function for dryland grain sorghum is estimated with ordinary least squares regression:

$$10)\ Y = 2674.46 + 27.88N - .323WN - .0804N^2$$
$$\qquad (16.99)\quad (5.51)\quad (5.73)\quad (3.07)$$
$$k = 32 \qquad R^2 = .71$$

where Y = grain sorghum yield in pounds per acre; N = nitrogen rate in pounds per acre; W = number of stress days in the growing season; k = degrees of freedom; and the values in parentheses are the t-statistics.

Under the standard regression assumptions made in estimating a response function of the form ($Y = \beta_0 + \beta_1 N + \beta_2 WN + \beta_3 N^2$), it can be shown that for a finite sample, $f_I(D|N,W)$ is distributed as Student's-t:

$$11)\ f_1(D\,N,W) = \frac{\Gamma[(k+1)/2]}{\Gamma(k/2)\sqrt{\pi k}}\left(1 + \frac{t^2}{k}\right)^{-(k+1)/2} \qquad -\infty < t < \infty$$

where:

$$t = \frac{(D-\mu)\sqrt{k}}{\sigma}$$

k = degrees of freedom

$$\mu = \hat\beta_1 N + \hat\beta_2 WN + \hat\beta_3 N^2$$

$$\sigma = N\hat{s}\begin{bmatrix} V(\hat\beta_1) + W^2 V(\hat\beta_2) + N^2 V(\hat\beta_3) \\ + 2W\cdot CV(\hat\beta_1,\hat\beta_2) + 2N\cdot CV(\hat\beta_1,\hat\beta_3) \\ + 2WN\cdot CV(\hat\beta_2,\hat\beta_3) \end{bmatrix}^{1/2}$$

with

s = standard error of the estimate
$V(\cdot)$ = variance
$CV(\cdot,\cdot)$ = covariance .

This gives the probability distribution relating to the uncertainty about yield response.

Weather Uncertainty

Weather uncertainty in the Texas Blacklands is appraised with estimates of the number of stress days for three ranges of soil moisture (Kissel, Ritchie, and Richardson). While it would be desirable to have more than three ranges of soil moisture, it was impossible to obtain the necessary data for the Texas Blacklands. To allow for more precise probability estimates, a continuous probability function was fitted to the stress-day data for each range of soil moisture.

Climatic and biological factors suggest that the probability density function be continuous, nonsymmetric in general, and concave or convex based on the value of its parameters. Nonsymmetry is needed because different biological effects are naturally linked to low W's compared to those linked to large W's. Photosynthesis, respiration availability of soil nutrients, probability of diseases, and pests are likely to be related to W in a nonsymmetrical way. Howell, et. al., computed empirical probability distribulions of grain crop yields as a function of soil moisture. They showed that under different levels of soil moisture, the distribution function is generally nonsymmetric, concave, or convex.

The Gamma density function possesses these characteristics, given by

$$12)\quad f(W) = \begin{cases} \frac{1}{\Gamma(a)b^a} W^{(a-1)} e^{-W/b} & \text{for } W \geq 0 \\ 0 & \text{for } W < 0 \end{cases}$$

with $E(W) = ab$ and $Var(W) = ab^2$, where a and b are parameters to be estimated, and $\Gamma(a)$ is the Gamma function (note that the Gamma density function is concave for $a>1$ and convex for $a<1$).[2]

The data in Kissel, et. al., are given in terms of cumulative distributions; hence, equation (11) must be integrated in order to estimate its parameters. This integration is carried out numerically by using an adaptive Romberg extrapolation discussed by de Boor. Parameters a and b were estimated by minimizing the sum of squared deviations using a non-linear optimization algorithm based on the Levenberg-Marquandt algorithm discussed by Brown. Since the number of stress days (W) is conditional on the available soil moisture (A), it was possible to estimate the following three conditional probability functions:

13) $f_2(W|0\leq A\leq 3.9) = \text{Gamma}(\hat{a}=2.45; \hat{b}=14.52)$ with $E(W)=35.6$, and mean squared error = 0.00089

14) $f_2(W|3.9<A\leq 6.5) = \text{Gamma}(\hat{a}=3.38; \hat{b}=6.54)$ with $E(W)=22.14$, and mean squared error = 0.00215

15) $f_2(W|A>6.5) = \text{Gamma}(\hat{a}=0.539; \hat{b}=18.71)$ with $E(W)=10.09$, and mean squared error = 0.00451.

If a no risk situation is assumed, equation (10) is known with certainty. This means that the maximization of equation (1) is no longer subject to equation (2), and the expected value of W could be used with certainty. Then, differentiating equation (1) with respect to N, and equating the derivative to zero provides the optimal value for N. These values are: for E(W) = 35.6, the no risk N* = 72.26 lbs/acre; for E(W) = 22.14, N* = 99.30 lbs/acre; and for E(W) = 10.09, N* = 123.50

To obtain the loss probability for this empirical problem, the response probability distribution (11) is combined with each of the weather probability distributions (13), (14), or (15) using rule (9). With the general weather distributor (12) we get:

$$16)\quad \Pr(R \leq mN|N,A) = \frac{1}{P}\int_{-\infty}^{mN/P}\int_0^{\infty} \frac{\Gamma[(k+1)/2]}{\Gamma(k/2)\sqrt{\pi k}} \left(1 + \frac{\left(\frac{(D-\mu)k}{\sigma}\right)^2}{k}\right)^{-(k+1)/2} \frac{W^{(a-1)}}{\Gamma(a)b^a} e^{-W/b} dWdD.$$

The complex mathematical form of this distribution requires that the integrations be done numerically using a procedure developed by Greville.

The optimal fertilization rates as related to soil moisture and the loss probability are shown in Figure 1. By specifying the value of δ acceptable to a farmer and by determining the level of soil moisture, one can use this figure to find the optimal fertilization rate. For example, suppose that soil moisture is between 3.9″ and 6.5″ and that δ is specified to be .50. From Figure 1, it can be seen that the "strict safety-first" level of fertilization (N*) is about 99 lbs/acre, which is also the

[2]Additional plausible justification for the application of the Gamma is two-fold. First, it can be shown that as a special case the Gamma distribution is symmetrical; hence, it is more general than the normal distribution. Second, a special form of the Gamma distribution (the Erlang distribution) is simply a summation of independent negative exponential random variates with parameter (1/b). The negative exponential was extensively used to describe random events corresponding to durations, which is what W essentially is [Phillips, et. al., pps. 218-220].

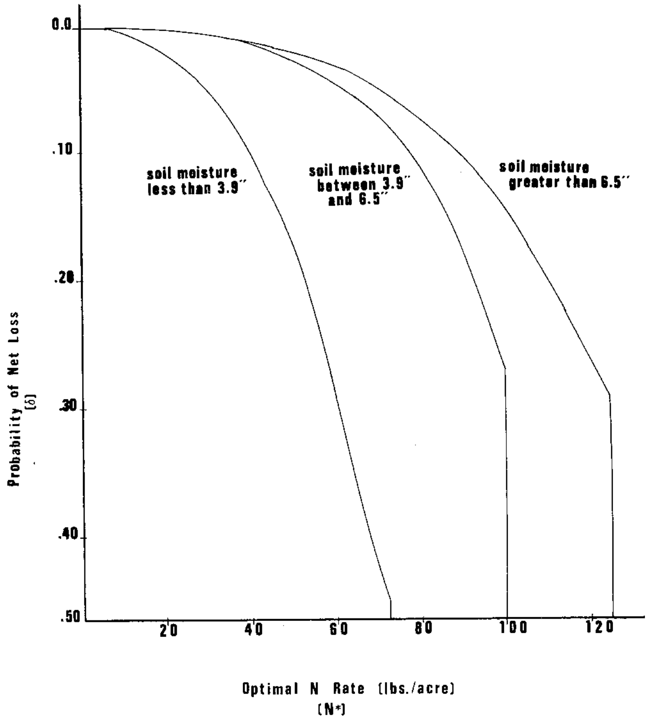

Figure 1. Optimal N rates where there is uncertainty about weather and yield response.

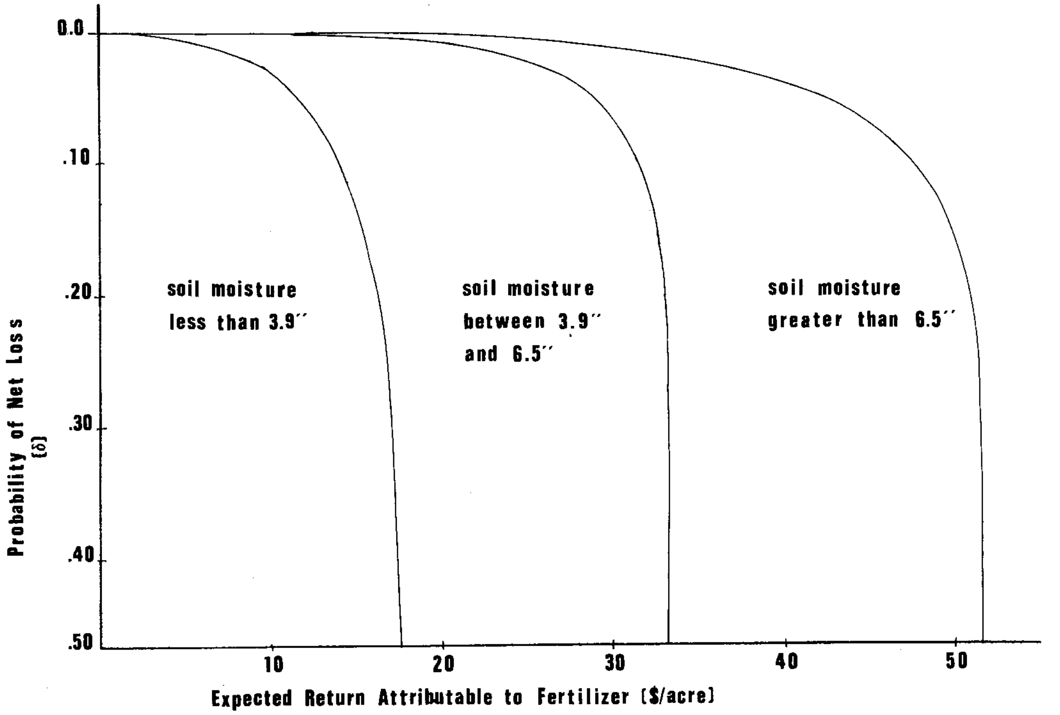

Figure 2. The relationship between expected profit and risk aversion for the case where there is uncertainty about weather and yield response

expected profit maximizing rate. Note that for this level of soil moisture, the expected profit maximizing rate should be used as long as the specified loss probability (δ) is greater than .27; that is, the loss constraint is not binding unless δ is less than .27. As another example, suppose that δ is specified to be .10 and soil moisture is between 3.9″ and 6.5″. In this case, the "strict safety-first" level of fertilizer is about 75 lbs/acre.

Figure 1 can also be used to find the loss probability associated with a specific fertilization rate. For example, if soil moisture is between 3.9″ and 6.5″ and 60 lbs/acre of fertilizer is applied, the associated loss probability is 0.05.

Figure 2 depicts the relationship between the expected return to fertilizer and the probability of a net loss for the three soil moisture conditions when "strict safety-first" fertilizer rates are applied. As an example, suppose that soil moisture is between 3.9″ and 6.5″ and that the acceptable loss probability (δ) is specified by the farmer to be .30. Under these conditions, the expected return to fertilizer is about $33.00 per acre. And if the loss probability is .05, the expected return is about $28.00 per acre. So, for soil moisture between 3.9″ and 6.5″, the farmer who wants to recover the cost of fertilizer 95 percent of the time rather than 70 percent of the time will give up an expected return of about $5.00 per acre per year. Trade-offs for other loss probabilities and soil moisture levels can be obtained from Figure 2.

Summary and Discussion

In this paper, the de Janvry model for finding the optimal fertilization level under risk is extended to include uncertainty about the response function. The critical assumptions for applying the model were complete knowledge on (a) the probability distributions for weather, (b) the mathematical form of the response function, and (c) prices of products and resources. If a long time-series of data were used to estimate the weather distribution (as in this paper), the first assumption is likely inconsequential. One way to overcome the weakness implied by the second assumption is to estimate various functional forms with a subjective probability assigned to each. If a fairly general form is used, the assumption would not appear to pose a serious problem.

The extension to two sources of risk may pave the way for dealing with the more generalized case of multiple sources of risk. Price uncertainty can be incorporated by extending the model. However, using probability distributions that are not easily integrated by analytical means will significantly increase the computational burden.

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