Multiperiod Optimization: Dynamic Programming vs. Optimal Control: Discussion

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The title of this session pitting Dynamic Programming against Control Theory is misleading since dynamic programming (DP) is an integral part of the discipline of control theory. However, it is timely to discuss the relative merits of DP and other empirical solution approaches to control problems in agricultural economics, and equally importantly, how control theory can be used to pose more useful and realistic problems. From this point in the paper I shall, like Burt, use the term Dynamic Programming (DP) to mean the classic solution procedure developed by Bellman. All the other solution approaches used to solve multiperiod problems, including Differential Dynamic Programming [Jacobson and Mayne], are termed Control Theory.

Burt takes the session title literally and rises to defend and extend the long history of DP research and application. I agree with him concerning the difficulty of teaching applied problem formulation, but like Zilberman, feel that lack of popularity of DP cannot be attributed to lack of exposure of graduate students.

In his section on "Obstacles to Implementation" Burt is unjustifiably pessimistic in his judgement of the practicality and theoretical basis of solutions based on the Pontryagin Maximum principle. Given the long history of Pontryagin based control applications in engineering and operations research, the problems cannot be categorized in general as "trivial" or without "theorems on the structure of the solution." A comprehensive survey of solution approaches to control problems may be found in Polak [1973], and some specific texts are Bryson and Ho, Canon *et al.*, Dyer and McReynolds, Jacobson and Mayne, Polak [1971]. In the context of applied economics, Zilberman cites many studies that most members of the profession would not classify as trivial. His citations include, of course, studies using DP solution approaches.

The applied economics literature has predominantly used the maximum principle in the analysis of the theoretical problems of dynamic economic systems. While there are special cases in which the first order conditions can be derived using Calculus of Variations, Dynamic Programming or the Maximum principle (and shown to be the same), the Maximum principle is both less restrictive in the form of the controls and constraints than calculus of variations, and easier to interpret than DP. The maximum principle has an overriding advantage for economic problems in that it explicitly specifies the intertemporal qualitative properties of the imputed value of state variables as costate variables. In addition, Lagrangian interpretations for binding constraints are incorporated in the first order conditions.

In the same section, Burt generalizes the major problem with DP — the "curse of dimensionality" to "an inherent characteristic of dynamic optimization problems in general when the number of state variables is large (more than 3 or 4)." However, much larger problems are routinely solved by analytic DP or programming approaches. The difference between the dimensionality increases for DP and nonlinear programming

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solutions is substantial. The DP memory requirement is the grid size raised to the power of the number of states, whereas the memory requirements for a nonlinear programming problem are proportional to the square of the product of state and time period dimensions. Clearly, the effect of adding additional state dimensions is very different. Intrilligator cites a very difficult DP problem of grid size/state of 100 and four states. Assuming 20 time periods and four controls with inequality constraints on all states and controls in all time periods, the same problem would yield a nonlinear programming problem with 160 columns and 240 rows, a routine operation using modern algorithms. A problem with nine states and thirty yearly time periods was solved as a quadratic program in Noel and Howitt.

Burt characterizes the dilemma of the applied analyst in the statement that "the primary objective in all modeling is to capture the essential aspects of the phenomenon under study and yet keep the model as simple as possible." The choice of solution approach thus depends on the appropriate model characteristics. I agree with Burt that for low dimension problems DP is a superior method, particularly if the functions are irregular and a range of stochastic values is important to the problem. However, for problems in which the simultaneity of many states is more important than smooth approximations to irregular functions there are several additional solution approaches. Furthermore given the structures of micro theory and limitations of many least squares and maximum likelihood approaches, most applied economic problems satisfy the requirements of non-DP approaches. Specifically: continuous and convex or concave functions, differentiability, and stochastic properties that can be characterized by sufficient statistics.

Considerable work has been done on the stochastic control problem since Kushner and Schweppe's cited article, a more recent article with illustrative examples is found in Haussman. The direct applicability of a range of econometric models to optimum control is shown in Rausser and Hochman, and Chow.

Burt provides a comprehensive review and extension of methods to reduce state vector decisions. The wheat storage study [Burt, Koo and Dudley] would make a good vehicle for study of alternative solution methods, being apparently solvable by the linear Quadratic Gaussian formulation for Differential Dynamic Programming or nonlinear programming, without any simplification of state varibles, a wider choice of controls, and the simplification of a stochastic error term.

Among the methods for reducing dimensionality in DP I would add that the regeneration point approach [Dryfus and Law] has a natural application to problem of capacity expansion and capital replacement. In these latter problems, the advantage that DP enjoys in integer problems makes it a logical choice. I am less convinced as to the value of DP in analyzing farm firms where crop alternatives, cash flows, asset stocks and updated rational expectations would probably expand the state dimension to an unmanageable level for DP.

Zilberman's paper is written from a more general view point which sees dynamic programming as an important part of the set of solution approaches to the general stochastic control problem. Zilberman skirts the details of solution procedures and concentrates on the values of optimal control theory in posing theoretical empirical and policy models in an extension of comparative statics. To his review of economic applications of control theory, I would add the areas of management science [Bensoussan, Kleindorfer and Tapiero], consumer demand [Houthakker and Taylor] and rational expectations [Taylor]. At the end of Zilberman's review of control applications to agricultural economics, he remarks correctly that "it is still a limited tool in its application and impacts." He attributes this outcome to the emphasis on solution techniques rather than policy results, a crime of which the first part of this paper is guilty. However, as Burt points out, empirical problems have to be formulated appropriately to be solvable, and I think it equally important that practitioners are familiar with enough solution methodolgies to minimize the "damage" to the theoretical problem.

The majority of Zilberman's paper develops optimal control solutions to problems in domains other than time. The results are stimulating and demonstrate the power of the maximum principle to extend microeconomics. In his two-stage optimal control development over time and nontime domains, Zilberman develops the microeconomic equivalent of Isard *et al.'s* specification. Given the fixed costs of irrigation adoption, it is not clear that the combined problem can be always decoupled to allow the two-stage optimization procedure suggested. For instance, a major problem facing irrigated agriculture is the decline in productivity from rising water tables, salinity build-up or both. Evidently the distribution of land class is in this case a dynamic phenomena which is affected by both the temporal allocation of water and the adoption rates of irrigation technology. Zilberman's development is attractive as it lays out a microeconomic foundation for rational technological change.

Throughout Zilberman's exposition of time and nontime domain optimization I had misgivings on the data availability to use the approaches. Zilberman is aware of these constraints, but is optimistic that they can be ameliorated by the rapid diffusion of minicomputers and networks in the industry. He is probably correct, although compatability of the variable data sources is likely to be troublesome.

To return to the central direction of this session the two widely differing papers each point to problems in the evolution of any new applied methodology. The first priority is to emphasize Zilberman's point that optimal control must first offer additional microeconomic insights over comparative statics. Once the value to policy questions of the additional time dimensions is clear, there will be an incentive to extend the comparative static models of traditional agricultural economics. The qualitative properties of comparative dynamic equilibria are often not easy to obtain in "nice" forms. An additional source not mentioned by Zilberman or Burt which works towards comparative dynamics is Kamien and Schwartz. Like Zilberman, I feel that there are valuable and relevant relationships to be derived using control theory.

Burt emphasizes the empirical intricacies of solution by Bellman's original DP method, however, the disparagment of other solution methods is unnecessary. The fundamental concept of dynamic programming, the principle of optimality, has been used to develop a wide variety of solution procedures that are not subject to the curse of dimensionality. The stage wise solution is maintained but the discretization of state and control space and its attendant curse is avoided by storing functional forms that characterize the optimal solution in the backward solution phase. This class of solution approaches is called differential dynamic programming DDP. A through treatment of DDP may be found in Jacobson and Mayne (1970), and convergence of the approximated problem to the true nonlinear problem has been shown to be quadratic.

The simplest example of DDP is the widely used Linear Quadratic Gaussian (LQG) procedure briefly referred to by Burt. This procedure while not without drawbacks, has immediate appeal for many economic problems where the system dynamics are estimated by several (usually greater than four) linear relationships with additive normal stochastic terms. The objective function is often characterized as a weighed combination of producer and consumer surplus, which given linear estimates of demand and supply over the relevant range, results in a quadratic objective function. An example of a similar problem specification is found in Burt, Koo and Dudley.

The problem of inequality constraints in DDP is illustrated by Murray and Yakowitz's application to the multireservoir control problem. In many other cases inequality constraints can be approximated by suboptimal truncation or penalty functions incorporated in the LQG approach.

An alternative solution approach that has considerable appeal and future potential is the solution of deterministic control problems by nonlinear programming. Efficient general algorithms for sparse problems that are nonlinear in both the objective function and the constraints are now generally available in MINOS, Murtagh and Saunders or Box and can be used to solve substantial empirical problems such as Richardson and Ray and Noel and Howitt. Further developments to combine this approach with stochastic simulations are likely to be forthcoming.

Where the constraint set can be successfully embedded in the objective function, very large control models can be solved by unconstrained optimization methods. Fair discusses methods for solving up to 100 state variable models and uses a nineteen state twenty time period model to compare alternative approaches. Methods to extend this type of solution to stochastic problems are developed in Tinsley, Craine and Havenner and are currently implemented (but not published) on the four hundred equation Federal Reserve model.

Inevitably there are trade-offs but no insurmountable barriers in empirically implementing control problems, particularly in the form that agricultural economists have traditionally specified them. What is now needed from advocates of control theory (that of course includes dynamic programming) is a clearer analytical exposition of the advantages of comparative dynamics and dynamic policy models over their static counterparts.

References

Bellman, R. *Dynamic Programming,* Princeton University Press, 1957.

Bensoussan, A., P. R. Kleindorfer and C. S. Tapiero. *Applied Optimal Control,* Studies in the Management Sciences, Vol. 9 North Holland, Amsterdam, 1978.

Box, M. J. "A New Method of Constrained Optimization and A Comparison with Other Methods," *Computer Journal* 8(1965):42-52.

Burt, O. R., W. W. Koo and N. J. Dudley. "Optimal Stochastic Control of U.S. Wheat Stocks and Exports," *American Journal of Agricultural Economics* 62(1980):173-87.

Bryson, A. E. and Y. C. Ho. *Applied Optimal Control,* John Wiley, New York, 1975.

Canon, M. D., C. D. Cullum and E. Polak. *Theory of Optimal Control and Mathematical Programming,* McGraw-Hill, New York, 1970.

Chow, G. C. *Analysis and Control of Dynamics Economics Systems,* John Wiley, New York, 1975.

Dryfus, S. E. and A. M. Law. *The Art and Theory of Dynamic Programming,* Academic Press, New York, 1977.

Dyer, P. and S. R. McReynolds. *The Computation and Theory of Optimal Control,* Academic Press, New York, 1970.

Fair, R. C. "On the Solution of Optimal Control Problems as Maximization Problems," *Annals of Economics and Social Measurement,* 311(1974):135-54.

Haussman, U. G. "Some Examples of Optimal Stochastic Controls Or: The Stochastic Maximum Principle at Work," *SIAM Review* 23(1981):292-307.

Houthakker, H. S. and L. D. Taylor. *Consumer Demand in the United States: Analyses and Projections,* Harvard University Press, 1970.

Intrilligator, M. D. *Mathematical Optimization and Economic Theory,* Prentice Hall, 1971.

Isard, W., P. Liossatos, Y. Kanemoto, and P. C. Kaniss. *Spatial Dynamics and Optimal Space-Time Development,* North Holland, New York, 1979.

Jacobson, D. H. and D. Q. Mayne. *Differential Dynamic Programming,* American Elsevier New York, 1970.

Kushner, H. J. and F. C. Schweppe. "A Maximum Principle for Stochastic Control Systems," *J. Mathematical Analysis and Applications* 8(1964):287-302.

Kamien, M. I. and N. L Schwartz. *Dynamic Optimzation,* North Holland, New York, 1981.

Murray, D. M. and S. J. Yakowitz. "Constrained Differential Dynamic Programming and its Application to

Multireservoir Control," *Water Resources Research*, 15(1979):1017-27.

Murtagh, B. A. and M. A. Saunders. "Large Scale Linearly Constrained Optimization," *Mathematical Programming* 14(1978):41-72.

Noel, J. E. and R. E. Howitt. "Conjunctive Multibasin Management: An Optimal Control Approach," *Water Resources Research* (forthcoming 1982).

Polak, E. *Computational Methods in Optimization* Academic Press, New York 1971.

______. "An Historical Survey of Computational Methods in Optimal Control," *SIAM Review* 15(1973):553-84.

Rausser, G. C. and E. Hochman. *Dynamic Agricultural Systems: Economic Prediction and Control* North Holland, New York, 1979.

Richardson, J. W. and D. E. Ray. "Commodity Programs and Control Theory," *American Journal of Agricultural Economics* 64(1982):28-38.

Taylor, J. B., Editor. "Issue on Rational Expectations Analysis." *Journal of Economic Dynamics and Control* 2:1 February 1980.

Tinsley, P., R. Craine, and A. Havenner. "Optimal Control of Large Nonlinear Stochastic Econometric Models," Proceedings Summer Computer Simulation Conference Chicago, IL, IEEE, 1977.