Borrowing Behavior under Financial Stress by the Proprietary Firm: A Theoretical Analysis

Lindon J. Robison, Peter J. Barry, and William G. Burghardt

This paper extends finance theory under risk to account for borrowing behavior under financial stress conditions. As the financial stress level for the firm increases, the role of credit or unused borrowing capacity changes. With a strong equity position, credit is valued as a reserve to avoid liquidation costs resulting from the sale of fixed assets to meet cash flow obligations. As the financial stress on the firm increases the model demonstrates the firm's willingness to reduce credit reserves and increase its financial leverage in order to increase its probability of survival. These results are derived in a tractable framework by describing risky alternatives in terms of expected values and variances.

Key words: bankruptcy, credit reserves, expected value-variance model, fixed assets, leverage ratios, liquidation costs, risk aversion.

A basic principle of finance for the single investor in a proprietary firm is that increases in financial leverage will increase the expected level and variability of returns to the investor's equity capital (e.g., Francis and Archer; Van Horne).[1] Optimal leverage then depends on the investor's attitude toward risk as reflected by the properties of his utility function and the level and variability of returns. Moreover, optimal leverage will change as changes occur in expected returns, interest costs, variances, and risk attitudes (Robison and Barry 1977; Adler; Levy). If, for example, a permanent increase in the variance of returns occurs, then one plausible response is to reduce leverage. However, a different response may occur in the short run if the greater risk, combined with high cash flow obligations, jeopardizes the firm's survival and pushes it toward bankruptcy. In this case, it is neither unusual nor irrational to observe incentives for greater borrowing as a short-term response to risk. A current example is the use of additional borrowing, debt restructuring, and other financial responses to risk by financially stressed farmers in the United States.

In this article, we extend finance theory under uncertainty to account for borrowing behavior under financial stress conditions, emphasizing the increased use of credit when survival is at stake. The major purpose is to highlight the "go for broke" phenomenon that arises from a willingness to incur greater financial risk as the proprietary firm's survival is threatened. The risk-averse investor's objective is expressed in terms of expected utility maximization where equity capital is the object of utility and the essential properties of equity capital are the expected value and variance of its earnings. Because financial stress conditions are complex, we begin by deriving the proprietary firm's optimal borrowing behavior under perfect market conditions. Then we introduce market imperfections arising from asset liquidation costs and financial constraints and show the impacts of these factors

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[1] Financial leverage is defined as the use of debt capital and other fixed-obligation financing relative to the use of equity capital. The positive effect on expected returns assumes, of course, that the returns on assets exceed the cost of borrowing.

on borrowing behavior with and without possible bankruptcy. We conclude by considering the empirical implications of the analysis.

Our analysis is related to, yet differs from, the financial analysis of large corporate firms in which highly efficient markets exist for trading equity claims. Typically, the corporate goals in those studies focus on financial and investment policies which maximize the present value of a corporation's wealth. In the latter case, the propositions of Modigliani and Miller suggest that the firm's value is independent of its capital structure.

Other studies have examined the financial and investment policies of corporations under imperfect capital market conditions. In such studies, transactions costs may occur as a result of liquidation costs, tax policies, or bankruptcy law limitations on liability (Kim, Chen and Kim; Haugen and Senbet; Galai and Masulis; Scott; Kraus and Litzenberger). The general conclusions of these corporate investment studies are generally well known and summarized in leading financial textbooks such as Brealey and Myers. Their conclusions are (*a*) as the potential for paying liquidations costs increases, borrowing decreases; and (*b*) if the corporation can shift its borrowing risks as a result of bankruptcy laws that limit liability, then optimal borrowings increase.

This study generally supports those described above. Moreover, we extend the results of the earlier studies in several ways. First, we focus on the proprietary firm maximizing the expected utility of ending period wealth. Second, this study focuses on how a credit reserve or borrowing limit influences the behavior of the proprietary firm. This inventory of potential borrowings along with the liquidity of firms' assets play a critical role for the investment and financial behavior of proprietary firms that make up much of commercial agriculture in the U.S. and elsewhere.

Optimal Borrowing under Uncertainty: No Constraints

The derivation of optimal borrowing under uncertainty with no financial constraints is based on the following assumptions. The major source of risk is the random return, $r + \epsilon$, earned on a risky asset A_o, where ϵ is a random variable with mean zero and variance σ_r^2. Asset A_o is a divisible durable that lasts beyond a single period and can be acquired in any quantity. The interest rate i paid on debt capital D_o is known with certainty, and the beginning period accounting identity $A_o = D_o + E_o$ must be met, where E_o is the beginning equity capital. Finally, the investor's risk attitude is expressed as a desired tradeoff (λ) between the expected level and variance of end-of-period equity capital.

Thus, the investor seeks a level of debt that maximizes the expected utility of ending equity, E_1, defined as the sum of the beginning equity E_o plus net profit (ΔE) earned during the period. This objective is equivalent to maximizing the expected value-variance model in equation (1):[2]

$$(1) \qquad \text{Max } EU(E_1) = E(E_1) - (\lambda/2)\sigma^2(E_1),$$

where E is the expectations operator. In turn, ending equity ($E_o + \Delta E$) is defined as the return on assets less the cost of debt less withdrawals W_d for consumption and other purposes.

$$(2) \qquad E(E_1) = (1 + r)A_o - (1 + i)D_o - W_d.$$

Variance of ending equity is

$$(3) \qquad \sigma^2(E_1) = \sigma_r^2 A_o^2.$$

Substituting (2) and (3) along with the accounting identity, $A_o = D_o + E_o$, into (1) yields the expected utility model:[3]

$$(4) \qquad \text{Max } EU(E_1) = r(D_o + E_o) - iD_o + E_o - W_d - (\lambda/2)(D_o + E_o)^2\sigma_r^2.$$

Considering the level of debt as the decision variable, the first-order condition is found by differentiating (4):

[2] It can be shown that the mean-variance model in equation (1) is consistent with the expected utility model under completely general conditions when the choice is between alternative combinations of a risky and safe asset because all possible choices are EV efficient. Then any particular solution (e.g., the one which maximizes $EU(E_1)$) can be found by the appropriate choice of λ. Thus, the expected value variance model and a more general expected utility model will yield the same solutions. Additional discussion and justification for the EV model's use as an expected utility-maximizing model are provided in Meyer and Robison and Robison and Barry (1986). Robison and Barry (1986, chap. 16) also use the mean-variance model and the results of an earlier draft of this paper to explain borrowing behavior under stress.

[3] Notice in (4) that the investor may save at the risk-free rate i. Savings occur in the model as negative values of D_o. If all the equity were "saved," with no leveraging, then the right-hand side of (4) would become $(1 + i)E_o$. The lower bound on D_o is E_o since savings are limited to the amount of equity. But since the interest rates on borrowing and "saving" are equal, no incentive exists to borrow solely to invest in savings.

$$\frac{dEU(E_1)}{dD_o} = (r - i) - \lambda(D_o + E_o)\sigma_r^2 = 0. \quad (5)$$

Equating (5) to zero and solving for D_o gives optimal debt of

$$D_o = \frac{r - i}{\lambda\sigma_r^2} - E_o. \quad (6)$$

Thus, optimal debt in the unconstrained case depends on the investor's risk attitude as well as on expected returns, interest costs, variance, and beginning equity. Moreover, comparative static results for this model show that optimal debt is positively related to changes in expected returns on assets and inversely related to changes in costs of borrowing, equity, variance of returns, and risk aversion (Barry, Baker, and Sanint). This is shown by differentiating (6) with respect to each of its components.

$$\frac{dD_o}{dr} = \frac{1}{\lambda\sigma_r^2} > 0 \quad (6a)$$

$$\frac{dD_o}{di} = \frac{-1}{\lambda\sigma_r^2} < 0 \quad (6b)$$

$$\frac{dD_o}{dE_o} = -1 < 0 \quad (6c)$$

$$\frac{dD_o}{d\lambda} = \frac{i - r}{\lambda^2\sigma_r^2} < 0 \quad (6d)$$

$$\frac{dD_o}{d\sigma_r^2} = \frac{i - r}{\lambda(\sigma_r^2)^2} < 0. \quad (6e)$$

The one-to-one trade-off between debt and equity in (6c) is consistent with the condition of constant absolute risk aversion. It implies that increasing wealth (E_o) allows a reduction in risk-free debt while holding constant the level of risky assets. Holdings of risky assets would only increase with increases in equity if risk aversion (λ) decreases or if some other parameter value changes accordingly.

Introducing Liquidation Costs and Constraints on Credit and Cash Flows

The analysis now is generalized by including the effects of asset liquidation costs and constraints on borrowing capacity and cash flows. The beginning period borrowing constraint for the proprietary firm is expressed as a maximum limit (α) on the debt-to-equity ratio:

$$D_o < \alpha E_o, \quad (7)$$

or, alternatively, as

$$C_o = \alpha E_o - D_o, \quad (8)$$

where C_o is the unused borrowing capacity held in reserve.

The asset liquidation cost, ρ, separates the asset's acquisition price, A_o, from its sale price, $(1 - \rho)A_o$, and, for $\rho > 0$, identifies an asset as fixed.[4] This liquidation cost also interacts with the borrowing constraint to itself impose a limit on indebtedness. This occurs because the lender takes the liquidation cost into consideration when determining the firm's credit limits. Because the recoverable value of assets pledged as loan collateral is $(1 - \rho)A_o$, the lender requires the firm to have sufficient equity to cover these costs.[5]

The cash flow constraint represents the firm's capacity to meet its known financial obligations, based on the returns earned from the risky assets. A positive cash flow allows the firm to service its debt commitments successfully and to provide for consumption and other withdrawals. A negative cash flow, however, triggers the need for either additional borrowing from the unused credit reserves or partial liquidation of assets.

A critical factor, then, in the firm's liquidity position is the random outcome ϵ and its effects on cash flow. Following the development in the preceding section, the firm's change in equity, ΔE which depends on ϵ, is expressed as

$$\Delta E = (r + \epsilon - i)D_o + (r + \epsilon)E_o - W_d. \quad (9)$$

In equation (9), the firm experiences the stochastic outcome ϵ after it has borrowed amount

[4] Assets, of course, vary in their degree of fixity. As a result, each class of assets may have a different liquidation value ρ. In our simple model with only one kind of risky asset, there is only one liquidation parameter ρ.

[5] With liquidation costs, the maximum limit on debt is found as follows:

$$D_{max} = (1 - \rho)A_o$$

Substituting for A_o gives

$$D_{max} = (1 - \rho)(D_{max} + E_o).$$

Then, solving for D_{max}

$$D_{max} = \frac{(1 - \rho)E_o}{\rho}.$$

If borrowing is less than D_{max}, unused borrowing capacity, C_o, exists:

$$C_o = \frac{(1 - \rho)E_o}{\rho} - D_o.$$

This expression for C_o is the same as given earlier in the text, thus implying that α has a value of $(1 - \rho)/\rho$.

D_o. If the level of ϵ yields a positive change in equity which is used to reduce its debt, then the cash flow requirement is satisfied and credit reserves in the next period will increase. However, if the level of ϵ yields a negative change in equity, then additional borrowing or asset liquidation must occur. Since the liquidation of assets is costly, it is plausible that the firm will first utilize unused borrowing capacity as a source of liquidity and then consider asset liquidation. In turn, the change in borrowing changes the credit reserve expression in (8) to

$$(10) \qquad (D_o - \Delta E) + C_1 = \alpha(E_o + \Delta E).$$

Under these conditions, the level of ϵ_o that forces the firm to fully deplete its credit reserve causes the value of C_1 in (10) to equal zero. This level of ϵ_o is found by specifying (9) as

$$(11) \quad \Delta E(\epsilon_o) = (r + \epsilon_o - i)D_o + (r + \epsilon_o)E_o - W_d.$$

Substituting (11) into (10) with $C_1 = 0$ and solving for ϵ_o yields

$$(12) \quad \epsilon_o = \frac{D_o - \alpha E_o}{(1 + \alpha)(D_o + E_o)} + \frac{W_d + iD_o}{D_o + E_o} - r.$$

If outcome ϵ_o exhausts both the firm's credit and its equity, then bankruptcy occurs. However, if the credit constraint is conservatively set so that equity is not driven to zero, then the firm can continue to function after partial liquidation. For now, we will assume that outcome ϵ_o does not cause bankruptcy.

The probability distribution for ϵ is shown in figure 1. Since ϵ_o exhausts credit, then any lower outcomes ($\epsilon < \epsilon_o$) will trigger partial liquidation of assets. In turn, partial liquidation reduces the firm's equity and thus increases the probability of an ϵ occurring in subsequent periods that exhausts credit. Moving ϵ_o to the left reduces the probability that the firm will be forced to liquidate assets. Moving ϵ_o to the right increases the probability that the firm will be forced to liquidate assets. In terms of expression (12), increasing i or W_d moves ϵ_o to the right, while increasing r, E_o, or α moves ϵ_o to the left. However, increasing D_o has an ambiguous effect on ϵ_o.

Accounting for the simultaneous effects of all these variables on ϵ_o would yield a complex theoretical model.[6] Thus, we adopt a simplifying, yet perhaps more realistic assumption that the stochastic outcome which triggers liquidation is based on predetermined values of the other variables which is defined here as ϵ_o^*.[7] Thus, the firm's initial level of borrowing in the present period still is limited by its liquidity characteristics and equity, as expressed in (9). However, additional credit to cover a negative cash flow is available only if the actual rate of return exceeds a minimum of $r - \epsilon_o^*$, as established by the lender. If ϵ is less than the lender-specified ϵ_o^*, then asset liquidation occurs to cover the cash deficit.

Borrowing Behavior and the Liquidation Model

Before solving the model for optimal data, expressions for the expected value and variance of ending equity must be found that include the effects of liquidation costs and the financial constraints. First, consider the expected value of ending equity. If $\Delta E(\epsilon_o^*)$ is the outcome level that exhausts credit, then worse outcomes $\Delta E(\epsilon < \epsilon_o^*)$ will require asset liquidations to cover the cash deficit. Moreover, the amount of the liquidated assets will exceed the cash deficit due to liquidation cost ρ. This is shown as follows, where ΔA_o is the amount of liquidated assets:

$$(13) \quad (1 - \rho)\Delta A_o = \Delta E(\epsilon_o^*) - \Delta E(\epsilon < \epsilon_o^*).$$

Liquidation cost $\rho \Delta A_o$ is found by dividing (13) by $1 - \rho$ and multiplying by ρ:

$$(14) \quad \rho \Delta A_o = \left[\frac{\rho}{(1 - \rho)}\right][\Delta E(\epsilon_o^*) - \Delta E(\epsilon < \epsilon_o^*)].$$

The expected value of the liquidation cost is then found by replacing ϵ_o with ϵ_o^* in (11) and by substituting the result along with (9) into (14) and integrating the difference in (14) over the range of $\epsilon < \epsilon_o^*$. The result is

$$(15) \quad E[\rho \Delta A_o] = \left[\frac{\rho}{(1 - \rho)}\right] \int_{-\infty}^{\epsilon_o^*} (\epsilon_o^* - \epsilon) \cdot (D_o + E_o) f(\epsilon)\, d\epsilon = \left[\frac{\rho}{(1 - \rho)}\right][\epsilon_o^* F(\epsilon_o^*) - \bar{\epsilon}] \cdot (D_o + E_o),$$

[6] Burghardt and Robison describe a computer simulation model which examines borrowing behavior in a much more complicated framework. Here, however, the focus is on the theoretical properties of the model.

[7] Specifying a predetermined lender limit on asset returns is plausible since lenders must make credit decisions based on expected outcomes in the context of a firm's current financial structure that, in turn, is based on past experiences.

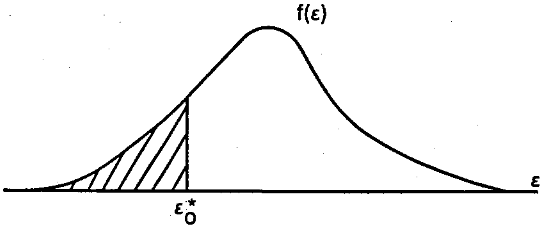

Figure 1. A probability distribution for ϵ with the credit exhaustion outcome equal to ϵ_o^*

where $\int_{-\infty}^{\epsilon_o^*} \epsilon f(\epsilon)\, d = \bar{\epsilon}$.

Since the expected value of ϵ is zero, then $\bar{\epsilon}$ which is a partial expectation of ϵ over the left tail values must be negative. Moreover, since $\bar{\epsilon}$ represents values of ϵ that are less than ϵ_o^*, but weighted by the same probability as ϵ_o^*, then $[\epsilon_o^* F(\epsilon_o) - \bar{\epsilon}] > 0$, and the expected liquidation cost in (15) is positive.

Now, a new expression is formulated for the expected value of ending equity, $E(E_1)$; it is equivalent to expression (2) less the expected liquidation cost, where $\rho/(1 - \rho)$ is replaced by $(1/\alpha)$.

$$(16) \quad E(E_1) = \left\{ r - \left[\frac{\epsilon_o^* F(\epsilon_o^*) - \bar{\epsilon}}{\alpha} \right] \right\} \cdot (D_o + E_o) - iD_o + E_o - W_d.$$

The variance of ending equity is found by separating the expression for E_1 into constants, that can be ignored, and functions of the random variable ϵ, both defined for outcomes above and below ϵ_o^*. If we let y be what remains after removing the constants, then y is

$$(17) \quad y = \begin{cases} \epsilon(D_o + E_o) & \epsilon > \epsilon_o^* \\ \left[\epsilon + \dfrac{(\epsilon - \epsilon_o^*)}{\alpha} \right](D_o + E_o) & \epsilon \le \epsilon_o^* \end{cases},$$

and its expected value is

$$(18) \quad E(y) = \left(\frac{1}{\alpha} \right)(D_o + E_o)[\bar{\epsilon} - \epsilon_o^* F(\epsilon_o^*)].$$

The variance of y, σ_y^2, is written as

$$(19a) \quad \sigma_y^2 = (D_o + E_o)^2 \sigma_{\epsilon_o^*}^2,$$

where $\sigma_{\epsilon_o^*}^2 = \sigma_r^2 + \sigma_l^2 + 2\sigma_{rl}$. The term σ_l^2 is variance of income associated with liquidation costs and σ_{rl} is the covariance between liquidation costs and the random return ϵ. The terms σ_l^2 and σ_{rl} can be written as

$$(19b) \quad \sigma_l^2 = \{\sigma_1^2 - 2\bar{\epsilon}\epsilon_o^* + \epsilon_o^{*2} F(\epsilon_o^*) - [\bar{\epsilon} - \epsilon_o^* F(\epsilon_o^*)]^2\}/\alpha^2, \text{ and}$$

$$(19c) \quad \sigma_{rl} = \sigma_1^2 - \bar{\epsilon}\epsilon_o^*,$$

where $\sigma_1^2 = \int_{-\infty}^{\epsilon_o^*} \epsilon^2 f(\epsilon)\, d\epsilon$. For later use, we define $\sigma_2^2 = \int_{\epsilon_o^*}^{\infty} \epsilon^2 f(\epsilon)\, d\epsilon$ where $\sigma_1^2 + \sigma_2^2 = \sigma_r^2$.

Given the variance expressions in (19a), the objective function with liquidation costs is

$$(20) \quad \text{Max } EU(E_1) = \left\{ r - \frac{[\epsilon_o^* F(\epsilon_o^*) - \bar{\epsilon}]}{\alpha} \right\} \cdot (D_o + E_o) - iD_o - E_o - W_d - \frac{\lambda}{2}(D_o + E_o)^2 \sigma_{\epsilon_o^*}^2$$
$$\text{s.t. } D_o \le \alpha E_o.$$

If the objective is to maximize the difference between expected wealth and variance where the trade-off between the two equals $\lambda/2$, then optimal debt is found by differentiating (20) with respect to D_o, equating the result to zero and solving for D_o:

$$(21) \quad D_o = \frac{\left\{ r - \dfrac{[\epsilon_o^* F(\epsilon_o^*) - \bar{\epsilon}]}{\alpha} - i \right\}}{\lambda \sigma_{\epsilon_o^*}^2} - E_o.$$

Compared to the optimal debt without the possibility of liquidation costs, solved for in equation (6), D_o in (21) will be less since liquidation costs are positive and $\sigma_{\epsilon_o^*}^2 > \sigma_r^2$.[8]

The variables r, i, λ, and $\sigma_{\epsilon_o^*}^2$ all have effects similar to those in the perfect market model of expression (6); however, the limit, α, on leverage enters (21) directly and influences op-

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[8] It can be shown that $\sigma_{\epsilon_o^*}^2 > \sigma_r^2$ by demonstrating that $\sigma_{rl} > 0$. To demonstrate $\sigma_{rl} > 0$, define ϵ_l as

$$\epsilon_l = \begin{cases} \dfrac{\epsilon - \epsilon_o^*}{\alpha} & \epsilon \le \epsilon_o^* \\ 0 & \epsilon > \epsilon_o^* \end{cases}.$$

Then σ_{rl} can be written as

$$\sigma_{rl} = \int_{-\infty}^{\epsilon_o^*} [\epsilon - \epsilon_o - \bar{\epsilon} + \epsilon_o^* F(\epsilon_o^*)] \epsilon f(\epsilon)\, d\epsilon - \int_{\epsilon_o^*}^{\infty} [\bar{\epsilon} - \epsilon_o^* F(\epsilon_o^*)] \epsilon f(\epsilon)\, d\epsilon$$
$$= \int_{-\infty}^{\epsilon_o^*} \epsilon(\epsilon - \epsilon_o^*) f(\epsilon)\, d\epsilon.$$

Consider two cases. If $\epsilon_o^* \le 0$, then ϵ and $(\epsilon - \epsilon_o^*)$ are negative, while the product is positive and $\sigma_{rl} > 0$. If $\epsilon_o^* > 0$, then σ_{rl} can be written as

$$\sigma_{rl} = \sigma_1^2 - \epsilon_o^* \bar{\epsilon} > 0 \qquad \text{since } \bar{\epsilon} < 0.$$

Therefore, $\sigma_{\epsilon_o^*}^2 = \sigma_r^2 + \sigma_l^2 + 2\sigma_{rl} > \sigma_r^2$.

timal debt even if borrowing has not reached the limit. Moreover, an increase in the leverage limit α increases the expected value of ending equity and decreases its variance. The net result is an increase in optimal debt ($dD_o/d\alpha > 0$) for investors with constant or decreasing absolute risk aversion. But if ϵ_o^* increases, the expected ending equity decreases while the variance of ending equity increases and D_o decreases

$$(22)\quad \frac{dD_o}{d\epsilon_o^*} = \frac{-\left[\dfrac{F(\epsilon_o)}{\alpha\lambda\sigma^2_{\epsilon_o^*}} + \{\cdot\}\lambda\dfrac{\partial\sigma^2_{\epsilon_o^*}}{\partial\epsilon_o^*}\right]}{(\lambda\sigma^2_{\epsilon_o^*})^2} < 0,$$

where $\{\cdot\}$ is the braced expression in the numerator in (21). The sign in (22) is unambiguously negative since $\partial\sigma^2_{\epsilon_o^*}/\partial\epsilon_o^* > 0$.[9] This response is consistent with intuitive expectations, since increasing ϵ_o^* increases the variance of liquidation costs. Moreover, introducing liquidation costs into the analysis shows the importance of unused borrowing capacity as a risk response.

Borrowing Behavior and Possible Bankruptcy

Now the condition that bankruptcy is unlikely is relaxed and we consider its effects on borrowing behavior. This essentially defines a new outcome $\hat{\epsilon}_o$ which is low enough to drive the firm's equity capital to zero. Moreover, to simplify the development, it is also assumed that liability is limited to the borrower's equity. As the analysis will show, the results yield the plausible condition that increased borrowing is a rational and appropriate response in the short run to forestall bankruptcy.

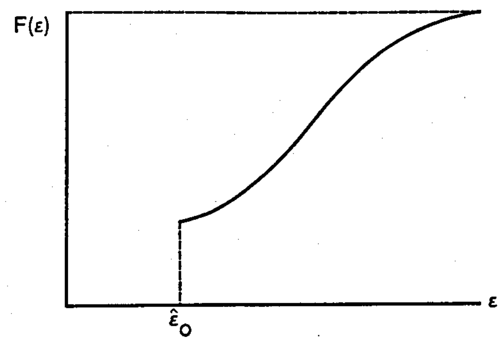

Figure 2. The cumulative probability density function for a firm facing bankruptcy at outcome ϵ_o

Allowing a "declaration of bankruptcy" significantly alters the firm's choice set. Figure 2 shows the cumulative probability distributions for E_1 with and without the possibility of bankruptcy. The distribution beginning with the dotted tail describes the no bankruptcy case. Since bankruptcy eliminates outcomes below $\hat{\epsilon}_o$, the distribution beginning at point $\hat{\epsilon}_o$ with the solid line reflects the cumulative probability density function with bankruptcy provisions. The truncated distribution which begins at point $\hat{\epsilon}_o$ has a lower variance and a higher expected value than the old distribution; thus it is less risky than the old one. As a result, the investor's optimal debt should be expected to increase in the short run and the responses of debt to changes in exogenous cash requirements should differ as well.

In formulating the bankruptcy model, the assumption is made that bankruptcy outcome $\hat{\epsilon}_o$ is determined exogeneously. In effect, the borrower (or lender) decides that rate of return $r - \hat{\epsilon}_o$ is so unacceptable, given the state of the firm, that recovery is impossible. To form the objective function, we must first find the expected level and variance of ending equity. The variance calculation is simplified by expressing variance as the sum of the variance of a random variable plus a constant. Thus, E_1 is expressed as:

$$(23)\quad E_1 = \begin{cases} (r+\epsilon)(D_o+E_o) - iD_o - W_d + E_o & \epsilon > \hat{\epsilon}_o \\ (r+\hat{\epsilon}_o)(D_o+E_o) - iD_o - W_d + E_o & \epsilon \le \hat{\epsilon}_o \end{cases}.$$

The constant in (23) is $r(D_o + E_o) - iD_o - W_d + E_o$. Let the remainder be random variable z:

[9] To show that $d\sigma^2_{\epsilon_o^*}/d\epsilon_o^* > 0$, which is required to sign (22), we write

$$\sigma^2_{\epsilon_o^*} = \sigma_r^2 + \frac{1}{\alpha^2}\int_{-\infty}^{\epsilon_o^*}[\epsilon - \epsilon_o^* - \bar{\epsilon} + \epsilon_o^*F(\epsilon_o^*)]^2 f(\epsilon)\,d\epsilon + \int_{\epsilon_o^*}^{\infty}[\epsilon_o^*F(\epsilon_o^*) - \bar{\epsilon}]^2 f(\epsilon)\,d\epsilon + \frac{2}{\alpha}\int_{-\infty}^{\epsilon_o^*}\epsilon(\epsilon - \epsilon_o^*)f(\epsilon)\,d\epsilon$$

$$\frac{d\sigma^2_{\epsilon_o^*}}{d\epsilon_o^*} = \left(\frac{2}{\alpha^2}\right)F(\epsilon_o^*)[1 - F(\epsilon_o^*)][\epsilon_o^*F(\epsilon_o^*) - \bar{\epsilon}] - \frac{2}{\alpha}\bar{\epsilon} > 0,$$

since $\epsilon_o^*F(\epsilon_o^*) - \bar{\epsilon} > 0$ and $\bar{\epsilon} < 0$.

The last term in the derivative of (22) needing to be signed is

$$\frac{1}{\lambda\sigma^2_{\epsilon_o^*}}\left\{-\epsilon_o^*\frac{\partial F(\epsilon_o^*)}{\partial\epsilon_o^*} - F(\epsilon_o^*) + \frac{\partial\bar{\epsilon}}{\partial\epsilon_o^*}\right\}/\alpha.$$

This, however, can be easily signed by recognizing that

$$\frac{\partial\bar{\epsilon}}{\partial\epsilon_o^*} = \frac{\epsilon_o^*\partial F(\epsilon_o^*)}{\partial\epsilon_o^*},$$

leaving only the term $-F(\epsilon_o^*)/\alpha\lambda\sigma^2_{\epsilon_o^*}$, which is unambiguously negative.

$$z = \begin{cases} \epsilon(D_o + E_o) & \epsilon > \hat{\epsilon}_o \\ \hat{\epsilon}_o(D_o + E_o) & \epsilon \leq \hat{\epsilon}_o \end{cases}$$

whose expected value is

$$E(z) = [\hat{\epsilon}_o F(\hat{\epsilon}_o) - \hat{\bar{\epsilon}}](D_o + E_o), \tag{24}$$

where $\int_{-\infty}^{\hat{\epsilon}_o} \epsilon(\epsilon)\, d\epsilon = \hat{\bar{\epsilon}}$.

Note the difference between $\hat{\epsilon}_o F(\hat{\epsilon}_o)$ and $\hat{\bar{\epsilon}}$. For $\hat{\epsilon}_o < 0$, $\hat{\epsilon}_o F(\hat{\epsilon}_o)$ is negative. Since $\hat{\bar{\epsilon}}$ is more negative than $\hat{\epsilon}_o F(\hat{\epsilon}_o)$, as indicated above, the term $\hat{\epsilon}_o F(\hat{\epsilon}_o) - \hat{\bar{\epsilon}}$ is positive.

The variance term for z is equal to

$$\begin{aligned} \sigma_z^2 &= (D_o + E_o)^2[\sigma_r^2 - \hat{\sigma}_1^2 + \hat{\epsilon}_o^2 F(\hat{\epsilon}_o) \\ &\quad - (\hat{\epsilon}_o F(\hat{\epsilon}_o) - \hat{\bar{\epsilon}})^2] \\ &= (D_o + E_o)^2 \sigma_{\hat{\epsilon}_o}^2, \end{aligned} \tag{25}$$

where $\sigma_{\hat{\epsilon}_o}^2 < \sigma_r^2$ is the transformation of σ_r^2 because of liability limits due to bankruptcy laws and where $\hat{\sigma}_1^2 = \int_{-\infty}^{\hat{\epsilon}_o} \epsilon^2 f(\epsilon)\, d\epsilon$.[10]

The objective function is now found as

$$\begin{aligned} \text{Max } EU(E_1) &= [r + \hat{\epsilon}_o F(\hat{\epsilon}_o) - \hat{\bar{\epsilon}}](D_o + E_o) \\ &\quad - iD_o - W_d + E_o \\ &\quad - \left(\frac{\lambda}{2}\right)(D_o + E_o)^2 \sigma_{\hat{\epsilon}_o}^2 \\ &\quad \text{s.t. } D_o < \alpha E_o. \end{aligned} \tag{26}$$

Optimal debt is derived as

$$D_o = \frac{r + \hat{\epsilon}_o F(\hat{\epsilon}_o) - \hat{\bar{\epsilon}} - i}{\lambda \sigma_{\hat{\epsilon}_o}^2} - E_o \qquad \text{s.t. } D_o < \alpha E_o. \tag{27}$$

The effects of truncating the distribution due to bankruptcy considerations are to increase the expected return and reduce the variance from investments in the risky assets. Expected returns increase because $[\hat{\epsilon}_o F(\hat{\epsilon}_o) - \hat{\bar{\epsilon}}] > 0$.

Another significant result is that the lender-imposed credit constraint does not appear in the equation for optimal debt. The credit constraint does place an upper bound on the firm's borrowings but has no other effect on the firm's response to possible bankruptcy.

The responses of optimal D_o to changes in r, λ, i, and E_o are similar to the responses in previous models. Moreover, an increase in W_d affects D_o through its effects on λ. As W_d increases, the firm's risk-free wealth decreases so that D_o decreases for investors with decreasing absolute risk aversion (that is, for $d\lambda/dW_d < 0$).

The most significant result for this analysis is the response of D_o to changes in $\hat{\epsilon}_o$. Optimal debt for the borrower increases as the probability of bankruptcy increases. In practice, borrowing often does increase as long as the lender is willing to supply the additional funds. Increasing $\hat{\epsilon}_o$ causes a greater truncation of the distribution $f(\epsilon)$ in the left tail, making the distribution more compact and "less risky." Consequently, the expected return on risky assets increases and variance decreases. In turn, these effects increase the likelihood of bankruptcy but also provide the incentives for greater borrowing and increased leverage, at least until the lender terminates financing, or until the events leading to the bankruptcy conditions have changed.

Concluding Comments

In this article, we have explored the proprietary firm's borrowing behavior under three sets of conditions. The first model derived optimal debt in a perfect financial market based on the combined effects of risk aversion, expected returns and variances of risky assets, and the cost of borrowing. The second model introduced imperfect asset markets represented by liquidation costs for risky assets, leverage and cash flows, and the use of credit reserves to avoid liquidation costs. The third model considered the effects of possible bankruptcy on borrowing behavior. An important theoretical result is the incentive to increase borrowing as a means of forestalling bankruptcy, which is consistent with the "go for broke" financial behavior often exhibited by highly stressed borrowers. The increased borrowing reflects the use of liquid credit reserves to see a firm through adversity by such practices as carrying over loans, deferring payments, refinancing high-debt loans, or otherwise utilizing those reserves during times of financial distress.

Clearly, the analysis in this article has not treated all of the issues associated with the management of credit reserves as a source of liquidity. Nor have we treated all the possible responses to risk that the firm might undertake to raise itself out of a difficult financial situation. But we have attempted to establish a the-

[10] It can be shown that $\sigma_{\hat{\epsilon}_o}^2 < \sigma_r^2$ by the procedures similar to those used to show that $\sigma_{\hat{\theta}}^2 > \sigma_r^2$.

oretical framework for explaining a proprietary firm's borrowing behavior under stress conditions so that extensions to more complex decision situations may occur. This should enhance the understanding of holding liquid reserves of credit and financial assets as a response to risk and provide a richer framework for empirical analyses of borrowing behavior.

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