

Borrowing Behavior of the Proprietary Firm: Do Some Risk-Averse Expected Utility Maximizers Plunge?

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When a proprietor's liability is limited, borrowing behavior for an expected utility maximizer may vary widely. Proprietors with little to lose may rationally choose very large debt levels while others may choose to finance with 100% equity. This article presents a theory to explain these widely observed variations in behavior.

Key words: debt, finance, risk.

An examination of a sample of farm financial records nearly always will reveal some farms that have very high levels of financial leverage. While this could be explained by risk-seeking behavior or a subliminal desire to commit financial suicide,¹ it may also be the result of rational risk-averse behavior. Although little is known about financial plunging, it has not been ignored in the literature.

Robison and Lev examined the effects of various forms of limited liability and found that they could explain a rational incentive to "go for broke" in these situations. Some of these points were cleverly illustrated with sports metaphors. Ahrendsen and Collender showed numerical simulations indicating that risk-averse decision makers could choose very large debt levels under certain conditions. Robison, Barry, and Burghardt examined the effects of financial stress and limitation of liability on optimal leverage choice. Using a mean-variance framework, they showed that given limited liability, the optimal debt choice increases with an increase in the probability of bank-

ruptcy.² Their result was independent of the size of the risk aversion parameter, the interest rate, and all other relevant parameters. This is a very strong conclusion. It suggests that any borrower maximizing a mean-variance utility function will wish to borrow more as the probability of bankruptcy increases. Since it is clear that borrowing more increases the probability of bankruptcy, if an increase in the probability of bankruptcy also causes the proprietor to borrow more, it would appear that this model suggests that all proprietors would always borrow as much as possible. The reason Robison, Barry, and Burghardt did not fall into this "Catch 22" was that they assumed that an increase in debt does not increase the probability of bankruptcy. They assumed that the level of income (loss) that would cause bankruptcy was exogenously determined.

The model presented here has tradeoffs. It makes the effect of debt choice on bankruptcy endogenous but uses a specific (although plausible) utility function and density function. The major results, however, compare favorably with empirical observation and have been confirmed numerically for a variety of combinations of distributions and utility functions. The model shows that some firms will choose to finance with all equity, some will borrow a moderate amount, and some will borrow to

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¹ Very large debt-asset ratios may also be noted after a farm has experienced a heavy loss of assets. Refusal to reduce the scale of the farm by selling assets and retiring debt, thereby reducing the debt-asset ratio, amounts to a choice of high leverage. Inability to sell assets could make it involuntary.

² We define bankruptcy as negative equity. We do not consider illiquidity problems.

the limit,³ i.e., plunge. The optimal choice depends on the farmer's aversion to risk, the cost of borrowing, and parameters of the density function for the rate of return on assets.

The Model

The probability density function (p.d.f.) of the rate of return on risky assets reflects the assumption of a subjective belief that the rate of return on farm assets has a worst possible outcome (a), a best possible outcome (b), and any rate of return between a and b is equally likely. We assume the farmer believes that the worst possible outcome will cause some loss of assets ($a < 0$) but does not expect a total loss of assets. Therefore, the worst possible rate of return on assets must be greater than -100% (or equivalently, $a > -1$). The farmer also believes that the best possible rate of return on risky assets exceeds the borrowing rate, k . Therefore, where $-1 < a < 0$ and $b > k > 0$, the subjective p.d.f. for the rate of return on assets (R) is:

$$g(R) = \begin{cases} \frac{1}{b - a}, & a \leq R \leq b \\ 0, & \text{otherwise.} \end{cases}$$

In order to make the leverage problem relevant, we assume that the farmer expects the rate of return on risky farm assets to exceed the borrowing rate,⁴ $E(R) = (a + b)/2 > k$.

Given initial equity of e_0 , terminal equity (e_1)⁵ is the sum of beginning equity and earnings on farm assets less the cost of debt ($D = \text{debt}$):

$$e_1 = e_0 + R(D + e_0) - kD \\ = e_0(1 + R) + (R - k)D.$$

Under these assumptions, terminal equity (wealth) also has a uniform distribution between the equity level resulting from the worst possible rate of return on assets, $e_a = e_0(1 + a) + (a - k)D$ and the equity level resulting from the best possible rate of return on assets, $e_b = e_0(1 + b) + (b - k)D$, with p.d.f.:

$$f(e_1) = \begin{cases} \frac{1}{e_b - e_a} = \frac{1}{(e_0 + D)(b - a)}, & e_a \leq e_1 \leq e_b \\ 0, & \text{otherwise.} \end{cases}$$

Since $a > -1$, bankruptcy cannot occur unless the farmer has some debt. If the worst possible rate of return on assets is realized, terminal equity will be $e_0(1 + a) > 0$ if there is no debt, but increases in debt move the lower limit of the p.d.f. to the left at the rate of $(a - k)$ per unit of D . Terminal equity equals zero if the worst possible rate of return on assets is realized when $D = e_0(1 + a)/(k - a) = D^*$ and becomes negative as D increases further. Therefore, bankruptcy is impossible if $D < D^*$, but the probability of bankruptcy increases for increases in D when $D \geq D^*$. The probability of bankruptcy is the area under the p.d.f. for negative terminal equity:

$$P(\text{bankruptcy}) = \begin{cases} \int_{e_a}^0 f(e_1) de_1 = -\frac{e_a}{(e_0 + D)(b - a)}, & D \geq D^* \\ 0, & D < D^*. \end{cases}$$

To illustrate the relationship between the debt level, the p.d.f. of terminal equity, and the probability of bankruptcy, let $a = -.1$, $b = .4$, $k = .1$, and $e_0 = \$100,000$. For these parameter values, the expected rate of return on assets is 15% and the debt level for which terminal equity is zero if the worst possible rate of return is realized is $D^* = \$450,000$. Figure 1 shows the p.d.f. of terminal equity for debt levels of \$0, \$450,000, and \$900,000. The shaded area in the bottom panel represents the probability of bankruptcy.

Since the expected rate of return on assets exceeds the cost of borrowing, the upper limit of the p.d.f. of terminal equity increases with debt. The expected terminal equity level is:

$$E(e_1) = e_0[1 + (a + b)/2] + [(a + b)/2 - k]D.$$

Therefore, expected terminal equity also increases with debt as long as the expected rate of return on assets exceeds the borrowing rate. Naturally the variance of terminal equity, $V(e_1) = [(e_0 + D)(b - a)]^2/12$, also increases with debt.

Even though the uniform p.d.f. is simplistic, it is a plausible description of subjective beliefs, and it captures all of the main features

³ We are examining the demand for credit. Lenders may not wish to supply all that is desired, thus creating a "credit limit."

⁴ Only a risk lover would wish to borrow if the cost of credit was higher than the expected rate of return on assets. We consider only risk-averse borrowers.

⁵ Throughout the article, e is equity, E is the expectation operator, \exp denotes exponentiation, D is debt, and d is the derivative operator.

of the leverage choice problem. Expected terminal equity increases with leverage capturing the well-known multiplier effect of financial leverage. In addition, risk increases with leverage, both in the sense of the variability of the outcome and the likelihood of bankruptcy. The remaining feature of the leverage choice problem that must be considered is the limitation of liability produced by the protection of bankruptcy.

The possibility of seeking the protection of bankruptcy may be modeled as a truncation of the p.d.f. of terminal equity at zero since a bankruptcy generally has the effect of forgiving any debt in excess of the proprietor's assets. The probability of bankruptcy then becomes a point mass in the distribution at a terminal equity of zero. This splits the expected utility integral into two parts: the probability of bankruptcy times the utility of the equity resulting from bankruptcy, and the conditional expected utility given bankruptcy does not occur:

$$\begin{aligned}
 E[u(e_1)] &= \int_{-\infty}^0 u(0)f(e_1)de_1 + \int_0^{\infty} u(e_1)f(e_1)de_1 \\
 &= u(0)P(\text{bankruptcy}) + \int_0^{\infty} u(e_1)f(e_1)de_1.
 \end{aligned}$$

For analytical convenience, we assume the utility function of terminal equity is negative exponential,

$$u(e_1) = 1 - \exp(-\gamma e_1).$$

For this and any other utility function where $u(0) = 0$,

$$E[u(e_1)] = \int_0^{\infty} u(e_1)f(e_1)de_1.$$

The integration for the uniform p.d.f., however, is complicated by the fact that when $D < e_0(1 + a)/(k - a) = D^*$, the lower endpoint of the p.d.f., e_a , is positive. For $D \geq D^*$ the expected utility of terminal equity is:

$$E[u(e_1)] = \int_0^{e_a} u(e_1)f(e_1)de_1,$$

and for $D \leq D^*$ it becomes:

$$E[u(e_1)] = \int_{e_a}^{e_b} u(e_1)f(e_1)de_1.$$

The two cases can be written more compactly as:

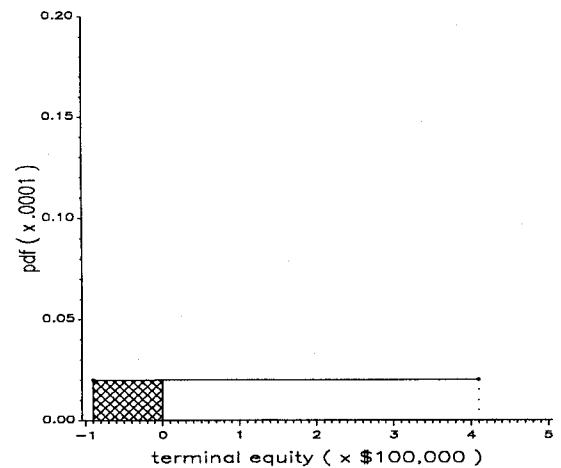
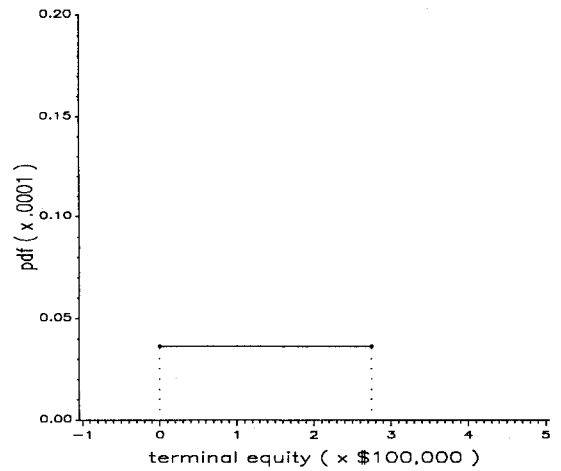
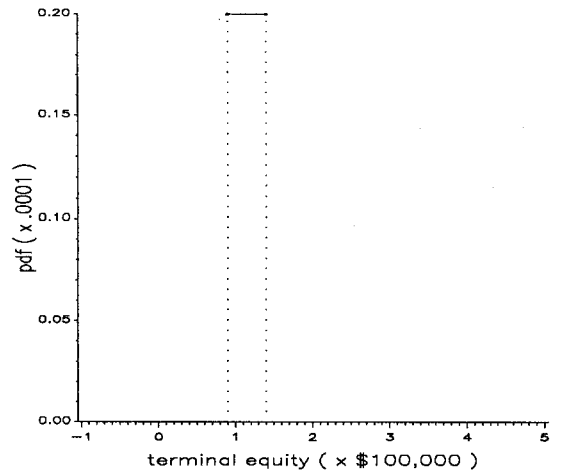


Figure 1. The p.d.f. of terminal equity for debt levels of \$0 (top panel), \$450,000 (middle panel), and \$900,000 (bottom panel)

$$E[u(e_1)] = \int_{\max\{0, e_a\}}^{e_b} [1 - \exp(-\gamma e_1)] / (e_b - e_a) de_1,$$

where $\max\{0, e_a\}$ indicates that the lower limit of integration is e_a when $e_a > 0$ and is zero when $e_a \leq 0$. This integration produces:

$$E[u(e_1)] = \frac{e_b - \max\{0, e_a\}}{e_b - e_a} + \frac{\exp(-\gamma e_b) - \exp(-\gamma \max\{0, e_a\})}{\gamma(e_b - e_a)}.$$

The expected utility function has two segments which can be defined in terms of e_a or D since $e_a \geq 0$ if and only if $D \leq D^*$. Thus,

$$E[u(e_1)] = \begin{cases} 1 + \frac{\exp(-\gamma e_b) - \exp(-\gamma e_a)}{\gamma(e_b - e_a)}, & 0 \leq D \leq D^* \\ \frac{\gamma e_b - [1 - \exp(-\gamma e_b)]}{\gamma(e_b - e_a)}, & D > D^* \end{cases}$$

The derivative is also in two segments but is a continuous function for $D \geq 0$:

$$\frac{dE[u(e_1)]}{dD} = \begin{cases} -\frac{(e_0 + D)y'_1(D) - y_1(D)}{\gamma(b - a)(e_0 + D)^2}, & 0 \leq D \leq D^* \\ \frac{1 - \gamma e_0(1 + k) - y_2(D)}{\gamma(b - a)(e_0 + D)^2}, & D > D^* \end{cases}$$

where $y_1(D) = \exp(-\gamma e_a) - \exp(-\gamma e_b)$, $y'_1(D)$ denotes the derivative of y_1 w.r.t. D , and $y_2(D) = [1 + \gamma(b - k)(e_0 + D)]\exp(-\gamma e_b)$.

Results of the Model

Analysis of the model produces conclusions that pass the test of casual empiricism but strongly conflict with existing models of optimal leverage choice. It is widely acknowledged that increasing financial leverage has the effect of increasing both expected return and risk for the proprietor. Most economic models of behavior under risk conclude that people will choose to take more risk if they are compensated for taking this risk by receiving enough additional expected return. Further, conventional models conclude that a risk-averse agent will not choose an arbitrarily large amount of risk when expected compensation is finite. This model, however, shows that there

are threshold levels of risk aversion, and the optimal behavior may change radically as the threshold is crossed.

Specific results depend on an analysis of the derivative of expected utility with respect to debt. Four propositions about the derivative are formally stated and proved in the appendix. These four propositions lead to the conclusion that the shape of expected utility as a function of debt depends on the size of the relative risk aversion parameter (γe_0) and whether the debt level exceeds D^* , the debt level at which a bankruptcy can occur if the worst possible outcome is realized for the rate of return on assets. It is important to note that even though the utility function of equity has constant absolute risk aversion (γ), behavior is determined by the coefficient of relative risk aversion (γe_0). Figure 2 shows examples of expected utility as a function of debt for the three relevant cases: $\gamma e_0 > 1/(k - a)$, $1/(1 + k) \leq \gamma e_0 < 1/(k - a)$, and $\gamma e_0 < 1/(1 + k)$. These three cases are examined in turn.

Optimal 100% Equity Financing

If the relative risk aversion parameter is "large" [i.e., $\gamma e_0 > 1/(k - a)$], proposition B in the appendix shows that expected utility decreases for increases in debt, $0 \leq D \leq D^*$. Since $-1 < a < 0$, however, $1/(k - a) > 1/(1 + k)$, and proposition A shows that expected utility also decreases with debt $D \geq D^*$ if $\gamma e_0 > 1/(k - a)$. Therefore, if $\gamma e_0 > 1/(k - a)$, the derivative of expected utility is always negative, and expected utility is maximized at zero debt.

The surprising thing is that this conclusion stands even if the cost of borrowing is free and the expected rate of return on risky assets is arbitrarily large.⁶ Therefore, the model suggests that no matter how high the potential return to leverage, there is a finite risk aversion parameter that will cause a rational choice maker to finance with 100% equity. Even though this result is theoretically novel, anyone who deals with farmers knows that non-borrowers exist. Indeed, the data show [U.S. Department of Agriculture (USDA)] that, as

⁶ For a given relative risk aversion parameter, whether a person falls on one side of the threshold or the other depends on a and k . In other words, a decline in k may induce a nonborrower to borrow.

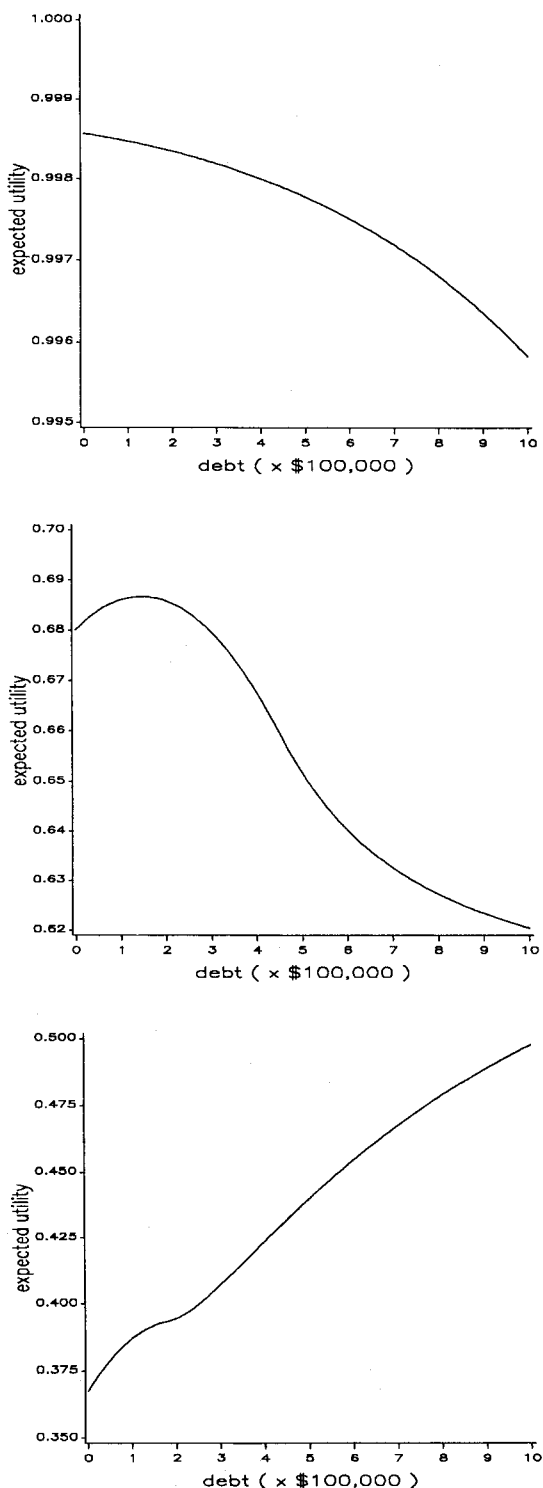


Figure 2. The expected utility of terminal equity as a function of debt level for a coefficient of relative risk aversion greater than $1/(k - a)$ (top panel), between $1/(k + 1)$ and $1/(k - a)$

of 1 January 1989, 47.9% of U.S. farms were financed with 100% equity.⁷

Moderate Debt Choice

If the coefficient of relative risk aversion is “moderate” [i.e., $1/(1 + k) \leq \gamma e_0 \leq 1/(k - a)$], the rational choice maker will use some financial leverage but never enough to put the farm in jeopardy of bankruptcy. If $\gamma e_0 < 1/(k - a)$ and b is sufficiently large, then proposition C in the appendix shows that expected utility increases with debt initially. However, proposition A shows that expected utility is negatively sloped for all debt levels that would cause bankruptcy should the worst possible outcome occur ($D \geq D^*$). Since expected utility is increasing at the origin and decreasing for $D \geq D^*$, the Mean Value Theorem guarantees a maximum between zero and D^* . Therefore, for a relative risk aversion parameter between $1/(1 + k)$ and $1/(k - a)$, some financial leverage will be chosen but never enough to ruin the firm should the worst possible outcome occur. Again, the threshold effect may be observed. Optimal debt will not exceed D^* even if the expected payoff of additional financial leverage is arbitrarily large. This range of risk aversion apparently characterizes a sizable minority of U.S. farmers who take advantage of credit to invest in profitable enterprises but limit borrowing to the point where they are confident that the firm will survive. As of 1 January 1989, 38.5% of U.S. farms had debt-asset ratios between 1% and 40% (USDA). Some would argue that most farms have a good chance of survival with a debt level in this range.

The Case of Plunging

The final case is when the relative risk aversion parameter is “small” [i.e., $\gamma e_0 < 1/(1 + k)$]

⁷ A reviewer correctly pointed out that this may overstate the case because USDA data show year-end debt. Some of these farms may have had an operating loan for part of the year. In addition, it is clear that none of these “nonborrowers” had an infinite rate of return on assets, so it must be emphasized that the empiricism is causal and is intended only to be illustrative.

←
(middle panel), and less than $1/(k + 1)$ (bottom panel)

Table 1. Risk Aversion Parameters Consistent with Optimal Choice of No Debt and Plunging Behavior for Various Utility and Probability Functions

Distribution of Rate of Return on Assets ^b	Utility of Equity ^a			
	Negative Exponential		Power	
	No debt	Plunging	No debt	Plunging
Uniform $a = -0.5$ $b = 3.0$	$\gamma = 0.01$	$\gamma = 1 \times 10^{-6}$	$\lambda = 0.01$	$\lambda = 0.20$
Normal $\mu = 0.15$ $\sigma = 0.15$	$\gamma = 1 \times 10^{-4}$	$\gamma = 1 \times 10^{-6}$	$\lambda = 0.10$	$\lambda = 0.50$
Gamma $\alpha = 10.0$ $\beta = 0.06$	$\gamma = 0.01^c$	$\gamma = 1 \times 10^{-6}$	$\lambda = 1 \times 10^{-6c}$	$\lambda = 0.40$
Beta $\alpha = 3.0$ $\beta = 3.0$	$\gamma = 0.10^c$	$\gamma = 1 \times 10^{-6}$	$\lambda = 1 \times 10^{-8c}$	$\lambda = 0.50$

^a The power utility is given by $u(e_t) = (1 + \lambda)e_t^\lambda$.

^b Probability density functions and parameter notation are as given in Hogg and Craig.

^c Expected utility appears to be a nonincreasing function in debt as the decision maker becomes increasingly risk averse for small levels of debt but decreases rapidly as debt becomes large. This may be due to the computer restriction on the number of significant digits which are retained. In any case, there would be no strong motivation to use debt.

and the expected rate of return on assets is "large." In this case, proposition D in the appendix shows that expected utility is an increasing function of wealth for the entire domain. This means that when farmers can choose to borrow any amount up to their credit limit at interest rate k , they will always borrow to the limit, even if the limit is arbitrarily large. Thus, financial plunging may be the rational choice of a risk-averse decision maker when investment opportunities are good and the relative risk aversion parameter is small. This behavior may also be observed in practice. As of 1 January 1989, the data show that 4.4% of U.S. farms had debt-asset ratios in excess of 70% (USDA).

The minimum level of b , the upper limit of the distribution of the rate of return on assets, that produces plunging behavior (b^*) increases with the agent's relative risk aversion parameter. As γe_0 approaches $1/(1+k)$, an increasingly large expected rate of return is required to produce plunging behavior. As the coefficient of absolute risk aversion (γ) or initial equity approaches zero, however, b^* approaches the value that is required to make the expected rate of return on risky assets greater than the borrowing rate. Therefore, given a coefficient of absolute risk aversion, the likelihood of demanding infinite debt increases as initial wealth becomes small.

Extensions of the Results

The uniform distribution and negative exponential utility function, which provide the basis for the results of the previous section, imply

several restrictive conditions about behavior and perception of the likelihood of various states of the world. The negative exponential utility implies constant absolute risk aversion. The uniform distribution does not allow for the existence of a mode or for skewness in the expected rate of return on assets.

We have shown, however, that the principal results of the previous section do not depend on either of these restrictive conditions by numerically evaluating the expected utility integral for a variety of combinations of utility functions and probability distributions. In all cases, reasonable sets of parameter values have been found that produce expected utility functions similar to those in figure 2. The specific relationship between parameter values and optimal debt levels which produces the three kinds of behavior depends on the functional form. No attempt was made to determine the threshold levels of risk aversion to move from case to case. For an optimal debt choice of zero and for plunging, table 1 summarizes the calculations for all combinations of the negative exponential and power utility functions with normal, gamma, uniform, and beta probability distributions. The gamma and beta distributions were transformed to allow for positive probabilities of negative rates of return on assets.⁸ For all calculations, $e_0 = \$100,000$ and $k = .10$. Debt levels ranged from \$0 to $50e_0 = \$5,000,000$. For purposes of numerical evaluation of expected utility integrals with infinite upper limits of integration, the mean plus six

⁸ The gamma distribution was defined for rates of return on assets greater than -0.4 and the beta distribution was defined for rates from -0.3 to 0.7 .

standard deviations was used as the upper limit for the normal distribution and the mean plus 10 standard deviations for the gamma distribution.

From table 1, for the negative exponential utility function, the optimal choice of no debt occurs for "large" values of the risk aversion parameter while plunging occurs for "small" values. The opposite is true for the power utility function. Risk aversion can be measured by $-u''(e_1)/u'(e_1)$ or by $-e_1 u''(e_1)/u'(e_1)$. Using either measure, increases in the risk aversion parameter under negative exponential utility correspond to increased risk aversion. For power utility, the opposite is true, i.e., increases in the risk aversion parameter correspond to decreased risk aversion. Thus, the numerical results in table 1 are consistent with accepted theory.

For both the negative exponential and power utility functions, continuity of the derivative of expected utility with respect to debt as a function of the risk aversion parameter, combined with the Mean Value Theorem, implies that there exist risk aversion parameter values between those for the no debt and plunging situations for which the choice of a finite amount of debt would be optimal. Thus, the major results of the previous section also appear to be valid for constant relative risk aversion and for unimodal symmetric and skewed distributions.

Conclusions

The model presented here supports the hypothesis of Robison, Barry, and Burghardt that financial stress may cause radical changes in borrowing behavior. Given a coefficient of constant absolute risk aversion (γ), the coefficient of relative risk aversion (γe_0) depends entirely on initial equity. For negative exponential utility and a uniform probability distribution, equity losses due to an unfavorable realization of rate of return on assets may cause the relative risk aversion parameter to cross the threshold level from $\gamma e_0 > 1/(1+k)$ to $\gamma e_0 < 1/(1+k)$ and cause optimal borrowing behavior to change from moderate levels to plunging. For example, suppose $a = -.1$, $b = .4$, $k = .1$, and $\gamma = .00001$. For these parameters, the expected rate of return on assets is 15% and at a borrowing interest rate of 10%, there is a positive return to leverage. Further

suppose $e_0 = \$100,000$. This means that $\gamma e_0 = 1$ which is greater than $1/(1+k) \approx .91$ and a debt level between zero and $D^* = \$450,000$ is chosen, producing an optimal asset level of between \$100,000 and \$550,000. If the worst possible outcome is realized, terminal equity will be reduced by at least \$10,000. This would cause γe_0 for the next period to be less than .9. This could, in turn, cause borrowing to increase to the limit in the next period, if the expected rate of return on assets is sufficiently large, since $.9 \leq 1/(1+k)$. Repeated losses guarantee that plunging will occur eventually.

Similar examples may be used to show the effect of government policies that affect agricultural interest rates and the risks and expected returns of farming. Although we did not derive formal comparative static results for this model, numerical examples with reasonable parameter values may be used to illustrate the Gabriel and Baker risk-balancing hypothesis, i.e., government policies that reduce the risk of farming will cause the farmer to use more leverage. In addition, the assertion by Featherstone et al. that subsidized credit and income support policies cause increased debt use by farmers and increased likelihood of failure may also be supported by this model. Finally, the Collins caveat about the effect of income support policies on optimal debt is also supported.

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Appendix: Proofs of Propositions A-D

The following lemma is used to prove propositions A-D. Its proof is straightforward and has been omitted.

- Lemma:* (i) $E[u(e_1)]$ is a continuous function of D for $D \geq 0$.
 (ii) $0 \leq E[u(e_1)] \leq 1$ for $D \geq 0$.
 (iii) $\lim_{D \rightarrow \infty} E[u(e_1)] = E_\infty = (b - k)/(b - a) < 1$.
 (iv) $\frac{dE[u(e_1)]}{dD}$ exists and is continuous for $D \geq 0$.

Proposition A: If $\gamma e_0 \geq 1/(1 + k)$, then $\frac{dE[u(e_1)]}{dD} < 0$ for $D \geq D^*$.

Proof:

If $D \geq D^*$, the derivative of expected utility w.r.t. debt is negative if $1 - \gamma e_0(1 + k) - y_2(D) < 0$. If $\gamma e_0 \geq 1/(1 + k)$, then $1 - \gamma e_0(1 + k) \leq 0$. Since $y_2(D)$ is strictly positive, expected utility is a strictly decreasing function of D for $D \geq D^*$ if $\gamma e_0 \geq 1/(1 + k)$. ■

Proposition B: If $\gamma e_0 > 1/(k - a)$, then $\frac{dE[u(e_1)]}{dD} < 0$ for $0 \leq D \leq D^*$.

Proof:

For $0 \leq D \leq D^*$, the derivative is negative if and only if $(e_0 + D)y'_1(D) - y_1(D) > 0$, or equivalently, if and only if $[1 + \gamma(e_0 + D)(b - k)]\exp(-\gamma e_b) - [1 - \gamma(e_0 + D)(k$

$- a)]\exp(-\gamma e_a) > 0$. A sufficient condition for the last inequality is that $[1 + \gamma(e_0 + D)(b - k)]\exp(-\gamma e_b) - [1 - \gamma e_0(k - a)]\exp(-\gamma e_a) > 0$ since $D \geq 0$. Since $\gamma e_0 > 1/(k - a)$, $1 - \gamma e_0(k - a) < 0$ and the left-hand side of the preceding inequality is the sum of a positive and a non-negative term. Therefore, expected utility is also a decreasing function of D , $0 \leq D \leq D^*$, if $\gamma e_0 > 1/(k - a)$. ■

Proposition C: If $\gamma e_0 < 1/(k - a)$ and b is "large," then $\frac{dE[u(e_1)]}{dD} > 0$ for $D = 0$.

Proof:

For $D = 0$, the derivative is positive if and only if $e_0 y'_1(0) - y_1(0) < 0$. But $e_0 y'_1(0) - y_1(0) = [1 + \gamma e_0(b - k)]\exp[-\gamma e_0(1 + b)] - [1 + \gamma e_0(a - k)]\exp[-\gamma e_0(1 + a)]$. Since $\gamma e_0 < 1/(k - a)$, the second term is positive. By application of l'Hôpital's Rule, $\lim_{b \rightarrow \infty} [1 + \gamma e_0(b - k)]\exp[-\gamma e_0(1 + b)] = 0$. Hence, for b sufficiently large, the first term will be smaller than the second and the difference will be negative. Therefore, the derivative will be positive and expected utility will be increasing at $D = 0$. ■

Proposition D: If $\gamma e_0 < 1/(1 + k)$ and b is "large," then $\frac{dE[u(w)]}{dD} > 0$ for $D \geq 0$.

Proof: The domain is considered in two sections, $0 \leq D \leq D^*$ and $D > D^*$. First, for $0 \leq D \leq D^*$, the derivative is positive if and only if $(e_0 + D)y'_1(D) - y_1(D) < 0$. However,

$$\begin{aligned} &(e_0 + D)y'_1(D) - y_1(D) \\ &= y_2(D) - [1 - \gamma(e_0 + D)(k - a)]\exp(-\gamma e_a) \\ &\leq y_2(D) - [1 - \gamma(e_0 + D^*)(k - a)]\exp[-\gamma e_0(1 + a)] \\ &= y_2(D) - [1 - \gamma e_0(1 + k)]\exp[-\gamma e_0(1 + a)]. \end{aligned}$$

Since $\gamma e_0 < 1/(1 + k)$, $[1 - \gamma e_0(1 + k)]\exp[-\gamma e_0(1 + k)]$ is always positive. By application of l'Hôpital's rule, $\lim_{b \rightarrow \infty} [y_2(D)] = 0$. Hence, $y_2(D)$ may be made arbitrarily small by the selection of a large b and the derivative will be positive.

For $D \geq D^*$, the derivative is positive if $1 - \gamma e_0(1 + k) - y_2(D)$ is positive. Again, by the application of l'Hôpital's Rule, $\lim_{b \rightarrow \infty} [y_2(D)] = 0$ so that the derivative will be positive if b is sufficiently large. ■